## Perturbative unitarity and the four-point vertices in the constructive standard model

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We find a complete set of four-point vertices in the constructive Standard Model (CSM). This set is smaller than in Feynman diagrams as the CSM does not need or allow any additional four-point vertices (or "contact" terms) beyond what is present in Feynman diagrams and, furthermore, it does *not* need or allow a four-point vertex for  $Z, Z, \overline{W}, W, W, W, \overline{W}, \overline{Y}, \gamma, Z, W, \overline{W}$  or  $\gamma, \gamma, W, \overline{W}$ , in addition to the already known absence of the four-gluon vertex. We show that with this set of four-point vertices, perturbative unitarity is satisfied in the CSM. Additionally, we show that many constructive diagrams are not Feynman diagrams rewritten in spinor form. In fact, we show that there is a significant rearrangement of contributions from the diagrams in constructive calculations relative to Feynman diagrams for some processes. In addition to the already known or expected rearrangement in diagrams involving external photons, we also find that diagrams involving four-vector bosons are also significantly different from their Feynman counterparts.

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#### I. INTRODUCTION

Alternate techniques for calculating scattering amplitudes "constructively," without the use of field theory or Feynman diagrams, have been gathering steam for several decades now [1-5]. Initially, most of the progress was for massless theories, but with [6] the method was extended to massive theories as well. Following this step, full calculations in the constructive Standard Model (CSM) were begun, beginning with a complete set of three-point vertices for the CSM [7], and other calculations [8–20]. Immediately following the publication of the three-point vertices, we set out to find the four-point vertices of the CSM. Our intention was to use perturbative unitarity [21–24] to determine them, but our attempts were blocked by challenges with diagrams with internal photons [25]. This last paper found a work-around using a massive photon and taking the massless limit, and [26] showed that the x factors had further structure that allowed them to be used with massless photons to obtain the correct amplitude in the case of  $f, \bar{f} \to f, \bar{f}, \bar{f}$ where f is a fermion. Moreover, Ref. [27] described the momentum shifts that enabled this procedure to work.

With the photon diagrams solved, we set out again to find the complete set of four-point vertices of the CSM. This culminated in the present paper where we find all the four-point vertices and establish perturbative unitarity in the CSM, in addition to two companion papers. In one, we use the four-point vertices along with the three-point vertices to find the four-point amplitudes in a comprehensive set of CSM processes [28]. In the other, we create a C++ package called SPINAS, designed to calculate any constructive amplitude [29]. Further, we use SPINAS to validate all the four-point amplitudes in the CSM, including those presented here.

In this paper, in greater detail, we analyze the highenergy-growth terms of all the four-point amplitudes involving longitudinal vector bosons. We find the processes where a four-point vertex is required to cancel the highenergy growth and achieve perturbative unitarity, and we make a complete list of these four-point vertices in Table I. We also include the four-Higgs vertex, for completeness and claim that this is a complete set of four-point vertices for the CSM. No other four-point vertices (or contact terms) are required or allowed. This includes the exclusion of fourpoint vertices for  $Z, Z, \overline{W}, W, W, W, \overline{W}, \overline{Y}, \gamma ZW\overline{W}$ , and

TABLE I. Complete list of four-point vertices in the CSM.

CSM four-point vertices		
Particles	Coupling	Vertex
hhhh	$-i \frac{3e^2 m_h^2}{4M_{2\nu}^2 s_{2\nu}^2}$	1
hhZZ	$i \frac{e^2}{2M^2 s^2}$	$[34]\langle 34  angle$
$hhWar{W}$	$irac{e^2}{2M_W^2 s_W^2}$	$[34]\langle 34  angle$

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 $\gamma\gamma W\bar{W}$  that are present in Feynman diagrams and the already well-known absence of a four-gluon vertex.

We further show that constructive diagrams are not always simply Feynman diagrams written in spinor notation. In fact, in some processes, the contributions to the amplitude are significantly rearranged, so that individual constructive diagrams do not equal individual Feynman diagrams. We find two classes where this is the case. The first is when there are external photons or gluons. This was already known for a small number of processes and expected in others, and we have enlarged the number of calculated processes and confirm a rearrangement in all of them. Additionally, we show in this paper that any process with a Feynman four-vectorboson vertex, such as  $Z, Z, W, \overline{W}$  and  $W, \overline{W}, W, \overline{W}$ , also have a significant rearrangement of contributions. This is clear to see since there is no four-point vertex in constructive calculations.

To find the four-point vertices of the CSM, we look in the same place we find them in Feynman diagrams. With the exception of the four-Higgs vertex and the four-gluon vertex, we find Feynman four-point vertices in processes where they are required to achieve perturbative unitarity. This occurs when a process has diagrams that grow in the limit of high energy (at tree level), which occurs with Feynman diagrams when a longitudinal vector boson is present. This is also where we look for them in the case of the CSM.

We can see this at a naive level by considering the contribution from the parts of a diagram. The propagator denominator grows as the energy squared ( $\mathcal{E}^2$ ), as usual. The growth of the numerator depends on the number of spinors and the number of momenta in it. Each spinor contributes  $\sqrt{\mathcal{E}}$  and each momentum  $\mathcal{E}$ . Therefore, we can count the expected number of each to get an idea. A four-fermion diagram would have four spinors creating  $\mathcal{E}^2$  growth that cancels with the  $\mathcal{E}^2$  in the denominator if no additional momenta are present. Therefore, we do not expect to need any four-fermion vertices and, indeed, we find that we do not in the companion to this paper [28]. The same is true if we replace two of these with Higgs bosons, which lowers the naive high-energy growth.

However, when we begin adding vector bosons, since they need two spinors each, we increase the chance that high-energy growth will appear and require a cancellation. For example, if we have two fermions and two vector bosons, we need six spinors, contributing  $\mathcal{E}^3$  growth (divided by  $\mathcal{E}^2$  from the propagator denominator), at a minimum, and requiring cancellation among the diagrams. If we have four vector bosons, we need eight spinors, contributing  $\mathcal{E}^4$  over the  $\mathcal{E}^2$  from the denominator. Again, even at the minimal level, we require high-energy cancellation. Moreover, these diagrams with vector bosons often have extra factors of momenta increasing their high-energy growth, as we will see in detail in this paper. On the other hand, external photons tend to decrease the high-energy growth (relative to vector bosons) due to extra propagator denominators. In fact, we will find in this paper that many processes involving external photons that have high-energy growth and cancellation in Feynman diagrams do not have any high-energy growth at all here in the CSM.

In this paper, we will show that the CSM fully satisfies perturbative unitarity, and we will establish a complete list of four-point vertices in the CSM. As mentioned earlier, we include the four-Higgs vertex for completeness, even though it is not required for perturbative unitarity of four-point amplitudes. It is trivially related to the Feynman four-Higgs vertex and has been validated in [28]. We break this paper up into four main sections. In Sec. II, we consider processes that require a four-point vertex, namely  $h, h \rightarrow Z, Z$  and  $h, h \to W, \overline{W}$ . In Sec. II, we consider processes that do have high-energy-growth terms that need to cancel, but do not require or allow a four-point vertex in the CSM, but do require one in Feynman diagrams. This includes the processes  $Z, Z \to W, \overline{W}, W, \overline{W} \to W, \overline{W}$ . In Sec. IV, we consider processes that do not have any high-energy growth and do not require or allow a four-point vertex in the CSM, but do require one in Feynman diagrams. This includes the processes  $\gamma^+, Z \to W, \overline{W}, \gamma^+, \gamma^+ \to W, \overline{W}$  and  $\gamma^+, \gamma^- \to W, \overline{W},$ with these choices of helicity representative of the other choices. In Sec. V, we look at a selection of the processes that do not require or allow a four-point vertex in either the CSM or in Feynman diagrams, but which include the potential of a cancellation of high-energy-growth terms. This includes the processes  $Z, Z \to Z, Z, t, \overline{t} \to W, \overline{W}, t, \overline{t} \to Z, Z$ , and  $\overline{t}, b \to Z, \overline{W}$ . Although we do not include every possible process with cancellations here, we have verified that they are all perturbatively unitary. To focus on some of the most significant cancellations, we use the third-generation quarks as examples since the high-energy-growth terms are typically proportional to the fermion mass and, therefore, processes with the top-quark mass will pose the largest potential challenge for perturbative unitarity. However, we find that perturbative unitarity is satisfied for the other fermions in a similar way. In Sec. VI, we conclude.

Additionally, we have included the following Appendixes. In Appendix A, we have included the full T- and U-channel contributions to the processes  $Z, Z \rightarrow W, \overline{W}$  and  $W, \overline{W} \rightarrow W, \overline{W}$ , which are very long and would hurt readability of the main text. In Appendix B, we give the complete list of high-energy-growth terms for the T-channel diagram for  $Z, Z \rightarrow W, \overline{W}$  as an example of how many terms there are before cancellation of most of them with the U-channel diagram. In the main body of this paper, we always combine the T- and U-channel diagrams before showing them for conciseness and readability. In Appendix C, we show the high-energy expansion of the Feynman four-point vertices for comparison with our CSM results. We find that, for most of them, there is a significant difference between the CSM four-point vertex and the

Feynman four-point vertex. This is a nice way to see that there is a rearrangement of contributions to the amplitude.

All calculations in this paper are performed in the center of momentum (CM) frame with

$$\mathcal{E} = \frac{\sqrt{s}}{2},\tag{1}$$

where  $s = (p_1 + p_2)^2$  is the Mandelstahm variable and

$$|\mathbf{p}_1| = \frac{1}{2} \sqrt{\frac{(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2)}{s}}, \quad (2)$$

$$|\mathbf{p}_3| = \frac{1}{2} \sqrt{\frac{(s - (m_3 + m_4)^2)(s - (m_3 - m_4)^2)}{s}}.$$
 (3)

The third particle travels at an angle  $\theta$  with respect to the first particle's direction. Details of the spinors can be found in various places [6,7,29]. After calculating the spinor products for a particular spin combination, we Taylor expanded in high  $\mathcal{E}$ .

#### II. PROCESSES WITH A CONSTRUCTIVE FOUR-POINT VERTEX

In this section, we will consider the processes that include high-energy growth in individual diagrams and that require a four-point vertex to cancel it both in constructive calculations and in Feynman diagrams. There are only two such processes, namely  $h, h \rightarrow Z, Z$ and  $h, h \rightarrow W, \overline{W}$ .

## A. $h,h \to Z,Z$ and $h,h \to W,\bar{W}$

Both these processes have very similar details. In fact, only the masses are changed between them. We will describe them together, but have done them both. The contribution to both amplitudes coming from an S-channel Higgs is

$$\mathcal{M}_h = -\frac{3e^2 m_h^2}{2M_W^2 s_W^2} \frac{[\mathbf{34}]\langle \mathbf{34} \rangle}{(s - m_h^2)}.$$
(4)

By inspection, we can see that there is no high-energy growth from this term. Both the numerator and denominator grow at the same  $\mathcal{E}^2$  rate, canceling at high energy.

The contributions from a Z boson in the T channel and U channel of  $h, h \rightarrow Z, Z$  are

$$\mathcal{M}_{TZ} = -\frac{e^2}{2M_W^2 s_W^2(t - M_Z^2)} (2M_Z^2 \langle \mathbf{34} \rangle [\mathbf{34}] + [\mathbf{3}|p_1|\mathbf{4}\rangle [\mathbf{4}|p_1|\mathbf{3}\rangle - ([\mathbf{34}][\mathbf{3}|p_1|\mathbf{4}\rangle + \langle \mathbf{34}\rangle [\mathbf{4}|p_1|\mathbf{3}\rangle) M_Z)$$
(5)

and

$$\mathcal{M}_{UZ} = -rac{e^2}{2M_W^2 s_W^2 (u - M_Z^2)} (2M_Z^2 \langle \mathbf{34} \rangle [\mathbf{34}]$$

+ 
$$[\mathbf{3}|p_1|\mathbf{4}\rangle[\mathbf{4}|p_1|\mathbf{3}\rangle + (\langle \mathbf{34}\rangle[\mathbf{3}|p_1|\mathbf{4}\rangle + [\mathbf{34}][\mathbf{4}|p_1|\mathbf{3}\rangle)M_Z).$$
 (6)

For the contributions of a W boson in the T and U channels of the process  $h, h \to W, \overline{W}$ , simply replace all the  $M_Z$  with  $M_W$ . Otherwise, the contributions look the same.

If we Taylor expand in high energy and combine the T and U channels, we are left with (in both cases)

$$\mathcal{M}_{Z/W}^{(0,0)} = \frac{e^2}{M_W^2 s_W^2} \mathcal{E}^2.$$
(7)

The four-point vertex could potentially have contributions from  $\langle 34 \rangle^2$ ,  $\langle 34 \rangle [34]$ , and  $[34]^2$ . By a simple process of elimination, we find that, in order to cancel the highenergy-growth terms and achieve perturbative unitarity, the four-point vertex is given in Table I and the contribution to this amplitude is given by

$$\mathcal{M}_4 = \frac{e^2}{2M_W^2 s_W^2} [\mathbf{34}] \langle \mathbf{34} \rangle. \tag{8}$$

Taylor expanding at high energy, we are left with

$$\mathcal{M}_4^{(0,0)} = -\frac{e^2}{M_W^2 s_W^2} \mathcal{E}^2.$$
(9)

We can see that, with this, the high-energy growth cancels. In fact, we find that the amplitude agrees at all energies.

