Low-mass enhancement of kaon pairs in $B^+ \to \bar{D}^{(*)0} K^+ \bar{K}^0$ and $B^0 \to D^{(*)-} K^+ \bar{K}^0$ decays

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(Received 25 March 2024; accepted 15 May 2024; published 10 June 2024)

Very recently, the Belle II collaboration presented a measurement for the decays $B^+ \to \bar{D}^{(*)0} K^+ \bar{K}^0$ and $B^0 \to D^{(*)-} K^+ \bar{K}^0$, with the bulk of observed $m(K^+ K_S^0)$ distributions showing low-mass structures in all four channels. In this work, we study the contributions of $\rho(770, 1450)^+$, $a_2(1320)^+$, and $a_0(980, 1450)^+$ resonances to these decay processes. The intermediate states $\rho(770, 1450)^+$ are found to dominate the low-mass distribution of kaon pairs roughly contributing to half of the total branching fraction in each of the four decay channels. The contribution of the tensor $a_2(1320)^+$ meson is found to be negligible. Near the threshold of the kaon pair, the state $a_0(980)^+$ turns out to be much less important than expected, not being able to account for the enhancement of events in that energy region observed in the $B^+ \to \bar{D}^{(*)0} K^+ \bar{K}^0$ decays. Further studies both from the theoretical and experimental sides are needed to elucidate the role of the nonresonant contributions governing the formation of $K^+ \bar{K}^0$ pairs near their threshold in these decay processes.

DOI: 10.1103/PhysRevD.109.116009

I. INTRODUCTION

Three-body hadronic B meson decay processes are regularly interpreted in terms of the contribution of various resonant states. The investigation of appropriate decay channels will help us to comprehend the properties and substructures of the related hadronic resonances involved in these decays. By employing the Dalitz plot amplitude analysis technique [1], the experimental efforts on relevant decay processes combined with the analysis within the isobar formalism have revealed valuable information on low-energy resonance dynamics [2,3]. Very recently, the Belle II collaboration presented a measurement for the decay channels $B^+ \to \bar{D}^{(*)0} K^+ K_S^0$ and $B^0 \to D^{(*)-} K^+ K_S^0$ [4,5]. In addition to the four branching fractions for these concerned decays, the $m(K^+ K_S^0)$ distribution of kaon pairs was also provided, showing relevant low-mass structures in all four channels [4].

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Given the presence of an open charm meson $D^{(*)}$ in the final state, these four decay processes measured by Belle II, which have also been previously searched by the Belle experiment [6], are relatively simple and clear from a theoretical point of view. One only has to consider the contributions from the tree-level W exchange operators O_1 and O_2 in the effective Hamiltonian $\mathcal{H}_{\rm eff}$ [7] within the framework of the factorization method [8]. In the low-mass region, the isospin $I = 1$ $K^+ \bar{K}^0$ kaon pair emitted in the $B^+ \to \bar{D}^{(*)0} K^+ \bar{K}^0$ and $B^0 \to D^{(*)-} K^+ \bar{K}^0$ decays can be originated from the charged intermediate states, $\rho(770)^+$, $a_0(980)^+$, $a_2(1320)^+$ and their excited states, via the quasi-two-body mechanism shown schematically in Fig. 1. The intermediate state R^+ in the figure, which decays into the final kaon pair, is generated in the hadronization of the light quark-antiquark pair $u\bar{d}$ or can be formed as a dynamically generated state through the meson-meson interactions.

The neutral states $\phi(1020)$, $\omega(782)$ and their excited states will not decay into $K^+ K_S^0$ as a result of charge conservation. The charged ρ resonances are then the expected intermediate states contributing to the $K^+ K_S^0$ system with spin-parity $J^P = 1^-$. In principle, the natural decay mode $\rho(770) \to K\bar{K}$ is blocked because the pole mass of the resonance $\rho(770)$ is below the threshold of the kaon pair. However, the virtual contribution [9–12] from the Breit-Wigner (BW) [13] tail effect of the $\rho(770)$ was

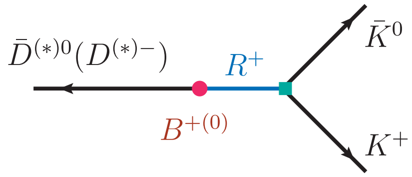

FIG. 1. Schematic view of the cascade decays $B^+ \to \bar{D}^{(*)0}R^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}R^+ \to D^{(*)-}K^+\bar{K}^0$, where R^+ stands for the intermediate states $\rho(770, 1450)^+$, $a_0(980, 1450)^+$, or $a_2(1320)^+$, which decays into $K^+\bar{K}^0$ in this work.

found to be indispensable for the productions of kaon pairs in the processes $\pi^- p \to K^- K^+ n$ and $\pi^+ n \to K^- K^+ p$ [14,15], $\bar{p}p \to K^+K^-\pi^0$ [16,17], $e^+e^- \to K^+K^-$ [18–26], and $e^+e^- \to K^0_S K^0_L$ [27–32]. Besides, the mesons $\rho(770)^\pm$ and $\rho(1450)^\pm$ are the important intermediate states for the hadronic τ decays with $K^\pm K^0_S$ in the final states [33–36]. In recent years, the contributions for kaon pairs originating from the ρ family of resonances have been explored in Refs. [37–40] for quasi-two-body B meson decays and in Refs. [41–44] for D meson decays.

The $a_0(980)$ is an experimentally well established scalar state, which has been primarily seen as an enhancement in the $\pi\eta$ channel [45], as well as in the $K\bar{K}$ system near threshold [10]. It has commonly been placed together with the states $f_0(500)$, $K^*_0(700)$, and $f_0(980)$ into a SU(3) flavor nonet. The quark-antiquark configuration in the naive quark model for their internal structure cannot explain its true nature. In this context, scenarios such as tetraquark states [46–50], molecular states [51,52] and dynamically generated states from meson-meson interactions [53–56] or a quark-antiquark seed [57–60] have been adopted to describe the mysterious properties of the $a_0(980)$; see Refs. [2,61–64] for reviews in this matter. Conversely, the state $a_0(1450)$, first observed from $\pi\eta$ pair [65], is usually described as a $q\bar{q}$ resonance in the phenomenological studies of Refs. [66–70]. This resonance $a_0(1450)^+$ is however expected to contribute to the kaon pair distribution from the $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}K^+\bar{K}^0$ decays with a small amount, in view of its tiny decay constant [66,71] and the small ratio between the $K\bar{K}$ decay channel and its dominant $\omega\pi\pi$ mode [2,72].

As for the contribution of the isovector tensor meson $a_2(1320)$, we note that it is the ground state of the a_2 family with quantum numbers $I^G J^{PC} = 1^- 2^{++}$ and it can be reasonably understood as a constituent quark-antiquark pair within the quark model [2]. The transition form factors for the B meson to the $a_2(1320)$ state have been obtained in Refs. [73–77] within various methods. Moreover, the hadronic B meson decays involving a tensor meson $a_2(1320)$ in the final state have been studied in Refs. [78–85] in recent years. The tensor meson $a_2(1700)$, assigned as the first radial excitation of the $a_2(1320)$ [2,86] state, will not be considered in this work in view of the negligible branching fraction of the decay of the $a_2(1700)$ into $K\bar{K}$ pairs [2,87].

This paper is organized as follows. In Sec. II, we briefly describe the theoretical framework for obtaining the resonance contributions to the decay rates of the $B^+ \to \bar{D}^{(*)0}R^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}R^+ \to D^{(*)-}K^+\bar{K}^0$ processes, relegating to Appendices A and B the specific details of the calculation of the decay amplitudes. In Sec. III, we present our numerical results of the branching fractions for the concerned quasi-two-body decay processes along with some necessary discussions. To test our model, we will also present results for the branching ratios of the B decay processes into a $D^{(*)}$ meson and a pair of pions in the final state. A summary and the conclusions of this work are given in Sec. IV.

II. FRAMEWORK

In the present paper, we analyze the low-mass enhancement in the distribution of kaon pairs in the final states of the $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}K^+\bar{K}^0$ decay processes within the factorization method. We will specifically concentrate on the resonances contributing to the invariant mass region of $m(K^+\bar{K}^0) < 1.7$ GeV, adopting the quasi-two-body framework for the relevant decays. The quasi-two-body framework based on the perturbative QCD (PQCD) [88–91] approach has been discussed in detail in Ref. [92] and has been applied to the study of B meson decays in Refs. [37–40,93–99] in recent years. Parallel analyses for the related three-body B meson decay processes within QCD factorization can be found in Refs. [100–112], and for other works employing relevant symmetry relations one is referred to Refs. [113–121].

