# Numerical simulation of quantum field fluctuations

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The quantum fluctuations of fields can exhibit subtle correlations in space and time. As the interval between a pair of measurements varies, the correlation function can change sign, signaling a shift between correlation and anticorrelation. A numerical simulation of the fluctuations requires a knowledge of both the probability distribution and the correlation function. Although there are widely used methods to generate a sequence of random numbers which obey a given probability distribution, the imposition of a given correlation function can be more difficult. Here we propose a simple method in which the outcome of a given measurement determines a shift in the peak of the probability distribution, to be used for the next measurement. We illustrate this method for three examples of quantum field correlation functions, and show that the resulting simulated function agrees well with the original, analytically derived function. We then discuss the application of this method to numerical studies of the effects of correlations on the random walks of test particles coupled to the fluctuating field.

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## I. INTRODUCTION

Although vacuum fluctuations of a quantized field, such as the electromagnetic field, are formally infinite, careful treatments of the contributions of high frequency modes lead to finite observable effects. Two examples are the Casimir effect and the Lamb shift. Welton [1] has given a simple argument which illustrates that the dominant contribution to the Lamb shift is due to the effects of Brownian motion of an electron responding to vacuum electric field fluctuations. In recent years, there has been interest in various physical effects which might be produced by the quantum fluctuations of linear operators, such as the electric field, or of quadratic operators, such as the energy density. For a recent review, see Ref. [2].

Although quantum field fluctuations are usually treated with analytic methods, such as the calculation of variances, numerical simulations can also play a role. This was done, for example, by Carlip *et al.* [3,4] to study the effects of quantum stress tensor fluctuations upon the focusing of light rays. Here a model in two-dimensional spacetime was used, with the fluctuations satisfying a non-Gaussian probability distribution given in Ref. [5]. However, as Carlip *et al.* note, their simulation ignores the possible correlations between fluctuations in different spacetime regions. More generally, it is desirable to find methods to include correlations in simulations, as quantum fields can exhibit subtle correlations and anticorrelations. These are discussed, for example, in Refs. [6,7].

There are statistical methods for introducing correlations into autoregression analyses. An example is the Monte Carlo method discussed in Ref. [8]. However, we are not aware of any use of these methods in the literature to study quantum field fluctuations. The purpose of this paper is to discuss the correlation functions for linear field operators and to propose a method for introducing these correlations into a numerical simulation. The outline of this paper is as follows: Sec. II will introduce and discuss three examples of quantum field correlation functions. Our proposed method for implementing correlations in a simulation will be introduced in Sec. III. The possible application of the method to the study of the effects of correlations on Brownian motion will be considered in Sec. IV. Our results will be summarized and discussed in Sec. V.

## II. CORRELATION FUNCTIONS

In this section, we will discuss correlation functions for quantum fields, and give three explicit examples of physical interest. In general, a correlation function may be defined as

$$C(t,\mathbf{x};t',\mathbf{x}') = \langle\psi(t,\mathbf{x})\psi(t',\mathbf{x}')\rangle - \langle\psi(t,\mathbf{x})\rangle\langle\psi(t',\mathbf{x}')\rangle. \tag{2.1}$$

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where $\psi(t,\mathbf{x})$ is a field operator at spacetime point $(t,\mathbf{x})$, and $\langle\rangle$ denotes an expectation value in a selected quantum state. This function describes the correlations in the fluctuations of $\psi$ between different spacetime regions.

