Vacuum instability in QED with an asymmetric x step: New example of an exactly solvable case

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We present a new exactly solvable case in strong-field QED with a one-dimensional step potential (x-step). The corresponding x-step is given by an analytic asymmetric with respect to the axis x reflection function. The step can be considered as a certain analytic "deformation" of the symmetric Sauter field. Moreover, it can be treated as a new regularization of the Klein step field. We study the vacuum instability caused by this x-step in the framework of a nonperturbative approach to strong-field QED. Exact solutions of the Dirac equation used in the corresponding nonperturbative calculations are represented in the form of stationary plane waves with special left and right asymptotics and identified as components of initial and final wave packets of particles. We show that in spite of the fact that the symmetry with respect to positive and negative bands of energies is broken, the distribution of created pairs and other physical quantities can be expressed via elementary functions. We consider the processes of transmission and reflection in the ranges of the stable vacuum and study physical quantities specifying the vacuum instability. We find the differential mean numbers of electron-positron pairs created from the vacuum, the components of current density and energy-momentum tensor of the created electrons and positrons leaving the area of the strong field under consideration. Besides, we study the particular case of the particle creation due to a weakly inhomogeneous electric field and obtain explicitly the total number, the current density and energy-momentum tensor of created particles. Unlike the symmetric case of the Sauter field the asymmetric form of the field under consideration causes the energy density and longitudinal pressure of created electrons to be not equal to the energy density and longitudinal pressure of created positrons.

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I. INTRODUCTION

The Schwinger effect, that is, creation of charged particles from the vacuum by strong external electriclike and gravitational fields (the vacuum instability) has been attracting attention already for a long time; see, e.g., monographs [1–4]. This is a nonperturbative effect of quantum field theory (QFT), which has not yet received a convincing experimental confirmation. However, recent progress in laser physics allows one to hope that the vacuum instability will be experimentally observed in the near future even in laboratory conditions. Recently, this, as well as the real possibility of observing an analogue of the Schwinger effect in condensed matter physics (in the graphene, topological insulators, 3D Dirac and Weyl semimetals, antiferromagnets, etc.) has increased theoretical interest in the problem and led to the development of various analytical and numeric approaches, see recent reviews [5–8]. From general quantum theory point of view, the most clear formulation of the problem of particle production from the vacuum by external fields is formulated for time-dependent external electric fields that are switched on and off at infinitely remote times $t \to \pm\infty$, respectively. The idealized problem statement described above was considered for uniform time-dependent external

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electric fields. Such kind of external fields are called the t-electric potential steps (t-steps). A complete nonperturbative with respect to the external background formulation of strong-field QED with such external fields was developed in Refs. [4,9]; it is based on the existence of exact solutions of the Dirac equation with time dependent external field (more exactly, complete sets of exact solutions). However, there exist many physically interesting situations in high-energy physics, astrophysics, and condensed matter where external backgrounds formally are time-independent. In our works; see Refs. [10–12], a nonperturbative approach in QED with the so-called x-potential steps, or simply x-steps, was developed. The x-steps represent time-independent inhomogeneous electriclike external fields of a constant direction. The latter approach is based on the existence of special exact solutions of the Dirac or Klein-Gordon equations with corresponding x-steps. In cases when such solutions can be found and all the calculations can be done analytically, we refer to these cases as to exactly solvable ones. Sauter potential and the Klein step, considered in the pioneer works [13–15], belong to the class of exactly solvable cases. Initially they were considered in the framework of the relativistic quantum mechanics, which gave rise to a rather long-lasting discussion about the Klein paradox (a detailed historical review can be found in Refs. [16,17]). In the work [10] it was pointed out that this paradox and other misunderstandings in considering quantum effects in fields of strong x-steps can be consistently solved as many particle effects of the QFT (QED) with unstable vacuum. Recently, a number of new exactly solvable cases were presented and studied in detail in the framework of general approach [10,11]. Particularly, interesting are the cases of a constant electric field between two capacitor plates (L-constant electric field) [18], a field of a piecewise form of continuous exponential functions [19], and a piecewise and a continuous configuration of an inverse-square step [20]. Exactly solvable cases are interesting not only in themselves, but also due to the fact that they allow you to develop and test new approximate and numerical methods for calculating quantum effects in strong-field QFT. One can find a number of application of these exactly solvable cases in high-energy physics and condensed matter physics; see, e.g., [21–24].

In this article, we present a new exactly solvable case for strong-field QED with x-step. For the generality, the field is considered in $d = D + 1$—dimensional Minkowski space-time, parametrized by the coordinates $X = (t, \mathbf{r})$, $\mathbf{r} = (x^1 = x, \mathbf{r}_\perp)$, $\mathbf{r}_\perp = x^2, \ldots, x^D$. The electric field is constant and has only one component along the x-axis, $\mathbf{E}(X) = (E^1(x) = E(x), 0, \ldots, 0)$. The field is given by a step potential $A_0(x)$, so that $E(x) = -A_0'(x)$.

We note that among the above exactly solvable cases only the Sauter electric field is given by an analytic function,

$$A_0^{(\mathrm{Sauter})}(x; L, E_S) = -LE_S \tanh(x/L),$$
$$E_{(\mathrm{Sauter})}(x; L, E_S) = E_S \cosh^{-2}(x/L),$$
$$E_S > 0, \qquad L > 0. \tag{1}$$

This field reaches its maximum value at $x = 0$ and is symmetric with respect to the origin. Unlike the above mentioned cases given by piecewise smooth x-steps, physical quantities calculated for the analytic Sauter field are presented by elementary functions, which makes this case especially convenient for physical interpretations. Here we present a new example of exactly solvable case in which the external field is given by the following analytic function:

$$A_0(x) = \frac{\sigma E_0}{\sqrt{1 + \exp(\frac{x}{\sigma})}}, \quad E_0 > 0, \quad \sigma > 0,$$
$$E(x) = \frac{E_0}{8}\sqrt{1 + \exp\left(\frac{x}{\sigma}\right)} \cosh^{-2}\left(\frac{x}{2\sigma}\right). \tag{2}$$

The potential energy of an electron (with the charge $q = -e$, $e > 0$) is $U(x) = -eA_0(x)$. It tends to be different in the general case constants values $U(-\infty)$ and $U(+\infty)$ as $x \to -\infty$ and $x \to +\infty$, respectively,

$$U(-\infty) \equiv U_\mathrm{L} = -eE_0\sigma, \quad U(+\infty) \equiv U_\mathrm{R} = 0. \tag{3}$$

The magnitude δU of the potential step is given by the difference

$$\delta U = U_\mathrm{R} - U_\mathrm{L} = eE_0\sigma \tag{4}$$

Note δU is equal to the increment of kinetic energy, if the particle retains the direction of motion and moves in the direction of acceleration, and if toward the opposite, then this increment changes sign. Depending on the magnitude δU, the step is called noncritical or critical one, see Ref. [10],

$$\begin{array}{ll} \delta U < \delta U_c = 2m, & \text{noncritical step} \\ \delta U > \delta U_c, & \text{critical step} \end{array}. \tag{5}$$

If the magnitude δU is large enough, the particle production from the vacuum could be essential.

The electric field (2) differs from the Sauter field (1) at $L = 2\sigma$ by the presence of an additional term in Eq. (2),

$$E(x) = \sqrt{1 + \exp\left(\frac{x}{\sigma}\right)} E_{(\mathrm{Sauter})}\left(x; 2\sigma, \frac{E_0}{8}\right). \tag{6}$$

There is no symmetry of the field $E(x)$ with respect to the point $x_\mathrm{M} = \sigma \ln 2$, in which the field has the maximum value $E_{\max} = E_0/(3\sqrt{3})$ (see Fig. 1). While the Sauter field

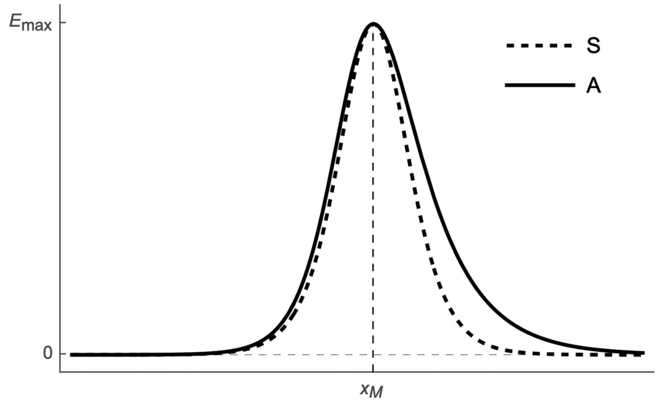

FIG. 1. The solid line labeled with “A” represents the asymmetric field E(x) given by Eq. (2). The dashed line labeled with “S” represent the Sauter field. The maximum values of the fields labeled with $E_{\max}$ are combined in the figure and the coordinate of this maximum is labeled as x_{M}.

exhibits this symmetry with respect to the point of its maximum. Moreover, they generally increase in a similar way, but the Sauter field decreases faster. One can say that field (2) for a given value of the parameter σ is a certain “deformation” of the Sauter field, which turns on at $x > 0$ and turns off at $x \to +\infty$. Finally, we note that distributions of created pairs by the Sauter field are symmetric with respect to the energy p_0. The latter symmetry is not inherent in realistic asymmetric fields. We demonstrate that in the case of the asymmetric analytic field (2), with broken symmetry with respect to p_0, the distributions of created pairs and other physical quantities can be still expressed in terms of elementary functions.

The article is organized as follows: In Sec. II, we construct exact solutions of the Dirac equation with a new example of x-step given by a analytic asymmetric function. These solutions are presented in the form of stationary plane waves with special left and right asymptotics and identified as components of initial and final wave packets of particles and antiparticles. We find coefficients of mutual decompositions of the initial and final solutions. In Sec. III, we consider the processes of transmission and reflection in ranges of the stable vacuum. In Sec. IV, we calculate physical quantities specifying the vacuum instability. We find differential mean numbers of electron-positron pairs created from the vacuum, as well as components of current density and energy-momentum tensor of the created electrons and positrons leaving the area of the strong external field. In Sec. V, we consider a particular case of the particle creation due to a weakly inhomogeneous electric field and obtain explicitly the total number, current density and energy-momentum tensor of the created particles. A new regularization of the Klein step is considered in Sec. VI, which is used then in calculating the corresponding vacuum instability. Section VII contains some concluding remarks. In Appendix IX A, we describe briefly basic elements of a nonperturbative approach to QED with x-steps. In Appendix IX B, we show that the density of created pairs and the probability of the vacuum to remain a vacuum obtained from exact formulas for the slowly varying field in the leading-term approximation are in agreement with results following in the framework of a locally constant field approximation (LCFA). In Appendix IX C, we list some useful properties of hypergeometric functions. We use the system of units, where $c = \hbar = 1$.

II. SOLUTIONS OF DIRAC EQUATION WITH ASYMMETRIC POTENTIAL x-STEP

A. General solution

Let us consider the Dirac equation with a x-step in the Hamiltonian form:

$$i\partial_0\psi(X) = \hat{H}\psi(X),\quad \hat{H} = \gamma^0(-i\gamma^j\partial_j + m) + U(x),\quad j = 1,\ldots,D. \tag{7}$$

The Dirac spinor $\psi(X)$ has $2^{[d/2]}$ components, $[d/2]$ denotes the integer part of $d/2$, and γ^μ are $2^{[d/2]} \times 2^{[d/2]}$ Dirac matrices in d dimensions, $[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}$, and $U(x) = -eA_0(x)$, where $A_0(x)$ is given by Eq. (2).