In Appendix C 3, we give the high-energy growth of the Feynman four-point vertex for this process. We can see that in this case, the high-energy growth is the same for the Feynman four-point vertex. Moreover, we have checked in SPINAS, and it agrees with the Feynman four-point vertex for all energies in this amplitude when squared. We have also checked the Higgs diagram and find that it agrees with the Feynman Higgs diagram at all energies when squared. Consequently, we find that each of these diagrams appears to be equivalent to Feynman diagrams, simply written in spinor notation. We have checked these statements at the squared-amplitude level in SPINAS.

## III. PROCESSES WITH NO CONSTRUCTIVE FOUR-POINT VERTICES I

In this section, we will consider processes with four massive vector bosons. Using Feynman diagrams, these include a four-point vertex. However, as we will show here, they do not involve a four-point vertex in constructive calculations. Although there is high-energy growth, it cancels among all the diagrams involving only three-point vertices. The fact that there is a four-point Feynman vertex,

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but no four-point constructive vertex, makes it immediately clear that constructive diagrams are not equivalent to Feynman diagrams and they are not simply Feynman diagrams rewritten in spinor notation. There is a significant rearrangement of contributions, especially when a fourpoint vertex is involved in one and not the other.

A. 
$$Z, Z \to W, \overline{W}$$

The contribution to the amplitude coming from an S-channel Higgs is

$$\mathcal{M}_h = -\frac{e^2}{M_W^2 s_W^2} \frac{\langle \mathbf{12} \rangle \langle \mathbf{34} \rangle [\mathbf{12}] [\mathbf{34}]}{(s - M_h^2)}.$$
 (10)

If we Taylor expand this in high energy, there is only one channel that contains energy growth. It is when all the external particles have helicity 0. We find

$$\mathcal{M}_{h}^{(0,0,0,0)} = -\frac{e^2}{M_W^2 s_W^2} \mathcal{E}^2.$$
(11)

All other channels have, at most, a constant. This agrees with the result from Feynman diagrams.

The contributions to the amplitude coming from a T- and U-channel W boson are quite complicated. We have included them in Appendix A 1. Each of these diagrams individually contributes energy growth in a large number of channels, with a maximum energy growth of  $\mathcal{E}^3$ . There are no quartic high-energy-growth terms in the constructive amplitude, unlike in Feynman diagrams. All of these energy-growth terms cancel except for one term. To keep this section legible, we give the energy growth after combining the T and U channels. The interested reader can see the terms for just the T channel in Appendix B. For the U channel, they are all equal in magnitude but opposite in sign, except for the single term that we show below. Therefore, after expansion and combining the T and U channels, we are left with

$$\mathcal{M}_{W}^{(0,0,0,0)} = \frac{e^2}{M_W^2 s_W^2} \mathcal{E}^2.$$
(12)

Comparing Eqs. (11) and (12), we see that all high-energy growth cancels. There is no need for a four-point vertex, and one is not allowed. If a four-point vertex were present, it would ruin perturbative unitarity at high energy. Moreover, we have found that the amplitude with only these contributions agrees with Feynman diagrams at all energies and for a variety of masses in SPINAS.

This is rather remarkable. It appears that the high-energy cancellation is better behaved in constructive amplitudes. There is no quartic energy growth at all, and all the high-energy growth that is present is canceled between the T- and U-channel *W* diagrams, with only a single quadratic energy-growth term left that cancels with the Higgs.

As we will see, the situation is similar for the four *W*-boson scattering and even better for amplitudes involving an external boson of helicity- $\pm 1$ .

It is remarkable for another reason. We see that constructive diagrams are *not* simply Feynman diagrams reduced to the spinor components. Constructive amplitudes rearrange the contributions. Even when the constructive diagrams have a resemblance to Feynman diagrams, they are not directly related for spin-1 and helicity- $\pm 1$  bosons. In this amplitude, only the Higgs contribution is the same as the Feynman-diagram contribution. This is not only true for high energy, but for all energies. We have checked this with SPINAS [29] at the squared amplitude level.

We further note that the cancellations that are present are under better control in constructive calculations. Feynman diagrams have a quartic high-energy growth that cancels between three diagrams (the T-channel W, the U-channel W, and four-point diagrams), which is absent here. There is no quartic energy growth in any diagram. Furthermore, the cancellations that do take place are mostly between just two diagrams (the T-channel and the U-channel diagrams), rather than the three Feynman diagrams.

## **B.** $W, W \rightarrow W, W$

The Higgs boson contributes in both the T and the U channels. The contributions to the amplitude are

$$\mathcal{M}_{Th} = -\frac{e^2}{M_W^2 s_W^2} \frac{\langle \mathbf{13} \rangle \langle \mathbf{24} \rangle [\mathbf{13}] [\mathbf{24}]}{(t - M_h^2)}, \qquad (13)$$

$$\mathcal{M}_{Uh} = -\frac{e^2}{M_W^2 s_W^2} \frac{\langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{14}] [\mathbf{23}]}{(u - M_h^2)}.$$
 (14)

After Taylor expanding in high energy, the combined result is

$$\mathcal{M}_{h}^{(0,0,0,0)} = \frac{e^2}{M_W^2 s_W^2} \mathcal{E}^2.$$
(15)

The individual channels do have high-energy growth in other polarization channels, but they all cancel except this one. This contribution to the amplitude agrees with Feynman diagrams.

The photon contributes in the T and U channels. Their full amplitude expressions can be found in Eqs. (A3) and (A4). As we can see in those expressions, the highest possible energy growth is cubic. Although that does appear in some polarization channels for individual diagrams, after combining the T and U channels, we are left with

$$\mathcal{M}_{\gamma}^{(0,0,0,0)} = -\frac{4e^2}{M_W^2} \mathcal{E}^2.$$
 (16)

The energy growth in all other channels cancels.

The Z boson also contributes in the T and U channels. Their full amplitude expressions can be found in Eqs. (A5) and (A6). After Taylor expanding in high energy, combining the T and U channels, and keeping only the highenergy-growth terms, we have

$$\mathcal{M}_{Z}^{(0,0,0,0)} = \frac{(3 - 4c_{W}^{2})e^{2}}{M_{W}^{2}s_{W}^{2}}\mathcal{E}^{2}.$$
 (17)

Combining the contributions, we have

$$\mathcal{M}^{(0,0,0,0)} = \frac{e^2 \mathcal{E}^2}{M_W^2 s_W^2} (1 - 4s_W^2 + 3 - 4c_W^2) = 0.$$
(18)

We find that all high-energy growth cancels. We have checked every polarization channel.

As in the case of  $Z, Z \rightarrow W, \overline{W}$ , there is no need for a four-point vertex in this process at all and, indeed, none is allowed. This is in contrast to Feynman rules where a four-point vertex is required. Once again, we see that constructive diagrams are significantly rearranged in some cases and are not Feynman diagrams written in a different form. We have included the high-energy growth for the Feynman four-point vertex contribution in Appendix C 2 to highlight the contrast. As in the previous case, only the Higgs diagrams contribute the same as Feynman diagrams. We have checked this with SPINAS [29] at the squared amplitude level for all energies.

As in the previous subsection, we again comment on the reduced dependence on precise cancellations between diagrams. There is no quartic high-energy growth here in any diagram, and no need for its cancellation. Moreover, the cancellations that do take place are mostly between just two diagrams at a time. The majority of them are between the T- and U-channel diagrams with one mediator. That is, most of the energy-growth cancellations are between the photon T- and U-channel diagrams and separately between the Z-boson T- and U-channel diagrams, whereas in Feynman diagrams, most of these cancellations are between five diagrams (including the four-point vertex diagram). This should lead to better numerical stability in phase-space integrations.

### IV. PROCESSES WITH NO CONSTRUCTIVE FOUR-POINT VERTICES II

In this section, we consider the processes which do *not* have any high-energy growth and which do *not* have any four-point vertices in the CSM, but *do* have both high-energy growth and four-point vertices in the Feynman diagram version of the SM. We will also see that for these processes, there is a significant rearrangement of contributions and that constructive diagrams are not equivalent to Feynman diagrams for these processes.

Furthermore, the property that these amplitudes do not have any high-energy growth and, therefore, do not need any cancellations in order to satisfy perturbative unitarity extends beyond the amplitudes shown here. We have found that it is true for any four-point amplitude with any number of external photons or gluons [28].

A. 
$$\gamma^+, Z \to W, \bar{W}$$

There is only one constructive diagram for this process. It could come from either the T or the U channels. They both gave the identical result, which is

$$\mathcal{M} = \frac{2e^2}{c_W s_W (t - M_W^2) (u - M_W^2)} (c_W^2 [14]^2 \langle \mathbf{23} \rangle^2 + (2c_W^2 - 1) [13] \langle \mathbf{24} \rangle [14] \langle \mathbf{23} \rangle + c_W [12] \langle \mathbf{34} \rangle [14] \langle \mathbf{23} \rangle + c_W^2 [13]^2 \langle \mathbf{24} \rangle^2 - c_W [12] [13] \langle \mathbf{24} \rangle \langle \mathbf{34} \rangle + c_W^2 [12]^2 \langle \mathbf{34} \rangle^2).$$
(19)

The high-energy Taylor expansion gives no energy-growth terms. In fact, this can be seen by inspection since there are no momentum terms in the numerator. There are four spinor products per term, each of which grows as  $\mathcal{E}^4$ . But, the denominator grows at the same rate. Thus, there is no highenergy growth. No four-point vertex is required to cancel any high-energy growth. Moreover, the amplitude in Eq. (19) is equal to the combined Feynman diagrams at all energies, not just at high energies, which we have checked with SPINAS. In other words, no four-point vertex is required or even allowed for any energies. This is another sign that the constructive diagrams are not simply the Feynman diagrams expanded in terms of spinors, in general. There is a rearrangement of contributions to the amplitude regarding the spin-1 boson contributions in some processes. In this case, it completely eliminates the need for a four-point vertex. Among other things, this means that one constructive diagram in this case accounts for three Feynman diagrams.

Furthermore, note that Feynman diagrams result in highenergy growth that must be canceled between terms. This results in large cancellations that must be carefully maintained and are a potential source of loss of precision. In the constructive case, on the other hand, there are not highenergy-growth terms with large cancellations at all. Even within the amplitude expression, each term does not grow at high energy. Thus, we find that constructive calculations result in better behaved numerical calculations and are under better control, especially in amplitudes, such as this one, where no high-energy growth exists at all.

Although there is no four-point vertex here to compare with, we nevertheless give the four-point vertex from Feynman diagrams in Appendix C 4 as well as its highenergy-growth terms for contrast. We can see that there are cancellations in every polarization channel that has at least one of the vector bosons in the longitudinal mode. In fact, in the channel where all three massive vector bosons are longitudinal, the energy growth is cubic ( $\mathcal{E}^3$ ). All of this is nonexistent in this constructive amplitude.

**B.** 
$$\gamma^+, \gamma^+ \to W, \bar{W}$$

The amplitude for this process is obtained from either the T- or the U-channel diagram. They gave identical results. It is

$$\mathcal{M}_W = \frac{2e^2[12]^2 \langle \mathbf{34} \rangle^2}{(t - M_W^2)(u - M_W^2)}.$$
 (20)

It can be seen by inspection that the numerator and denominator grow at the same rate at high energy; therefore, the amplitude does not have any high-energy-growth terms. Once again, there is no need for a four-point vertex and, indeed, *none is allowed*. This amplitude agrees with Feynman diagrams for all energies, as we checked with SPINAS. We again see that this diagram is *not* a Feynman diagram written in spinor form. There is a significant rearrangement of contributions in constructive diagrams, resulting in a much simpler result.

Again, for contrast, we show the Feynman four-point vertex contribution in Appendix C 5, along with its highenergy-growth terms. All five longitudinal polarization channels grow at either linear or quadratic order at high energy. These require cancellations against the other Feynman diagrams. In the constructive case, no high-energy growth exists at all, and there is no cancellation. As before, the calculation is much better behaved and under much better numerical control. In fact, this process is special in that there are no cancellations at all, as there is only one term.

C. 
$$\gamma^+, \gamma^- \to W, \bar{W}$$

Once again, the amplitude for this process is obtained from either the T- or the U-channel diagram. They gave identical results. It is

$$\mathcal{M}_{W} = \frac{2e^{2}(\langle 2\mathbf{4} \rangle [1\mathbf{3}] + \langle 2\mathbf{3} \rangle [1\mathbf{4}])^{2}}{(t - M_{W}^{2})(u - M_{W}^{2})}.$$
 (21)

It can be seen by inspection that the numerator and denominator grow at the same rate at high energy; therefore, the amplitude does not have any high-energy-growth terms. Once again, there is no need for a four-point vertex and *none is allowed*. This simple amplitude agrees with the combined Feynman diagrams for all energies, as we show with SPINAS. Again, we see that this one diagram is equivalent to all three Feynman diagrams, and there is no direct diagram-for-diagram relationship. The contributions are rearranged.

As before, we include the Feynman four-point vertex contribution in Appendix C 5, along with its high-energy-growth terms. As in the previous subsection, all five longitudinal polarization channels contribute high-energy growth that must be canceled against the other Feynman

diagrams. Once again, the constructive amplitude is not only simpler, but under better numerical control, since there are no high-energy-growth terms to cancel.