In this paper, we restrict our attention to field operators at a fixed point $\mathbf{x}$ in space, and drop explicit mention of this point. We also assume that the mean field value vanishes, so

$$\langle\psi(t,\mathbf{x})\rangle = \langle\psi(t)\rangle = 0. \tag{2.2}$$

We also consider cases where the correlation function depends only upon the time difference, $t-t'$, and write

$$C = C(t-t') = \langle\psi(t)\psi(t')\rangle. \tag{2.3}$$

### A. Massless scalar field

In this subsection, we treat the vacuum fluctuations of a massless scalar field in four-dimensional Minkowski spacetime, where

$$C_0(t-t') = -\frac{1}{4\pi^2(t-t')^2}. \tag{2.4}$$

The $t=t'$ limit, $C_0(0)$, is singular due to the short distance divergences of a quantum field. Here we adopt the viewpoint that measurements of vacuum fluctuations at a single spacetime point are unphysical, and all measurements require an average over a finite spacetime region, or at least a finite time or space interval. For this purpose, define a time averaged field operator by

$$\bar{\psi}(t_0) = \int_{-\infty}^{\infty} dt\psi(t)g(t-t_0), \tag{2.5}$$

where $g(t)$ is a sampling function peaked at $t=t_0$ which is normalized by $\int_{-\infty}^{\infty} dtg(t) = 1$. As a convenient example, we use a Lorentzian form

$$g(t) = \frac{\tau}{\pi(t^2+\tau^2)}, \tag{2.6}$$

where $\tau$ is the characteristic duration of the time averaging. The effect of the averaging will be to suppress the contributions of high frequency modes, those whose periods are short compared to $\tau$.

Let

$$C(t_0) = \langle\bar{\psi}(0)\bar{\psi}(t_0)\rangle = \int_{-\infty}^{\infty} dtdt' g(t')g(t-t_0)C_0(t-t') \tag{2.7}$$

be the correlation between two measurements separated in time by $t_0$. Here the peaks of the sampling functions of these measurements are displaced by $t_0$. This integral contains a second order pole at $t=t'$, which may be treated by an integration by parts procedure. Use

$$C_0(t-t') = -\frac{1}{8\pi^2}\frac{\partial^2}{\partial t\partial t'}\log\left[(t-t')^2\alpha^2\right] \tag{2.8}$$

to find

$$C(t_0) = -\frac{1}{8\pi^2}\int dtdt'\dot{g}(t')\dot{g}(t-t_0)\log\left[(t-t')^2\alpha^2\right], \tag{2.9}$$

which is independent of the value of the arbitrary constant $\alpha$. This expression is finite, and may be explicitly evaluated for the case of the Lorentzian sampling function, Eq. (2.6), to find

$$C(t_0) = \frac{(4-t_0^2)}{4\pi^2(t_0^2+4)^2}. \tag{2.10}$$

This function is plotted in Fig. 2. Here units in which $\tau=1$ are used, so the time separation $t_0$ is given as a multiple of $\tau$. Note that $C(t_0)>0$ if $t_0<2$, so a pair of measurements with a small temporal separation are positively correlated, but larger separations $t_0>2$, lead to anticorrelations, $C(t_0)<0$. The latter result is expected from the minus sign in Eq. (2.4). The mathematical origin of the former result is more subtle, but it incorporates the physically reasonable result that two measurements made in rapid succession are positively correlated.

The probability distribution for operators linear in a free quantum field is a Gaussian

$$P(\bar{\psi}) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{\bar{\psi}^2}{2\sigma^2}\right) \tag{2.11}$$

with variance $\sigma^2 = C(0) = \frac{1}{4\pi^2}$. This means that if we consider a pair of measurements with separation $t_0$, the outcome of the first measurement is equally likely to be either positive or negative. However, the outcome of the second measurement is more likely to have the same sign as the first if $t_0<2$, and more likely to have the opposite sign if $t_0>2$. A method for implementing this bias in a numerical simulation will be a key topic to be treated in Sec. III.

### B. The electromagnetic and similar fields

The massless scalar field has dimensions of inverse length in units where $\hbar=c=1$. Some other massless fields of physical interest have dimensions of inverse length squared, including the electromagnetic field, and first derivatives of the massless scalar field. In all of these cases, Eq. (2.4) is replaced by a vacuum correlation function of the form

$$C_0(t-t') = \frac{\kappa}{4\pi^2(t-t')^4}, \tag{2.12}$$

where $\kappa > 0$ is a constant which depends upon the specific case. Linear field operators will still have a probability distribution of the form of Eq. (2.11). Note that if we rescale an averaged field by $\bar{\psi} \to \sigma\bar{\psi}$ then the rescaled field has variance one. Here it is convenient to suppose that this has been done, so that we can discuss several cases with different actual variances at once.