There exist solutions of Eq. (7) in the form of stationary plane waves propagating along the space-time directions t and $\mathbf{r}_\perp$. In this case the Dirac spinors labeled by quantum numbers n have the form:

$$\psi_n(X) = \exp(-ip_0t + i\mathbf{p}_\perp\mathbf{r}_\perp)\psi_n(x),\qquad n = (p_0, \mathbf{p}_\perp, \sigma),$$
$$\psi_n(x) = \{\gamma^0[p_0 - U(x)] + i\gamma^1\partial_x - \boldsymbol{\gamma}_\perp\mathbf{p}_\perp + m\}\varphi_n(x) \times v_{\chi,\sigma}, \tag{8}$$

where the spinors $\psi_n(x)$ and the scalar functions $\varphi_n(x)$ depend exclusively on x while $v_{\chi,\sigma}$ is a set of constant orthonormalized spinors, satisfying the following conditions:

$$\gamma^0\gamma^1 v_{\chi,\sigma} = \chi v_{\chi,\sigma},\quad v^\dagger_{\chi,\sigma}v_{\chi',\sigma'} = \delta_{\chi,\chi'}\delta_{\sigma,\sigma'},$$
$$\chi = \pm 1,\quad \sigma = (\sigma_s = \pm 1, s = 1,2,\ldots,[d/2]-1). \tag{9}$$

Quantum numbers s and χ describe the spin polarization (if $d \leq 3$ there are no spin degrees of freedom that are described by the quantum numbers s). Solutions $\psi_n^{(\chi)}(X)$ and $\psi_n^{(\chi')}(X)$ given by Eqs. (8) that differ only by values of χ are linearly dependent if $d > 3$. Therefore, it suffices to work with solutions corresponding to one of possible values of χ, and sometimes we omit the subscript χ, supposing that the spin quantum number χ is fixed in a certain way. Due to the same reason, there exists, in fact, only $J_{(d)} = 2^{[d/2]-1}$ different spin states (labeled by quantum numbers σ) for a given set of $p_0, \mathbf{p}_\perp$. Substituting

Eq. (8) into Eq. (7), one finds that scalar functions $\varphi_n(x)$ obey the following second-order ordinary differential equation:

$$\left\{\frac{d^2}{dx^2} + [p_0 - U(x)]^2 - \pi_\perp^2 + i\chi\partial_x U(x)\right\}\varphi_n(x) = 0,$$
$$\pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}. \tag{10}$$

Solutions of a similar type of equation

$$\left\{\frac{d^2}{dt^2} + [p_x - \tilde{U}(t)]^2 + \pi_\perp^2 - i\tilde{\chi}\partial_t \tilde{U}(t)\right\}\tilde{\varphi}_n(t) = 0, \tag{11}$$

where $\tilde{U}(t) = -A_x(t)$, were recently found in Ref. [25] for the case of a nonperturbative treatment of the vacuum instability due to a time-dependent electric field $E(t)$, given by the potential:

$$A_x(t) = \frac{\sigma E_0}{\sqrt{1 + \exp(t/\sigma)}}.$$

It is quite obvious that Eq. (10) can be obtained from Eq. (11) by a substitution

$$t \to x, \qquad p_x \to p_0, \qquad \pi_\perp^2 \to -\pi_\perp^2, \qquad \tilde{\chi} \to -\chi.$$

Therefore, solutions of Eq. (10) can be obtained from solutions of Eq. (11) using the same substitution. As the result, the general solution of Eq. (10) can be represented as a linear combination of the functions $\varphi_{n,i}(x)$,

$$\varphi_{n,i}(x) = (1+z)^{i\alpha_1}(1-z)^{i\alpha_2}\hat{M}_n w_{n,i}\left(\frac{z+1}{2}\right)\Big|_{z=z(x)},$$
$$\hat{M}_n = \frac{bz - (\alpha_1 - \alpha_2 + 2\chi e E_0 \sigma^2)}{(ia-1)b}\frac{d}{dz} + 1,$$
$$z(x) = \sqrt{1 + \exp\left(\frac{x}{\sigma}\right)},$$
$$a = \alpha_1 + \alpha_2 - \sqrt{2(\alpha_1^2 + \alpha_2^2) - (2eE_0\sigma^2)^2},$$
$$b = \alpha_1 + \alpha_2 + \sqrt{2(\alpha_1^2 + \alpha_2^2) - (2eE_0\sigma^2)^2},$$
$$\alpha_1 = \sigma\sqrt{\pi_\perp^2 - (p_0 - eE_0\sigma)^2},$$
$$\alpha_2 = \sigma\sqrt{\pi_\perp^2 - (p_0 + eE_0\sigma)^2}. \tag{12}$$

In the above combination, we use two pairs of linearly independent solutions $w_{n,i}(\xi)$ with additional indices $i = 1, \ldots, 4$. The first pair reads:

$$w_{n,1}(\xi) = \xi^{-(ia-1)}F\left(ia-1, i(a-2\alpha_1); 2i\alpha_2; 1-\xi^{-1}\right),$$
$$w_{n,2}(\xi) = \xi^{i(a-2\alpha_1)-1}(1-\xi)^{1-2i\alpha_2}$$
$$\times F\left(i(2\alpha_1 - a), -ia; 2(1-i\alpha_2); 1-\xi^{-1}\right), \tag{13}$$

where $F(\alpha, \beta; \gamma; \xi)$ are Gaussian hypergeometric functions [26]. Solutions (13) are well-defined in a vicinity of the singular point $\xi = 1$, which corresponds to $x \to -\infty$. The second pair reads:

$$w_{n,3}(\xi) = (-\xi)^{-(ia-1)}$$
$$\times F\left(ia-1, i(a-2\alpha_1)-1; i(a-b); \xi^{-1}\right),$$
$$w_{n,4}(\xi) = (-\xi)^{-ib}$$
$$\times F\left(ib, i(b-2\alpha_1)+1; i(b-a)+2; \xi^{-1}\right). \tag{14}$$

Solutions (14) are well-defined in a vicinity of the singular point $\xi = \infty$, which corresponds to $x \to +\infty$. Using functions (13) and (14) one can construct four complete sets $\varphi_{n,i}(x)$, $i = 1, 2, 3, 4$, of solutions of Eq. (10).

B. Solutions with special left and right asymptotics

Unlike the explicitly time-dependent solutions of Eq. (11) the solutions given by Eqs. (8) and (12) are stationary plane waves. In the treatment of the vacuum instability in the x-case under consideration, they describe qualitatively different cases depending on the ranges of quantum numbers. By this reason their role in calculations and interpretation of physics of the vacuum instability in the x-case is quite different compared to similar in appearance the above cited time-dependent solutions. Further, we carry out a nonperturbative study of the vacuum instability using solutions (12) within the framework of the approach formulated in Refs. [10,11].

Due to local properties of equation (10) at $x \to \mp\infty$ (where the electric field is zero), the scalar functions $\varphi_n(x)$ have definite left "L" and right "R" asymptotics:

$$_\zeta\varphi_n(x) = {}_\zeta\mathcal{N}e^{i\zeta|p^{\mathrm{L}}|x} \quad \text{as } x \to -\infty,$$
$$^\zeta\varphi_n(x) = {}^\zeta\mathcal{N}e^{i\zeta|p^{\mathrm{R}}|x} \quad \text{as } x \to +\infty. \tag{15}$$

Here $_\zeta\mathcal{N}$ and $^\zeta\mathcal{N}$ are some normalization constants, and $p^{\mathrm{L/R}} = \zeta|p^{\mathrm{L/R}}|$, $\zeta = \pm = \mathrm{sgn}(p^{\mathrm{L}}) = \mathrm{sgn}(p^{\mathrm{R}})$, denotes real asymptotic momenta along the x-axis,

$$|p^{\mathrm{L/R}}| = \sqrt{\pi_0(\mathrm{L/R})^2 - \pi_\perp^2},$$
$$\pi_0(\mathrm{L/R}) = p_0 - U_{\mathrm{L/R}}, \tag{16}$$

where $U_{\mathrm{L/R}}$ is given by Eq. (3). Then for the corresponding Dirac spinors the following relations hold:

$$\hat{p}_x{}_\zeta\psi_n(X) = \zeta|p^{\mathrm{L}}|{}_\zeta\psi_n(X) \quad \text{as } x \to -\infty,$$
$$\hat{p}_x{}^\zeta\psi_n(X) = \zeta|p^{\mathrm{R}}|{}^\zeta\psi_n(X) \quad \text{as } x \to +\infty. \tag{17}$$

Note that the electric field under consideration can be neglected at sufficiently big $|x|$, let say, in the macroscopic regions S_{L} on the left of $x = x_{\mathrm{L}} < 0$ and S_{R} on the right of $x = x_{\mathrm{R}} > 0$. To this end one can choose finite $x_{\mathrm{L/R}}$ such that

$$1 - \frac{U(x_{\mathrm{R}}) - U(x_{\mathrm{L}})}{\delta U} \ll 1. \tag{18}$$

We assume that the asymptotic behavior (17) is sufficiently good approximation for all the functions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ if $x < x_{\mathrm{L}}$ and $x > x_{\mathrm{R}}$, which means that particles are free in the regions S_{L} and S_{R}.

Nontrivial sets of Dirac spinors $\{{}_\zeta\psi_n(X)\}$ and $\{{}^\zeta\psi_n(X)\}$, that are key elements of the above-mentioned approach, do exist for the quantum numbers n satisfying the conditions:

$$\pi_0(\mathrm{L/R})^2 > \pi_\perp^2 \Leftrightarrow \begin{cases} \pi_0(\mathrm{L/R}) > \pi_\perp \\ \pi_0(\mathrm{L/R}) < -\pi_\perp \end{cases}. \tag{19}$$

Note that $\pi_0(\mathrm{L}) > \pi_0(\mathrm{R})$. As a result of these inequalities, the complete set of the quantum numbers n can be divided in some ranges Ω_k, where the index k labels the ranges and the corresponding quantum numbers, $n_k \in \Omega_k$. For critical steps, $\delta U > \delta U_c$, there are five ranges of the quantum numbers, Ω_k, $k = 1, \ldots, 5$, where the solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ have similar forms and properties for given perpendicular momenta $p_\perp$ and any spin polarizations σ. The ranges Ω_1 and Ω_5 are characterized by energies bounded from the below, $\Omega_1 = \{n\colon p_0 \geq U_{\mathrm{R}} + \pi_\perp\}$, and by energies bounded from the above $\Omega_5 = \{n\colon p_0 \leq U_{\mathrm{L}} - \pi_\perp\}$. The ranges Ω_2 and Ω_4 are characterized by bounded energies, namely $\Omega_2 = \{n\colon U_{\mathrm{R}} - \pi_\perp < p_0 < U_{\mathrm{R}} + \pi_\perp\}$ and $\Omega_4 = \{n\colon U_{\mathrm{L}} - \pi_\perp < p_0 < U_{\mathrm{L}} + \pi_\perp\}$ if $\delta U \geq 2\pi_\perp$ or $\Omega_2 = \{n\colon U_{\mathrm{L}} + \pi_\perp < p_0 < U_{\mathrm{R}} + \pi_\perp\}$ and $\Omega_4 = \{n\colon U_{\mathrm{L}} - \pi_\perp < p_0 < U_{\mathrm{R}} - \pi_\perp\}$ if $\delta U < 2\pi_\perp$. In the ranges Ω_2 and Ω_4 we deal with standing waves $\psi_n(X)$ completed by linear superpositions of solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ with corresponding longitudinal fluxes that are equal in magnitude for a given n. The range Ω_3 is nontrivial only for critical steps and perpendicular momenta $\mathbf{p}_\perp$ restricted by the inequality $2\pi_\perp \leq \delta U$. This range is characterized by bounded energies, $\Omega_3 = \{n\colon U_{\mathrm{L}} + \pi_\perp \leq p_0 \leq U_{\mathrm{R}} - \pi_\perp\}$. For noncritical steps $\delta U < \delta U_c$, the range Ω_3 is absent.

Stationary plane waves, ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$, are subjected to the following orthonormality conditions on the $x = \mathrm{const}$ hyperplane:

$$\left({}_\zeta\psi_n, {}_{\zeta'}\psi_{n'}\right)_x = \zeta\eta_{\mathrm{L}}\delta_{\zeta,\zeta'}\delta_{n,n'},$$
$$\left({}^\zeta\psi_n, {}^{\zeta'}\psi_{n'}\right)_x = \zeta\eta_{\mathrm{R}}\delta_{\zeta,\zeta'}\delta_{n,n'};$$
$$(\psi, \psi')_x = \int \psi^\dagger(X)\gamma^0\gamma^1\psi'(X)dtd\mathbf{r}_\perp, \tag{20}$$

where $\eta_{\mathrm{L/R}} = \mathrm{sgn}\pi_0(\mathrm{L/R})$ is sign of $\pi_0(\mathrm{L/R})$. We consider our theory in a large space-time box that has a spatial volume $V_\perp = \prod_{j=2}^{D} K_j$ and the time dimension T, where all K_j and T are macroscopically large. It is supposed that all the solutions $\psi(X)$ are periodic under transitions from one box to another. The integration over the transverse coordinates is fulfilled from $-K_j/2$ to $+K_j/2$, and over the time t from $-T/2$ to $+T/2$. Under these suppositions, one can verify, integrating by parts, that the inner product (20) does not depend on x. We assume that the macroscopic time T is the system surveillance time.