For the cascade decays $B^+ \to \bar{D}^{(*)0}R^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}R^+ \to D^{(*)-}K^+\bar{K}^0$, where the intermediate state R^+ stands for $\rho(770, 1450)^+$, $a_0(980, 1450)^+$, or $a_2(1320)^+$, the related effective weak Hamiltonian $\mathcal{H}_{\rm eff}$ accounting for the $\bar{b} \to \bar{c}$ transition is written as [7]

$$\mathcal{H}_{\rm eff} = \frac{G_F}{\sqrt{2}} V^*_{cb} V_{ud} [C_1(\mu) O^c_1(\mu) + C_2(\mu) O^c_2(\mu)], \quad (1)$$

where $G_F = 1.1663788(6) \times 10^{-5}$ GeV^{-2} [2] is the Fermi coupling constant, $C_{1,2}(\mu)$ are the Wilson coefficients at scale μ, and V_{cb} and V_{ud} are the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. The four-quark operators $O^c_{1,2}$ are products of two $V-A$ currents, namely $O^c_1 = (\bar{b}d)_{V-A}(\bar{u}c)_{V-A}$ and $O^c_2 = (\bar{b}c)_{V-A}(\bar{u}d)_{V-A}$.

With the factorization ansatz, the decay amplitudes for $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}K^+\bar{K}^0$ are given as [122]

$$\mathcal{M}(\bar{D}^{(*)0}K^+\bar{K}^0) = \frac{G_F}{\sqrt{2}}V_{cb}^*V_{ud}[a_2\langle\bar{D}^{(*)0}|(\bar{b}c)_{V-A}|B^+\rangle\langle K^+\bar{K}^0|(\bar{u}d)_{V-A}|0\rangle + a_1\langle K^+\bar{K}^0|(\bar{b}d)_{V-A}|B^+\rangle\langle\bar{D}^{(*)0}|(\bar{u}c)_{V-A}|0\rangle], \quad (2)$$

$$\mathcal{M}(D^{(*)-}K^+\bar{K}^0) = \frac{G_F}{\sqrt{2}}V_{cb}^*V_{ud}a_2\langle D^{(*)-}|(\bar{b}c)_{V-A}|B^0\rangle\langle K^+\bar{K}^0|(\bar{u}d)_{V-A}|0\rangle, \quad (3)$$

where the effective Wilson coefficients are expressed as $a_1 = C_1 + C_2/3$ and $a_2 = C_2 + C_1/3$.

The differential branching fraction ($\mathcal{B}$) for the considered decays is written as [2,105,123]

$$\frac{d\mathcal{B}}{\sqrt{s}d\sqrt{s}} = \tau_B\frac{|\mathbf{p}_K||\mathbf{p}_D|}{4(2\pi)^3m_B^3}\left[|\mathcal{M}_S|^2 + \frac{1}{3}|\mathbf{p}_K|^2|\mathbf{p}_D|^2|\mathcal{M}_V|^2\right], \quad (4)$$

where the amplitudes $\mathcal{M}_V$ and $\mathcal{M}_S$ are related to the vector $\rho(770,1450)^+$ and scalar $a_0(980,1450)^+$ intermediate states, respectively, with the help of the Eqs. (2) and (3). Here, $\tau_B(m_B)$ is the mean lifetime (mass) for the B meson, $s = m^2_{K^+\bar{K}^0}$ is the invariant mass square, and

$$|\mathbf{p}_K| = \frac{\sqrt{[s-(m_{K^+}+m_{\bar{K}^0})^2][s-(m_{K^+}-m_{\bar{K}^0})^2]}}{2\sqrt{s}}, \quad (5)$$

$$|\mathbf{p}_D| = \frac{\sqrt{[m_B^2-(\sqrt{s}+m_D)^2][m_B^2-(\sqrt{s}-m_D)^2]}}{2\sqrt{s}}, \quad (6)$$

correspond, respectively, to the magnitude of the momentum of each kaon and that of the bachelor meson $\bar{D}^{(*)0}$ or $D^{(*)-}$, with mass m_D, in the rest frame of the intermediate resonance.

By combining various contributions from the relevant Feynman diagrams at quark level in Fig. 2, the total decay amplitudes for the concerned quasi-two-body decays in the PQCD approach are written as

$$\mathcal{A}_V(B^+\to\bar{D}^0[\rho^+\to]hh') = \frac{G_F}{\sqrt{2}}V_{cb}^*V_{ud}[a_1F_{T\rho} + C_2M_{T\rho} + a_2F_{TD} + C_1M_{TD}], \quad (7)$$

$$\mathcal{A}_V(B^0\to D^-[\rho^+\to]hh') = \frac{G_F}{\sqrt{2}}V_{cb}^*V_{ud}[a_2F_{TD} + C_1M_{TD} + a_1F_{a\rho} + C_2M_{a\rho}], \quad (8)$$

where $hh'\in\{\pi^+\pi^0, K^+\bar{K}^0\}$. The label F (M) denotes that the corresponding decay amplitude comes from the factorizable (nonfactorizable) Feynman diagrams, the subscripts $T\rho$ and TD stand for the transition $B\to\rho$ and $B\to D$, respectively, and the subscript $a\rho$ is related to the annihilation Feynman diagram of Fig. 2(c). The specific expressions in the PQCD approach for these general amplitudes F and M in these decay amplitudes are found in Appendix A. One should note that the $\mathcal{A}$s here have a constant factor $(2/m_B)^2$ different from $\mathcal{M}_V$ in Eq. (4) because of the different definitions between PQCD and QCD factorization, see the corresponding expression for the differential branching fraction of the former in [38].

The quasi-two-body decay amplitudes (7)–(8) are related to the corresponding two-body decay amplitude M_{2B} for the $B\to\bar{D}\rho^+$ transition via the relation

$$\mathcal{A}_V = M_{2B}\cdot\frac{\langle hh'|\rho^+\rangle}{\mathcal{D}_{\rho^+}(s)}, \quad (9)$$

where $\langle hh'|\rho^+\rangle$ stands for the coupling between the ρ^+ and the hh' pair. Note that the former equation is effectively incorporating the electromagnetic form factor associated to the subprocesses $\rho(770,1450)^+\to\pi^+\pi^0$ and $\rho(770,1450)^+\to K^+\bar{K}^0$ in the corresponding quasi-two-body decays, given by [124–127]

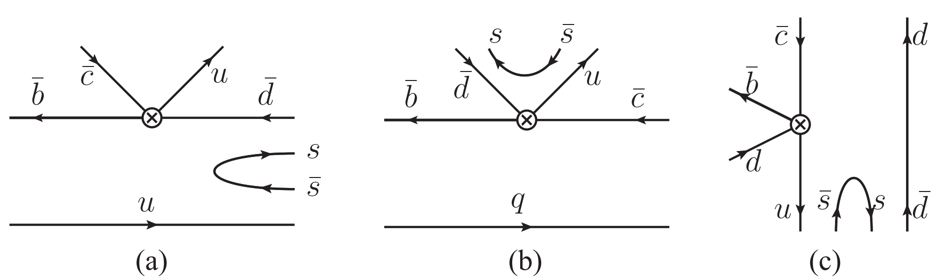

FIG. 2. Typical Feynman diagrams for the decays $B^+\to\bar{D}^{(*)0}R^+\to\bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0\to D^{(*)-}R^+\to D^{(*)-}K^+\bar{K}^0$ at quark level, where (a) and (b) are the emission diagrams, (c) is the annihilation one, the quark $q=u$ and d for the B^+ and B^0 processes, respectively, and the symbol $\otimes$ stands for the weak vertex.

$$F^R_{\pi,K}(s) = c_R^{\pi,K}\mathrm{BW}_R(s) \equiv c_R^{\pi,K}\frac{m_R^2}{\mathcal{D}_R(s)}, \tag{10}$$

where the label R represents the resonance, $\rho(770)$ or $\rho(1450)$, and the coefficient $c_R^\pi = f_R g_{R\pi\pi}/(\sqrt{2}m_R)$ [124] depends on the corresponding decay constant f_R, the coupling constant $g_{R\pi\pi}$ and the mass m_R. To obtain the coefficient c_R^K we relay on flavor SU(3) symmetry, which establishes $g_{\rho(770)^0K^+K^-} = g_{\rho(770)^0\pi^+\pi^-}/2$ [124]. The function $\mathrm{BW}_R(s)$ stands for a Breit-Wigner shape of the form [124,127,128]

$$\mathrm{BW}_R \equiv \frac{m_R^2}{\mathcal{D}_R(s)} = \frac{m_R^2}{m_R^2 - s - im_R\Gamma_R(s)}, \tag{11}$$

with the s-dependent width given by

$$\Gamma_R(s) = \Gamma_R\frac{m_R}{\sqrt{s}}\frac{|\mathbf{p}_h|^3}{|\mathbf{p}_{h0}|^3}X^2(|\mathbf{p}_h|r^R_{\mathrm{BW}}), \tag{12}$$