### V. PROCESSES WITH NO CONSTRUCTIVE FOUR-POINT VERTICES III

In this section, we will consider a selection of processes that do not have four-point vertices, either in constructive calculations or in Feynman diagrams. For the processes in this section, there is agreement between constructive and Feynman contributions for each internal particle.

### A. $Z, Z \rightarrow Z, Z$

The contribution of the Higgs boson in the S, T, and U channels are

$$\mathcal{M}_{S} = -\frac{e^{2}}{M_{W}^{2}s_{W}^{2}} \frac{\langle \mathbf{12} \rangle \langle \mathbf{34} \rangle [\mathbf{12}] [\mathbf{34}]}{(s - m_{h}^{2})}, \qquad (22)$$

$$\mathcal{M}_T = -\frac{e^2}{M_W^2 s_W^2} \frac{\langle \mathbf{13} \rangle \langle \mathbf{24} \rangle [\mathbf{13}] [\mathbf{24}]}{(t - m_h^2)}, \qquad (23)$$

$$\mathcal{M}_U = -\frac{e^2}{M_W^2 s_W^2} \frac{\langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{14}] [\mathbf{23}]}{(u - m_h^2)}.$$
 (24)

Taylor expanding, we have for the S channel

$$\mathcal{M}_{S}^{(0,0,0)} = -\frac{e^{2}}{M_{W}^{2} s_{W}^{2}} \mathcal{E}^{2}.$$
 (25)

The combined contribution from the T and U channels is

$$\mathcal{M}_{TU}^{(0,0,0,0)} = \frac{e^2}{M_W^2 s_W^2} \mathcal{E}^2, \tag{26}$$

which is exactly the opposite and cancels it, so there is no high-energy growth.

This occurs, as we can see, without the need for a fourpoint vertex, and none is allowed. In this case, there is also no Feynman four-point vertex contributing, so these cases are similar in this respect. We have found agreement with the Feynman-diagram amplitude for all energies with SPINAS.

## **B.** $t, \bar{t} \to W, \bar{W}$

We will also consider processes with two fermions that have a longitudinal vector boson. For concreteness and conciseness, we will discuss it in the context of the thirdgeneration quarks, but the processes with other generations of quarks or leptons are similar.

We begin with the process  $t, \overline{t} \to W, \overline{W}$ . It has a contribution from the Higgs in the S channel, which is

$$\mathcal{M}_{h} = \frac{-e^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}} \frac{\langle \langle \mathbf{12} \rangle + [\mathbf{12}] \rangle \langle \mathbf{34} \rangle [\mathbf{34}]}{(s - m_{h}^{2})}.$$
 (27)

Taylor expanding in high energy results in

$$\mathcal{M}_{h}^{(-\frac{1}{2},-\frac{1}{2},0,0)} = \frac{e^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}}\mathcal{E},$$
(28)

$$\mathcal{M}_{h}^{(\frac{1}{2},\frac{1}{2},0,0)} = -\frac{e^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}}\mathcal{E}.$$
(29)

The contribution from the b quark in the T channel is

$$\mathcal{M}_{b} = \frac{-e^{2}}{M_{W}^{2} s_{W}^{2}} \frac{\langle \mathbf{13} \rangle [\mathbf{24}] (M_{W} \langle \mathbf{34} \rangle - [\mathbf{3}|p_{1}|\mathbf{4} \rangle)}{(t - m_{b}^{2})}.$$
 (30)

Taylor expanding at high energy results in

$$\mathcal{M}_{b}^{(-\frac{1}{2},-\frac{1}{2},0,0)} = -\frac{e^{2}m_{t}(1+\cos(\theta))}{2M_{W}^{2}s_{W}^{2}}\mathcal{E},$$
 (31)

$$\mathcal{M}_{b}^{(-\frac{1}{2},\frac{1}{2},-1,0)} = -\frac{\sqrt{2}e^{2}\sin^{2}(\frac{\theta}{2})}{M_{W}s_{W}^{2}}\mathcal{E},$$
(32)

$$\mathcal{M}_{b}^{(-\frac{1}{2},\frac{1}{2},0,-1)} = \frac{e^{2}(1+\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
(33)

$$\mathcal{M}_{b}^{(-\frac{1}{2}\frac{1}{2},0,0)} = -\frac{e^{2}\sin(\theta)}{M_{W}^{2}s_{W}^{2}}\mathcal{E}^{2},$$
(34)

$$\mathcal{M}_{b}^{(-\frac{1}{2},\frac{1}{2},0,1)} = \frac{e^{2}(1-\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
(35)

$$\mathcal{M}_{b}^{(-\frac{1}{2},\frac{1}{2},1,0)} = -\frac{e^{2}(1+\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
(36)

$$\mathcal{M}_{b}^{(\frac{1}{2},\frac{1}{2},0,0)} = \frac{e^{2}m_{t}(1+\cos(\theta))}{2M_{W}^{2}s_{W}^{2}}\mathcal{E}.$$
 (37)

The contribution from the photon in the S channel is

$$\mathcal{M}_{\gamma} = \frac{-2e^2Q}{M_W^2s} (M_W(\langle \mathbf{34} \rangle + [\mathbf{34}]) \\ \times (\langle \mathbf{24} \rangle [\mathbf{13}] + \langle \mathbf{23} \rangle [\mathbf{14}] + \langle \mathbf{14} \rangle [\mathbf{23}] + \langle \mathbf{13} \rangle [\mathbf{24}]) \\ + (\langle \mathbf{34} \rangle [\mathbf{34}] [\mathbf{1}|p_3|\mathbf{2} \rangle + \langle \mathbf{34} \rangle [\mathbf{34}] [\mathbf{2}|p_3|\mathbf{1} \rangle)), \quad (\mathbf{38})$$

where Q = 2/3. After Taylor expanding, we have

$$\mathcal{M}_{\gamma}^{(\frac{-1}{2},-\frac{1}{2},0,0)} = \frac{2e^2 Q m_t \cos(\theta)}{M_W^2} \mathcal{E},$$
(39)

$$\mathcal{M}_{\gamma}^{(-\frac{1}{2},\frac{1}{2},-1,0)} = \frac{\sqrt{2}e^2Q(1-\cos(\theta))}{M_W}\mathcal{E},$$
 (40)

$$\mathcal{M}_{\gamma}^{(-\frac{1}{2}\frac{1}{2},0,-1)} = -\frac{\sqrt{2}e^2Q(1+\cos(\theta))}{M_W}\mathcal{E},\qquad(41)$$

$$\mathcal{M}_{\gamma}^{(-\frac{1}{2},\frac{1}{2},0,0)} = \frac{2e^2Q\sin(\theta)}{M_W^2}\mathcal{E}^2,$$
(42)

$$\mathcal{M}_{\gamma}^{(-\frac{1}{2},\frac{1}{2},0,1)} = \frac{\sqrt{2}e^2Q(-1+\cos(\theta))}{M_W}\mathcal{E},$$
 (43)

$$\mathcal{M}_{\gamma}^{(-\frac{1}{2},\frac{1}{2},1,0)} = \frac{\sqrt{2}e^2Q(1+\cos(\theta))}{M_W}\mathcal{E},$$
 (44)

$$\mathcal{M}_{\gamma}^{(\frac{1}{2},-\frac{1}{2},-1,0)} = -\frac{\sqrt{2}e^2Q(1+\cos(\theta))}{M_W}\mathcal{E},\qquad(45)$$

$$\mathcal{M}_{\gamma}^{(\frac{1}{2},-\frac{1}{2},0,-1)} = \frac{\sqrt{2}e^2Q(1-\cos(\theta))}{M_W}\mathcal{E},$$
 (46)

$$\mathcal{M}_{\gamma}^{(\frac{1}{2},-\frac{1}{2},0,0)} = \frac{2e^2Q\sin(\theta)}{M_W^2}\mathcal{E}^2,$$
(47)

$$\mathcal{M}_{\gamma}^{(\frac{1}{2},-\frac{1}{2},0,1)} = \frac{\sqrt{2}e^2Q(1+\cos(\theta))}{M_W}\mathcal{E},$$
 (48)

$$\mathcal{M}_{\gamma}^{(\frac{1}{2},-\frac{1}{2},1,0)} = \frac{\sqrt{2}e^2Q(-1+\cos(\theta))}{M_W}\mathcal{E},$$
 (49)

$$\mathcal{M}_{\gamma}^{(\frac{1}{2},\frac{1}{2},0,0)} = -\frac{2e^2 Q m_t \cos(\theta)}{M_W^2} \mathcal{E}.$$
 (50)

The contribution from the Z boson in the S channel is

$$\mathcal{M}_{Z} = \frac{-e^{2}}{2M_{W}^{2}s_{W}^{2}(s - M_{Z}^{2})} (2\langle \mathbf{34} \rangle [\mathbf{34}] (g_{R}[\mathbf{1}|p_{3}|\mathbf{2}) + g_{L}[\mathbf{2}|p_{3}|\mathbf{1}\rangle) + m_{t}(\langle \mathbf{12} \rangle - [\mathbf{12}]) \langle \mathbf{34} \rangle [\mathbf{34}] + 2M_{W}(\langle \mathbf{34} \rangle + [\mathbf{34}]) [g_{R}(\langle \mathbf{24} \rangle [\mathbf{13}] + \langle \mathbf{23} \rangle [\mathbf{14}]) + g_{L}(\langle \mathbf{14} \rangle [\mathbf{23}] + \langle \mathbf{13} \rangle [\mathbf{24}])]),$$
(51)

where  $g_L = -2Qs_W^2 + 1$  and  $g_R = -2Qs_W^2$ . After Taylor expanding, we have

$$\mathcal{M}_{Z}^{(\frac{1}{2},-\frac{1}{2},0,0)} = \frac{(g_{L}+g_{R})e^{2}m_{l}\cos(\theta)}{2M_{W}^{2}s_{W}^{2}}\mathcal{E},$$
 (52)

$$\mathcal{M}_{Z}^{(-\frac{1}{2}\frac{1}{2},-1,0)} = \frac{g_{L}e^{2}(1-\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
(53)

$$\mathcal{M}_{Z}^{(-\frac{1}{2},\frac{1}{2},0,-1)} = -\frac{g_{L}e^{2}(1+\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
 (54)

$$\mathcal{M}_{Z}^{(-\frac{1}{2},\frac{1}{2},0,0)} = \frac{g_{L}e^{2}\sin(\theta)}{M_{W}^{2}s_{W}^{2}}\mathcal{E}^{2},$$
(55)

$$\mathcal{M}_{Z}^{(-\frac{1}{2},\frac{1}{2},0,1)} = -\frac{\sqrt{2}g_{L}e^{2}\sin^{2}(\frac{\theta}{2})}{M_{W}s_{W}^{2}}\mathcal{E},$$
(56)

$$\mathcal{M}_{Z}^{(-\frac{1}{2}\frac{1}{2},1,0)} = \frac{g_{L}e^{2}(1+\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
(57)

$$\mathcal{M}_{Z}^{(\frac{1}{2},-\frac{1}{2},-1,0)} = -\frac{g_{R}e^{2}(1+\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
 (58)

$$\mathcal{M}_{Z}^{(\frac{1}{2},-\frac{1}{2},0,-1)} = \frac{g_{R}e^{2}(1-\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
(59)

$$\mathcal{M}_{Z}^{(\frac{1}{2},-\frac{1}{2},0,0)} = \frac{g_{R}e^{2}\sin(\theta)}{M_{W}^{2}s_{W}^{2}}\mathcal{E}^{2},$$
(60)

$$\mathcal{M}_{Z}^{(\frac{1}{2},-\frac{1}{2},0,1)} = \frac{g_{R}e^{2}(1+\cos(\theta))}{\sqrt{2}M_{W}s_{W}^{2}}\mathcal{E},$$
 (61)

$$\mathcal{M}_{Z}^{(\frac{1}{2},-\frac{1}{2},1,0)} = -\frac{\sqrt{2}g_{R}e^{2}\sin^{2}(\frac{\theta}{2})}{M_{W}s_{W}^{2}}\mathcal{E},$$
(62)

$$\mathcal{M}_{Z}^{(\frac{1}{2},\frac{1}{2},0,0)} = -\frac{(g_{L} + g_{R})e^{2}m_{t}\cos(\theta)}{2M_{W}^{2}s_{W}^{2}}\mathcal{E},$$
 (63)

where  $g_L = -2Qs_W^2 + 1$  and  $g_R = -2Qs_W^2$  for the top quark.

After combining the contributions, the  $(-\frac{1}{2}, -\frac{1}{2}, 0, 0)$  and  $(\frac{1}{2}, \frac{1}{2}, 0, 0)$  polarization combinations are proportional to

$$(4Qs_W^2 + g_L + g_R - 1) = 0. (64)$$

The  $(-\frac{1}{2}, \frac{1}{2}, \pm 1, 0)$ ,  $(-\frac{1}{2}, \frac{1}{2}, 0, \pm 1)$ , and  $(-\frac{1}{2}, \frac{1}{2}, 0, 0)$  polarization combinations are proportional to

$$(2Qs_W^2 + g_L - 1) = 0. (65)$$

The  $(\frac{1}{2}, -\frac{1}{2}, \pm 1, 0)$ ,  $(\frac{1}{2}, -\frac{1}{2}, 0, \pm 1)$ , and  $(\frac{1}{2}, -\frac{1}{2}, 0, 0)$  polarization combinations are proportional to

$$(2Qs_W^2 + g_R) = 0. (66)$$

All the high-energy-growth terms cancel. We did this for the top quark, but a similar relation holds for all the fermions.