Solutions (12) with the asymptotic conditions (15) have the following form:

$$\begin{aligned}
{}_+\varphi_n(x) &= {}_+\mathcal{N}U_{n,1}(1+z)^{i\alpha_1}(1-z)^{i\alpha_2} \\
&\quad \times \hat{M}_n w_{n,1}\left(\frac{z+1}{2}\right), \\
{}_-\varphi_n(x) &= {}_-\mathcal{N}U_{n,2}(1+z)^{i\alpha_1}(1-z)^{i\alpha_2} \\
&\quad \times \hat{M}_n w_{n,2}\left(\frac{z+1}{2}\right), \\
{}^+\varphi_n(x) &= {}^+\mathcal{N}U_{n,3}(1+z)^{i\alpha_1}(1-z)^{i\alpha_2} \\
&\quad \times \hat{M}_n w_{n,3}\left(\frac{z+1}{2}\right), \\
{}^-\varphi_n(x) &= {}^-\mathcal{N}U_{n,4}(1+z)^{i\alpha_1}(1-z)^{i\alpha_2} \\
&\quad \times \hat{M}_n w_{n,4}\left(\frac{z+1}{2}\right),
\end{aligned} \tag{21}$$

where the constants $U_{n,i}$, $i = 1, 2, 3, 4$, and the normalization constants ${}_\zeta\mathcal{N}$ and ${}^\zeta\mathcal{N}$ are

$$\begin{aligned}
U_{n,1} &= \frac{2^{2-i(\alpha_1-\alpha_2)}\alpha_2}{a - (\alpha_1 - \alpha_2 - 2\chi eE_0\sigma^2)} e^{\pi\alpha_2}, \\
U_{n,2} &= \frac{2^{1-i(\alpha_1+3\alpha_2)}b(ia-1)}{(2i\alpha_2 - 1)[b - (\alpha_1 - \alpha_2 + 2\chi eE_0\sigma^2)]} e^{-\pi\alpha_2}, \\
U_{n,3} &= \frac{2^{1-ia}b(b-a)}{a(\alpha_1 - \alpha_2 + 2\chi eE_0\sigma^2) + b(\alpha_1 - \alpha_2 - 2\chi eE_0\sigma^2)} \\
&\quad \times e^{\pi(\alpha_2 - a)}, \\
U_{n,4} &= \frac{2^{-ib}(1-ia)}{1 + i(b-a)} e^{\pi(\alpha_2 - b)};
\end{aligned} \tag{22}$$

$$
{}_\zeta\mathcal{N} = {}_\zeta CY, \quad {}^\zeta\mathcal{N} = {}^\zeta CY, \quad Y = (V_\perp T)^{-1/2},
$$
$$
{}_\zeta C = \left[2\left|p^{\mathrm{L}}\right|\left|\pi_0(\mathrm{L}) - \chi p^{\mathrm{L}}\right|\right]^{-1/2},
$$
$$
{}^\zeta C = \left[2\left|p^{\mathrm{R}}\right|\left|\pi_0(\mathrm{R}) - \chi p^{\mathrm{R}}\right|\right]^{-1/2}. \tag{23}
$$

Stationary plane waves in the ranges Ω_k, $k = 1, 2, 4, 5$ are usually used in the potential scattering theory. In each of these ranges $\mathrm{sign}\eta_{\mathrm{L}}$ and $\mathrm{sign}\eta_{\mathrm{R}}$ coincide ($\eta_{\mathrm{L/R}} = 1$ for particles and $\eta_{\mathrm{L/R}} = -1$ for antiparticles). We stress that definitions of particle and antiparticle in the framework of one-particle quantum theory and QFT are in agreement. In the ranges Ω_1 and Ω_2 there exist only states of particles whereas in the ranges Ω_4 and Ω_5 there exist only states of antiparticles. In these ranges particles and antiparticles are subjected to the scattering and the reflection only. In fact, $\psi_n(X)$ for $m \in \Omega_2$ are wave functions that describe an unbounded motion of particles (electrons) in $x \to -\infty$ direction while $\psi_n(X)$ for $n \in \Omega_4$ are wave functions that describe an unbounded motion of antiparticles (positrons) toward $x = +\infty$. Such one-particle interpretation does not exist in the range Ω_3, where $\mathrm{sign}\eta_{\mathrm{L}}$ is opposite to $\mathrm{sign}\eta_{\mathrm{R}}$, here one must take a many-particle QFT consideration into account, in particular, the vacuum instability, see Appendix IX A for details. Note that the range Ω_3 is often referred to as the Klein zone and the pair creation from the vacuum occurs in this range, whereas the vacuum is stable in the ranges Ω_k, $k = 1, 2, 4, 5$.

It was demonstrated in Ref. [10] (see Secs. V and VII and Appendices C1 and C2) by using one-particle mean currents and the energy fluxes that the plane waves ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ are unambiguously identified as components of initial and final wave packets of particles and antiparticles,

$$
\begin{aligned}
&\text{in} - \text{solutions:} \\
&{}_+\psi_{n_1}, {}^-\psi_{n_1}; \quad {}_-\psi_{n_5}, {}^+\psi_{n_5}; \quad {}_-\psi_{n_3}, {}^-\psi_{n_3}; \\
&\text{out} - \text{solutions:} \\
&{}_-\psi_{n_1}, {}^+\psi_{n_1}; \quad {}_+\psi_{n_5}, {}^-\psi_{n_5}; \quad {}_+\psi_{n_3}, {}^+\psi_{n_3},
\end{aligned} \tag{24}
$$

where $n_k \in \Omega_k$. In the ranges Ω_2 and Ω_4 we deal with a total reflection. The complete sets of in- and out-solutions must include solutions $\psi_{n_2}(X)$ and $\psi_{n_4}(X)$.

Since each pair of solutions ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ with quantum numbers $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$ are complete, there exist mutual decompositions:

$$
\eta_{\mathrm{L}}{}^\zeta\psi_n(X) = {}_+\psi_n(X) g({}_+|^\zeta) - {}_-\psi_n(X) g({}_-|^\zeta),
$$
$$
\eta_{\mathrm{R}}{}_\zeta\psi_n(X) = {}^+\psi_n(X) g({}^+|_\zeta) - {}^-\psi_n(X) g({}^-|_\zeta), \tag{25}
$$

where the decomposition coefficients g are

$$
g({}^{\zeta'}|_\zeta)^* = g({}_\zeta|^{\zeta'}) = ({}_\zeta\psi_n, {}^{\zeta'}\psi_n)_x, \quad n \in \Omega_1 \cup \Omega_3 \cup \Omega_5. \tag{26}
$$

These coefficients satisfy the following unitary relations:

$$
|g({}_-|^+)|^2 = |g({}_+|^-)|^2, |g({}_+|^+)|^2 = |g({}_-|^-)|^2,
$$
$$
\frac{g({}_+|^-)}{g({}_-|^-)} = \frac{g({}^+|_-)}{g({}^+|_+)},
$$
$$
|g({}_+|^-)|^2 - |g({}_+|^+)|^2 = -\eta_{\mathrm{L}}\eta_{\mathrm{R}}. \tag{27}
$$

One can see that all the coefficients g can be expressed via only one of them, e.g., via $g({}^-|_+)$. Using the Kummer relations (C9) for the hypergeometric equation; see Ref. [26], this coefficient can be found to be

$$
g({}^-|_+) = i\eta_{\mathrm{R}} \frac{{}_+\mathcal{N}}{{}^-\mathcal{N}} \times \frac{2^{i(b-\alpha_1+\alpha_2)+1}\Gamma[i(a-b)]\Gamma(1+2i\alpha_2)}{(a-\alpha_1+\alpha_2+2\chi eE_0\sigma^2)\Gamma(ia)\Gamma[-i(b-2\alpha_2)]}, \tag{28}
$$

where $\Gamma(x)$ is the gamma function. Then

$$
|g({}_+|^-)|^{-2} = \frac{\sinh(2\pi|p^{\mathrm{L}}|\sigma)\sinh(4\pi|p^{\mathrm{R}}|\sigma)}{|\beta_+\beta_-|},
$$
$$
\beta_\pm = \sinh\left\{\pi\sigma\left[\sqrt{2\delta U^2 + 2|p^{\mathrm{R}}|^2 - |p^{\mathrm{L}}|^2} \pm (2|p^{\mathrm{R}}| - |p^{\mathrm{L}}|)\right]\right\}, \tag{29}
$$

where δU and $|p^{\mathrm{L/R}}|$ are given by Eqs. (4) and (16).

Relation (29) holds true for the quantum numbers $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$. However, interpretations of this relation in the range Ω_3 and in the ranges Ω_1 and Ω_5 are quite different. Note that there exists a useful relation between absolute values of the momenta p^{R} and p^{L},

$$
|p^{\mathrm{L}}| = \sqrt{|p^{\mathrm{R}}|^2 + 2\eta_{\mathrm{R}}\delta U\sqrt{|p^{\mathrm{R}}|^2 + \pi_\perp^2} + \delta U^2},
$$
$$
|p^{\mathrm{R}}| = \sqrt{|p^{\mathrm{L}}|^2 - 2\eta_{\mathrm{L}}\delta U\sqrt{|p^{\mathrm{L}}|^2 + \pi_\perp^2} + \delta U^2}, \tag{30}
$$

As follows from Eqs. (29) and (30), if either $|p^{\mathrm{R}}|$ or $|p^{\mathrm{L}}|$ tends to zero, one of the following limits takes place:

$$
|g({}_-|^+)|^{-2} \sim |p^{\mathrm{R}}| \to 0,
$$
$$
|g({}_-|^+)|^{-2} \sim |p^{\mathrm{L}}| \to 0, \quad \forall\ \pi_\perp^2 \neq 0. \tag{31}
$$

These properties are essential for the justification of in- and out-particle interpretation in the general construction given in Ref. [10].

III. PROCESSES IN STABLE VACUUM RANGES

In the ranges Ω_2 and Ω_4 we deal with a total reflection. In the adjacent ranges, Ω_1 and Ω_5, a particle can be reflected and transmitted. For example, in the range Ω_1, the total $\tilde{R}$ and the relative R amplitudes of an electron reflection, and the total $\tilde{T}$ and the relative T amplitudes of an electron transmission can be presented via the following matrix elements:

$$\begin{aligned} R_{+,n} &= \tilde{R}_{+,n} c_v^{-1}, \\ \tilde{R}_{+,n} &= \langle 0,\text{out}|{}_{-}a_n(\text{out})_{+}a_n^{\dagger}(\text{in})|0,\text{in}\rangle, \\ T_{+,n} &= \tilde{T}_{+,n} c_v^{-1}, \\ \tilde{T}_{+,n} &= \langle 0,\text{out}|{}^{+}a_n(\text{out})_{+}a_n^{\dagger}(\text{in})|0,\text{in}\rangle, \\ R_{-,n} &= \tilde{R}_{-,n} c_v^{-1}, \\ \tilde{R}_{-,n} &= \langle 0,\text{out}|{}^{+}a_n(\text{out})^{-}a_n^{\dagger}(\text{in})|0,\text{in}\rangle, \\ T_{-,n} &= \tilde{T}_{-,n} c_v^{-1}, \\ \tilde{T}_{-,n} &= \langle 0,\text{out}|{}_{-}a_n(\text{out})^{-}a_n^{\dagger}(\text{in})|0,\text{in}\rangle, \end{aligned} \quad (32)$$

where initial creation $a^{\dagger}(\text{in})$ and final annihilation $a(\text{out})$ operators, initial $|0,\text{in}\rangle$ and final $|0,\text{out}\rangle$ vacua, and the vacuum-to-vacuum transition amplitude $c_v = \langle 0,\text{out}|0,\text{in}\rangle$ are defined in Appendix IX A. Note that the partial vacua are stable in Ω_k, $k = 1, 2, 4, 5$, and the vacuum instability with $|c_v| \neq 1$ is due to the partial vacuum-to-vacuum transition amplitude formed in Ω_3. Using a linear canonical transformation between in and out-operators in Eq. (32) (see Eq. (4.33) in Ref. [10]) one find that the relative reflection $|R_{\zeta,n}|^2$ and transmission $|T_{\zeta,n}|^2$ probabilities are

$$\begin{aligned} |T_{\zeta,n}|^2 &= 1 - |R_{\zeta,n}|^2, \\ |R_{\zeta,n}|^2 &= [1 + |g({}_{+}|{}^{-})|^{-2}]^{-1}, \quad \zeta = \pm, \end{aligned} \quad (33)$$

where $|g({}_{+}|{}^{-})|^{-2}$ is given by Eq. (29). Similar expressions can be derived for positron amplitudes in the range Ω_5. In particular, relation (33) holds true literally for the positrons in the range Ω_5.