Again we assume time averaging of the field operators with the Lorenztian function, Eq. (2.6) with $\tau = 1$. Now the correlation function with unit variance becomes

$$K(t_0) = \langle\bar{\psi}(0)\bar{\psi}(t_0)\rangle = \frac{1-6t_0^2+t_0^4}{(1+t_0^2)^4}, \tag{2.13}$$

which satisfies $K(0) = 1$, as required.

The function $K(t_0)$ is plotted in Fig. 3, and in more detail in Fig. 3 of Ref. [9]. It is qualitatively similar in form to $C(t_0)$ in that it is positive for $t_0 \lesssim 0.25$ and negative for $0.25 \lesssim t_0 \lesssim 0.8$, representing regions of correlation and anticorrelation, respectively.

### C. Baths of photons or gravitons in a squeezed state

Sections II A and II B have dealt with the vacuum fluctuations of a quantized field, where averaging is essential to define finite fluctuations. There is another source of quantum fluctuations arising from particles in a nonclassical state, such as a squeezed vacuum. Photons or gravitons in such a state can give rise to Brownian motion of test particles. The operator whose fluctuations cause this motion will be the electric field for photons and the linearized Riemann tensor for gravitons. In either case, the field fluctuations are described by a correlation function of the form of Eq. (2.3). However, now we are interested in an excited state with a large occupation number and wish to ignore the vacuum contribution, so we replace the expectation value by the difference between an expectation value in the squeezed state and that in the vacuum. If only a finite number of modes are excited, this difference is finite when $t = t'$, and time averaging is not needed. In the case that a single mode of wave number $k$ is occupied, the correlation function with unit variance may be written as

$$C_1(t-t') = \cos[k(t-t')], \tag{2.14}$$

an oscillatory function exhibiting alternating correlations and anticorrelations. The Brownian motion of test particles in squeezed states of photons or gravitons will be discussed further in Ref. [10].

## III. NUMERICAL SIMULATIONS WITH CORRELATIONS

There are well-known numerical methods to generate a sequence of otherwise random numbers which obey a specified probability distribution, such as the Gaussian of Eq. (2.11). An example is the command RandomVariate in the program *Mathematica*. However, generating a sequence which satisfies a given correlation function is more difficult, but can be done by Monte Carlo methods [8], for example. These methods are used in autoregression analyses in many applications.

### A. Basic method

Here we propose an alternative method for use in simulating quantum fluctuations that will involve shifting the origin of the probability distribution from which a given outcome will be drawn, in a way depending upon a previous outcome. Suppose that we make a measurement of a field value for which the probability distribution $P(x)$ is of the form of Eq. (2.11). The first outcome, $x_1$, is equally likely to have either sign. However, if the correlation function is non-zero, the value of $x_1$ will bias the outcome $x_2$, of a subsequent measurement, and the magnitude of $x_1$ will influence the degree of bias. If the correlation is positive, $C > 0$, then $x_2$ is more likely than not to have the same sign as $x_1$. Similarly, if $C < 0$, then $x_1$ and $x_2$ are more likely to have opposite signs.