Furthermore, we find agreement at all energies in SPINAS. There is no need or tolerance for a four-point vertex here. We have also compared separately the contributions from the internal Higgs, bottom quark, photon, and Z boson, and we find agreement in all cases. In this process, the diagrams are in one-to-one correspondence.

C.  $t, \bar{t} \rightarrow Z, Z$ 

The contribution from the Higgs in the S channel is

$$\mathcal{M}_{h} = -\frac{e^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}} \frac{\langle \mathbf{34} \rangle [\mathbf{34}](\langle \mathbf{12} \rangle + [\mathbf{12}])}{(s - m_{h}^{2})}.$$
 (67)

Taylor expanding gives

$$\mathcal{M}_{h}^{(-\frac{1}{2},-\frac{1}{2},0,0)} = \frac{e^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}}\mathcal{E},$$
(68)

$$\mathcal{M}_{h}^{(\frac{1}{2},0,0)} = -\frac{e^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}}\mathcal{E}.$$
(69)

The contributions from the top quark in the T and U channels are

$$\mathcal{M}_{Tt} = -\frac{e^2}{2M_W^2 s_W^2 (t - m_t^2)} (g_L^2 (M_Z \langle \mathbf{34} \rangle - [\mathbf{3}|p_1|\mathbf{4} \rangle) \langle \mathbf{13} \rangle [\mathbf{24}] + g_R^2 (M_Z [\mathbf{34}] - [\mathbf{4}|p_1|\mathbf{3} \rangle) \langle \mathbf{24} \rangle [\mathbf{13}] - g_L g_R m_t (\langle \mathbf{34} \rangle [\mathbf{13}] [\mathbf{24}] + \langle \mathbf{13} \rangle \langle \mathbf{24} \rangle [\mathbf{34}])),$$
(70)

$$\mathcal{M}_{Ut} = -\frac{e^2}{2M_W^2 s_W^2 (u - m_t^2)} (-g_R^2 (M_Z [\mathbf{34}] + [\mathbf{3}|p_1|\mathbf{4}\rangle) \\ \times \langle \mathbf{23} \rangle [\mathbf{14}] - g_L^2 (M_Z \langle \mathbf{34} \rangle + [\mathbf{4}|p_1|\mathbf{3}\rangle) \langle \mathbf{14} \rangle [\mathbf{23}] \\ + g_L g_R m_t (\langle \mathbf{34} \rangle [\mathbf{14}] [\mathbf{23}] + \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{34}])).$$
(71)

After Taylor expanding in high energy and combining the T and U channels, we have

$$\mathcal{M}_{t}^{(-\frac{1}{2},-\frac{1}{2},0,0)} = -\frac{e^{2}(g_{L}-g_{R})^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}}\mathcal{E},$$
(72)

$$\mathcal{M}_{t}^{(\frac{1}{2},\frac{1}{2},0,0)} = \frac{e^{2}(g_{L}-g_{R})^{2}m_{t}}{2M_{W}^{2}s_{W}^{2}}\mathcal{E}.$$
(73)

Combining all the contributions, we see that the highenergy-textgrowth term is proportional to  $(g_L - g_R)^2 - 1$ , which vanishes.

Moreover, we have compared the constructive amplitude with Feynman diagrams for all energies and found agreement in SPINAS. No four-point vertex is needed or allowed here. We have also tested and find the contribution from an internal Higgs and separately from an internal top quark agree with Feynman diagrams. No rearrangement of contributions is present here.

# **D.** $\bar{t}, b \to Z, \bar{W}$

The contribution from the W boson in the S channel is

$$\mathcal{M}_{W} = \frac{-e^{2}}{\sqrt{2}M_{W}^{2}M_{Z}^{2}s_{W}^{2}(s - M_{W}^{2})} (2M_{W}^{3}\langle\langle\mathbf{24}\rangle\langle\mathbf{34}\rangle[\mathbf{13}] + \langle\mathbf{23}\rangle[\mathbf{14}][\mathbf{34}]) + 2M_{W}^{2}M_{Z}\langle\langle\mathbf{23}\rangle\langle\mathbf{34}\rangle[\mathbf{14}] + \langle\mathbf{24}\rangle[\mathbf{13}][\mathbf{34}]) + m_{t}(M_{Z}^{2} - 2M_{W}^{2})\langle\mathbf{12}\rangle\langle\mathbf{34}\rangle[\mathbf{34}] - m_{b}(M_{Z}^{2} - 2M_{W}^{2})\langle\mathbf{34}\rangle[\mathbf{12}][\mathbf{34}] + 2M_{W}^{2}\langle\mathbf{34}\rangle[\mathbf{34}][\mathbf{1}|p_{3}|\mathbf{2}\rangle).$$
(74)

Taylor expanding gives us

$$\mathcal{M}_{W}^{(\frac{-1}{2},-\frac{1}{2},0,0)} = \frac{e^{2}m_{t}(1-c_{W}^{2}+c_{W}^{2}\cos(\theta))}{\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}, \quad (75)$$

$$\mathcal{M}_{W}^{(\frac{1}{2},-\frac{1}{2},-1,0)} = -\frac{e^{2}(1+\cos(\theta))\mathcal{E}}{M_{Z}s_{W}^{2}},$$
(76)

$$\mathcal{M}_{W}^{(\frac{1}{2},-\frac{1}{2},0,-1)} = \frac{e^{2}c_{W}(1-\cos(\theta))\mathcal{E}}{M_{Z}s_{W}^{2}},$$
(77)

$$\mathcal{M}_{W}^{(\frac{1}{2},-\frac{1}{2},0,0)} = \frac{\sqrt{2}e^{2}\sin(\theta)}{M_{Z}^{2}s_{W}^{2}}\mathcal{E}^{2},$$
(78)

$$\mathcal{M}_{W}^{(\frac{1}{2},-\frac{1}{2},0,1)} = \frac{e^{2}c_{W}(1+\cos(\theta))\mathcal{E}}{M_{Z}s_{W}^{2}},$$
(79)

$$\mathcal{M}_{W}^{(\frac{1}{2},-\frac{1}{2},1,0)} = \frac{e^{2}(-1+\cos(\theta))\mathcal{E}}{M_{Z}s_{W}^{2}},$$
(80)

$$\mathcal{M}_{W}^{(\frac{1}{2},\frac{1}{2},0,0)} = \frac{e^{2}m_{b}(1-c_{W}^{2}-c_{W}^{2}\cos(\theta))}{\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}.$$
 (81)

The contribution from a top quark in the T channel is

$$\mathcal{M}_{t} = \frac{e^{2} \langle \mathbf{24} \rangle}{\sqrt{2} M_{W}^{2} s_{W}^{2}(t - m_{t}^{2})} (g_{Ru} m_{t} \langle \mathbf{13} \rangle [\mathbf{34}] - g_{Lu} [\mathbf{13}] (M_{Z} [\mathbf{34}] - [\mathbf{4}|p_{1}|\mathbf{3}\rangle)).$$
(82)

After Taylor expanding,

$$\mathcal{M}_{t}^{(-\frac{1}{2},-\frac{1}{2},0,0)} = -\frac{e^{2}m_{t}(g_{Lu} - 2g_{Ru} + g_{Lu}\cos(\theta))}{2\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}, \quad (83)$$

$$\mathcal{M}_{t}^{(\frac{1}{2},-\frac{1}{2},-1,0)} = \frac{e^{2}g_{Lu}(1+\cos(\theta))}{2c_{W}^{2}M_{Z}s_{W}^{2}}\mathcal{E},$$
(84)

$$\mathcal{M}_{t}^{(\frac{1}{2},-\frac{1}{2},0,-1)} = -\frac{e^{2}g_{Lu}\sin^{2}(\frac{\theta}{2})}{c_{W}M_{Z}s_{W}^{2}}\mathcal{E},$$
(85)

$$\mathcal{M}_{t}^{(\frac{1}{2},-\frac{1}{2},0,0)} = -\frac{e^{2}g_{Lu}\sin(\theta)}{\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}^{2},$$
(86)

$$\mathcal{M}_{t}^{(\frac{1}{2},-\frac{1}{2},0,1)} = -\frac{e^{2}g_{Lu}(1+\cos(\theta))}{2c_{W}M_{Z}s_{W}^{2}}\mathcal{E},$$
(87)

$$\mathcal{M}_{t}^{(\frac{1}{2},-\frac{1}{2},1,0)} = \frac{e^{2}g_{Lu}(1-\cos(\theta))}{2c_{W}^{2}M_{Z}s_{W}^{2}}\mathcal{E},$$
(88)

$$\mathcal{M}_{t}^{(\frac{1}{2},\frac{1}{2},0,0)} = \frac{e^{2}g_{Lu}m_{b}(1+\cos(\theta))}{2\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}.$$
(89)

The contribution from the b quark in the U channel is

$$\mathcal{M}_{b} = \frac{-e^{2}[\mathbf{14}]}{\sqrt{2}M_{W}^{2}s_{W}^{2}(u-m_{b}^{2})}(g_{Rd}m_{b}\langle\mathbf{34}\rangle[\mathbf{23}] - g_{Ld}\langle\mathbf{23}\rangle(M_{W}[\mathbf{34}] + [\mathbf{3}|p_{1}|\mathbf{4}\rangle)).$$
(90)

Taylor expanding, we have

$$\mathcal{M}_{b}^{(-\frac{1}{2},-\frac{1}{2},0,0)} = -\frac{e^{2}g_{Ld}m_{t}\sin^{2}(\frac{\theta}{2})}{\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E},$$
(91)

$$\mathcal{M}_{b}^{(\frac{1}{2},-\frac{1}{2},-1,0)} = -\frac{e^{2}g_{Ld}(1+\cos(\theta))}{2c_{W}^{2}M_{Z}s_{W}^{2}}\mathcal{E},$$
(92)

$$\mathcal{M}_{b}^{(\frac{1}{2},-\frac{1}{2},0,-1)} = \frac{e^{2}g_{Ld}(1-\cos(\theta))}{2c_{W}M_{Z}s_{W}^{2}}\mathcal{E},$$
(93)

$$\mathcal{M}_{b}^{(\frac{1}{2},-\frac{1}{2},0,0)} = \frac{e^{2}g_{Ld}\sin(\theta)}{\sqrt{2}c_{W}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}^{2},$$
(94)

$$\mathcal{M}_{b}^{(\frac{1}{2},-\frac{1}{2},0,1)} = \frac{e^{2}g_{Ld}(1+\cos(\theta))}{2c_{W}M_{Z}s_{W}^{2}}\mathcal{E},$$
(95)

$$\mathcal{M}_{b}^{(\frac{1}{2},-\frac{1}{2},1,0)} = -\frac{e^{2}g_{Ld}\sin^{2}(\frac{\theta}{2})}{c_{W}^{2}M_{Z}s_{W}^{2}}\mathcal{E},$$
(96)

$$\mathcal{M}_{b}^{(\frac{1}{2}\frac{1}{2}0,0)} = -\frac{e^{2}m_{b}(2g_{Rd} - g_{Ld} + g_{Ld}\cos(\theta))}{2\sqrt{2}c_{w}^{2}M_{Z}^{2}s_{W}^{2}}\mathcal{E}.$$
 (97)

After combining, every term is proportional to one of

$$2c_W^2 + g_{Ld} - g_{Lu} = 2c_W^2 + 2(Q_u - Q_d)s_W^2 - 2, \quad (98)$$

$$2c_W^2 + g_{Ld} + g_{Lu} - 2g_{Ru} - 2 = 2c_W^2 + 2(Q_u - Q_d)s_W^2 - 2,$$
(99)

$$2c_W^2 - g_{Ld} - g_{Lu} + 2g_{Rd} - 2 = 2c_W^2 + 2(Q_u - Q_d)s_W^2 - 2,$$
(100)

and they all vanish. We have also validated this process at all energies and find agreement with Feynman diagrams. No fourpoint vertex is needed or allowed. Further, we have tested the diagrams individually, and they agree with Feynman diagrams, so there is no rearrangement of contributions here.

#### **VI. CONCLUSIONS**

In this paper, we have analyzed perturbative unitarity in all the processes involving a longitudinal vector boson. We have found the high-energy expansion of all the CSM contributions coming from S-, T-, and U-channel diagrams using constructive techniques. In the cases that these do not cancel on their own, we have identified the four-point vertex required to cancel them. Ultimately, we have shown that perturbative unitarity is satisfied in all these processes, and we have found a complete set of four-point vertices in the CSM (shown in Table I) in order to achieve these cancellations. We have included the four-point vertex for four Higgs bosons for completeness.