where h stands for the pion and kaon, respectively, in the $\pi^+\pi^0$ and $K^+\bar{K}^0$ final state pairs. The magnitude $|\mathbf{p}_{h0}|$ corresponds to the value of $|\mathbf{p}_h|$ at $s = m_R^2$, while $|\mathbf{p}_\pi|$ can be obtained from Eq. (5) with the replacement of $m_{K^{+,0}}$ by $m_{\pi^{+,0}}$. The Blatt-Weisskopf barrier factor [129] with barrier radius $r^R_{\mathrm{BW}} = 4.0\ \mathrm{GeV}^{-1}$ [128] is given by

$$X(z) = \sqrt{\frac{1+z_0^2}{1+z^2}}. \tag{13}$$

As for the other two decay amplitudes corresponding to the B^+ and B^0 decays into final vector mesons $\bar{D}^{*0}$ and D^{*-}, respectively, they are obtained from Eqs. (7) and (8) with the replacement of the D meson wave function by the D^* one. As has been done in the study of B decays into two vector mesons in the final state, the two-body decay amplitudes for $B \to \bar{D}^*\rho^+$ in this work can be decomposed as [130]

$$\begin{aligned} M_{2B}^{(\lambda)} &= \epsilon^*_{\bar{D}^*\mu}(\lambda)\epsilon^*_{\rho\nu}(\lambda)\left[ag^{\mu\nu} + \frac{b}{m_D\sqrt{s}}P_B^\mu P_B^\nu + i\frac{c}{m_D\sqrt{s}}\epsilon^{\mu\nu\alpha\beta}P_\alpha P_{3\beta}\right], \\ &\equiv M_L + M_N\epsilon^*_{\bar{D}^*}(\lambda = T)\cdot\epsilon^*_\rho(\lambda = T) + i\frac{M_T}{m_B^2}\epsilon^{\alpha\beta\gamma\eta}\epsilon^*_{\rho\alpha}(\lambda)\epsilon^*_{D^*\beta}(\lambda)P_\gamma P_{3\eta}, \end{aligned} \tag{14}$$

with three kinds of polarizations of the vector meson, namely, longitudinal (L), normal (N), and transverse (T). According to the polarized decay amplitudes, one has the total decay amplitude $|\mathcal{A}_V|^2 = |A_L|^2 + |A_\parallel|^2 + |A_\perp|^2$, and a longitudinal polarization fraction $\Gamma_L/\Gamma = |A_L|^2/|\mathcal{A}_V|^2$, where the amplitudes $A_L, A_\parallel$, and $A_\perp$ are related to the two-body amplitudes M_L, M_N, and M_T, respectively, via Eq. (9). For a detailed discussion, one is referred to Refs. [130–134].

III. RESULTS AND DISCUSSIONS

In this section we present our results for the branching ratios of the decay of B mesons into a charm D or D^* meson and a pair of light pseudoscalar mesons. In the numerical calculations, we adopt the decay constants $f_{\rho(770)} = 0.216 \pm 0.003$ GeV [135] and $f_{\rho(1450)} = 0.185^{+0.030}_{-0.035}$ GeV [92,136] for the $\rho(770)$ and $\rho(1450)$ resonances, respectively, and the mean lives $\tau_{B^\pm} = 1.638 \times 10^{-12}$ s and $\tau_{B^0} = 1.519 \times 10^{-12}$ s for the initial states $B^\pm$ and B^0 [2], respectively. The masses for the particles in the relevant decay processes, the decay constants for B, D, and D^* mesons (in units of GeV), and the Wolfenstein parameters for the CKM matrix elements, A and λ, are presented in Table I. We adopt the full widths $\Gamma_{\rho(770)} = 149.1 \pm 0.8$ MeV, $\Gamma_{\rho(1450)} = 400 \pm 60$ MeV, $\Gamma_{a_0(1450)} = 265 \pm 13$ MeV, and $\Gamma_{a_2(1320)} = 107 \pm 5$ MeV for the intermediate states involved in this work.

To illustrate the capabilities of the PQCD approach, we first obtain the branching fractions for the quasi-two-body decays $B^+ \to \bar{D}^{(*)0}[\rho(770)^+ \to]\pi^+\pi^0$ and $B^0 \to D^{(*)-}[\rho(770)^+ \to]\pi^+\pi^0$. Our results, displayed in Table II, employ the P-wave two-pion distribution amplitudes of Ref. [92], the $D^{(*)}$ meson wave functions of Refs. [99,138] and consider $\mathcal{B}_{\rho(770)^+\to\pi^+\pi^0} \approx 100\%$ [2].

TABLE I. Masses, decay constants (in units of GeV) for relevant states, as well as the Wolfenstein parameters for the CKM matrix elements from the *Review of Particle Physics* [2]. The value of f_{D^*} is taken from [137].

$m_{B^\pm} = 5.279$	$m_{B^0} = 5.280$	$m_{D^{*\pm}} = 2.010$	$m_{D^{*0}} = 2.007$
$m_{D^\pm} = 1.870$	$m_{D^0} = 1.865$	$m_{\pi^\pm} = 0.140$	$m_{\pi^0} = 0.135$
$m_{K^\pm} = 0.494$	$m_{K^0} = 0.498$	$m_{\rho(770)} = 0.775$	$m_{\rho(1450)} = 1.465$
$m_{a_0(980)} = 0.980$	$m_{a_0(1450)} = 1.474$	$m_{a_2(1320)} = 1.318$	$f_B = 0.190$
$f_D = 0.212$	$f_{D^*} = 0.2235$	$A = 0.826$	$\lambda = 0.225$

TABLE II. PQCD results for the branching fractions of the quasi-two-body decays $B^+ \to \bar{D}^{(*)0}[\rho(770)^+ \to]\pi^+\pi^0$ and $B^0 \to D^{(*)-}[\rho(770)^+ \to]\pi^+\pi^0$.

Decay modes	Units	PQCD
$B^+ \to \bar{D}^0[\rho(770)^+ \to]\pi^+\pi^0$	%	$1.21^{+0.16+0.10+0.05}_{-0.16-0.12-0.06}$
$B^0 \to D^-[\rho(770)^+ \to]\pi^+\pi^0$	10^{-3}	$7.63^{+0.58+0.97+0.34}_{-0.58-0.73-0.21}$
$B^+ \to \bar{D}^{*0}[\rho(770)^+ \to]\pi^+\pi^0$	10^{-3}	$9.03^{+1.55+0.73+0.51}_{-1.55-0.64-0.46}$
$B^0 \to D^{*-}[\rho(770)^+ \to]\pi^+\pi^0$	10^{-3}	$8.15^{+1.31+0.64+0.03}_{-1.31-0.62-0.07}$

The calculated branching fractions agree well with the corresponding data,

$$\mathcal{B}(B^+ \to \bar{D}^0\rho(770)^+) = (1.34 \pm 0.18)\%, \quad (15)$$

$$\mathcal{B}(B^0 \to D^-\rho(770)^+) = (7.6 \pm 1.2) \times 10^{-3}, \quad (16)$$

$$\mathcal{B}(B^+ \to \bar{D}^{*0}\rho(770)^+) = (9.8 \pm 1.7) \times 10^{-3}, \quad (17)$$

$$\mathcal{B}(B^0 \to D^{*-}\rho(770)^+) = (6.8 \pm 0.9) \times 10^{-3}, \quad (18)$$

in the *Review of Particle Physics* [2], indicating that the framework employed and the inputs adopted in this work are adequate. The branching fraction (15) for $B^+ \to \bar{D}^0\rho(770)^+$ was averaged in [2] from the data $(1.35 \pm 0.12 \pm 0.15)\%$ and $(1.3 \pm 0.4 \pm 0.4)\%$ presented by CLEO and ARGUS in Refs. [139] and [140], respectively. Very recently, the Belle II collaboration measured the decay $B^- \to D^0\rho(770)^-$ using data collected with the Belle II detector, its branching fraction was determined to be $(0.939 \pm 0.021(\mathrm{stat}) \pm 0.050(\mathrm{syst}))\%$ by restricting the $\pi^-\pi^0$ invariant mass to a 300 MeV range centered at the $\rho(770)^-$ mass pole [141]. This measurement is smaller than the value in Eq. (15), but it is still in agreement with our result in Table II within the uncertainties.

With the help of the kaon form factor $F_{K^+\bar{K}^0}(s)$ discussed in detail in [38], we obtain the concerned branching fractions of the B mesons into a D or D^* meson and a pair of kaons for the quasi-two-body processes $\rho(770)^+ + \rho(1450)^+ \to K^+\bar{K}^0$. Our results are displayed in Table III.

TABLE III. PQCD predictions for the branching fractions of the concerned quasi-two-body decays with the subprocess $\rho^+ \to K^+\bar{K}^0$, here $\rho^+ = \rho(770)^+ + \rho(1450)^+$.