In the ranges Ω_1 and Ω_5 we meet a realization of rules of the potential scattering theory in the framework of QFT and can see that relative probabilities of the reflection and the transmission coincide with mean currents of reflected particles $J_R = |R_m|^2$ and transmitted particles $J_T = |T_m|^2$. The correct result $J_R + J_T = 1$ follows from the unitary relation (27).

Limits (31) imply the following properties of the coefficients $|g({}_{+}|{}^{-})|$: $|g({}_{+}|{}^{-})|^{-2} \to 0$ in the range Ω_1 if n tends to the boundary with the range Ω_2 ($|p^{\text{R}}| \to 0$); $|g({}_{+}|{}^{-})|^{-2} \to 0$ in the range Ω_5 if n tends to the boundary with the range Ω_4 ($|p^{\text{L}}| \to 0$). Thus, in the latter cases the relative reflection probabilities $|R_{\zeta,n}|^2$ tend to the unity; i.e., they are continuous functions of the quantum numbers n on the boundaries. In addition, it follows from Eq. (28) that $|g({}_{+}|{}^{-})|^{-2} \to 0$ and, therefore, $|R_{\zeta,n}|^2 \to 0$ as $p_0 \to \pm\infty$, as it is expected.

IV. PHYSICAL QUANTITIES SPECIFYING THE VACUUM INSTABILITY

The vacuum instability is due to contributions formed in the range Ω_3. In this range the important characteristic of all the processes are differential mean numbers N_n^{cr} of electron-positron pairs created from the vacuum. The differential mean numbers of electrons and positrons created from the vacuum are equal and related to the mean numbers N_n^{cr} of created pairs,

$$\begin{aligned} N_n^a(\text{out}) &= \langle 0,\text{in}|\hat{N}_n^a(\text{out})|0,\text{in}\rangle = |g({}_{-}|{}^{+})|^{-2}, \\ N_n^b(\text{out}) &= \langle 0,\text{in}|\hat{N}_n^b(\text{out})|0,\text{in}\rangle = |g({}_{+}|{}^{-})|^{-2}, \\ N_n^{\text{cr}} &= N_n^b(\text{out}) = N_n^a(\text{out}), \qquad n \in \Omega_3. \end{aligned} \quad (34)$$

Here $\hat{N}_n^a(\text{out})$ and $N_n^b(\text{out})$ are operators of the number of the final electrons and positrons, given by Eq. (A10) (see Appendix IX A for details), and the quantity $|g({}_{+}|{}^{-})|^{-2}$ is given by Eq. (28). The probabilities of a particle reflection (transmission is impossible) and a pair creation and annihilation in the Klein zone can be expressed via differential mean numbers of created pairs N_n^{cr}; see Eq. (7.22) in Ref. [10]).

Unlike the case of uniform time-dependent electric fields, in the constant inhomogeneous electric fields, there is a critical surface in space of particle momenta, which separates the Klein zone Ω_3 from the adjacent ranges Ω_2 and Ω_4. In the ranges Ω_2 and Ω_4, the work of the electric field is sufficient to ensure the total reflection for electrons and positrons, respectively, but is not sufficient to produce pairs from the vacuum. Accordingly, it is expected that for any nonpathological field configuration, the pair creation vanishes close to this critical surface. Limits (31) imply that $N_n^{\text{cr}} \to 0$ if n tends to the boundary with either the range Ω_2 ($|p^{\text{R}}| \to 0$) or the range Ω_4 ($|p^{\text{L}}| \to 0$),

$$N_n^{\text{cr}} \sim |p^{\text{R}}| \to 0, \quad N_n^{\text{cr}} \sim |p^{\text{L}}| \to 0, \quad \forall\ \pi_{\perp} \neq 0. \quad (35)$$

Standard integral characteristics of the vacuum instability are sums over the range Ω_3 (see Appendix IX A for details), are the total number N^{cr} of pairs created from the vacuum, and the vacuum-to-vacuum transition probability P_v,

$$N^{\text{cr}} = \sum_{n \in \Omega_3} N_n^{\text{cr}}, \quad N_n^{\text{cr}} = |g({}_{+}|{}^{-})|^{-2}, \quad (36)$$

$$P_v = |c_v|^2 = \exp\left(\sum_{n \in \Omega_3} \ln\left(1 - N_n^{\text{cr}}\right)\right). \quad (37)$$

The summations over Ω_3 can be converted into integrals in the standard way,

$$(V_\perp T)^{-1}\sum_{p_0,\mathbf{p}_\perp\in\Omega_3}\leftrightarrow(2\pi)^{1-d}\int dp_0 d\mathbf{p}_\perp,$$

in which $V_\perp$, T are macroscopically large. It follows from Eq. (37) that $\ln P_v\approx -N^{\rm cr}$ if all $N_n^{\rm cr}\ll 1$.

Under approximation (18) the electric field under consideration can be neglected in the macroscopic regions $S_{\rm L}$ (at $x<x_{\rm L}$) and $S_{\rm R}$ (at $x>x_{\rm R}$), that is, particles are free in these regions. We note that near all the work δU is performed by the electric field situated in a region $S_{\rm int}$ between two planes $x=x_{\rm L}$ and $x=x_{\rm R}$. Assuming that the areas $S_{\rm L}$ and $S_{\rm R}$ are much wider than the area $S_{\rm int}$, this part of the field affects only coefficients g entering into the mutual decompositions of the solutions given by Eq. (25). Created electrons and positrons leaving the area $S_{\rm int}$ enter the areas $S_{\rm L}$ and $S_{\rm R}$, respectively, and continue to move with constant velocities. The positron of a pair created with quantum number n moves in the x direction with a velocity $v^{\rm R}=|p^{\rm R}/\pi_0({\rm R})|$ while the electron belonging to the same pair moves in the opposite direction with a velocity $-v^{\rm L}$, $v^{\rm L}=|p^{\rm L}/\pi_0({\rm L})|$. It is shown that the microscopical parameter T can be interpreted as the time of the observation of the created particles leaving the area $S_{\rm int}$; see Ref. [11].

Following the way used in Ref. [11], we can calculate the current densities and the energy flux densities of electrons and positrons, after the instant when these fluxes become completely separated and already have left the region $S_{\rm int}$. The motion of the positrons forms the flux density

$$\langle j_x\rangle_n = N_n^{\rm cr}(TV_\perp)^{-1} \tag{38}$$

in the area $S_{\rm R}$, while the electron motion forms the flux density $-(j_x)_n$ in the area $S_{\rm L}$. Here it is taken into account that differential mean numbers of created electrons and positrons with a given n are equal. The total flux densities of the positrons and electrons are

$$\langle j_x\rangle = \sum_{n\in\Omega_3}\langle j_x\rangle_n = N^{\rm cr}(TV_\perp)^{-1} \tag{39}$$

and $-\langle j_x\rangle$, respectively. The current density of both created electrons and positrons is $J_x^{\rm cr}=e\langle j_x\rangle$. It is conserved in the x-direction.

During the time T, the created positrons carry the charge $e\langle j_x\rangle_n T$ over the unit area $V_\perp$ of the surface $x=x_{\rm R}$. This charge is evenly distributed over the cylindrical volume of the length $v^{\rm R}T$. Thus, the charge density of the positrons created with a given n is $ej_n^0({\rm R})$, where $j_n^0({\rm R})=\langle j_x\rangle_n/v^{\rm R}$ is the number density of the positrons. During the time T, the created electrons carry the charge $e\langle j_x\rangle_n T$ over the unit area $V_\perp$ of the surface $x=x_{\rm L}$. Taking into account that this charge is evenly distributed over the cylindrical volume of the length $v^{\rm L}T$, we can see that the charge density of the electrons created with a given n is $-ej_n^0({\rm L})$, where $j_n^0({\rm L})=\langle j_x\rangle_n/v^{\rm L}$ is the number density of the electrons. The total charge density of the created particles reads:

$$J_0^{\rm cr}(x) = e\begin{cases}-\sum_{n\in\Omega_3} j_n^0({\rm L}), & x\in S_{\rm L}\\ \sum_{n\in\Omega_3} j_n^0({\rm R}), & x\in S_{\rm R}\end{cases}. \tag{40}$$

Due to a relation between the velocities $v^{\rm L}$ and $v^{\rm R}$, the total number densities of the created electrons and positrons are the same,

$$\sum_{n\in\Omega_3} j_n^0({\rm L}) = \sum_{n\in\Omega_3} j_n^0({\rm R}).$$

The created electrons and positrons are spatially separated and carry a charge that tends to weaken the external electric field.

In the same manner, one can derive some representation for the nonzero components of energy-momentum tensor of the created particles:

$$T_{\rm cr}^{00}(x) = \begin{cases}\sum_{n\in\Omega_3} j_n^0({\rm L})\pi_0({\rm L}), & x\in S_{\rm L}\\ \sum_{n\in\Omega_3} j_n^0({\rm R})|\pi_0({\rm R})|, & x\in S_{\rm R}\end{cases},$$

$$T_{\rm cr}^{11}(x) = \begin{cases}\sum_{n\in\Omega_3}\langle j_x\rangle_n|p^{\rm L}|, & x\in S_{\rm L}\\ \sum_{n\in\Omega_3}\langle j_x\rangle_n|p^{\rm R}|, & x\in S_{\rm R}\end{cases},$$

$$T_{\rm cr}^{kk}(x) = \begin{cases}\sum_{n\in\Omega_3}\langle j_x\rangle_n(p_k)^2/|p^{\rm L}|, & x\in S_{\rm L}\\ \sum_{n\in\Omega_3}\langle j_x\rangle_n(p_k)^2/|p^{\rm R}|, & x\in S_{\rm R}\end{cases},\qquad k\neq 1,$$

$$T_{\rm cr}^{10}(x) = \begin{cases}-\sum_{n\in\Omega_3}\langle j_x\rangle_n\pi_0({\rm L}), & x\in S_{\rm L}\\ \sum_{n\in\Omega_3}\langle j_x\rangle_n|\pi_0({\rm R})|, & x\in S_{\rm R}\end{cases}. \tag{41}$$

Here $T_{\rm cr}^{00}(x)$ and $T_{\rm cr}^{kk}(x)$, $k=1,\ldots,D$, (there is no summation over k) are energy density and components of the pressure of the particles created in the areas $S_{\rm L}$ and $S_{\rm R}$ respectively, whereas $T_{\rm cr}^{10}(x)v_s$, for $x\in S_{\rm L}$ or $x\in S_{\rm R}$, is the energy flux density of the created particles through the surfaces $x=x_{\rm L}$ or $x=x_{\rm R}$ respectively. In a strong field, or in a field with the sufficiently large potential step δU, the energy density and the pressure along the direction of the axis x are near equal, $T_{\rm cr}^{00}(x)\approx T_{\rm cr}^{11}(x)$, in the areas $S_{\rm L}$ and $S_{\rm R}$ respectively.

V. PARTICLE CREATION DUE TO A WEAKLY INHOMOGENEOUS ELECTRIC FIELD

A. Intensity of the particle creation over the Klein zone

The above study of the vacuum instability caused by the asymmetric x-step can be useful for a consideration of the particle creation by a weakly inhomogeneous electric field between two capacitor plates separated by a sufficiently large length. Indeed, if the parameter σ is taken to be sufficiently large,

$$\sigma \gg (eE_0)^{-1/2} \max\{1, m^2/eE_0\}, \tag{42}$$

the step can be considered as a regularization (like the Sauter potential with appropriate parameters) of a weakly inhomogeneous constant electric field between the plates. For example, for such big σ we can consider the behavior of mean numbers of electron-positron pairs created over the Klein zone Ω_3. For this purpose, consider arguments of the functions $\beta_\pm$ in the denominator of expression (29). Absolute values of $|p^{\mathrm{R}}|$ and $|p^{\mathrm{L}}|$ are related by Eq. (30). One can see from Eq. (30) that $d|p^{\mathrm{L}}|/d|p^{\mathrm{R}}| < 0$, and at any given $\mathbf{p}_\perp$ these quantities are restricted inside the range Ω_3 as

$$0 \le |p^{\mathrm{R/L}}| \le p^{\max}, \qquad p^{\max} = \sqrt{\delta U(\delta U - 2\pi_\perp)}. \tag{43}$$

These relations for big σ are

$$\pi\sigma\left(\sqrt{\delta U(\delta U - \pi_\perp)} - p^{\max}\right) \gg 1,$$
$$\pi\sigma\left(\sqrt{\delta U(\delta U + 2\pi_\perp)} - p^{\max}\right) \gg 1, \tag{44}$$

and imply that

$$\pi\sigma\left[\sqrt{2\delta U^2 + 2|p^{\mathrm{R}}|^2 - |p^{\mathrm{L}}|^2} \pm (2|p^{\mathrm{R}}| - |p^{\mathrm{L}}|)\right] \gg 1. \tag{45}$$

We get from (29) that

$$N_n^{\mathrm{cr}} = |g({}_+|{}^-)|^{-2} \approx \frac{4\sinh(2\pi|p^{\mathrm{L}}|\sigma)\sinh(4\pi|p^{\mathrm{R}}|\sigma)}{\exp\left[2\pi\sigma\sqrt{2\delta U^2 + 2|p^{\mathrm{R}}|^2 - |p^{\mathrm{L}}|^2}\right]}. \tag{46}$$

It follows from Eq. (46) that the quantities N_n^{cr} are exponentially small,

$$N_n^{\mathrm{cr}} \approx 2(4\pi\sigma)^2 |p^{\mathrm{L}} p^{\mathrm{R}}| \exp(-2\sqrt{2}\pi\delta U\sigma), \tag{47}$$

if the range Ω_3 is small enough

$$eE\sigma - 2\pi_\perp \to 0 \Rightarrow \pi\sigma p^{\max} \ll 1 \Rightarrow |p^{\mathrm{R/L}}| \ll 1. \tag{48}$$

Then we can consider the opposite case of big ranges Ω_3 when

$$\pi\sigma p^{\max} \gg 1 \tag{49}$$

and the quantities N_n^{cr} are not small. Such ranges do exist if

$$eE\sigma \gg m \tag{50}$$

and

$$\sigma\pi_\perp < K_\perp, \tag{51}$$

where $K_\perp$ is a given arbitrary number, restricted as $m\sigma \ll K_\perp \ll eE\sigma^2$.