We have shown that, not only do we not need more fourpoint vertices in the CSM relative to Feynman diagrams, we actually need fewer. It was already well known that the CSM does not require a four-point gluon vertex, but we additionally show in this paper that we also do not need a Z, Z,  $\overline{W}$ , W vertex, a W, W,  $\overline{W}$ ,  $\overline{W}$  vertex, a  $\gamma$ , Z,  $\overline{W}$ , W vertex, or a  $\gamma$ ,  $\gamma$ ,  $\overline{W}$ , W vertex. In other words, we explicitly find that we do not need any "contact" terms at four points, especially if we note that we did not need any other fourpoint vertices (or contact terms) for any other four-point amplitudes in the CSM that were given in [28]. In fact, we claim that this is what we should expect in a renormalizable theory such as the SM, even when constructive, and we do not expect to need any extra contact terms at any order in external particles or loops in the CSM. If we did need further four-point vertices (or contact terms) at a higher loop level, then those four-point vertices would arguably ruin the tree-level four-point amplitudes that do not allow them. Showing this is a goal of future research.

We have further shown that there is a significant rearrangement of contributions to the amplitude in many cases, especially those involving a four-point vertex in Feynman diagrams but do not have a four-point vertex in constructive calculations. In particular, we see that the constructive calculations for  $Z, Z \rightarrow W, \overline{W}, W, W \rightarrow W, W,$  $\gamma, Z, W, \overline{W}$ , and  $\gamma, \gamma, W, \overline{W}$  do not even allow a four-point vertex in the CSM, but it is required for Feynman diagrams. This shows that constructive diagrams are not in one-to-one correspondence with Feynman diagrams in general, even when there is a superficial resemblance. In many cases, the contributions to the complete amplitude are rearranged. There was already a hint that this was true with processes with external photons or gluons, and we have found agreement with this more generally; but we have additionally shown that it is also true for some processes that do not include external photons or gluons. In particular, we have shown it for amplitudes with four external Z or W bosons. We expect this to compound as the number of the external number of particles grows, and understanding this will be one of the goals of future projects. We have shown it explicitly in their high-energy expansions here. But, we have also found it for all energies using SPINAS.

Finally, we have noted that the cancellations are under much better control in these constructive calculations compared to Feynman diagrams. First, there is no quartic energy growth at all in the constructive amplitudes for  $Z, Z \to W, \overline{W}$ and  $W, W \rightarrow W, W$ , unlike in Feynman diagrams. Further, we have found that the cancellations that do take place are mostly between pairs of diagrams involving the same internal particle, whereas in Feynman diagrams, a larger set of particles involving a four-point vertex and different internal particles are typically involved. The final cancellation between diagrams with different particles is only quadratic and only in the all-longitudinal channel. Furthermore, if the process involves an external photon or gluon, there is no high-energy growth at all. Therefore, there is no cancellation at all. In fact, there is only one term in the expression with both propagator denominators, and every term does not have energy growth. All of this should lead to better numerical stability in phase-space integrators.

## APPENDIX A: LONG CONTRIBUTIONS TO $Z, Z \rightarrow W, \overline{W}$ AND $W, \overline{W} \rightarrow W, \overline{W}$

In this appendix, we will give the full expressions for the longer contributions to  $Z, Z \rightarrow W, \overline{W}$  and  $W, \overline{W} \rightarrow W, \overline{W}$ . The expressions when the triple-boson vertices are involved are quite lengthy, so we have included them here to improve the readability of the main text.

## **1.** Long contribution to $Z, Z \rightarrow W, \overline{W}$

The contributions to the amplitude for  $Z, Z \rightarrow W, \overline{W}$  coming from a T- and a U-channel W boson are

$$\mathcal{M}_{TW} = \frac{e^2}{2M_W^3 s_W^2 (t - M_W^2)} ((-4c_W^3 \langle 24 \rangle \langle 34 \rangle [12] [13] + 3c_W^3 \langle 13 \rangle \langle 23 \rangle \langle 24 \rangle [14] - 4c_W^2 \langle 23 \rangle \langle 24 \rangle [13] [14] \\ - 3c_W^4 \langle 14 \rangle \langle 24 \rangle [13] [23] - 3c_W^2 \langle 13 \rangle \langle 24 \rangle [14] [23] - c_W^4 \langle 13 \rangle \langle 24 \rangle [14] [23] + c_W^3 \langle 24 \rangle [13] [14] [23] \\ + 3c_W^3 \langle 13 \rangle \langle 14 \rangle \langle 23 \rangle [24] - 3c_W^2 \langle 14 \rangle \langle 23 \rangle [13] [24] - c_W^4 \langle 14 \rangle \langle 23 \rangle [13] [24] - 2 \langle 13 \rangle \langle 24 \rangle [13] [24] \\ + 6c_W^2 \langle 13 \rangle \langle 24 \rangle [13] [24] - 2c_W^4 \langle 13 \rangle \langle 24 \rangle [13] [24] - 2c_W^2 \langle 12 \rangle \langle 34 \rangle [13] [24] + 2c_W^4 \langle 12 \rangle \langle 34 \rangle [13] [24]$$

$$-c_{W}^{4}\langle \mathbf{13}\rangle\langle \mathbf{23}\rangle[\mathbf{14}][\mathbf{24}] - 4c_{W}^{2}\langle \mathbf{13}\rangle\langle \mathbf{14}\rangle[\mathbf{23}][\mathbf{24}] + c_{W}^{3}\langle \mathbf{14}\rangle[\mathbf{13}][\mathbf{23}][\mathbf{24}] - 4c_{W}^{3}\langle \mathbf{12}\rangle\langle \mathbf{13}\rangle[\mathbf{24}][\mathbf{34}]\rangle M_{W} + (-3c_{W}^{4}\langle \mathbf{13}\rangle\langle \mathbf{24}\rangle[\mathbf{23}] - 3c_{W}^{4}\langle \mathbf{13}\rangle\langle \mathbf{23}\rangle[\mathbf{24}] + c_{W}^{3}\langle \mathbf{23}\rangle[\mathbf{13}][\mathbf{24}] + c_{W}^{3}\langle \mathbf{13}\rangle[\mathbf{23}][\mathbf{24}]\rangle[\mathbf{1}|p_{3}|\mathbf{4}\rangle + (3c_{W}^{3}\langle \mathbf{23}\rangle\langle \mathbf{24}\rangle[\mathbf{13}] + 3c_{W}^{3}\langle \mathbf{13}\rangle\langle \mathbf{24}\rangle[\mathbf{23}] - c_{W}^{4}\langle \mathbf{24}\rangle[\mathbf{13}][\mathbf{23}] - c_{W}^{4}\langle \mathbf{23}\rangle[\mathbf{13}][\mathbf{24}]\rangle[\mathbf{4}|p_{3}|\mathbf{1}\rangle\rangle,$$
(A1)

$$\mathcal{M}_{UW} = \frac{e^2}{2M_W^3 s_W^2 (u - M_W^2)} ((3c_W^3 \langle 14 \rangle \langle 23 \rangle \langle 24 \rangle [13] + 4c_W^3 \langle 23 \rangle \langle 34 \rangle [12] [14] - 4c_W^2 \langle 23 \rangle \langle 24 \rangle [13] [14] + 3c_W^3 \langle 13 \rangle \langle 14 \rangle \langle 24 \rangle [23] - c_W^4 \langle 14 \rangle \langle 24 \rangle [13] [23] - 2 \langle 14 \rangle \langle 23 \rangle [14] [23] + 6c_W^2 \langle 14 \rangle \langle 23 \rangle [14] [23] - 2c_W^4 \langle 14 \rangle \langle 23 \rangle [14] [23] - 3c_W^2 \langle 13 \rangle \langle 24 \rangle [14] [23] - c_W^4 \langle 13 \rangle \langle 24 \rangle [14] [23] + 2c_W^2 \langle 12 \rangle \langle 34 \rangle [14] [23] - 2c_W^4 \langle 12 \rangle \langle 34 \rangle [14] [23] - 3c_W^2 \langle 13 \rangle \langle 24 \rangle [13] [24] - c_W^4 \langle 14 \rangle \langle 23 \rangle [13] [24] - 3c_W^4 \langle 13 \rangle \langle 23 \rangle [14] [24] + c_W^3 \langle 23 \rangle [13] [14] [24] - 4c_W^2 \langle 13 \rangle \langle 14 \rangle [23] [24] + c_W^3 \langle 13 \rangle [14] [23] [24] + 4c_W^3 \langle 12 \rangle \langle 14 \rangle [23] [34] \rangle M_W + (-3c_W^4 \langle 14 \rangle \langle 24 \rangle [23] + c_W^3 \langle 24 \rangle [14] [23] - 3c_W^4 \langle 14 \rangle \langle 23 \rangle [24] + c_W^3 \langle 14 \rangle [23] [24] ) [1|p_4|3 \rangle + (3c_W^3 \langle 23 \rangle \langle 24 \rangle [14] - 1c_W^4 \langle 24 \rangle [14] [23] + 3c_W^3 \langle 14 \rangle \langle 23 \rangle [24] - c_W^4 \langle 23 \rangle [14] [24] ) [3|p_4|1 \rangle).$$
(A2)

## **2.** Long contributions to $W, \overline{W} \to W, \overline{W}$

The contributions coming from the photon in the T and U channels are

$$\mathcal{M}_{T\gamma} = \frac{e^2}{M_W^3 t} ((2\langle 24 \rangle \langle 34 \rangle [12] [13] - \langle 13 \rangle \langle 23 \rangle \langle 24 \rangle [14] + 2\langle 23 \rangle \langle 24 \rangle [13] [14] + \langle 14 \rangle \langle 24 \rangle [13] [23] + 2\langle 13 \rangle \langle 24 \rangle [14] [23] - \langle 24 \rangle [13] [14] [23] - \langle 13 \rangle \langle 14 \rangle \langle 23 \rangle [24] + 2\langle 14 \rangle \langle 23 \rangle [13] [24] - 2\langle 13 \rangle \langle 24 \rangle [13] [24] + \langle 12 \rangle \langle 34 \rangle [13] [24] + \langle 13 \rangle \langle 23 \rangle [14] [24] + 2\langle 13 \rangle \langle 14 \rangle [23] [24] - \langle 14 \rangle [13] [23] [24] + \langle 13 \rangle \langle 24 \rangle [12] [34] + 2\langle 12 \rangle \langle 13 \rangle [24] [34] ) M_W + (\langle 13 \rangle \langle 24 \rangle [23] + \langle 13 \rangle \langle 23 \rangle [24] - \langle 23 \rangle [13] [24] - \langle 13 \rangle [23] [24] ) [1|p_3|4 \rangle + (-\langle 23 \rangle \langle 24 \rangle [13] - \langle 13 \rangle \langle 24 \rangle [23] + \langle 24 \rangle [13] [23] + \langle 23 \rangle [13] [24] ) [4|p_3|1 \rangle)$$
(A3)

and

$$\mathcal{M}_{U\gamma} = \frac{e^2}{M_W^3 u} ((-\langle 14 \rangle \langle 23 \rangle \langle 24 \rangle [13] - 2 \langle 23 \rangle \langle 34 \rangle [12] [14] + 2 \langle 23 \rangle \langle 24 \rangle [13] [14] - \langle 13 \rangle \langle 14 \rangle \langle 24 \rangle [23] + \langle 14 \rangle \langle 24 \rangle [13] [23] - 2 \langle 14 \rangle \langle 23 \rangle [14] [23] + 2 \langle 13 \rangle \langle 24 \rangle [14] [23] - \langle 12 \rangle \langle 34 \rangle [14] [23] + 2 \langle 14 \rangle \langle 23 \rangle [13] [24] + \langle 13 \rangle \langle 23 \rangle [14] [24] - \langle 23 \rangle [13] [14] [24] + 2 \langle 13 \rangle \langle 14 \rangle [23] [24] - \langle 13 \rangle [14] [23] [24] - \langle 14 \rangle \langle 23 \rangle [12] [34] - 2 \langle 12 \rangle \langle 14 \rangle [23] [34] \rangle M_W + (\langle 14 \rangle \langle 24 \rangle [23] - \langle 24 \rangle [14] [23] + \langle 14 \rangle \langle 23 \rangle [24] - \langle 14 \rangle [23] [24] ) [1| p_4 | 3 \rangle + (-\langle 23 \rangle \langle 24 \rangle [14] + \langle 24 \rangle [14] [23] - \langle 14 \rangle \langle 23 \rangle [24] + \langle 23 \rangle [14] [23] p_4 | 1 \rangle).$$
(A4)