Our specific proposal is to shift the probability distribution for $x_2$ by an amount proportional to $x_1$:

$$P(x) \to P(x - fx_1), \tag{3.1}$$

where $f$ is a constant with $|f| \le 1$. Thus, $f > 0$ leads to positive correlation, and $f < 0$ to anticorrelation. We next create a correlation function $C_f(f)$ by the following procedure: Let $P(x)$ be a Gaussian with unit variance, that is, Eq. (2.11) with $\sigma = 1$, then draw an outcome $x_1$. Next, use the distribution $P(x - fx_1)$ to draw the next outcome, $x_2$. Repeat this procedure many times, so the $n$th outcome is drawn from the distribution $P(x - fx_{n-1})$. Now we define $C_f(f)$ as the numerical average of the sequence of products of successive outcomes:

$$C_f(f) = \langle x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\rangle. \tag{3.2}$$

Finally, we fit the result to the following trial form for $C_f$:

$$C_f(f) = a\tan(bf), \tag{3.3}$$

where $a$ and $b$ are constants, which have the fitted values

$$a = 0.067 \pm 0.003 \quad b = 1.58 \pm 0.02. \tag{3.4}$$

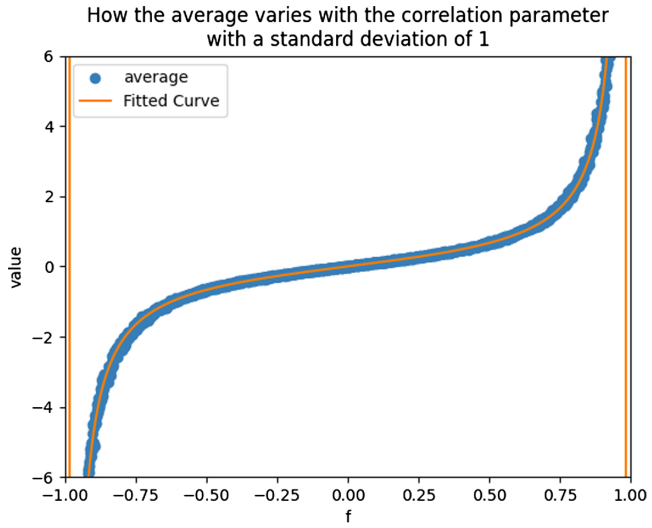


FIG. 1. The fitted form of $C(f)$. The blue dots represent the average of pairs of outcomes with a given value of the parameter $f$. The orange curve is the functional form given in Eq. (3.3).

Both the analytic form, Eq. (3.3), (orange curve) and the numerical results (blue dots) are plotted in Fig. 1. For each value of $f$, of the order of $10^4$ products were averaged. That is, $n \approx 10^4$ in Eq. (3.2). This functional form has the following properties: (1) $C_f(0) = 0$, so $f = 0$ describes lack of correlation; (2) The range of $C$ is infinite, while that of $f$ is finite; $-\infty < C < \infty$ and $-1 < 2bf/\pi < 1$; (3) It is an odd function of $f$, which allows the possibility to treat negative correlations as well as positive ones. At this time, the functional form of $C_f$ given in Eq. (3.3) is strictly empirical. However, we hope to find deeper reasons for it in future research.

To obtain numerical simulations of $C(t_0)$, a correlation function of a time separation, such as those discussed in Sec. II, we define a function $f(t_0)$ such that

$$C_f(f(t_0)) = C(t_0), \tag{3.5}$$

where $C(t_0)$ has a given functional form. If its variance, $C(0) \neq 1$, then we rescale the constant $a$ by

$$a \to C(0)a. \tag{3.6}$$

For each value of $t_0$, and hence of $f$, we repeat the procedure described just before Eq. (3.2), again with $n \approx 10^4$.

For each of the three cases given in Sec. II, we have carried numerical calculations of the correlation function with the results below:

### 1. Massless scalar field

Here the correlation function, $C(t_0)$, is given by Eq. (2.10), and we use

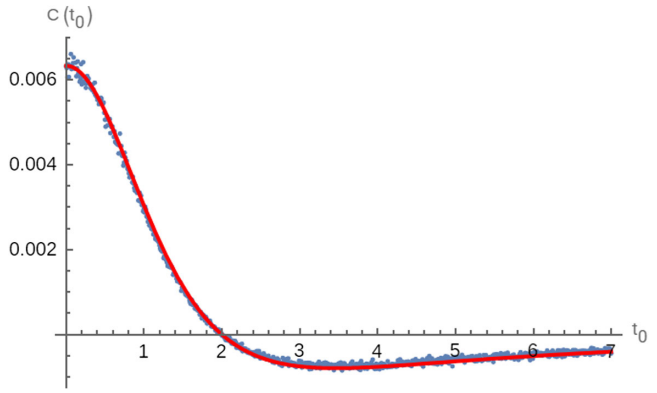


FIG. 2. The correlation function $C(t_0)$ given by Eq. (2.10) (red) is plotted against a simulated function (blue).

$$f(t_0) = \frac{1}{b}\arctan\left(\frac{4 - t_0^2}{4\pi a(4 + t_0^2)^2}\right). \tag{3.7}$$

In this case $C(0) = 1/(16\pi^2)$, so we use $a \approx 4.3 \times 10^{-4}$, but $b$ as given in Eq. (3.4). The analytic and simulated forms of this correlation function are plotted together in Fig. 2.

### 2. The electromagnetic and similar fields

Here the correlation function, $K(t_0)$ is given by Eq. (2.13), and we use

$$f(t_0) = \frac{1}{b}\arctan\left(\frac{1 - 6t_0^2 + t_0^4}{a(1 + t_0^2)^4}\right). \tag{3.8}$$

The analytic and simulated forms of this correlation function are plotted together in Fig. 3.

### 3. Baths of photons or gravitons in a squeezed state

Here the correlation function, $C_1(t_0)$ is given by Eq. (2.14), and we use

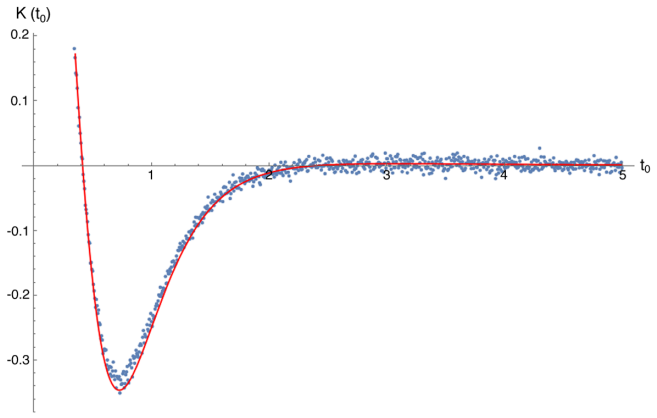


FIG. 3. The correlation function $K(t_0)$ (red) and its simulated form (blue) are plotted.

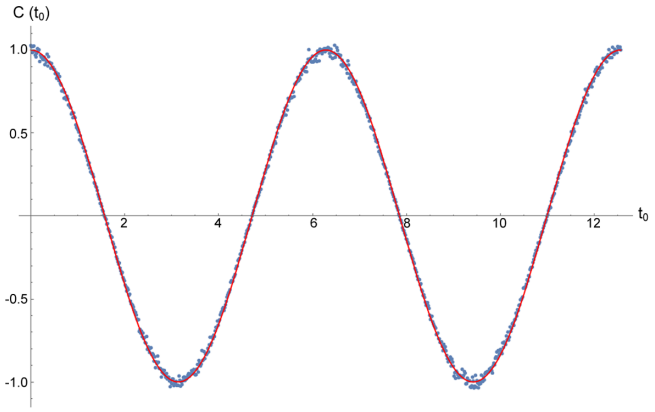


FIG. 4. The correlation function $C_1(t_0)$ (red) and its simulated form (blue) are plotted.

$$f(t_0) = \frac{1}{b}\arctan\left(\frac{\cos(kt_0)}{a}\right). \tag{3.9}$$

The analytic and simulated forms of this correlation function are plotted together in Fig. 4.