Mode	Unit	$\mathcal{B}$
$B^+ \to \bar{D}^0[\rho^+ \to]K^+\bar{K}^0$	10^{-4}	$1.68^{+0.20+0.17+0.12}_{-0.20-0.15-0.12}$
$B^0 \to D^-[\rho^+ \to]K^+\bar{K}^0$	10^{-4}	$0.98^{+0.06+0.13+0.06}_{-0.06-0.12-0.06}$
$B^+ \to \bar{D}^{*0}[\rho^+ \to]K^+\bar{K}^0$	10^{-4}	$1.33^{+0.21+0.11+0.05}_{-0.21-0.11-0.07}$
$B^0 \to D^{*-}[\rho^+ \to]K^+\bar{K}^0$	10^{-4}	$1.16^{+0.19+0.08+0.02}_{-0.19-0.09-0.02}$

In the results for the branching fractions shown in Tables II–III, the first source of the error corresponds to the uncertainties of the shape parameter $\omega_B = 0.40 \pm 0.04$ of the $B^{\pm,0}$ wave functions, while the Gegenbauer moments $C_D = 0.6 \pm 0.15$ or $C_{D^*} = 0.5 \pm 0.10$ present in the D or D^* wave functions [99] contribute to the second source of error. The third one is induced by the Gegenbauer moments $a_R^0 = 0.25 \pm 0.10$, $a_R^t = -0.60 \pm 0.20$ and $a_R^s = 0.75 \pm 0.25$ [92] present in the wave functions of the intermediate states. The other errors for the PQCD predictions in this work, which come from the uncertainties of the masses and the decay constants of the initial and final states and from the uncertainties of the Wolfenstein parameters, etc., are small and have been neglected.

Comparing our calculated branching rations of Table III with the measured results (in units of 10^{-4}) [4]

$$\mathcal{B}(B^+ \to \bar{D}^0K^+K_S^0) = 1.89 \pm 0.16 \pm 0.10, \quad (19)$$

$$\mathcal{B}(B^0 \to D^-K^+K_S^0) = 0.85 \pm 0.11 \pm 0.05, \quad (20)$$

$$\mathcal{B}(B^+ \to \bar{D}^{*0}K^+K_S^0) = 1.57 \pm 0.27 \pm 0.12, \quad (21)$$

$$\mathcal{B}(B^0 \to D^{*-}K^+K_S^0) = 0.96 \pm 0.18 \pm 0.06, \quad (22)$$

and taking into account that half of the $\bar{K}^0$ or K^0 goes to K_S^0, we conclude that an important fraction of the decays $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}K^+\bar{K}^0$ proceeds through the intermediate states $\rho(770)^+$ and $\rho(1450)^+$, but there is still room for other contributions.

One could argue that the resonance $\rho(770)^+$, as a virtual bound state [9,10], will not completely exhibit its properties in a quasi-two-body cascade decay like $B^0 \to D^-[\rho(770)^+ \to]K^+\bar{K}^0$, since the invariant masses of the emitted kaon pairs exclude the region around the $\rho(770)$ pole mass. However, as we will show below, the width of this resonance renders its contribution quite sizable in the energy region of interest. It is therefore important to consider explicitly the subthreshold resonances in the analysis of the branching ratios, even if they contribute via the tail of their mass distribution. In other words, experimental analyses or theoretical studies of three-body B meson decay process should not attribute as nonresonant $K\bar{K}$ invariant mass strength the specific known contribution from a certain resonant state like the $\rho(770)$.

To make this point more evident, we show in Fig. 3 the differential branching fraction for the quasi-two-body decay $B^0 \to D^-\rho^+ \to D^-K^+\bar{K}^0$. The dashed line with a peak at about 1.465 GeV reveals the contribution from the $\rho(1450)^+$, while the dash-dot line, depicting the contribution of the $\rho(770)^+$, presents a bump around 1.2 GeV, which shall not be claimed experimentally as a resonant state with quite a large decay width. This bump is actually formed by the BW tail of the $\rho(770)^+$ along with the phase

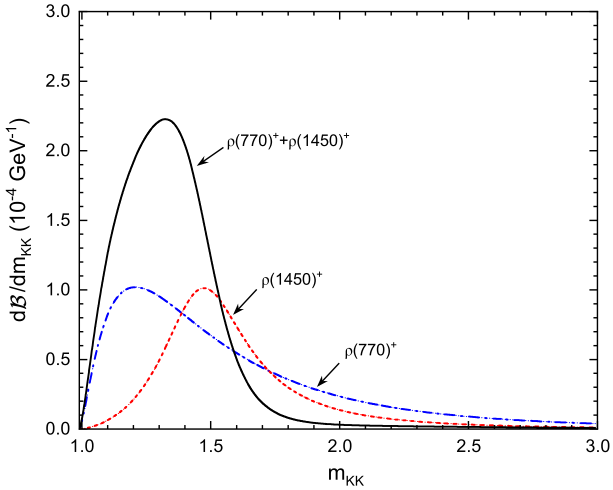

FIG. 3. The differential branching fraction for the quasi-two-body decay $B^0 \to D^-\rho^+ \to D^-K^+\bar{K}^0$, with the intermediate $\rho^+ \in \{\rho(770)^+, \rho(1450)^+, \rho(770)^+ + \rho(1450)^+\}$.

space factor of Eq. (4). The interference between the BW expressions for $\rho(770)^+$ and $\rho(1450)^+$ is constructive in the region before the pole mass of the $\rho(1450)^+$ and destructive after it as a result of the sign difference between $c^K_{\rho(770)} = 1.247 \pm 0.019$ and $c^K_{\rho(1450)} = -0.156 \pm 0.015$ [38] in Eq. (10). Note that the theoretical distribution has the same pattern in the low-mass region of the kaon pair as that shown in the bottom panel of Fig. 4 for the three-body decay $\bar{B}^0 \to D^+K^-K^0_S$. This comparison reflects the dominant contributions for this decay coming from the intermediate states $\rho(770)^+$ and $\rho(1450)^+$.

The higher mass resonance $\rho(1700)^+$ as an intermediate state could also decay into $K^+\bar{K}^0$ [37,38,40] and hence contribute to the $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}K^+\bar{K}^0$ decays. Take the case $B^0 \to D^-\rho(1700)^+ \to D^-K^+\bar{K}^0$ as an example. With the coefficient $c^K_{\rho(1700)} = -0.083 \pm 0.019$ [38], its branching fraction was predicted to be $\mathcal{B} = 4.00^{+3.36}_{-2.26} \times 10^{-5}$ in [40], which represents about a 20% of the total branching fraction for $B^0 \to D^-K^+\bar{K}^0$ when comparing with the result of Eq. (20) and neglecting the large error from the PQCD prediction. However, the $m(K^-K^0_S)$ distributions from the $B^- \to D^0K^-K^0_S$ and $\bar{B}^0 \to D^+K^-K^0_S$ decays measured by Belle II [4], see Fig. 4, show no prominent enhancement around the mass of the $\rho(1700)^+$. In view of the fact that $\mathcal{B}(a_2(1700) \to K\bar{K}) = (1.9 \pm 1.2)\%$ [2] in the same region, the explanation for the lack of structure possibly lies in (i) the interference between the $\rho(1700)^+$ and the $\rho(770)^+ + \rho(1450)^+$ contributions being destructive around 1.7 GeV and (ii) the coefficient $c^K_{\rho(1700)}$ in [38] being highly overrated, since one can also find sensibly smaller values in the literature, namely -0.015 ± 0.022 in [124] and -0.028 ± 0.012 in [125].

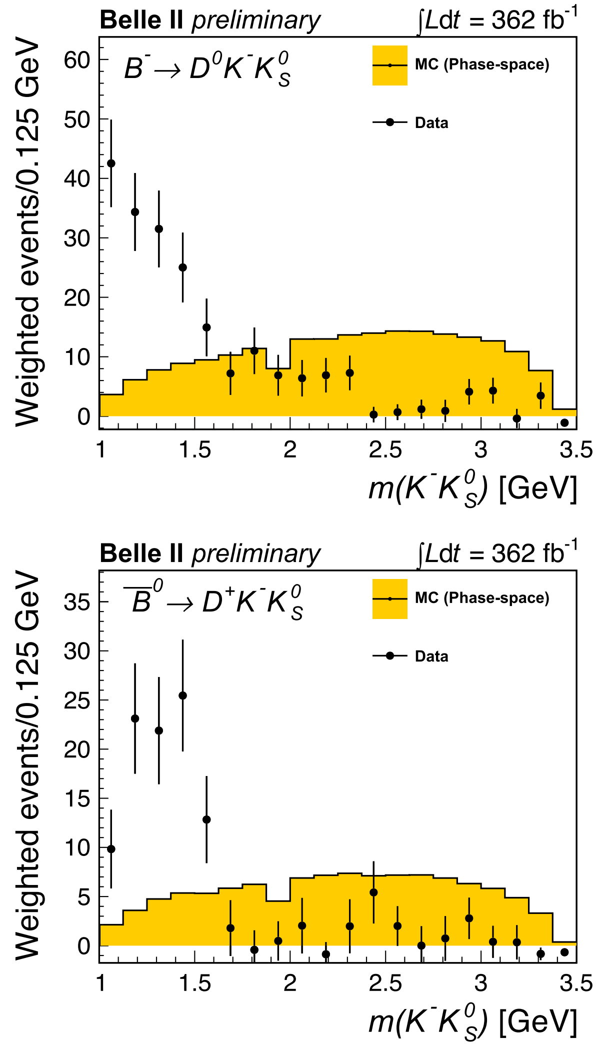

FIG. 4. The distribution of $m(K^-K^0_S)$ for $B^- \to D^0K^-K^0_S$ (top) and $\bar{B}^0 \to D^+K^-K^0_S$ (bottom) decays from Belle II [4].