Let us study the behavior of N_n^{cr} on the boundaries of the Klein domain Ω_3, when $|p^{\mathrm{R}}| \to 0$ or $|p^{\mathrm{L}}| \to 0$. Let $\pi\sigma|p^{\mathrm{R/L}}| < K_1$, where $K_1 \ge 1$ is some arbitrary number satisfying the inequality

$$K_1 \ll eE_0\sigma^2. \tag{52}$$

In close proximity to these boundaries, $\pi\sigma|p^{\mathrm{L/R}}| < K_0$, $K_0 < 1$, we obtain that the value N_n^{cr} is exponentially small,

$$N_n^{\mathrm{cr}} \approx 2e^{-4\pi m\sigma}, \qquad \pi\sigma|p^{\mathrm{R}}| < K_0;$$
$$N_n^{\mathrm{cr}} \approx 2e^{-2\pi m\sigma}, \qquad \pi\sigma|p^{\mathrm{L}}| < K_0. \tag{53}$$

For boundary regions located closer to the center of the Klein zone Ω_3, $1 \lesssim \pi\sigma|p^{\mathrm{L/R}}| < K_1$, $K_1 > 1$ the following approximation is valid

$$N_n^{\mathrm{cr}} \approx \exp\left\{-4\pi\sigma\left[\sqrt{(p^{\mathrm{R}})^2 + \pi_\perp^2} - |p^{\mathrm{R}}|\right]\right\},$$
$$1 \lesssim \pi\sigma|p^{\mathrm{R}}| \lesssim K_1,$$
$$N_n^{\mathrm{cr}} \approx \exp\left\{-2\pi\sigma\left[\sqrt{(p^{\mathrm{L}})^2 + \pi_\perp^2} - |p^{\mathrm{L}}|\right]\right\},$$
$$1 \lesssim \pi\sigma|p^{\mathrm{L}}| \lesssim K_1. \tag{54}$$

The numbers N_n^{cr} increase as n moves away from the boundaries of the Klein region Ω_3 and for any fixed value of $\pi_\perp$, they reach their maximum value when $\pi\sigma|p^{\mathrm{L/R}}| \to K_1$. In turn, this maximum value increases as $\pi_\perp \to m$. In this range we can estimate N_n^{cr} as

$$N_n^{\mathrm{cr}} < \exp\left\{-4\pi\sigma m\left[\frac{\pi\sigma m}{2K_1} - \left(\frac{\pi\sigma m}{2K_1}\right)^3\right]\right\},$$
$$\pi\sigma|p^{\mathrm{R}}| \to K_1, \tag{55}$$

Similarly, in the domain $\pi\sigma|p^{\mathrm{L}}| \to k$ we have the estimate

$$N_n^{\mathrm{cr}} < \exp\left\{-2\pi\sigma m\left[\frac{\pi\sigma m}{2K_1} - \left(\frac{\pi\sigma m}{2K_1}\right)^3\right]\right\},$$
$$\pi\sigma|p^{\mathrm{L}}| \to K_1. \tag{56}$$

The right sides of inequalities (55) and (56) are exponentially small if

$$K_1 \ll \pi m\sigma/2 \tag{57}$$

for any K_1.

Consequently, the main contribution to the pairs creation is formed in the subrange of the Klein zone $D \subset \Omega_3$, where the energy $\pi_\perp$ is restricted by inequality (51) and the energy p_0 is restricted as:

$$-eE_0\sigma^2 + K < \sigma p_0 < -K, \quad K = \sqrt{K_1^2 + (\sigma\pi_\perp)^2}. \tag{58}$$

Inequalities (51) and (52) imply that $K \ll eE\sigma^2$. In such a case, we can approximate numbers (46) as follows

$$N_n^{\rm cr} \approx N_{p_0,\mathbf{p}_\perp}^{\rm as} = e^{-\pi\tau},$$
$$\tau = 2\sigma\left(\sqrt{2\delta U^2 + 2|p^{\rm R}|^2 - |p^{\rm L}|^2} - |p^{\rm L}| - 2|p^{\rm R}|\right). \tag{59}$$

In the point $x_{\rm M} = \sigma \ln 2$ the electric field (2) has the maximum value $E_{\rm max} = E_0/(3\sqrt{3})$ and the corresponding kinetic energy reads:

$$\pi_0' = p_0 - U(x_{\rm M}) = p_0 + \frac{eE_0\sigma}{\sqrt{3}}.$$

The function τ takes its minimum value at the point $\pi_0' = 0$,

$$\min \tau = \tau|_{\pi_0'=0} = \lambda = \frac{\pi_\perp^2}{eE_{\rm max}}. \tag{60}$$

Further, τ monotonically increases with $|\pi_0'|$ approaching the boundary of the subrange D. On the one side of the center of the Klein zone Ω_3, the function τ takes the maximum value

$$\tau_{\rm max}^- = \tau|_{\sigma|p_0| \to eE_0\sigma^2 - K} \simeq 2\left(K - K_1/\sigma + \frac{\lambda}{4\sqrt{3}}\right),$$

while on the other side, the maximum value is

$$\tau_{\rm max}^+ = \tau|_{\sigma|p_0| \to K} \simeq 4(K - K_1/\sigma).$$

In the wide range D, where

$$K/\sigma - eE_0\sigma(1 - 1/\sqrt{3}) < \pi_0' < eE_0\sigma/\sqrt{3} - K/\sigma, \tag{61}$$

the numbers $N_n^{\rm cr}$ practically do not depend on the parameter σ and have the form of the differential number of created particles in an uniform electric field [27,28],

$$N_n^{\rm cr} \approx e^{-\pi\lambda}. \tag{62}$$

B. Integral quantities

The total number of created pairs is given by integral (36). The main contribution to the integral is due to the subrange $D \subset \Omega_3$, defined by Eqs. (51) and (58). In this subrange the functions $N_n^{\rm cr}$ can be approximated by Eq. (59), and integral (36) can be represented as:

$$N^{\rm cr} \approx \frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \int_{\alpha\pi_\perp < K_\perp} (I_{p_\perp}^+ + I_{p_\perp}^-) d\mathbf{p}_\perp,$$
$$I_{p_\perp}^+ = \int_0^{eE_0\sigma/\sqrt{3} - K/\sigma} e^{-\pi\tau} d\pi_0',$$
$$I_{p_\perp}^- = \int_{-[eE_0\sigma(1-1/\sqrt{3}) - K/\sigma]}^0 e^{-\pi\tau} d\pi_0', \tag{63}$$

where τ is given by Eq. (59). To calculate $I_{p_\perp}^+$ and $I_{p_\perp}^-$, it is convenient to use the representation $\tau = \lambda(q+1)$ and to pass from the integration over π_0' to the integration over the parameter q (the transition to such a variable provides exponential decrease of the integrand with increasing q, and the expansion of the preexponential factor in powers of q has a form of an asymptotic series).

Finding π_0' as a function of q, one has to take into account that in the region D the following expansions are valid up to linear terms in reciprocal powers of large parameters:

$$|p^{\rm L}| \approx \frac{1}{\epsilon_1} - \frac{\epsilon_1 \pi_\perp^2}{2} + O(\epsilon_1),$$
$$\epsilon_1 = \left[\pi_0' + eE_0\sigma\left(1 - 1/\sqrt{3}\right)\right]^{-1},$$
$$|p^{\rm R}| \approx \frac{1}{\epsilon_2} - \frac{\epsilon_2 \pi_\perp^2}{2} + O(\epsilon_2),$$
$$\epsilon_2 = \left[eE_0\sigma/\sqrt{3} - \pi_0'\right]^{-1},$$
$$\sqrt{2\delta U^2 + 2|p^{\rm R}|^2 - |p^{\rm L}|^2} \approx \frac{1}{\epsilon_3} - \frac{\epsilon_3 \pi_\perp^2}{2} + O(\epsilon_3),$$
$$\epsilon_0 = \left[eE_0\sigma\left(1 + 1/\sqrt{3}\right) - \pi_0'\right]^{-1}.$$

Then we obtain the following approximation for τ:

$$\tau \approx \frac{2\sqrt{3}\lambda}{2\sqrt{3} + 9(r - \sqrt{3})r^2}, \quad r = \frac{\pi_0'}{eE_0\sigma}. \tag{64}$$

Besides, we have to find π_0' as a function of q using Eq. (64). Such a function can be found from the cubic equation

$$r^3 - \sqrt{3}r^2 + \frac{2q}{3\sqrt{3}(q+1)} = 0. \tag{65}$$

Note that when $\pi_0' \to eE_0\sigma/\sqrt{3} - K/\sigma$ and $\pi_0' \to -[eE_0\sigma(1 - 1/\sqrt{3}) - K/\sigma]$, the parameter τ reaches the limiting values $\tau^{\pm}_{\max} = \lambda(q^{\pm}_{\max} + 1)$, respectively. However, since contributions of the factor $\exp(-\pi\tau)$ to integrals (63) outside of range D are exponentially small, one can extend limits of the integration over q to $\pm\infty$. Equation (65) has three real roots:

$$r_1 = \frac{2}{\sqrt{3}}\cos\frac{\alpha(q)}{3} + 1/\sqrt{3}, \qquad \alpha(q) = \arccos\left[(q+1)^{-1}\right],$$
$$r_2 = -\frac{2}{\sqrt{3}}\cos\left[\frac{\alpha(q)}{3} + \frac{\pi}{3}\right] + 1/\sqrt{3},$$
$$r_3 = -\frac{2}{\sqrt{3}}\cos\left[\frac{\alpha(q)}{3} - \frac{\pi}{3}\right] + 1/\sqrt{3}; \tag{66}$$

see, e.g., [29].

Since $0 < q < +\infty$, the inequality $0 \le \alpha(q) \le \pi/2$ holds true, which implies:

$$0 \le \frac{\alpha(q)}{3} \le \frac{\pi}{6}, \qquad \frac{\pi}{3} \le \left[\frac{\alpha(q)}{3} + \frac{\pi}{3}\right] \le \frac{\pi}{2},$$
$$-\frac{\pi}{3} \le \left[\frac{\alpha(q)}{3} - \frac{\pi}{3}\right] \le -\frac{\pi}{6}. \tag{67}$$

Then

$$1 \le r_1 \le \sqrt{3}, \quad 0 \le r_2 \le \frac{1}{\sqrt{3}}, \quad \left(\frac{1}{\sqrt{3}} - 1\right) \le r_3 \le 0 \tag{68}$$

such that roots r_2 and r_3 represent the quantity π_0' in the subranges $\pi_0' \in (-eE_0\sigma(1 - 1/\sqrt{3}), 0)$ and $\pi_0' \in (0, eE_0\sigma/\sqrt{3})$, respectively.