In all three cases, we see that the simulations and analytic forms of the correlation functions agree well. We have also used a modified version of our procedure in which we start with the derivative $C'(t_0)$ of the analytically derived correlation function, obtain a numerical simulation of this derivative, and then numerically integrate the result to find simulated functions which agree reasonably well with the simulations plotted in Figs. 2–4.

## IV. APPLICATION TO THE BROWNIAN MOTION OF TEST PARTICLES

One possible application of the method developed in Sec. II could be to the study of the effects of field fluctuations on Brownian motion, the fluctuations in test particle trajectories. This topic was discussed using analytic correlation functions in many papers, including Refs. [6,7]. Quantum electric field fluctuations create a fluctuating force on a test charge and hence cause the linear momentum of the charge to fluctuate. In the absence of an external source of energy, the mean squared momentum cannot on average grow in time. In this case, energy conservation is enforced by the anticorrelations of the electric field fluctuations. The particle may temporarily acquire energy from a fluctuation, but soon an anticorrelated fluctuation will tend to take this energy away. This feature is encoded in the correlation function by the vanishing of its integral over all time. For the case of the quantized electric field,

$$\int_0^\infty K(t_0)dt_0 = 0. \tag{4.1}$$

Numerical simulations may give insight into the behavior of test particles in baths of squeezed photons or squeezed gravitons, and will be a topic of future research. For the purpose of this paper, we consider a simple example of a test particle coupled to the fluctuating massless scalar field. At each step the value of the field $\bar{\psi}$ found determines the sign and magnitude of the particle's displacement for that step. If the field fluctuations were uncorrelated, this would constitute a random walk in which the mean squared displacement grows proportionally with the square root of the number of steps. Correlations are expected to either enhance or retard this rate of growth, depending upon the sign of the correlation. We have performed simulations for the cases $f = -0.5$, $f = 0$, and $f = 0.5$, and found square root growth in all cases for step number $N$ in the range $0 < N < 100$. Specifically

$$y \approx \sqrt{0.68N + 0.039} \quad f = -0.5, \tag{4.2}$$

$$y \approx \sqrt{1.32N + 0.092} \quad f = 0, \tag{4.3}$$

and

$$y \approx \sqrt{5.35N + 8.13} \quad f = 0.5. \tag{4.4}$$

We see that the uncorrelated case, $f = 0$, grows faster than that of anticorrelation, $f = -0.5$, but slower than the case of positive correlation, $f = 0.5$. The fact that we still find some growth in the case of anticorrelation is probably due to the limited nature of this simulation. Here we are assuming that each step is correlated only with the preceding step, and not with earlier steps. A more realistic multistep simulation is a topic for future research.

## V. SUMMARY

In this paper we have proposed a method to numerically simulate quantum field fluctuations with a given temporal correlation. This potentially allows simulations of the subtle effects of correlation and anticorrelation which quantum fields exhibit. Here our attention was restricted to linear fields with a Gaussian probability distribution, and our method involves a displacement of the Gaussian which depends upon the outcome of a previous measurement. We were able to use this method to numerically implement correlations in several explicit cases. We also used this method for a simplified treatment of the effects of correlations on a random walk. That treatment involved only one-step correlations between an event and the immediately prior event. We hope to generalize our treatment to include multistep correlations with several prior events.

We also plan to extend our method to cases of non-Gaussian probability distribution. This will be of interest in simulating quantum stress tensor fluctuations, which can have a non-Gaussian probability distribution which falls more slowly than exponentially, and depends upon

the details of the measurement sampling function $f(t)$ [5,11]. Stress tensor fluctuations pose the complication that the probability distribution need not be symmetric, although it can be for flux components [12]. For components that are non-negative in classical physics, such as the energy density, the distribution has a lower bound but no upper bound. In these cases, the symmetric procedure used in Sec. III A seems likely to require modification. Even for symmetric non-Gaussian probability distributions, it is not clear whether the empirical ansatz used in Eq. (3.3) will still hold. This is a topic for future work.

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