Near the threshold of the kaon pair, one finds remarkable enhancements in the $m(K^-K^0_S)$ distributions for the decays $B^+ \to \bar{D}^0K^+K^0_S$ and $B^+ \to \bar{D}^{*0}K^+K^0_S$ from Belle II [4], but not for $B^0 \to D^-K^+\bar{K}^0$ or $B^0 \to D^{*-}K^+\bar{K}^0$. The invariant kaon pair mass around 1 GeV is the energy region of the state $a_0(980)$, but we do not expect the same strength of the $a_0(980)$ contributions in the $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and the $B^0 \to D^{(*)-}K^+\bar{K}^0$ processes, since their decay mechanisms proceed through different quark-type Feynman diagrams, shown in Fig. 2, as explained in the following. The annihilation Feynman diagrams represented by Fig. 2(c) will only contribute to the decays $B^0 \to D^{(*)-}R^+ \to D^{(*)-}K^+\bar{K}^0$, and the contributions are highly suppressed when comparing with the those from the emission diagrams of Figs. 2(a) and 2(b).

For the decays $B^0 \to D^{(*)-}K^+\bar{K}^0$ with subprocess $a_0(980)^+ \to K^+\bar{K}^0$, the contribution of the Feynman diagrams of Fig. 2(b) is small. This is because the matrix element $\langle K^+\bar{K}^0|(\bar{u}d)_{V-A}|0\rangle$ with $a_0(980)^+$ as the intermediate state depends on the tiny decay constant $f_{a_0(980)} \approx 1.1$ MeV [71,142–144]. This qualitatively explains why we do not see a remarkable enhancement of events around 1 GeV in the $m(K^-K_S^0)$ distribution from the decay $B^0 \to D^{(*)-}K^+\bar{K}^0$ [4], depicted in the bottom panel of Fig. 4. But the amplitudes for the decay $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ receive contribution not only from Fig. 2(b), with an $a_0(980)^+$ being emitted but also from the diagram of Fig. 2(a), which is a $B^+ \to a_0(980)^+$ transition with an emitted $\bar{D}^{(*)0}$. Despite being a color suppressed Feynman diagram [145], the hierarchical relation $f_{\bar{D}^{(*)0}} \gg f_{a_0(980)}$ makes the contribution from Fig. 2(a) much larger than that from Fig. 2(b) for the decays $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ with the subprocess $a_0(980)^+ \to K^+\bar{K}^0$.

Let us now proceed to the explicit numerical calculation. Within the naive factorization approach, the evaluation of the decay amplitude for the $B^+ \to \bar{D}^0K^+\bar{K}^0$ decay with the subprocesses $a_0(980, 1450)^+ \to K^+\bar{K}^0$ can be found in Appendix B. With Eq. (B5) and the inputs from the *Review of Particle Physics* [2], we obtain a branching fraction $\mathcal{B} = 1.56 \times 10^{-5}$ for the quasi-two-body decay $B^+ \to \bar{D}^0a_0(980)^+ \to \bar{D}^0K^+\bar{K}^0$, which corresponds to a value $\mathcal{B}(B^+ \to \bar{D}^0a_0(980)^+) = 1.07 \times 10^{-4}$ for the two-body decay. Likewise, we obtain $\mathcal{B} = 0.72 \times 10^{-5}$ for $B^+ \to \bar{D}^0a_0(1450)^+ \to \bar{D}^0K^+\bar{K}^0$, where we have employed $F_0^{B\to a_0(1450)}(0) = 0.26$ [146] and $\Gamma(a_0(1450) \to K\bar{K})/\Gamma(a_0(1450) \to \omega\pi\pi) \approx 0.082$ [2]. In order to check the reliability of the method we adopted here, the measured channel $B^0 \to D_s^+a_0(980)^-$ is studied as a reference. This is a process with a $B^0 \to a_0(980)^-$ transition and an emitted D_s^+ state. Within naive factorization, we find $\mathcal{B}(B^0 \to D_s^+a_0(980)^-) = 1.93 \times 10^{-5}$. This branching fraction is very close to the upper limit 1.9×10^{-5} at 90% CL presented by the *BABAR* collaboration in Ref. [147] assuming $\mathcal{B}(a_0(980)^+ \to \eta\pi^+)$ to be 100%, but it is much smaller than the prediction $\mathcal{B} = 4.81^{+2.19}_{-1.79} \times 10^{-5}$ in [148] within PQCD for the decay $B^0 \to D_s^+a_0(980)^-$. However, taking into account $\Gamma(a_0(980) \to K\bar{K})/\Gamma(a_0(980) \to \pi\eta) = 0.172 \pm 0.019$ [2], one has $\mathcal{B}(a_0(980)^+ \to \eta\pi^+) \approx 0.85$ and this will change the upper limit in [147] for $B^0 \to D_s^+a_0(980)^-$ up to 2.24×10^{-5} at 90% CL, which is still much smaller than the prediction in [148], hinting that the PQCD approach is possibly not appropriate for the study of the $B^0 \to D_s^+a_0(980)^-$ decay with the $B \to a_0(980)$ transition.

When we put the contributions from $a_0(980, 1450)^+ \to K^+\bar{K}^0$ and $\rho(770, 1450)^+ \to K^+\bar{K}^0$ for the decay $B^+ \to \bar{D}^0K^+\bar{K}^0$ together, the resulting differential branching fraction does not have the shape shown in the top panel of Fig. 4. The contribution from the scalar intermediate state $a_0(980)^+$ is far from what would be required to overcome the peak of the $\rho(770, 1450)^+$ distribution in order to reproduce the enhancement near the threshold of $K^+\bar{K}^0$ pairs measured experimentally. The shape of the measured $B^+ \to \bar{D}^0K^+\bar{K}^0$ differential branching fraction would only be obtained with a branching fraction $\mathcal{B} \approx 4.5 \times 10^{-4}$ for the quasi-two-body decay $B^+ \to \bar{D}^0a_0(980)^+ \to \bar{D}^0K^+\bar{K}^0$, which is beyond the total branching fraction for $B^+ \to \bar{D}^0K^+\bar{K}^0$ decay. This situation is probably indicating the existence of large nonresonant contributions to the $B^+ \to \bar{D}^0K^+\bar{K}^0$ decay around the threshold of the kaon pair or other unknown sources. Note that the interference between $\rho(770)^+$ and $\rho(1450)^+$ could reduce the corresponding branching fractions in Table III through an appropriate complex phase difference between their respective BW expressions. This would alleviate the requirement of an enhanced contribution from the $a_0(980)^+$. For example, a factor of $e^{i\pi/4}$ before the BW of the $\rho(1450)^+$ will produce half of the $B^+ \to \bar{D}^0[\rho^+ \to]K^+\bar{K}^0$ branching fraction listed in Table III. But such an universal phase difference will also make the branching fractions of the decays $B^0 \to D^{(*)-}[\rho^+ \to]K^+\bar{K}^0$ decrease by half in Table III, which is not desirable.

Let us mention that, in the amplitude analysis of the decay $D_s^+ \to \pi^+\pi^0\eta$, an unexpected large branching fraction $(1.46 \pm 0.15_{\rm stat} \pm 0.23_{\rm sys})\%$ was measured for $D_s^+ \to a_0(980)^{+(0)}\pi^{0(+)}$, $a_0(980)^{+(0)} \to \pi^{+(0)}\eta$ by the BESIII collaboration [149], which was successfully interpreted via the rescattering processes $K\bar{K} \to a_0(980) \to \pi\eta$ or $\pi\eta^{(\prime)} \to a_0(980) \to \pi\eta$ with the triangle diagrams suppression in Refs. [150–153]. But one should note that the above decay process is quite different when compared with the $B^+ \to \bar{D}^{(*)0}K^+\bar{K}^0$ and $B^0 \to D^{(*)-}K^+\bar{K}^0$ decays studied here. For the three-body decay $D_s^+ \to \pi^+\pi^0\eta$, the intermediate state $a_0(980)$ can only be generated by the final state interactions; the $c\bar{s}$ quark pair in the initial state D_s^+ cannot produce $D_s^+ \to a_0(980)$ transitions directly.