The contributions coming from the Z boson in the T and U channels are

$$\mathcal{M}_{TZ} = \frac{e^2}{2M_W^2 M_Z s_W^2(t - M_Z^2)} ((4c_W^2 \langle 24 \rangle \langle 34 \rangle [12][13] - 3c_W^2 \langle 13 \rangle \langle 23 \rangle \langle 24 \rangle [14] + 4c_W^2 \langle 23 \rangle \langle 24 \rangle [13][14] + 3c_W^2 \langle 14 \rangle \langle 24 \rangle [13][23] + \langle 13 \rangle \langle 24 \rangle [14][23] + 3c_W^2 \langle 13 \rangle \langle 24 \rangle [14][23] - c_W^2 \langle 24 \rangle [13][14][23] - 3c_W^2 \langle 13 \rangle \langle 14 \rangle \langle 23 \rangle [24] + \langle 14 \rangle \langle 23 \rangle [13][24] + 3c_W^2 \langle 14 \rangle \langle 23 \rangle [13][24] - 2c_W^2 \langle 13 \rangle \langle 24 \rangle [13][24] - 2\langle 12 \rangle \langle 34 \rangle [13][24] + 2c_W^2 \langle 12 \rangle \langle 34 \rangle [13][24] + c_W^2 \langle 13 \rangle \langle 23 \rangle [14][24] + 4c_W^2 \langle 13 \rangle \langle 14 \rangle [23][24] - c_W^2 \langle 14 \rangle [13][23][24] + 4c_W^2 \langle 12 \rangle \langle 13 \rangle [24][34]) M_Z + (3c_W \langle 13 \rangle \langle 24 \rangle [23] + 3c_W \langle 13 \rangle \langle 23 \rangle [24] - c_W \langle 23 \rangle [13][24] - c_W \langle 13 \rangle [23][24]) [1|p_3|4 \rangle + (-3c_W \langle 23 \rangle \langle 24 \rangle [13] - 3c_W \langle 13 \rangle \langle 24 \rangle [23] + c_W \langle 24 \rangle [13][23] + c_W \langle 23 \rangle [13][24]) [4|p_3|1 \rangle)$$
(A5)

and

$$\mathcal{M}_{UZ} = \frac{e^2}{2M_W^2 M_Z s_W^2 (u - M_Z^2)} \left( \left( -3c_W^2 \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle \langle \mathbf{24} \rangle [\mathbf{13}] - 4c_W^2 \langle \mathbf{23} \rangle \langle \mathbf{34} \rangle [\mathbf{12}] [\mathbf{14}] + 4c_W^2 \langle \mathbf{23} \rangle \langle \mathbf{24} \rangle [\mathbf{13}] [\mathbf{14}] \right. \\ \left. - 3c_W^2 \langle \mathbf{13} \rangle \langle \mathbf{14} \rangle \langle \mathbf{24} \rangle [\mathbf{23}] + c_W^2 \langle \mathbf{14} \rangle \langle \mathbf{24} \rangle [\mathbf{13}] [\mathbf{23}] - 2c_W^2 \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{14}] [\mathbf{23}] + \langle \mathbf{13} \rangle \langle \mathbf{24} \rangle [\mathbf{14}] [\mathbf{23}] \right. \\ \left. + 3c_W^2 \langle \mathbf{13} \rangle \langle \mathbf{24} \rangle [\mathbf{14}] [\mathbf{23}] + 2 \langle \mathbf{12} \rangle \langle \mathbf{34} \rangle [\mathbf{14}] [\mathbf{23}] - 2c_W^2 \langle \mathbf{12} \rangle \langle \mathbf{34} \rangle [\mathbf{14}] [\mathbf{23}] + \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{13}] [\mathbf{24}] \right. \\ \left. + 3c_W^2 \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{13}] [\mathbf{24}] + 3c_W^2 \langle \mathbf{13} \rangle \langle \mathbf{23} \rangle [\mathbf{14}] [\mathbf{24}] - c_W^2 \langle \mathbf{23} \rangle [\mathbf{13}] [\mathbf{14}] [\mathbf{24}] + 4c_W^2 \langle \mathbf{13} \rangle \langle \mathbf{14} \rangle [\mathbf{23}] [\mathbf{24}] \right. \\ \left. - c_W^2 \langle \mathbf{13} \rangle [\mathbf{14}] [\mathbf{23}] [\mathbf{24}] - 4c_W^2 \langle \mathbf{12} \rangle \langle \mathbf{14} \rangle [\mathbf{23}] [\mathbf{34}] \rangle M_Z \right. \\ \left. + \left( 3c_W \langle \mathbf{14} \rangle \langle \mathbf{24} \rangle [\mathbf{23}] - c_W \langle \mathbf{24} \rangle [\mathbf{14}] [\mathbf{23}] + 3c_W \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{24}] - c_W \langle \mathbf{14} \rangle [\mathbf{23}] [\mathbf{24}] \right) [\mathbf{1}| p_4 | \mathbf{3} \rangle \right. \\ \left. + \left( -3c_W \langle \mathbf{23} \rangle \langle \mathbf{24} \rangle [\mathbf{14}] + c_W \langle \mathbf{24} \rangle [\mathbf{14}] [\mathbf{23}] - 3c_W \langle \mathbf{14} \rangle \langle \mathbf{23} \rangle [\mathbf{24}] + c_W \langle \mathbf{23} \rangle [\mathbf{14}] [\mathbf{24}] \right) [\mathbf{3}| p_4 | \mathbf{1} \rangle \right).$$

## APPENDIX B: T-CHANNEL HIGH-ENERGY-GROWTH TERMS FOR $Z, Z \rightarrow W, \overline{W}$

In this appendix, we show the high-energy-growth terms separately for the *W* boson in the T channel for the process  $Z, Z \rightarrow W, \overline{W}$ . This serves as an example for the other processes. Since there are many canceling terms, it would be too much to show this for every diagram. But, we calculated the same terms for every diagram and every process discussed in this paper, although we only show combined T and U diagrams in the main body of this paper. In the case of this process, for the U-channel diagram, each term is the same magnitude but opposite sign except for the terms that do not cancel and combine to give the result in Eq. (12):

$$\mathcal{M}_{TW}^{(-1,-1,-1,0)} = \frac{-3c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B1)

$$\mathcal{M}_{TW}^{(-1,-1,0,-1)} = \frac{-3c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B2)

$$\mathcal{M}_{TW}^{(-1,-1,0,0)} = \frac{-3c_W^2 e^2 \cos(\theta)}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B3}$$

$$\mathcal{M}_{TW}^{(-1,-1,0,1)} = \frac{7c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B4)

$$\mathcal{M}_{TW}^{(-1,-1,1,0)} = \frac{7c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B5)

$$\mathcal{M}_{TW}^{(-1,0,-1,-1)} = \frac{3c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B6)

$$\mathcal{M}_{TW}^{(-1,0,-1,0)} = \frac{-3c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B7}$$

$$\mathcal{M}_{TW}^{(-1,0,0,-1)} = \frac{3c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B8}$$

$$\mathcal{M}_{TW}^{(-1,0,0,0)} = \frac{(1 - 2c_W^2 - c_W^4)e^2\sin(\theta)}{2\sqrt{2}c_W M_W s_W^2} \mathcal{E} + \frac{-3c_W^3 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3, \tag{B9}$$

$$\mathcal{M}_{TW}^{(-1,0,0,1)} = \frac{4c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B10}$$

$$\mathcal{M}_{TW}^{(-1,0,1,0)} = \frac{-4c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B11)$$

$$\mathcal{M}_{TW}^{(-1,0,1,1)} = \frac{5c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B12)

$$\mathcal{M}_{TW}^{(0,-1,-1,-1)} = \frac{3c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B13)

$$\mathcal{M}_{TW}^{(0,-1,-1,0)} = \frac{3c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B14}$$

$$\mathcal{M}_{TW}^{(0,-1,0,-1)} = \frac{-3c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B15)$$

$$\mathcal{M}_{TW}^{(0,-1,0,0)} = \frac{(1 - 2c_W^2 - c_W^4)e^2\sin(\theta)}{2\sqrt{2}c_W M_W s_W^2} \mathcal{E} + \frac{-3c_W^3 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3, \tag{B16}$$

$$\mathcal{M}_{TW}^{(0,-1,0,1)} = \frac{-4c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B17)$$

$$\mathcal{M}_{TW}^{(0,-1,1,0)} = \frac{4c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B18)$$

$$\mathcal{M}_{TW}^{(0,-1,1,1)} = \frac{5c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B19)

$$\mathcal{M}_{TW}^{(0,0,-1,-1)} = \frac{-3c_W^4 e^2 \cos(\theta)}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B20)$$

$$\mathcal{M}_{TW}^{(0,0,-1,0)} = \frac{(-1+5c_W^2+2c_W^4)e^2\sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E} + \frac{3c_W^4 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3,$$
(B21)

$$\mathcal{M}_{TW}^{(0,0,0,-1)} = \frac{(-1 + 5c_W^2 + 2c_W^4)e^2\sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E} + \frac{3c_W^4 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3,$$
(B22)

$$\mathcal{M}_{TW}^{(0,0,0,0)} = \frac{(1 + (-1 + 8c_W^2 + 4c_W^4)\cos(\theta))e^2}{2M_W^2 s_W^2} \mathcal{E}^2, \quad (B23)$$

$$\mathcal{M}_{TW}^{(0,0,0,1)} = \frac{(1 + -7c_W^2 + 2c_W^4)e^2\sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E} + \frac{-c_W^4 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3,$$
(B24)

$$\mathcal{M}_{TW}^{(0,0,1,0)} = \frac{(1 - 7c_W^2 + 2c_W^4)e^2\sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E} + \frac{-c_W^4 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3,$$
(B25)

$$\mathcal{M}_{TW}^{(0,0,1,1)} = \frac{-c_W^4 e^2 \cos(\theta)}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B26}$$

$$\mathcal{M}_{TW}^{(0,1,-1,-1)} = \frac{-7c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B27)

$$\mathcal{M}_{TW}^{(0,1,-1,0)} = \frac{4c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B28}$$

$$\mathcal{M}_{TW}^{(0,1,0,-1)} = \frac{-4c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B29)$$

$$\mathcal{M}_{TW}^{(0,1,0,0)} = \frac{(-1+2c_W^2+3c_W^4)e^2\sin(\theta)}{2\sqrt{2}c_W M_W s_W^2} \mathcal{E} + \frac{c_W^3 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3,$$
(B30)

$$\mathcal{M}_{TW}^{(0,1,0,1)} = \frac{-c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B31}$$

$$\mathcal{M}_{TW}^{(0,1,1,0)} = \frac{c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B32}$$

$$\mathcal{M}_{TW}^{(0,1,1,1)} = \frac{-c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B33)

$$\mathcal{M}_{TW}^{(1,0,-1,-1)} = \frac{-7c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B34)

$$\mathcal{M}_{TW}^{(1,0,-1,0)} = \frac{-4c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B35)$$

$$\mathcal{M}_{TW}^{(1,0,0,-1)} = \frac{4c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \qquad (B36)$$

$$\mathcal{M}_{TW}^{(1,0,0,0)} = \frac{(-1+2c_W^2+3c_W^4)e^2\sin(\theta)}{2\sqrt{2}c_W M_W s_W^2} \mathcal{E} + \frac{c_W^3 e^2\sin(\theta)}{\sqrt{2}M_W^3 s_W^2} \mathcal{E}^3,$$
(B37)

$$\mathcal{M}_{TW}^{(1,0,0,1)} = \frac{c_W^3 e^2 \cos^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2,$$
(B38)

$$\mathcal{M}_{TW}^{(1,0,1,0)} = \frac{-c_W^3 e^2 \sin^2(\frac{\theta}{2})}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B39}$$

$$\mathcal{M}_{TW}^{(1,0,1,1)} = \frac{-c_W^3 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B40)

$$\mathcal{M}_{TW}^{(1,1,-1,0)} = \frac{-5c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B41)

$$\mathcal{M}_{TW}^{(1,1,0,-1)} = \frac{-5c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B42)

$$\mathcal{M}_{TW}^{(1,1,0,0)} = \frac{-c_W^2 e^2 \cos(\theta)}{M_W^2 s_W^2} \mathcal{E}^2, \tag{B43}$$

$$\mathcal{M}_{TW}^{(1,1,0,1)} = \frac{c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E},$$
 (B44)

$$\mathcal{M}_{TW}^{(1,1,1,0)} = \frac{c_W^2 e^2 \sin(\theta)}{2\sqrt{2}M_W s_W^2} \mathcal{E}.$$
 (B45)

## APPENDIX C: HIGH-ENERGY GROWTH OF THE FEYNMAN FOUR-POINT VERTEX

The polarization vectors are given by

$$\epsilon^{(+1)\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -\cos\theta\cos\phi + i\sin\phi \\ -\cos\theta\sin\phi - i\cos\phi \\ \sin\theta \end{pmatrix}, \quad (C1)$$

$$\epsilon^{(0)\mu} = \frac{1}{M} \begin{pmatrix} |\mathbf{p}| \\ E\sin\theta\cos\phi \\ E\sin\theta\sin\phi \\ E\cos\theta \end{pmatrix}, \quad (C2)$$

$$\epsilon^{(-1)\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0\\ \cos\theta\cos\phi + i\sin\phi\\ \cos\theta\sin\phi - i\cos\phi\\ -\sin\theta \end{pmatrix}.$$
 (C3)

# 1. $Z, Z \rightarrow W, \overline{W}$

The contribution from the Feynman four-point vertex to the process  $Z, Z \rightarrow W, \overline{W}$  is

$$\mathcal{M}_{4\text{Feyn}} = -\frac{e^2 c_W^2}{s_W^2} \times (2\epsilon_1 \cdot \epsilon_2 \epsilon_3^* \cdot \epsilon_4^* - \epsilon_1 \cdot \epsilon_3^* \epsilon_2 \cdot \epsilon_4^* - \epsilon_1 \cdot \epsilon_4^* \epsilon_2 \cdot \epsilon_3^*).$$
(C4)

If we expand this in high energy and keep only the highenergy-growth terms, we get contributions whenever any of the bosons is longitudinal. In fact, the power of  $\mathcal{E}$  is equal to the number of longitudinal modes.