Thus, the integrals $I^{+}_{p_\perp}$ and $I^{-}_{p_\perp}$ take the forms:

$$I^{\pm}_{p_\perp} = \pm\frac{2\delta U}{3\sqrt{3}}\int_0^{+\infty}\left\{\frac{(q+1)^{-2}}{\sqrt{1-(q+1)^{-2}}} \times \sin\left[\frac{\alpha(q)}{3} \pm \frac{\pi}{3}\right]\exp\left[-\pi\lambda(q+1)\right]\right\}dq, \tag{69}$$

and their sum can be represented as:

$$I^{-}_{p_\perp} + I^{+}_{p_\perp} = \frac{2\delta U}{3}\int_0^{+\infty}\left\{\frac{(1+q)^{-2}}{\sqrt{1-(1+q)^{-2}}} \times \cos\frac{\alpha(q)}{3}\exp\left[-\pi\lambda(q+1)\right]\right\}dq. \tag{70}$$

Substituting Eq. (70) into Eq. (63) and integrating over $dp_\perp^{(d-2)}$, we obtain:

$$N^{\mathrm{cr}} \approx V_\perp T\rho^{\mathrm{D}}, \qquad \rho^{\mathrm{D}} = \beta\frac{\delta U}{eE_{\max}}w,$$
$$\beta = \frac{J_{(d)}[eE_{\max}]^{d/2}}{(2\pi)^{d-1}}\exp\left[-\frac{\pi m^2}{eE_{\max}}\right],$$
$$w(d) = \frac{2}{3}\int_0^{+\infty}\left\{(q^2+2q)^{-1/2}(q+1)^{-d/2} \times \cos\frac{\alpha(q)}{3}\exp\left[-\frac{\pi m^2}{eE_{\max}}q\right]\right\}dq. \tag{71}$$

The corresponding probability P_{v} of the vacuum to remain a vacuum reads:

$$P_{\mathrm{v}} = \exp\left[-\mu N^{\mathrm{cr}}\right],$$
$$\mu = \sum_{l=0}^{\infty}(l+1)^{-d/2}\exp\left(-l\frac{\pi m^2}{eE_{\max}}\right). \tag{72}$$

In general, the current density J_x^{cr} of created particles is given by Eq. (39), while the charge polarization due to the separation of created positrons and electrons is presented by the charge density $\rho^{\mathrm{cr}}(x)$ given by Eq. (40). The nonzero components of energy-momentum tensor of the created particles are given by Eq. (41). In the subrange $D \subset \Omega_3$ the velocities v^{L} and v^{R} tend to the speed of the light $c = 1$ such that $|p^{\mathrm{L/R}}| \approx |\pi_0'(\mathrm{L/R})| \approx |U'_{\mathrm{L/R}}|$, with $U'_{\mathrm{L}} = -eE_0\sigma(1 - 1/\sqrt{3})$ and $U'_{\mathrm{R}} = eE_0\sigma/\sqrt{3}$ according to inequality (61). Using representation (71), we find:

$$J_x^{\mathrm{cr}} \approx e\rho^{\mathrm{D}}, \qquad J_0^{\mathrm{cr}}(x) \approx \begin{cases} -J_x^{\mathrm{cr}}, & x \in S_{\mathrm{L}} \\ J_x^{\mathrm{cr}}, & x \in S_{\mathrm{R}} \end{cases};$$
$$T^{00}_{\mathrm{cr}}(x) \approx T^{11}_{\mathrm{cr}}(x) \approx \begin{cases} \rho^{\mathrm{D}}|U'_{\mathrm{L}}|, & x \in S_{\mathrm{L}} \\ \rho^{\mathrm{D}}|U'_{\mathrm{R}}|, & x \in S_{\mathrm{R}} \end{cases},$$
$$T^{kk}_{\mathrm{cr}}(x) \approx \begin{cases} \tilde{\rho}^{\mathrm{D}}|U'_{\mathrm{L}}|^{-1}, & x \in S_{\mathrm{L}} \\ \tilde{\rho}^{\mathrm{D}}|U'_{\mathrm{R}}|^{-1}, & x \in S_{\mathrm{R}} \end{cases}, \qquad k \neq 1,$$
$$T^{10}_{\mathrm{cr}}(x) \approx \begin{cases} -\rho^{\mathrm{D}}|U'_{\mathrm{L}}|, & x \in S_{\mathrm{L}} \\ \rho^{\mathrm{D}}|U'_{\mathrm{R}}|, & x \in S_{\mathrm{R}} \end{cases},$$
$$\tilde{\rho}^{\mathrm{D}} = \beta\frac{\delta U\sigma}{2\pi}w(d+2). \tag{73}$$

It can be seen that the transversal components of the pressure of the created particles are much less than the longitudinal pressure in the areas S_{L} and S_{R}, respectively, $T^{kk}_{\mathrm{cr}}(x) \ll T^{11}_{\mathrm{cr}}(x)$. It is in accordance with the relation $T^{00}_{\mathrm{cr}}(x) \approx T^{11}_{\mathrm{cr}}(x)$.

Unlike the case with the Sauter field, the energy density and the longitudinal component of the pressure in the areas S_{L} and S_{R}, given by Eq. (73), are not equal, $T^{00}_{\mathrm{cr}}(x_{\mathrm{L}}) = 2T^{00}_{\mathrm{cr}}(x_{\mathrm{R}})$ and $T^{11}_{\mathrm{cr}}(x_{\mathrm{L}}) = 2T^{11}_{\mathrm{cr}}(x_{\mathrm{R}})$, that is, the energy density and longitudinal pressure of created electrons is two times more than the energy density and longitudinal pressure of created positrons due to the asymmetric form of the field (2). Furthermore, the magnitudes of energy flux density of the created particles through the surfaces $x = x_\mathrm{L}$ or $x = x_\mathrm{R}$ are not equal, $|T^{10}_{\mathrm{cr}}(x_\mathrm{L})| = 2|T^{10}_{\mathrm{cr}}(x_\mathrm{R})|$, even though the current density of both created electrons and positrons are the same, J^{cr}_x. This is due to the fact the main contribution to the components of energy-momentum tensor of the created particles comes from particles with different kinetic energies $|\pi'_0(\mathrm{L/R})|$ in the areas S_L and S_R. In the framework of semiclassical interpretation (appropriate in the case of a weakly inhomogeneous electric field) one can explain that pair of particle and antiparticle are created with small kinetic energies in the region of the strongest field, which is located asymmetrically with respect to the point of the field maximum x_M. Then the electron and positron move under the influence of the electric field in opposite directions, passing through different potential differences before leaving the region of the strong field. As a result, electrons and positrons acquire different kinetic energies by the time they exit into the areas S_L and S_R, respectively. In the case under consideration, created electrons acquire higher kinetic energies than created positrons due to the choice of direction of the field. Note that the direction of this field is chosen along the x-axis. Choosing the opposite direction of the field (choosing the opposite sign of the potential), one finds that electrons and positrons switch places. Then the energy density, longitudinal pressure, and energy flux density of created positrons is more than the corresponding characteristics of created electrons.

One can show that representations (71)–(73) can be obtained in the framework of a new kind of a locally constant field approximation (LCFA) constructed in Ref. [30]; see Appendix IX B. This can be considered as an additional argument in favor of the validity of such an approximation.

Besides, it turns out that LCFA allows one to see relation to the Heisenberg-Euler effective action method. Note that in the framework of the LCFA the probability P_v, given by Eq. (72), can be represented via the imaginary part of a one-loop effective action S by the seminal Schwinger formula [31],

$$P_\mathrm{v} = \exp(-2\mathrm{Im}S). \tag{74}$$

Thus, all the vacuum mean values, obtained in the framework of the LCFA, can be associated with the effective action approach. If the total number of created particles is small, $N^{\mathrm{cr}} \ll 1$, then $1 - P_v \approx N^{\mathrm{cr}}$. Therefore, knowledge of the probability P_v allows one to estimate the total number of created particles N^{cr}. It is, in this case, the effective action approach to calculating P_v turns out to be useful. We note that this approach is a base of a number of approximation methods; see, e.g., Ref. [7] for a review. In this relation, it should be noted that the probability P_v by itself is not very useful in the case of strong fields when $P_v \ll 1$. In the latter case it is necessary to directly calculate the vacuum mean values of physical quantities, using either exact solutions, as it is done above, or the new kind of the LCFA [30].

VI. THE KLEIN STEP

The Klein paradox is known from the work by Klein [13] who considered, in the framework of the one-particle relativistic theory, reflection and transmission probabilities of charged relativistic particles incident on a sufficiently high rectangular potential step (the Klein step) of the form

$$qA_0(x) = \begin{cases} U_\mathrm{L}, & x < 0 \\ U_\mathrm{R}, & x > 0 \end{cases}, \tag{75}$$

where U_R and U_L are some constants. The field (75) represents a kind of x-step. According to calculations of Klein and other authors, for certain energies and sufficient high magnitude $U = U_\mathrm{R} - U_\mathrm{L}$ of the Klein step, it seems that there are more reflected fermions than incident. This fact many articles and books where treated as a paradox (the Klein paradox); see Refs. [16,17]) for historical review. This paradox and other misunderstandings in considering quantum effects in fields of strong x-steps can be consistently solved as many particle effects in the *QED* with an unstable vacuum; see Ref. [10]. Obviously that the Klein step is a limiting case of a very sharp peak field. It is important to have in hands examples of potentials representing very sharp peak field that can be considered as their regularizations. Such an example given by Sauter potential (1) was presented in Ref. [10]; see Refs. [19,20] for regularizations by piecewise forms of analytic functions. The x-step (2) under consideration represents a new example of such regularizations given by an analytic function.

Let us study characteristics of the vacuum instability caused by the field (2) with σ sufficiently small, $\sigma \to 0$. If $U_\mathrm{R} = 0$ and $U_\mathrm{L} = -\delta U = -eE_0\sigma$ are given constant and

$$\delta U\sigma \ll 1 \tag{76}$$

the field imitates sufficiently well the asymmetric Klein step (75) and coincides with the latter as $\sigma \to 0$.

In the ranges Ω_1 and Ω_5 the energy $|p_0|$ is not restricted from the above, that is why, in what follows, we consider only the subranges, where $\max\{\sigma|p^\mathrm{L}|, \sigma|p^\mathrm{R}|\} \ll 1$. In the leading-term approximation in σ it follows from Eqs. (28) that

$$|g({}_+|^-)|^{-2} \approx \frac{4k}{(1-k)^2}, \quad k = \frac{|p^\mathrm{R}|}{|p^\mathrm{L}|}\frac{\pi_0(\mathrm{L}) + \pi_\perp}{\pi_0(\mathrm{R}) + \pi_\perp}, \tag{77}$$

where k is called the kinematic factor.

Note that k are positive and do not achieve the unit values, $k \neq 1$ in the ranges Ω_1 and Ω_5. In these ranges, the coefficients g satisfy the same relations,

$$|g(_+|^+)|^2 = |g(_+|^-)|^2 + 1. \tag{78}$$

Therefore, reflection and transmission probabilities derived from Eqs. (77) have the same forms

$$|T_{\zeta,n}|^2 = |g(_+|^+)|^{-2} = \frac{4k}{(1+k)^2},$$
$$|R_{\zeta,n}|^2 = |g(_+|^-)|^2 |g(_+|^+)|^{-2} = \frac{(1-k)^2}{(1+k)^2}. \tag{79}$$

To compare our exact results with results of the nonrelativistic consideration obtained in any textbook for one dimensional quantum motion, we set $p_\perp = 0$, then $\pi_\perp = m$, $\pi_0(\mathrm{L}) = m + E$, and $\pi_0(\mathrm{R}) = m + E - \delta U$. In the nonrelativistic limit, when $\delta U, E \ll m$, we obtain

$$k = k^{\mathrm{NR}} = \sqrt{\frac{E - \delta U}{E}},$$

which can be identified with the nonrelativistic results, e.g., see Ref. [32].