We finally discuss the contribution of the tensor resonance $a_2(1320)$. We note that this resonance decays into a kaon pair with a small branching fraction of about 5% [2]. Taking into account that the tensor states cannot be generated from a $V-A$ current [154], we do not expect to have considerable contributions from the subprocess $a_2(1320)^+ \to K^+\bar{K}^0$ for the decays $B^0 \to D^{(*)-}K^+\bar{K}^0$ by the related decay amplitudes in Eq. (3). In Ref. [155], the quasi-two-body decay $B_s^0 \to \bar{D}^0[\bar{K}_2^*(1430)^0 \to]K^-\pi^+$ was measured with the branching fraction $\mathcal{B} = (3.7 \pm 1.4) \times 10^{-5}$, which means $\mathcal{B} = (1.1 \pm 0.4) \times 10^{-4}$ [2] for the corresponding two-body process $B_s^0 \to \bar{D}^0\bar{K}_2^*(1430)^0$. With this measured branching fraction and the replacement of a s quark by a u quark, it is easy to predict the branching fraction $\mathcal{B} = (0.99 \pm 0.37) \times 10^{-4}$ for the decay $B^+ \to \bar{D}^0a_2(1320)^+$ within SU(3) flavor symmetry and employing the form factors A_0 in [73] for the

$B \to a_2$ and $B_s \to K_2^*$ transitions. This predicted value is consistent with the corresponding theoretical results in [81,84,156]. However, when taking into account the branching fraction $\mathcal{B}(a_2(1320)^+ \to K^+\bar{K}^0) = 4.9 \pm 0.8\%$ [2], the contribution from $a_2(1320)^+$ to the $B^+ \to \bar{D}^0 K^+\bar{K}^0$ process is negligible. The decay $B^+ \to \bar{D}^{*0} a_2(1320)^+$ shares the same Feynman diagrams with $B^+ \to \bar{D}^0 a_2(1320)^+$ at quark level, and it is reasonable to infer a insignificant branching fraction for the decay $B^+ \to \bar{D}^{*0} K^+\bar{K}^0$ with the subprocess $a_2(1320)^+ \to K^+\bar{K}^0$ as well.

IV. SUMMARY

To sum up, the Belle II collaboration presented a measurement for the $B^+ \to \bar{D}^{(*)0} K^+\bar{K}^0$ and $B^0 \to D^{(*)-} K^+\bar{K}^0$ decays very recently, where the bulk of the observed $m(K^+K_S^0)$ distribution was located in the low-mass region of the kaon pair, showing structures in all four decay channels. In this work we have presented a theoretical calculation of these decays within the factorization method. We have focused on exploring the region of kaon pair invariant masses $m(K^+\bar{K}^0) < 1.7$ GeV. The resonance contributions from vector intermediate states $\rho(770, 1450)^+$ have been found to dominate the branching fractions for the three-body decays $B^+ \to \bar{D}^{(*)0} K^+\bar{K}^0$ and $B^0 \to D^{(*)-} K^+\bar{K}^0$, representing roughly half of the total branching fractions of the corresponding decay channels. The role of the tensor $a_2(1320)^+$ was analyzed and found to give negligible contributions to the branching fractions of these four decay processes and the contribution of the state $a_0(980)^+$ turned out to be less important than expected in the $m(K^+K_S^0)$ region near the threshold of the kaon pair. As a result of our study, we conclude that the enhancement of events in the kaon pair distribution near threshold observed in the $B^+ \to \bar{D}^0 K^+\bar{K}^0$ and $B^+ \to \bar{D}^{*0} K^+\bar{K}^0$ decay processes cannot be interpreted as the resonance contributions from the $a_0(980)^+$ meson. The nonresonant contributions are probably governing the formation of the kaon pair in $B^+ \to \bar{D}^{(*)0} K^+\bar{K}^0$ near the threshold of $K^+\bar{K}^0$, and hence deserve further examination both from the theoretical and the experimental sides.

ACKNOWLEDGMENTS

We thank V. Bertacchi, K. Trabelsi, and A. Feijoo for valuable discussions. This work was supported in part by the National Natural Science Foundation of China under Grants No. 12205148 and No. 11947011, the Fund for Shanxi "1331 Project" Key Subjects Construction, the Natural Science Foundation of Jiangsu Province under Grant No. BK20191010, and the Qing Lan Project of Jiangsu Province. This research has also been supported from the Projects No. CEX2019-000918-M (Unidad de Excelencia "María de Maeztu") and No. PID2020–118758GB-I00, financed by the Spanish Grant No. MCIN/ AEI/10.13039/501100011033/, as well as by the EU STRONG-2020 project, under the program H2020-INFRAIA-2018-1 Grant Agreement No. 824093.

APPENDIX A: GENERAL AMPLITUDES FOR $B \to \bar{D}^{(*)}\rho \to \bar{D}^{(*)} K^+\bar{K}^0$ DECAYS

The wave functions for B, D, and D^* mesons and the corresponding inputs are the same as they in Ref. [99]. The kaon and pion timelike form factors are referred to the Sec. II of Ref. [38]. With the subprocesses $\rho^+ \to K^+\bar{K}^0$, where ρ is $\rho(770)$ or $\rho(1450)$, the specific expressions in PQCD approach for the Lorentz invariant decay amplitudes of these general amplitudes F and M for $B \to \bar{D}^{(*)}\rho \to \bar{D}^{(*)} K^+\bar{K}^0$ decays are given as follows.

The amplitudes from Fig. 2(a) for the decays with a pseudoscalar $\bar{D}^0$ or D^- meson in the final states are given as

$$\begin{aligned} F_{T\rho} = {} & 8\pi C_F m_B^4 f_D \int dx_B dx \int b_B db_B b db \phi_B \{[[r^2 - \bar{\zeta}(x(r^2-1)^2+1)]\phi^0 - \sqrt{\zeta}[(r^2+\bar{\zeta} \\ & + 2\bar{\zeta}x(r^2-1))\phi^s - (r^2-1)\bar{\zeta}(2x(r^2-1)+1) - r^2)\phi^t]E_e(t_a)h_a(x_B, x, b, b_B)S_t(x) \\ & + [(r^2-1)[\zeta\bar{\zeta} - r^2(\zeta - x_B)]\phi^0 - 2\sqrt{\zeta}[\bar{\zeta} - r^2(x_B - 2\zeta + 1)]\phi^s] \\ & \times E_e(t_b)h_b(x_B, x, b_B, b)S_t(|x_B - \zeta|)\}, \end{aligned} \tag{A1}$$

$$\begin{aligned} M_{T\rho} = {} & 16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b_3 db_3 \phi_B \phi_D \{[-[(\bar{\zeta}+r^2)((r^2-1)(x_3\bar{\zeta}+x_B) \\ & + r^2(\zeta x - 1) - \zeta(x+1)+1) + r r_c(r^2 - \bar{\zeta})]\phi^0 - \sqrt{\zeta}[(r^2(\bar{\zeta}(x_3+x-2)+x_B) - x\bar{\zeta} + 4 r r_c)\phi^s \\ & + (r^2-1)(r^2(\bar{\zeta}(x-x_3)-x_B) - x\bar{\zeta})\phi^t]]E_n(t_c)h_c(x_B, x, x_3, b_B, b_3) \\ & + [x(r^2-1)[(r^2-\bar{\zeta})\phi^0 + \sqrt{\zeta}\bar{\zeta}(\phi^s - (r^2-1)\phi^t)] - (x_3\bar{\zeta} - x_B)[(r^2-\bar{\zeta})\phi^0 \\ & + \sqrt{\zeta}r^2((r^2-1)\phi^t + \phi^s)]]E_n(t_d)h_d(x_B, x, x_3, b_B, b_3)\}, \end{aligned} \tag{A2}$$

where the symbol $\bar{\zeta} = 1 - \zeta$, the mass ratios $r = m_{D^{(*)}}/m_B$ and $r_c = m_c/m_B$. The amplitudes from Fig. 2(b) are written as

$$\begin{aligned}F_{TD} = {}& 8\pi C_F m_B^4 f_\rho \int dx_B dx_3 \int b_B db_B b_3 db_3 \phi_B \phi_D \{[(r+1)[r^2 - \bar{\zeta} - x_3\bar{\zeta}(r-1)(2r-\bar{\zeta})]] \\ & \times E_e(t_m) h_m(x_B, x_3, b_3, b_B) S_t(x_3) + [(r^2-\bar{\zeta})[2r(r_c+1) - r^2\bar{\zeta} - r_c] - \zeta x_B(2r-\bar{\zeta})] \\ & \times E_e(t_n) h_n(x_B, x_3, b_B, b_3) S_t(x_B)\},\end{aligned} \tag{A3}$$

$$\begin{aligned}M_{TD} = {}& 16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b db \phi_B \phi_D \phi^0 \{[x_B[\bar{\zeta}^2 - \bar{\zeta}r^2 + \zeta r] + \bar{\zeta}x_3 r(\zeta r + (r+1)(r-1)^2) \\ & - \zeta(r-1)^2(r+1)[(r+2)x - 2(r+1)] + \zeta^2[x - r^2(x-2) - 1] \\ & + (x-1)(r^2-1)^2] E_n(t_o) h_o(x_B, x, x_3, b_B, b) + [(r-1)(\bar{\zeta}+r)[x_B + (r^2-1)x] \\ & + \bar{\zeta}x_3[(r-1)^2(r+1) - \zeta]] E_n(t_p) h_p(x_B, x, x_3, b_B, b)\}.\end{aligned} \tag{A4}$$