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,-1,0)} = \frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},\tag{C5}$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,0,-1)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},$$
 (C6)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,0,0)} = \frac{c_W^2 e^2 (s_\theta^2 - 4) \mathcal{E}^2}{M_W^2 s_W^2},$$
 (C7)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,0,1)} = \frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},\tag{C8}$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,1,0)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},$$
(C9)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,-1,-1)} = \frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
 (C10)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,-1,0)} = -\frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C11)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,-1,1)} = -\frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}}, \qquad (\text{C12})$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,0,-1)} = \frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C13)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,0,0)} = \frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\,\mathcal{E}^{3}}{M_{W}^{2}M_{Z}s_{W}^{2}},\tag{C14}$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,0,1)} = \frac{c_W^2 e^2 \mathcal{E}^2 (c_\theta - s_\theta^2 + 1)}{M_W M_Z s_W^2}, \qquad (C15)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,1,-1)} = -\frac{(c_{\theta}-1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}}, \qquad (C16)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,1,0)} = \frac{c_W^2 e^2 \mathcal{E}^2 (c_\theta + s_\theta^2 - 1)}{M_W M_Z s_W^2}, \qquad (C17)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,1,1)} = \frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
(C18)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,-1,0)} = -\frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}}, \qquad (C19)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,0,-1)} = \frac{(c_{\theta} - 1)c_W^2 e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2}, \qquad (C20)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,0,0)} = -\frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W^2 s_W^2},$$
 (C21)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,0,1)} = -\frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},\qquad(\text{C22})$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,1,0)} = \frac{(c_{\theta} - 1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},$$
 (C23)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,-1,-1)} = \frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
 (C24)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,-1,0)} = -\frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C25)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,-1,1)} = -\frac{(c_{\theta}-1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},\qquad(\text{C26})$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,0,-1)} = \frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C27)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,0,0)} = \frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}^{3}}{M_{W}^{2}M_{Z}s_{W}^{2}},$$
 (C28)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,0,1)} = -\frac{c_W^2 e^2 \mathcal{E}^2 (c_\theta + s_\theta^2 - 1)}{M_W M_Z s_W^2}, \quad (C29)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,1,-1)} = -\frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}}, \qquad (C30)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,1,0)} = \frac{c_W^2 e^2 \mathcal{E}^2 (-c_\theta + s_\theta^2 - 1)}{M_W M_Z s_W^2}, \quad (C31)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,1,1)} = \frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
(C32)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,-1,-1)} = -\frac{c_W^2 e^2 (s_\theta^2 - 4) \mathcal{E}^2}{M_Z^2 s_W^2}, \qquad (C33)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,-1,0)} = -\frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}^{3}}{M_{W}M_{Z}^{2}s_{W}^{2}},$$
 (C34)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,-1,1)} = \frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_Z^2 s_W^2},$$
 (C35)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,0,-1)} = \frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}^{3}}{M_{W}M_{Z}^{2}s_{W}^{2}},$$
 (C36)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,0,0)} = \frac{2c_W^2 e^2 \mathcal{E}^2 (M_W^2 + M_Z^2)}{M_W^2 M_Z^2 s_W^2} - \frac{2c_W^2 e^2 (s_\theta^2 + 2) \mathcal{E}^4}{M_W^2 M_Z^2 s_W^2},$$
(C37)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,0,1)} = -\frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}^{3}}{M_{W}M_{Z}^{2}s_{W}^{2}},$$
 (C38)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,1,-1)} = \frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_Z^2 s_W^2},$$
 (C39)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,1,0)} = \frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}^{3}}{M_{W}M_{Z}^{2}s_{W}^{2}}, \qquad (C40)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,1,1)} = -\frac{c_W^2 e^2 (s_\theta^2 - 4)\mathcal{E}^2}{M_Z^2 s_W^2}, \qquad (C41)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,-1,-1)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
 (C42)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,-1,0)} = \frac{c_W^2 e^2 \mathcal{E}^2 (-c_\theta + s_\theta^2 - 1)}{M_W M_Z s_W^2}, \quad (C43)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,-1,1)} = \frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
 (C44)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,0,-1)} = -\frac{c_W^2 e^2 \mathcal{E}^2 (c_\theta + s_\theta^2 - 1)}{M_W M_Z s_W^2}, \quad (C45)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,0,0)} = -\frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\,\mathcal{E}^{3}}{M_{W}^{2}M_{Z}s_{W}^{2}},\tag{C46}$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,0,1)} = \frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C47)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,1,-1)} = \frac{(c_{\theta} - 1)c_W^2 e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_Z s_W^2},$$
 (C48)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,1,0)} = -\frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C49)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,1,1)} = -\frac{c_{\theta}c_W^2 e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_Z s_W^2},$$
 (C50)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,-1,0)} = -\frac{(c_{\theta}-1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}}, \qquad (\text{C51})$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,-1)} = \frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},$$
 (C52)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,0)} = -\frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W^2 s_W^2},$$
 (C53)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,1)} = -\frac{(c_{\theta}-1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}}, \qquad (\text{C54})$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,1,0)} = \frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},$$
 (C55)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,-1)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}\,s_{W}^{2}},\tag{C56}$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,0)} = \frac{c_W^2 e^2 \mathcal{E}^2 (c_\theta + s_\theta^2 - 1)}{M_W M_Z s_W^2}, \qquad (C57)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,1)} = \frac{(c_{\theta} - 1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
 (C58)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,-1)} = \frac{c_W^2 e^2 \mathcal{E}^2 (c_\theta - s_\theta^2 + 1)}{M_W M_Z s_W^2}, \qquad (\text{C59})$$

$$\mathcal{M}_{4\text{Feyn}}^{(1.0,0,0)} = -\frac{\sqrt{2}c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}^{3}}{M_{W}^{2}M_{Z}s_{W}^{2}},$$
 (C60)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,1)} = \frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C61)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,-1)} = \frac{(c_{\theta}+1)c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
 (C62)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,0)} = -\frac{c_W^2 e^2 s_\theta^2 \mathcal{E}^2}{M_W M_Z s_W^2},$$
 (C63)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,1)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{Z}s_{W}^{2}},$$
(C64)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,-1,0)} = \frac{c_{\theta}c_W^2 e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2},\tag{C65}$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,-1)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}},\tag{C66}$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,0)} = \frac{c_W^2 e^2 (s_\theta^2 - 4) \mathcal{E}^2}{M_W^2 s_W^2},$$
 (C67)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,1)} = \frac{c_{\theta}c_W^2 e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2},$$
(C68)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,1,0)} = -\frac{c_{\theta}c_{W}^{2}e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}s_{W}^{2}}.$$
(C69)

#### 2. $W, W \rightarrow W, W$

The contribution from the Feynman four-point vertex to the process  $W, W \rightarrow W, W$  is

$$\mathcal{M}_{4\text{Feyn}} = \frac{e^2}{s_W^2} (2\epsilon_1 \cdot \epsilon_4^* \epsilon_2 \cdot \epsilon_3^* - \epsilon_1 \cdot \epsilon_3^* \epsilon_2 \cdot \epsilon_4^* - \epsilon_1 \cdot \epsilon_2 \epsilon_3^* \cdot \epsilon_4^*).$$
(C70)

If we expand this in high energy and keep only the highenergy-growth terms, we get contributions whenever any of the bosons is longitudinal. Once again, the power of  $\mathcal{E}$  is equal to the number of longitudinal modes:

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,-1,0)} = \frac{(c_{\theta}-3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C71)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,0,-1)} = -\frac{(c_{\theta}-3)e^2s_{\theta}\mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C72)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,0,0)} = \frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C73)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,0,1)} = \frac{(c_{\theta} - 1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C74)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,-1,1,0)} = -\frac{(c_{\theta}-1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C75)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,-1,-1)} = \frac{(c_{\theta} - 3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C76)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,-1,0)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C77)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,-1,1)} = -\frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C78)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,0,-1)} = \frac{e^2(s_{\theta}^2 - 4)\mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C79)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,0,0)} = \frac{(c_{\theta}+3)e^2s_{\theta}\mathcal{E}^3}{\sqrt{2}M_W^3s_W^2} - \frac{3e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_Ws_W^2}, \quad (C80)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,0,1)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C81)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,1,-1)} = -\frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C82)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,1,0)} = \frac{e^2 \mathcal{E}^2 (-2c_\theta + s_\theta^2 + 2)}{2M_W^2 s_W^2}, \qquad (C83)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,0,1,1)} = \frac{(c_{\theta} - 1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C84)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,-1,0)} = -\frac{(c_{\theta}+1)e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}^{2}},$$
 (C85)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,0,-1)} = \frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C86)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,0,0)} = \frac{e^2 \mathcal{E}^2 (4c_\theta - s_\theta^2 + 4)}{2M_W^2 s_W^2}, \qquad (C87)$$

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,0,1)} = -\frac{(c_{\theta}+1)e^2s_{\theta}\mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C88)

$$\mathcal{M}_{4\text{Feyn}}^{(-1,1,1,0)} = \frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C89)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,-1,-1)} = \frac{(c_{\theta}-3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C90)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,-1,0)} = -\frac{e^2(s_{\theta}^2 - 4)\mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C91)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,-1,1)} = -\frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C92)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,0,-1)} = \frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C93)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,0,0)} = \frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}^3}{\sqrt{2}M_W^3 s_W^2} - \frac{3e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2}, \quad (C94)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,0,1)} = \frac{e^2 \mathcal{E}^2 (2c_\theta - s_\theta^2 - 2)}{2M_W^2 s_W^2}, \qquad (C95)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,1,-1)} = -\frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C96)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,1,0)} = \frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C97)

$$\mathcal{M}_{4\text{Feyn}}^{(0,-1,1,1)} = \frac{(c_{\theta}-1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C98)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,-1,-1)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C99)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,-1,0)} = \frac{3e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2} - \frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}^3}{\sqrt{2}M_W^3 s_W^2}, \quad (C100)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,-1,1)} = \frac{e^2 \mathcal{E}^2 (-4c_\theta + s_\theta^2 - 4)}{2M_W^2 s_W^2}, \quad (C101)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,0,-1)} = \frac{(c_{\theta}+3)e^2s_{\theta}\mathcal{E}^3}{\sqrt{2}M_W^3 s_W^2} - \frac{3e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_W s_W^2}, \quad (C102)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,0,0)} = \frac{e^2 \mathcal{E}^4 (6c_\theta - s_\theta^2 - 2)}{M_W^4 s_W^2} + \frac{2(1 - 3c_\theta) e^2 \mathcal{E}^2}{M_W^2 s_W^2},$$
(C103)

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,0,1)} = \frac{3e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2} - \frac{(c_{\theta} + 3)e^2 s_{\theta} \mathcal{E}^3}{\sqrt{2}M_W^3 s_W^2}, \qquad (C104)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,1,-1)} = \frac{e^2 \mathcal{E}^2 (-4c_\theta + s_\theta^2 - 4)}{2M_W^2 s_W^2}, \quad (C105)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,1,0)} = \frac{(c_{\theta}+3)e^2s_{\theta}\mathcal{E}^3}{\sqrt{2}M_W^3s_W^2} - \frac{3e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_Ws_W^2}, \qquad (C106)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,0,1,1)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C107)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,-1,-1)} = -\frac{(c_{\theta}-1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C108)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,-1,0)} = \frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C109)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,-1,1)} = \frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C110)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,0,-1)} = \frac{e^2 \mathcal{E}^2 (2c_\theta - s_\theta^2 - 2)}{2M_W^2 s_W^2}, \qquad (C111)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,0,0)} = \frac{3e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2} - \frac{(c_{\theta} + 3)e^2 s_{\theta} \mathcal{E}^3}{\sqrt{2}M_W^3 s_W^2}, \quad (C112)$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,0,1)} = \frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2} \tag{C113}$$

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,1,-1)} = \frac{(c_{\theta}+3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C114)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,1,0)} = -\frac{e^2(s_{\theta}^2 - 4)\mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C115)

$$\mathcal{M}_{4\text{Feyn}}^{(0,1,1,1)} = -\frac{(c_{\theta} - 3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C116)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,-1,0)} = -\frac{(c_{\theta}+3)e^2s_{\theta}\mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C117)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,-1)} = \frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C118)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,0)} = \frac{e^2 \mathcal{E}^2 (4c_\theta - s_\theta^2 + 4)}{2M_W^2 s_W^2}, \qquad (\text{C119})$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,1)} = -\frac{(c_{\theta}+3)e^2s_{\theta}\mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C120)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,1,0)} = \frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C121)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,-1)} = -\frac{(c_{\theta}-1)e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}^{2}}, \qquad (\text{C122})$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,0)} = \frac{e^2 \mathcal{E}^2 (-2c_\theta + s_\theta^2 + 2)}{2M_W^2 s_W^2}, \quad (C123)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,1)} = \frac{(c_{\theta}+3)e^2 s_{\theta}\mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C124)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,-1)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C125)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0)} = \frac{3e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W s_W^2} - \frac{(c_{\theta} + 3)e^2 s_{\theta} \mathcal{E}^3}{\sqrt{2}M_W^3 s_W^2}, \quad (C126)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,1)} = \frac{e^2(s_{\theta}^2 - 4)\mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C127)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,-1)} = \frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C128)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,0)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C129)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,1)} = -\frac{(c_{\theta} - 3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2},$$
 (C130)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,-1,0)} = \frac{(c_{\theta}-1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C131)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,-1)} = -\frac{(c_{\theta}-1)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}, \qquad (C132)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,0)} = \frac{e^2 s_{\theta}^2 \mathcal{E}^2}{2M_W^2 s_W^2},$$
 (C133)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,1)} = \frac{(c_{\theta} - 3)e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}^{2}},$$
 (C134)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,1,0)} = -\frac{(c_{\theta} - 3)e^2 s_{\theta} \mathcal{E}}{2\sqrt{2}M_W s_W^2}.$$
 (C135)