Let us consider the range Ω_3. Here the quantum numbers $\mathbf{p}_\perp$ are restricted by the inequality $2\pi_\perp \leq \delta U$ and for any of such $\pi_\perp$ the quantum numbers p_0 obey the strong inequality $U_\mathrm{L} + \pi_\perp \leq p_0 \leq U_\mathrm{R} - \pi_\perp$. In this range the quantity $|g(_+|^-)|^{-2}$ represents the differential mean numbers of electron-positron pairs created from the vacuum, $N_n^{\mathrm{cr}} = |g(_+|^-)|^{-2}$. In this range for any given $\pi_\perp$ the absolute values of $|p^\mathrm{R}|$ and $|p^\mathrm{L}|$ are restricted from the above, see (43). Therefore, condition (76) implies $\max\{\sigma|p^{\mathrm{L/R}}|\} \ll 1$. It follows from Eq. (28) that in the leading approximation the following equation holds true

$$|g(_+|^-)|^{-2} \approx \frac{4|p^\mathrm{L}||p^\mathrm{R}|}{\delta U^2 - (|p^\mathrm{L}| - |p^\mathrm{R}|)^2} = \frac{4|k|}{(1+|k|)^2}. \tag{80}$$

Note that expression (80) differs from expression (77) only by the sign of the kinematic factor k. This factor is positive in the ranges Ω_1 and Ω_5, and it is negative in the range Ω_3. In the range Ω_3, the difference $|p^\mathrm{L}| - |p^\mathrm{R}|$ may be zero at $p_0 = U_\mathrm{L}/2$, which corresponds to $k = -(\delta U + 2\pi_\perp)/(\delta U - 2\pi_\perp)$. Namely in this case the quantity $|g(_+|^-)|^{-2}$ has a maximum at a given $\pi_\perp$,

$$\max |g(_+|^-)|^{-2} = 1 - (2\pi_\perp/\delta U)^2. \tag{81}$$

Note that expressions (77), (79) and (80) coincide up to the redesigning of the constants $U_{\mathrm{L/R}}$ with expressions, corresponding to other regularizations of the Klein step; see Ref. [10,19,20]. We see that the Klein step is (in a sense) a limiting case of various sharp peak fields under condition that the magnitude δU is a given finite constant and the peak of the fields are sufficiently sharp.

VII. CONCLUDING REMARKS

A new exactly solvable case in strong-field *QED* with x-step is presented. This step can be seen as a certain analytic "deformation" of the Sauter field. In contrast to the Sauter field the potential field under consideration is asymmetric with respect of the axis x reflection. Bearing in mind numerous examples of using the results of the exactly solvable problem with the Sauter field in physical applications related to the problem of vacuum instability, we believe that the new exact solvable case will also be useful in such applications. It can be treated as a new regularization of the Klein step. Exact solutions of the Dirac equation used in the above nonperturbative calculations, are presented in the form of stationary plane waves with special left and right asymptotics and identified as components of initial and final wave packets of particles and antiparticles in the framework of the strong-field *QED* [10]. We show that in spite of the fact that the symmetry with respect to positive and negative bands of energies is broken, distribution of created pairs and other physical quantities can be presented by elementary functions. We consider the processes of transmission and reflection in the ranges of the stable vacuum and study physical quantities specifying the vacuum instability as well. We find the differential mean numbers of electron-positron pairs created from the vacuum, the components of current density and energy-momentum tensor of the created electrons and positrons leaving the area of the strong field under consideration. We study the particular case of the particle creation due to a weakly inhomogeneous electric field and obtain explicitly the total number, the current density and energy-momentum tensor of created particles. Unlike the symmetric case of the Sauter field the asymmetric form of the field under consideration causes the energy density and longitudinal pressure of created electrons to be not equal to the energy density and longitudinal pressure of created positrons.

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APPENDIX A: BASIC ELEMENTS OF A NONPERTURBATIVE APPROACH TO *QED* WITH x-STEPS

In this appendix, we briefly present some basic constructions of quantization in terms of particles for QED with x-steps; see Secs. IV–VII in Ref. [10] for details.

The time-independent inner product for any pair of solutions of the Dirac equation, $\psi_n(X)$ and $\psi'_{n'}(X)$, is defined on the $t = \mathrm{const}$ hyperplane as follows:

$$(\psi_n, \psi'_{n'}) = \int_{V_\perp} d\mathbf{r}_\perp \int_{-K^{(\mathrm{L})}}^{K^{(\mathrm{R})}} \psi_n^\dagger(X)\psi'_{n'}(X)dx, \tag{A1}$$

where the integral over the spatial volume $V_\perp$ is completed by an integral over the interval $[K^{(\mathrm{L})}, K^{(\mathrm{R})}]$ in the x direction. The parameters $K^{(\mathrm{L/R})}$ are assumed sufficiently large in final expressions. Assuming that the principal value of integral (A1) is determined by integrals over the areas where the field $E(x)$ is negligibly small it is possible to evaluate this value using only the asymptotic behavior (15) of functions in the regions where particles are free. The field $E(x)$ in the area where it is strong enough affects only coefficients g entering into the mutual decompositions of the solutions given by Eq. (25). One can see that the norms of the plane waves ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$ with respect to the inner product (A1) are proportional to the macroscopically large parameters $\tau^{(\mathrm{L})} = K^{(\mathrm{L})}/v^{\mathrm{L}}$ and $\tau^{(\mathrm{R})} = K^{(\mathrm{R})}/v^{\mathrm{R}}$, where $v^{\mathrm{L}} = |p^{\mathrm{L}}/\pi_0(\mathrm{L})| > 0$ and $v^{\mathrm{R}} = |p^{\mathrm{R}}/\pi_0(\mathrm{R})| > 0$ are absolute values of the longitudinal velocities of particles in the regions where particles are free; see Sec. III C.2 and Appendix B in Ref. [10] for details.

It was shown (see Appendix B in Ref. [10]) that the following couples of plane waves are orthogonal with respect to the inner product (A1)

$$\begin{aligned} ({}_\zeta\psi_n, {}^{-\zeta}\psi_n) &= 0, \qquad n \in \Omega_1 \cup \Omega_5; \\ ({}_\zeta\psi_n, {}^{\zeta}\psi_n) &= 0, \qquad n \in \Omega_3, \end{aligned} \tag{A2}$$

if the parameters of the volume regularization $\tau^{(\mathrm{L/R})}$ satisfy the condition

$$\tau^{(\mathrm{L})} - \tau^{(\mathrm{R})} = O(1), \tag{A3}$$

where $O(1)$ denotes terms that are negligibly small in comparison with the macroscopic quantities $\tau^{(\mathrm{L/R})}$. One can see that $\tau^{(\mathrm{R})}$ and $\tau^{(\mathrm{L})}$ are macroscopic times of motion of particles and antiparticles in the areas S_{R} and S_{L}, respectively and they are equal,

$$\tau^{(\mathrm{L})} = \tau^{(\mathrm{R})} = \tau. \tag{A4}$$

It allows one to introduce an unique time of motion τ for all the particles in the system under consideration. This time can be interpreted as a system monitoring time during its evolution and as such is fixed as $\tau = T$ in the framework of the renormalization procedure; see Ref. [11] for details.

The renormalization and volume regularization procedures are associated with the introduction of a modified inner product and a parameter τ of the regularization. Based on physical considerations, we fix this parameter. It turns out that in the Klein range this parameter can be interpreted as the time of the observation of the pair production process.

Under condition (A3) the following orthonormality relations on the $t = \mathrm{const}$ hyperplane are

$$\begin{aligned} &({}_\zeta\psi_n, {}_\zeta\psi_{n'}) = ({}^\zeta\psi_n, {}^\zeta\psi_{n'}) = \delta_{n,n'}\mathcal{M}_n, \\ &\qquad n \in \Omega_1 \cup \Omega_3 \cup \Omega_5; \\ &(\psi_n, \psi_{n'}) = \delta_{n,n'}\mathcal{M}_n, n \in \Omega_2 \cup \Omega_4, \\ &\mathcal{M}_n = 2\frac{\tau}{T}|g({}_+|^+)|^2, \quad n \in \Omega_1 \cup \Omega_5, \\ &\mathcal{M}_n = 2\frac{\tau}{T}|g({}_+|^-)|^2, \quad n \in \Omega_3, \\ &\mathcal{M}_n = 2\frac{\tau}{T}, \quad n \in \Omega_2, \quad \mathcal{M}_n = 2\frac{\tau}{T}, \quad n \in \Omega_4. \end{aligned} \tag{A5}$$

All the wave functions having different quantum numbers n are orthogonal, and

$$\begin{aligned} &({}_\zeta\psi_n, {}^{-\zeta}\psi_n) = 0, \qquad n \in \Omega_1 \cup \Omega_5, \\ &{}_\zeta\psi_n \quad \text{and} \quad {}^{-\zeta}\psi_n \text{ independent;} \\ &({}_\zeta\psi_n, {}^{\zeta}\psi_n) = 0, \qquad n \in \Omega_3, \\ &{}_\zeta\psi_n \quad \text{and} \quad {}^{\zeta}\psi_n \text{ independent.} \end{aligned} \tag{A6}$$

We denote the corresponding quantum numbers by n_k, so that $n_k \in \Omega_k$. Then we identify components of the initial and final wave packets of particles and antiparticles in Eq. (24).

We decompose the Heisenberg operator $\hat{\Psi}(X)$ in two sets of solutions $\{{}_\zeta\psi_n(X)\}$ and $\{{}^\zeta\psi_n(X)\}$ of the Dirac equation (7) complete on the $t = \mathrm{const}$ hyperplane. Operator-valued coefficients in such decompositions do not depend on coordinates. Our division of the quantum numbers n in five ranges Ω_k, implies the representation for $\hat{\Psi}(X)$ as a sum of five operators $\hat{\Psi}_k(X)$, $k = 1, 2, 3, 4, 5$,

$$\hat{\Psi}(X) = \sum_{k=1}^{5} \hat{\Psi}_k(X). \tag{A7}$$

For each of three operators $\hat{\Psi}_k(X), k = 1, 3, 5$, there exist two possible decompositions (24) according to the existence of two different complete sets of solutions with the same quantum numbers n in the ranges Ω_1, Ω_3, and Ω_5,

$$\begin{aligned}
\hat{\Psi}_1(X) &= \sum_{n_1} \mathcal{M}_{n_1}^{-1/2}[{}_+a_{n_1}(\mathrm{in})_+\psi_{n_1}(X) \\
&\quad + {}^-a_{n_1}(\mathrm{in})^-\psi_{n_1}(X)] \\
&= \sum_{n_1} \mathcal{M}_{n_1}^{-1/2}[{}^+a_{n_1}(\mathrm{out})^+\psi_{n_1}(X) \\
&\quad + {}_-a_{n_1}(\mathrm{out})_-\psi_{n_1}(X)], \\
\hat{\Psi}_3(X) &= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2}[{}^-a_{n_3}(\mathrm{in})^-\psi_{n_3}(X) \\
&\quad + {}_-b^\dagger_{n_3}(\mathrm{in})_-\psi_{n_3}(X)] \\
&= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2}[{}^+a_{n_3}(\mathrm{out})^+\psi_{n_3}(X) \\
&\quad + {}_+b^\dagger_{n_3}(\mathrm{out})_+\psi_{n_3}(X)], \\
\hat{\Psi}_5(X) &= \sum_{n_5} \mathcal{M}_{n_5}^{-1/2}[{}^+b^\dagger_{n_5}(\mathrm{in})^+\psi_{n_5}(X) \\
&\quad + {}_-b^\dagger_{n_5}(\mathrm{in})_-\psi_{n_5}(X)] \\
&= \sum_{n_5} \mathcal{M}_{n_5}^{-1/2}[{}_+b^\dagger_{n_5}(\mathrm{out})_+\psi_{n_5}(X) \\
&\quad + {}^-b^\dagger_{n_5}(\mathrm{out})^-\psi_{n_5}(X)]. \qquad \text{(A8)}
\end{aligned}$$

There may exist only one complete set of solutions with the same quantum numbers n_2 and n_4. Therefore, we have only one possible decomposition for each of the two operators $\hat{\Psi}_i(X)$, $i = 2, 4$,

$$\begin{aligned}
\hat{\Psi}_2(X) &= \sum_{n_2} \mathcal{M}_{n_2}^{-1/2} a_{n_2}\psi_{n_2}(X), \\
\hat{\Psi}_4(X) &= \sum_{n_4} \mathcal{M}_{n_4}^{-1/2} b^\dagger_{n_4}\psi_{n_4}(X). \qquad \text{(A9)}
\end{aligned}$$

We interpret all a and b as annihilation and all $a^\dagger$ and $b^\dagger$ as creation operators. All a and $a^\dagger$ are interpreted as describing electrons and all b and $b^\dagger$ as describing positrons. All the operators labeled by the argument in are interpreted as in-operators, whereas all the operators labeled by the argument out as out-operators. This identification is confirmed by a detailed mathematical and physical analysis of solutions of the Dirac equation with subsequent QFT analysis of correctness of such an identification in Ref. [10].

Taking into account the orthogonality and orthonormalization relations, we find that the standard anticommutation relations for the Heisenberg operator (A7) yield the standard anticommutation rules for the introduced creation and annihilation in- or out-operators.

We define two vacuum vectors $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$, one of which is the zero-vector for all in-annihilation operators and the other is zero-vector for all out-annihilation operators. Besides, both vacua are zero-vectors for the annihilation operators a_{n_2} and b_{n_4}. One can verify that the introduced vacua have minimum (zero by definition) kinetic energy and zero electric charge and all the excitations above the vacuum have positive energies. Then we postulate that the state space of the system under consideration is the Fock space constructed, say, with the help of the vacuum $|0,\mathrm{in}\rangle$ and the corresponding creation operators. This Fock space is unitarily equivalent to the other Fock space constructed with the help of the vacuum $|0,\mathrm{out}\rangle$ and the corresponding creation operators if the total number of particles created by the external field is finite.