The amplitudes from Fig. 2(c) the annihilation diagrams are written as

$$\begin{aligned}F_{A\rho} = {}& 8\pi C_F m_B^4 f_B \int dx_3 dx \int b db b_3 db_3 \phi_D \{[((2rr_c - 1)(r^2 - \bar{\zeta}) - (r^2-1)^2 x\bar{\zeta})\phi^0 + \sqrt{\zeta} \\ & \times [(r^2-1)(r_c(r^2-\bar{\zeta}) - 2r(r^2-1)x)\phi^t + (r_c(r^2-\zeta+1) + 2r(x - xr^2 - 2))\phi^s]] \\ & \times E_a(t_e) h_e(x, x_3, b, b_3) S_t(x) + [(r^2-1)[x_3\bar{\zeta}^2 - \zeta(r^2-\bar{\zeta})]\phi^0 + 2\sqrt{\zeta}r(x_3\bar{\zeta} + \zeta - r^2 + 1)\phi^s] \\ & \times E_a(t_f) h_f(x, x_3, b_3, b) S_t(|\bar{\zeta}x_3 + \zeta|)\},\end{aligned} \tag{A5}$$

$$\begin{aligned}M_{A\rho} = {}& -16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b db \phi_B \phi_D \{[(r^2-1)[r^2(x_B + x_3 - 1) + x_B + x_3]\phi^0 \\ & + \zeta[r^4 x - (r^2-1)x_B + \zeta((r^2-1)x_3 - xr^2 + x + 1) - (r^4 + r^2 - 2)x_3 - x - 1]\phi^0 \\ & + \zeta^{3/2} r(1-x_3)[(r^2-1)\phi^t + \phi^s] + \sqrt{\zeta}r[\phi^s(x_B + r^2(x-1) + x_3 - x + 3) \\ & + (r^2-1)(x_B - xr^2 + r^2 + x_3 + x - 1)\phi^t]] E_n(t_g) h_g(x_B, x, x_3, b, b_B) \\ & + [(r^2-\bar{\zeta})[r^2(x_B - x_3 - x + 1) + \zeta(r^2(x_3 + x - 2) - x + 1) + x - 1]\phi^0 \\ & + \sqrt{\zeta}r[(x_B - x_3\bar{\zeta} - \zeta + (r^2-1)(1-x))\phi^s + (1-r^2)(x_B - x_3\bar{\zeta} - \zeta + (r^2-1)(x-1))\phi^t]] \\ & \times E_n(t_h) h_h(x_B, x, x_3, b, b_B)\}.\end{aligned} \tag{A6}$$

Where the $T\rho$, TD, and $A\rho$ in the subscript of above expressions stand for $B \to \rho$, $B \to D$ transitions and the annihilation Feynman diagrams, respectively.

The longitudinal polarization amplitudes from Fig. 2(a) for the decays with a vector $\bar{D}^{*0}$ or D^{*-} meson in the final state are written as

$$\begin{aligned}F_{T\rho,L} = {}& 8\pi C_F m_B^4 f_{D^*} \int dx_B dx \int b_B db_B b db \phi_B \{[[\bar{\zeta} + \bar{\zeta}x(r^2-1)^2 + (2\zeta-1)r^2]\phi^0 \\ & + \sqrt{\zeta}[(1-r^2)(2\bar{\zeta}x(r^2-1) + \bar{\zeta} + r^2)\phi^t + (2\bar{\zeta}x(r^2-1) + \bar{\zeta} - r^2)\phi^s]] \\ & \times E_e(t_a) h_a(x_B, x, b, b_B) S_t(x) + [(r^2-1)[r^2 x_B + \zeta^2 - \zeta(r^2+1)]\phi^0 \\ & - 2\sqrt{\zeta}[r^2(1-x_B) - \bar{\zeta}]\phi^s] E_e(t_b) h_b(x_B, x, b_B, b) S_t(|x_B - \zeta|)\},\end{aligned} \tag{A7}$$

$$
\begin{aligned}
M_{T\rho,L} = {}& 16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b_3 db_3 \phi_B \phi_{D^*} \{[[r r_c(1-\bar{\zeta}r^2-\zeta^2)-(r^2-\bar{\zeta}) \\
& \times(\bar{\zeta}x_3(r^2-1)+x_B(r^2-1)+(\zeta x-1)r^2-\zeta(x+1)+1)]\phi^0-\sqrt{\zeta}[(r^2(x_3\bar{\zeta}-\bar{\zeta}x+x_B) \\
& -\zeta x+x)\phi^s+(r^2-1)(\bar{\zeta}x(1-r^2)-r^2((x_3-2)\bar{\zeta}+x_B))\phi^t]] \\
& \times E_n(t_c)h_c(x_B,x,x_3,b_B,b_3)+[x_B[(\bar{\zeta}+(2\zeta-1)r^2)\phi^0+\sqrt{\zeta}r^2((r^2-1)\phi^t+\phi^s)] \\
& -\bar{\zeta}x_3[(\bar{\zeta}+(2\zeta-1)r^2)\phi^0+\sqrt{\zeta}r^2((r^2-1)\phi^t+\phi^s)]+x(r^2-1)[(\bar{\zeta}+(2\zeta-1)r^2)\phi^0 \\
& -\sqrt{\zeta}\bar{\zeta}(\phi^s-(r^2-1)\phi^t)]]E_n(t_d)h_d(x_B,x,x_3,b_B,b_3)\}. \qquad \text{(A8)}
\end{aligned}
$$

The longitudinal polarization amplitudes from Fig. 2(b) are

$$
\begin{aligned}
F_{TD^*,L} = {}& 8\pi C_F m_B^4 f_\rho \int dx_B dx_3 \int b_B db_B b_3 db_3 \phi_B \phi_{D^*} \{[\bar{\zeta}+(2r-1)(r^2-1)x_3\bar{\zeta}^2+r \\
& \times[\zeta(r^2+2r-\zeta)-r^2-r+1]]E_e(t_m)h_m(x_B,x_3,b_3,b_B)S_t(x_3)+[r^2[r_c(2\zeta-1) \\
& -\zeta^2+1]-\bar{\zeta}(\zeta x_B-r_c+r^4)]E_e(t_n)h_n(x_B,x_3,b_B,b_3)S_t(x_B)\}, \qquad \text{(A9)}
\end{aligned}
$$

$$
\begin{aligned}
M_{TD^*,L} = {}& -16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b db \phi_B \phi_{D^*} \phi^0 \{[\bar{\zeta}x_B(1-r)(\bar{\zeta}+r)-\bar{\zeta}x_3 r \\
& \times(r^3-\bar{\zeta}(r^2+r-1))-\zeta^2(2r^3-x(r+1)(r-1)^2-2r+1)+(x-1)(r^2-1)^2 \\
& -\zeta(r+1)(r-1)^2(rx+2x-2)]E_n(t_o)h_o(x_B,x,x_3,b_B,b)+[\bar{\zeta}x_3[r^2(r\bar{\zeta}+2\zeta-1)+\bar{\zeta}-r\bar{\zeta}] \\
& -(x_B+(r^2-1)x)[\bar{\zeta}-\bar{\zeta}\zeta r+(2\zeta-1)r^2]]E_n(t_p)h_p(x_B,x,x_3,b_B,b)\}. \qquad \text{(A10)}
\end{aligned}
$$