# 3. $h,h \to Z,Z$ and $h,h \to W,\bar{W}$

The contribution from the Feynman four-point vertex to the process  $h, h \rightarrow Z, Z$  is

$$\mathcal{M}_{4\text{Feyn}} = -\frac{e^2}{2c_W^2 s_W^2} \epsilon_3^* \cdot \epsilon_4^*.$$
 (C136)

The contribution from the Feynman four-point vertex to the process  $h, h \rightarrow W, \overline{W}$  is

$$\mathcal{M}_{4\text{Feyn}} = -\frac{e^2}{2s_W^2}\epsilon_3^* \cdot \epsilon_4^*. \tag{C137}$$

If we expand these in high energy and keep only the high-energy-growth terms, we get the following nonzero contributions. The result is the same for both processes, namely,

$$\mathcal{M}_{4\text{Feyn}}^{(0,0)} = -\frac{e^2 \mathcal{E}^2}{M_W^2 s_W^2}.$$
 (C138)

## 4. $\gamma^+, Z \to W, \overline{W}$

The contribution from the Feynman four-point vertex to the process  $\gamma^+, Z \to W, \overline{W}$  is

$$\mathcal{M}_{4\text{Feyn}} = -\frac{e^2 c_W}{s_W} (2\epsilon_1 \cdot \epsilon_4^* \epsilon_2 \cdot \epsilon_3^* - \epsilon_1 \cdot \epsilon_3^* \epsilon_2 \cdot \epsilon_4^* - \epsilon_1 \cdot \epsilon_2 \epsilon_3^* \cdot \epsilon_4^*).$$
(C139)

If we expand this in high energy and keep only the highenergy-growth terms, we get contributions whenever any of the bosons is longitudinal:

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,-1,0)} = \frac{c_W e^2 s_\theta \mathcal{E}(2(c_\theta + 1)M_W + (c_\theta - 1)M_Z)}{2\sqrt{2}M_W M_Z s_W},$$
(C140)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,-1)} = \frac{(c_{\theta}+1)c_{W}e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}}, \qquad (C141)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,0)} = \frac{c_W e^2 \mathcal{E}^2 (M_W (2s_\theta^2 - 4c_\theta - 4) + M_Z s_\theta^2)}{2M_W^2 M_Z s_W},$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,1)} = -\frac{(c_{\theta}+3)c_{W}e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}}, \qquad (C143)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,1,0)} = -\frac{(c_{\theta}+1)c_{W}e^{2}s_{\theta}\mathcal{E}(2M_{W}+M_{Z})}{2\sqrt{2}M_{W}M_{Z}s_{W}}, \quad (C144)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,-1)} = -\frac{(c_{\theta}-1)c_{W}e^{2}s_{\theta}\mathcal{E}(M_{W}+2M_{Z})}{2\sqrt{2}M_{W}M_{Z}s_{W}}, \quad (C145)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,0)} = \frac{c_W e^2 \mathcal{E}^2 (2c_\theta - s_\theta^2 - 2)}{2M_W M_Z s_W}, \qquad (C146)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,-1,1)} = \frac{c_W e^2 s_\theta \mathcal{E}((c_\theta - 1)M_W + 2(c_\theta + 1)M_Z)}{2\sqrt{2}M_W M_Z s_W},$$
(C147)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,-1)} = -\frac{c_W e^2 s_\theta^2 \mathcal{E}^2 (M_W + 2M_Z)}{2M_W^2 M_Z s_W}, \qquad (C148)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,0)} = \frac{(c_{\theta} + 3)c_{W}e^{2}s_{\theta}\mathcal{E}^{3}}{\sqrt{2}M_{W}^{2}M_{Z}s_{W}} - \frac{c_{W}e^{2}s_{\theta}\mathcal{E}(6M_{W}^{2} - (c_{\theta} - 3)M_{Z}^{2})}{4\sqrt{2}M_{W}^{2}M_{Z}s_{W}}, \quad (C149)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,0,1)} = \frac{c_W e^2 \mathcal{E}^2 (M_W s_\theta^2 + 2M_Z (s_\theta^2 - 2))}{2M_W^2 M_Z s_W}, \quad (C150)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,-1)} = \frac{(c_{\theta}+1)c_{W}e^{2}s_{\theta}\mathcal{E}(M_{W}+2M_{Z})}{2\sqrt{2}M_{W}M_{Z}s_{W}}, \quad (C151)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,0)} = \frac{c_W e^2 s_\theta^2 \mathcal{E}^2}{2M_W M_Z s_W},$$
 (C152)

$$\mathcal{M}_{4\text{Feyn}}^{(1,0,1,1)} = -\frac{c_W e^2 s_\theta \mathcal{E}((c_\theta + 1)M_W + 2(c_\theta - 1)M_Z)}{2\sqrt{2}M_W M_Z s_W},$$
(C153)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,-1,0)} = -\frac{(c_{\theta}-1)c_{W}e^{2}s_{\theta}\mathcal{E}(2M_{W}+M_{Z})}{2\sqrt{2}M_{W}M_{Z}s_{W}}, \quad (C154)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,-1)} = -\frac{(c_{\theta} - 1)c_{W}e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}}, \qquad (C155)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,0)} = -\frac{c_W e^2 s_{\theta}^2 \mathcal{E}^2 (2M_W + M_Z)}{2M_W^2 M_Z s_W}, \qquad (C156)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,1)} = \frac{(c_{\theta} - 3)c_{W}e^{2}s_{\theta}\mathcal{E}}{2\sqrt{2}M_{W}s_{W}},$$
 (C157)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,1,0)} = \frac{c_W e^2 s_\theta \mathcal{E}(2(c_\theta - 1)M_W + (c_\theta + 1)M_Z)}{2\sqrt{2}M_W M_Z s_W}.$$
(C158)

5. 
$$\gamma^+, \gamma^\pm \to W, \bar{W}$$

The contribution from the Feynman four-point vertex to the process  $\gamma, \gamma \to W, \overline{W}$  is

$$\mathcal{M}_{4\text{Feyn}} = -e^2(2\epsilon_1 \cdot \epsilon_2\epsilon_3^* \cdot \epsilon_4^* - \epsilon_1 \cdot \epsilon_3^*\epsilon_2 \cdot \epsilon_4^* - \epsilon_1 \cdot \epsilon_4^*\epsilon_2 \cdot \epsilon_3^*).$$
(C159)

We first expand this in high energy when both photons have positive helicity:

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,-1,0)} = \frac{c_{\theta}e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_W},$$
(C160)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,-1)} = -\frac{c_{\theta}e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_W},$$
(C161)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,0)} = \frac{e^2(s_{\theta}^2 - 4)\mathcal{E}^2}{M_W^2},$$
 (C162)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,0,1)} = \frac{c_{\theta}e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_W},$$
(C163)

$$\mathcal{M}_{4\text{Feyn}}^{(1,1,1,0)} = -\frac{c_{\theta}e^2s_{\theta}\mathcal{E}}{\sqrt{2}M_W}.$$
(C164)

We next expand this when the first photon has positive helicity and the second has negative helicity:

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,-1,0)} = -\frac{(c_{\theta}-1)e^{2}s_{\theta}\mathcal{E}}{\sqrt{2}M_{W}}, \qquad (C165)$$

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,-1)} = \frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W},$$
 (C166)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,0)} = -\frac{e^2 s_{\theta}^2 \mathcal{E}^2}{M_W^2},$$
 (C167)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,0,1)} = -\frac{(c_{\theta}-1)e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W},$$
 (C168)

$$\mathcal{M}_{4\text{Feyn}}^{(1,-1,1,0)} = \frac{(c_{\theta}+1)e^2 s_{\theta} \mathcal{E}}{\sqrt{2}M_W}.$$
 (C169)

- [1] S. J. Parke and T. R. Taylor, An amplitude for *n* gluon scattering, Phys. Rev. Lett. **56**, 2459 (1986).
- [2] R. Gastmans and T. T. Wu, The ubiquitous photon: Helicity method for QED and QCD, Int. Ser. Monogr. Phys. 80, 1 (1990).
- [3] L. J. Dixon, Calculating scattering amplitudes efficiently, in *Boulder 1995, QCD and Beyond*, pp. 539–582, arXiv: hep-ph/9601359.
- [4] R. Britto, F. Cachazo, B. Feng, and E. Witten, Direct proof of tree-level recursion relation in Yang-Mills theory, Phys. Rev. Lett. 94, 181602 (2005).
- [5] H. Elvang and Y. t. Huang, Scattering amplitudes, arXiv: 1308.1697.
- [6] N. Arkani-Hamed, T. C. Huang, and Y. t. Huang, Scattering amplitudes for all masses and spins, J. High Energy Phys. 11 (2021) 070.
- [7] N. Christensen and B. Field, Constructive Standard Model, Phys. Rev. D 98, 016014 (2018).
- [8] H. Liu, T. Ma, Y. Shadmi, and M. Waterbury, An EFT Hunter's guide to two-to-two scattering: HEFT and SMEFT on-shell amplitudes, J. High Energy Phys. 05 (2023) 241.

- [9] A. Ochirov, Helicity amplitudes for QCD with massive quarks, J. High Energy Phys. 04 (2018) 089.
- [10] R. Aoude and C. S. Machado, The rise of SMEFT on-shell amplitudes, J. High Energy Phys. 12 (2019) 058.
- [11] G. Durieux, T. Kitahara, Y. Shadmi, and Y. Weiss, The electroweak effective field theory from on-shell amplitudes, J. High Energy Phys. 01 (2020) 119.
- [12] R. Franken and C. Schwinn, On-shell constructibility of Born amplitudes in spontaneously broken gauge theories, J. High Energy Phys. 02 (2020) 073.
- [13] B. Bachu and A. Yelleshpur, On-shell electroweak sector and the Higgs mechanism, J. High Energy Phys. 08 (2020) 039.
- [14] R. Balkin, G. Durieux, T. Kitahara, Y. Shadmi, and Y. Weiss, On-shell higgsing for EFTs, J. High Energy Phys. 03 (2022) 129.
- [15] P. Baratella, C. Fernandez, and A. Pomarol, Renormalization of higher-dimensional operators from on-shell amplitudes, Nucl. Phys. **B959**, 115155 (2020).
- [16] G. Durieux, T. Kitahara, C. S. Machado, Y. Shadmi, and Y. Weiss, Constructing massive on-shell contact terms, J. High Energy Phys. 12 (2020) 175.

- [17] G. F. S. Alves, E. Bertuzzo, and G. M. Salla, On-shell approach to neutrino oscillations, Phys. Rev. D 106, 036028 (2022).
- [18] N. Christensen, B. Field, A. Moore, and S. Pinto, Two-, three-, and four-body decays in the constructive standard model, Phys. Rev. D 101, 065019 (2020).
- [19] B. Bachu, Spontaneous symmetry breaking from an on-shell perspective, J. High Energy Phys. 02 (2024) 098.
- [20] D. Liu and Z. Yin, Gauge invariance from on-shell massive amplitudes and tree-level unitarity, Phys. Rev. D 106, 076003 (2022).
- [21] B. W. Lee, C. Quigg, and H. B. Thacker, The strength of weak interactions at very high-energies and the Higgs boson mass, Phys. Rev. Lett. 38, 883 (1977).
- [22] T. Appelquist and M. S. Chanowitz, Unitarity bound on the scale of fermion mass generation, Phys. Rev. Lett. 59, 2405 (1987); 60, 1589(E) (1988).
- [23] R. S. Chivukula, N. D. Christensen, B. Coleppa, and E. H. Simmons, Unitarity and bounds on the scale of

fermion mass generation, Phys. Rev. D **75**, 073018 (2007).

- [24] R. S. Chivukula, N. D. Christensen, and E. H. Simmons, Low-energy effective theory, unitarity, and non-decoupling behavior in a model with heavy Higgs-triplet fields, Phys. Rev. D 77, 035001 (2008).
- [25] N. Christensen, H. Diaz-Quiroz, B. Field, J. Hayward, and J. Miles, Challenges with internal photons in constructive QED, Nucl. Phys. **B993**, 116278 (2023).
- [26] H. Y. Lai, D. Liu, and J. Terning, The constructive method for massive particles in QED, arXiv:2312.11621.
- [27] Y. Ema, T. Gao, W. Ke, Z. Liu, K. F. Lyu, and I. Mahbub, Momentum shift and on-shell constructible massive amplitudes, arXiv:2403.15538.
- [28] N. Christensen, A complete set of 4-point amplitudes in the constructive standard model, arXiv:2403.07978.
- [29] N. Christensen, SPINAS: Spinor amplitude subroutines for constructive diagram evaluations, arXiv:2403.07981.