Because any annihilation operators with quantum numbers n_k corresponding to different k anticommute between themselves, we can represent the introduced vacua $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$ as tensor products of the corresponding partial vacua in the five ranges Ω_k, $k = 1,\ldots,5$. The partial vacua are stable in Ω_k, $k = 1,2,4,5$, and the vacuum instability with $|c_v| \neq 1$ is due to the partial vacuum-to-vacuum transition amplitude formed in Ω_3. In the range Ω_3 operators of the number of final electrons and positrons are

$$\begin{aligned}
\hat{N}^a_n(\mathrm{out}) &= {}^+a^\dagger_n(\mathrm{out})^+a_n(\mathrm{out}), \\
N^b_n(\mathrm{out}) &= {}_+b^\dagger_n(\mathrm{out})_+b_n(\mathrm{out}). \qquad \text{(A10)}
\end{aligned}$$

Using the linear canonical transformation between in and out-operators of creation and annihilation (see Eq. (7.4) in Ref. [10]) one sees that the differential mean numbers of electrons and positrons created from vacuum are presented by Eq. (34). The vacuum-to-vacuum transition amplitude, given by Eq. (7.21) in Ref. [10]), is

$$c_v = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle = \prod_{n\in\Omega_3} g({}^-|_-)g({}^-|_+)^{-1}. \qquad \text{(A11)}$$

Then the probability for a vacuum to remain a vacuum can be presented by Eq. (37).

APPENDIX B: LOCALLY CONSTANT FIELD APPROXIMATION

A new kind of a locally constant field approximation (LCFA) was formulated in Ref. [30]. Here we pretend to show that the density of created pairs (71) and the probability of the vacuum to remain a vacuum (72), obtained from exact equations for the slowly varying field in the leading-term approximation, are in agreement with results following in the framework of LCFA.

We call the electric field $E(x)$ a weakly inhomogeneous electric field on a spatial interval Δl if the following condition holds true:

$$\left|\frac{\overline{\partial_x E(x)\Delta l}}{\overline{E(x)}}\right| \ll 1, \qquad \Delta l/\Delta l^{\mathrm{m}}_{\mathrm{st}} \gg 1, \qquad \text{(B1)}$$

where $\overline{E(x)}$ and $\overline{\partial_x E(x)}$ are the mean values of $E(x)$ and $\partial_x E(x)$ on the spatial interval Δl, respectively, and Δl is significantly larger than the length scale $\Delta l_{\rm st}^{\rm m}$, which is

$$\Delta l_{\rm st}^{\rm m} = \Delta l_{\rm st} \max\{1, m^2/e\overline{E(x)}\},$$
$$\Delta l_{\rm st} = [e\overline{E(x)}]^{-1/2}. \tag{B2}$$

Note that the length scale $\Delta l_{\rm st}^{\rm m}$ appears in Eq. (B1) as the length scale when the perturbation theory with respect to the electric field breaks down and the Schwinger (nonperturbative) mechanism is primarily responsible for the pair creation. In what follows, we show that this condition is sufficient. We are primarily interested in strong electric fields, $m^2/e\overline{E(x)} \lesssim 1$. In this case, the second inequality in Eq. (B1) is simplified to the form $\Delta l/\Delta l_{\rm st} \gg 1$, in which the mass m is absent. In such cases, the potential of the corresponding electric step hardly differs from the potential of a uniform electric field,

$$U(x) = -eA_0(x) \approx U_{\rm const}(x) = e\overline{E(x)}x + U_0, \tag{B3}$$

on the interval Δl, where U_0 is a given constant.

For an arbitrary weakly inhomogeneous strong electric field, in the leading-term approximation, were derived universal formulas for the total density of created pairs

$$\rho^{\Omega} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int_{x_{\rm L}}^{x_{\rm R}} dx eE(x) \int d\mathbf{p}_\perp N_n^{\rm uni},$$
$$N_n^{\rm uni} = \exp\left[-\pi \frac{\pi_\perp^2}{eE(x)}\right], \tag{B4}$$

and an expression for the probability $P_{\rm v}$ given by Eq. (37) of the vacuum to remain a vacuum,

$$P_{\rm v} \approx \exp\left\{-\frac{V_\perp T J_{(d)}}{(2\pi)^{d-1}} \times \sum_{l=1}^{\infty} \int_{x_{\rm L}}^{x_{\rm R}} dx \frac{[eE(x)]^{d/2}}{l^{d/2}} \exp\left[-\pi \frac{lm^2}{eE(x)}\right]\right\}; \tag{B5}$$

see Ref. [30]. In Eqs. (B4) and (B5) integration limits are specified over the region $S_{\rm int} = (x_{\rm L}, x_{\rm R})$, in which the electrical the field is not zero. In our case $x_{\rm L} = -\infty$, $x_{\rm R} = +\infty$.

Let us compare Eqs. (B4) and (B5) with the results obtained above in (71). To do this, let us represent (B4) in the form

$$\rho^{\Omega} \approx \frac{J_{(d)}}{(2\pi)^{d-1}} \int d\mathbf{p}_\perp (J_{p_\perp}^{+} + J_{p_\perp}^{-}),$$
$$J_{p_\perp}^{+} = \int_{x_{\rm M}}^{\infty} dx [eE(x)] N_n^{\rm univ} = -e \int_{x_{\rm M}}^{\infty} dA_0(x) N_n^{\rm univ},$$
$$J_{p_\perp}^{-} = \int_{-\infty}^{x_{\rm M}} dx [eE(x)] N_n^{\rm univ} = -e \int_{x_{\rm M}}^{\infty} dA_0(x) N_n^{\rm univ}. \tag{B6}$$

Note that the functions $A_0(x)$ and $E(x)$ can be related to each other using the cubic equation

$$y^3 - y - \frac{2}{3\sqrt{3}(q+1)} = 0,$$
$$y = -A_0(x)/(E_0\sigma), \qquad q = \left(3\sqrt{3}\frac{E(x)}{E_0}\right)^{-1} - 1. \tag{B7}$$

We can express $A_0(x)$ as a function of the field $E(x)$ or as a function of the variable q using solutions of equation (B7). This equation has three real solutions,

$$y_1 = \frac{2}{\sqrt{3}} \cos\frac{\alpha(q)}{3},$$
$$y_2 = -\frac{2}{\sqrt{3}} \cos\left[\frac{\alpha(q)}{3} + \frac{\pi}{3}\right],$$
$$y_3 = -\frac{2}{\sqrt{3}} \cos\left[\frac{\alpha(q)}{3} - \frac{\pi}{3}\right],$$
$$\alpha(q) = \arccos[(q+1)^{-1}], \tag{B8}$$

see, e.g., [29]. Since $A_0(x)$ is negative, only the solutions $y_{2,3}$ are relevant. One can see that for solutions $y_{2,3}$ the differential $dA_0(x)$ takes the form:

$$dA(x) = -2(E_{\max}\sigma) \sin\left[\frac{\alpha(q)}{3} + \frac{\pi}{3}\right] \times \frac{(q+1)^{-2}}{\sqrt{1-(q+1)^{-2}}} dq, \qquad x > 0,$$
$$dA(x) = -2(E_{\max}\sigma) \sin\left[\frac{\alpha(q)}{3} - \frac{\pi}{3}\right] \times \frac{(q+1)^{-2}}{\sqrt{1-(q+1)^{-2}}} dq \qquad x \le 0,$$
$$E(x) = -dA(x) = \frac{E_{\max}}{q+1}. \tag{B9}$$

Passing from the integration over x to the integration over the parameter q in Eq. (B6), we find:

$$J^{\pm}_{p_\perp} = I^{\pm}_{p_\perp}, \tag{B10}$$

where the quantities $I^{\pm}_{p_\perp}$ are given by Eq. (69).

It follows from Eq. (B10) that the density of the created pairs (B4) and the probability of the vacuum to remain a vacuum (B5) obtained with the help of the approximation for weakly inhomogeneous strong electric field coincide with expressions (71) and (72), respectively.

APPENDIX C: SOME PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

The hypergeometric function $F(a,b,c;z) = {}_2F_1(a,b,c;z)$ (here and in what follows it is supposed that parameters a and b are not equal to $0, -1, -2, \ldots$) is defined by series

$$F(a,b,c;z) = \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{+\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}. \tag{C1}$$

for $|z| < 1$. Note that in the solutions (13) and (14) the arguments $1 - \xi^{-1}$ and ξ^{-1} in the corresponding hypergeometric functions are less than unity and the series (C1) converges.

At $|z| = 1$ the series (C1) converges absolutely when $\mathrm{Re}(c - a - b) > 0$. The integral representation

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt, \times (\mathrm{Re}c > \mathrm{Re}b > 0) \tag{C2}$$

gives an analytical continuation for the function $F(a,b,c;z)$ to the complex z-plane with a cut along the real axis from 1 to ∞ (since the right-hand side is an unambiguous analytic function in the domain $|\arg(1-z)| \le \pi$). From the integral representation (C2) it is easy to see that $\lim_{z\to 0} F(a,b,c;z) = 1$. The formula for differentiating the hypergeometric function has the form:

$$\frac{d}{dz} F(a,b,c;z) = \frac{ab}{c} F(a+1, b+1, c+1; z). \tag{C3}$$

It is follows from (C2) that

$$F(a,b,c;z) = (1-z)^{c-a-b} F(c-a, c-b, c; z), \quad |z| < 1. \tag{C4}$$

Hypergeometric function can be transformed as

$$\begin{aligned} F(a,b,c;z) = {} & \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \\ & \times F(a,b,a+b-c+1; 1-z) \\ & + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ & \times F(c-a, c-b, c-a-b+1; 1-z), \\ & \times (|\arg(1-z)| < \pi), \end{aligned} \tag{C5}$$

The hypergeometric equation in its general form,

$$z(1-z)w''(z) + [c - (a+b+1)z]w'(z) - abw(z) = 0, \tag{C6}$$

has three regular singular points $z = 0, 1, \infty$. When none of the numbers c, $c - a - b$, $a - b$ is integer, the general solution $w(z)$ of the hypergeometric equation (C6) can be obtained as

$$\begin{aligned} w(z) &= c_1 w_1(z) + c_2 w_2(z), & z &\to 0, \\ w(z) &= c_1 w_3(z) + c_2 w_4(z), & z &\to 1, \\ w(z) &= c_1 w_5(z) + c_2 w_6(z), & z &\to \infty. \end{aligned} \tag{C7}$$

where c_1 and c_2 are some constants, and the functions $w_j(z)$, $j = 1, \ldots, 6$, have the form:

$$\begin{aligned} w_1(z) &= F(a,b,c;z), \\ w_2(z) &= z^{1-c} F(a-c+1, b-c+1, 2-c; z), \\ w_3(z) &= F(a,b,a+b+1-c; 1-z), \\ w_4(z) &= (1-z)^{c-a-b} F(c-b, c-a, c-a-b+1, 1-z), \\ w_5(z) &= z^{-a} F(a, a-c+1, a-b+1, z^{-1}), \\ w_6(z) &= z^{-b} F(b, b-c+1, b-a+1, z^{-1}). \end{aligned} \tag{C8}$$

The Kummer relations and for the hypergeometric equation [26] allow us to represent the functions $w_1(z)$ and $w_2(z)$ via the functions $w_3(z)$ and $w_4(z)$,

$$\begin{aligned} w_1(z) = {} & e^{i\pi(2\alpha_1 - b)} \frac{\Gamma(2(\alpha_1+1) - a - b)\Gamma(b-a+1)}{\Gamma(2-a)\Gamma(2\alpha_1 - a + 1)} \\ & \times w_4(z) - e^{i\pi(2\alpha_1 - a)} \\ & \times \frac{\Gamma(2(\alpha_1+1) - a - b)\Gamma(a-b-1)}{\Gamma(1-b)\Gamma(2\alpha_1 - b)} w_3(z), \\ w_2(z) = {} & e^{i\pi(a-1)} \frac{\Gamma(a+b-2\alpha_1)\Gamma(b-a+1)}{\Gamma(b-2\alpha_1+1)\Gamma(b)} w_4(z) \\ & + e^{i\pi b} \frac{\Gamma(a+b-2\alpha_1)\Gamma(a-b-1)}{\Gamma(a-2\alpha_1)\Gamma(a-1)} w_3(z). \end{aligned} \tag{C9}$$

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