The longitudinal polarization amplitudes from Fig. 2(c) are

$$
\begin{aligned}
F_{A\rho,L} = {}& -8\pi C_F m_B^4 f_B \int dx_3 dx \int b db b_3 db_3 \phi_{D^*} \{[\sqrt{\zeta}r_c[(r^4-\zeta r^2-\bar{\zeta})\phi^t+(r^2-\bar{\zeta})\phi^s] \\
& +[\bar{\zeta}(1-x(r^2-1)^2)+r^2(2\zeta-1)]\phi^0]E_a(t_e)h_e(x,x_3,b,b_3)S_t(x) \\
& +[2r\sqrt{\zeta}\bar{\zeta}((x_3-1)\bar{\zeta}+r^2)\phi^s+(r^2-1)[\zeta(r^2+\bar{\zeta}(1-x_3)-x_3)+x_3]\phi^0] \\
& \times E_a(t_f)h_f(x,x_3,b_3,b)S_t(|\bar{\zeta}x_3+\zeta|)\}, \qquad \text{(A11)}
\end{aligned}
$$

$$
\begin{aligned}
M_{A\rho,L} = {}& 16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b db \phi_B \phi_{D^*} \{[-(x_3\bar{\zeta}+x_B)[(r^2-1)(r^2-\bar{\zeta})\phi^0 \\
& -\sqrt{\zeta}\bar{\zeta}r(r^2-1)\phi^t-\sqrt{\zeta}\bar{\zeta}r\phi^s]-\zeta\bar{\zeta}(x+1)\phi^0+\sqrt{\zeta}\bar{\zeta}r^5(x-1)\phi^t+\sqrt{\zeta}\bar{\zeta}r^3((\zeta-2x)\phi^t \\
& -(x-1)\phi^s)+\sqrt{\zeta}\bar{\zeta}r((x-\bar{\zeta})\phi^s+(x+\bar{\zeta})\phi^t)-r^4(\zeta x-1)\phi^0-r^2(\zeta x(\zeta-2)+1)\phi^0] \\
& \times E_n(t_g)h_g(x_B,x,x_3,b,b_B)-[(r^2-\bar{\zeta})(r^2(x_B-x_3\bar{\zeta})+r^2(\bar{\zeta}x-1)-\bar{\zeta}(x-1))\phi^0 \\
& -\sqrt{\zeta}\bar{\zeta}r[((x_3-1)\bar{\zeta}+(1-x)r^2+x-x_B)\phi^s-(r^2-1)(x_3\bar{\zeta}+\zeta-(1-x)r^2-x-x_B+1)\phi^t]] \\
& \times E_n(t_h)h_h(x_B,x,x_3,b,b_B)\}. \qquad \text{(A12)}
\end{aligned}
$$

The normal along with transverse polarization amplitudes from Fig. 2 for the decays with a vector $\bar{D}^{*0}$ or D^{*-} are written as

$$\begin{aligned}
F_{T\rho,\|\perp} = {}& 8\pi C_F m_B^4 f_{D^*} r \int dx_B dx \int b_B db_B b db \phi_B \{[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[\sqrt{\zeta}(x(r^2-1)(\phi^a-\phi^v)+2\phi^v) \\
&+\zeta(2x(r^2-1)+1)\phi^T+(1-r^2)\phi^T]-i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[\sqrt{\zeta}((x(r^2-1)-2)\phi^a \\
&-x(r^2-1)\phi^v)+\zeta(2x(r^2-1)+1)\phi^T+(r^2-1)\phi^T]]E_e(t_a)h_a(x_B,x,b,b_B)S_t(x) \\
&+\sqrt{\zeta}[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[(\zeta-x_B-r^2+1)\phi^v+(\bar{\zeta}+x_B-r^2)\phi^a]+i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[(\zeta-x_B-r^2+1) \\
&\times\phi^a-(\zeta-x_B+r^2-1)\phi^v]]E_e(t_b)h_b(x_B,x,b_B,b)S_t(|x_B-\zeta|)\},
\end{aligned} \tag{A13}$$

$$\begin{aligned}
M_{T\rho,\|\perp} = {}& 16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b_3 db_3 \phi_B \phi_{D^*}\{[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[\zeta^{3/2}r_c(\phi^a-\phi^v) \\
&+\sqrt{\zeta}r_c((r^2-1)\phi^a+(r^2+1)\phi^v)+r(r^2-1)(x_B+x_3-1)\phi^T-\zeta r((r^2-1) \\
&\times(x_3+x)-2r^2+1)\phi^T]-i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[\zeta^{3/2}r_c(\phi^a-\phi^v)-\sqrt{\zeta}r_c((r^2+1)\phi^a \\
&+(r^2-1)\phi^v)-r(r^2-1)(x_B+x_3-1)\phi^T+\zeta r((x_3-x)(r^2-1)+1)\phi^T]] \\
&\times E_n(t_c)h_c(x_B,x,x_3,b_B,b_3)+r[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[2\sqrt{\zeta}(x_B+x(r^2-1)-x_3\bar{\zeta})\phi^v \\
&+(r^2-1)(x_B-x\zeta-x_3\bar{\zeta})\phi^T]+i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[2\sqrt{\zeta}(x_B+x(r^2-1)-x_3\bar{\zeta})\phi^a \\
&+(r^2-1)(x_B+x\zeta-x_3\bar{\zeta})\phi^T]]E_n(t_d)h_d(x_B,x,x_3,b_B,b_3)\},
\end{aligned} \tag{A14}$$

$$\begin{aligned}
F_{TD^*,\|\perp} = {}& 8\pi C_F m_B^4 f_\rho \sqrt{\zeta}\int dx_B dx_3 \int b_B db_B b_3 db_3 \phi_B \phi_{D^*}\{[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[x(r^2-1)(2\bar{\zeta}-r)+\bar{\zeta}+r^2+2r] \\
&-i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[x(r^2-1)(r-2\bar{\zeta})-\bar{\zeta}+r^2]]E_e(t_m)h_m(x_B,x_3,b_3,b_B) \\
&\times S_t(x_3)+r[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[\zeta-x_B+2r_c-r^2+1]-i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[\bar{\zeta}+x_B-r^2]] \\
&\times E_e(t_n)h_n(x_B,x_3,b_B,b_3)S_t(x_B)\},
\end{aligned} \tag{A15}$$

$$\begin{aligned}
M_{TD^*,\|\perp} = {}& 16\sqrt{\frac{2}{3}}\pi C_F m_B^4\sqrt{\zeta}\int dx_B dx dx_3 \int b_B db_B b db \phi_B \phi_{D^*}\{[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[r^2(\bar{\zeta}((2-x)\phi^v-x\phi^a) \\
&+(\bar{\zeta}x_3+x_B)(\phi^a-\phi^v))+\bar{\zeta}x(\phi^a+\phi^v)]-i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[r^2((\bar{\zeta}(x-x_3)-x_B)\phi^v \\
&+(\bar{\zeta}(x_3+x-2)+x_B)\phi^a)-\bar{\zeta}x(\phi^a+\phi^v)]]E_n(t_o)h_o(x_B,x,x_3,b_B,b) \\
&+[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[(r^2(x_B-x_3\bar{\zeta})-x\bar{\zeta}(r^2-1))\phi^a+(x(r^2-1)(2r-\bar{\zeta})-(r-2) \\
&\times r(x_B-x_3\bar{\zeta}))\phi^v]+i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[(x(r^2-1)(2r-\bar{\zeta})-r(r-2)(x_B-x_3\bar{\zeta}))\phi^a \\
&+(r^2(x_B-x_3\bar{\zeta})-\bar{\zeta}x(r^2-1))\phi^v]]E_n(t_p)h_p(x_B,x,x_3,b_B,b)\},
\end{aligned} \tag{A16}$$

$$\begin{aligned}
F_{A\rho,\|\perp} = {}& 8\pi C_F m_B^4 f_B r\int dx_3 dx \int b db b_3 db_3 \phi_{D^*}\{[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[\sqrt{\zeta}(x(r^2-1)(\phi^a-\phi^v)-2\phi^v) \\
&-r_c(r^2-\zeta-1)\phi^T]+i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[\sqrt{\zeta}(x(r^2-1)\phi^v-(x(r^2-1)+2)\phi^a)+(r^2-\bar{\zeta})r_c\phi^T]] \\
&\times E_a(t_e)h_e(x,x_3,b,b_3)S_t(x)+\sqrt{\zeta}[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[(\bar{\zeta}x_3+\zeta-r^2+1)\phi^v+(\bar{\zeta}x_3+\zeta+r^2-1)\phi^a] \\
&+i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[(\bar{\zeta}x_3+\zeta+r^2-1)\phi^v+(\bar{\zeta}x_3+\zeta-r^2+1)\phi^a]] \\
&\times E_a(t_f)h_f(x,x_3,b_3,b)S_t(|\bar{\zeta}x_3+\zeta|)\},
\end{aligned} \tag{A17}$$

$$
\begin{aligned}
M_{A\rho,\parallel\perp} = {} & 16\sqrt{\frac{2}{3}}\pi C_F m_B^4 \int dx_B dx dx_3 \int b_B db_B b db \phi_B \phi_{D^*} \{[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[(r^2 x_B(r^2-1) \\
& + \bar{\zeta} r^2((r^2-1)(x_3-1)+\zeta) - \bar{\zeta}\zeta x(r^2-1))\phi^T - 2\sqrt{\zeta} r\phi^v] + i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}}[(r^2(\bar{\zeta}r^2 \\
& - \bar{\zeta}^2 - (\bar{\zeta}x_3 + x_B)(r^2-1)) - \bar{\zeta}\zeta x(r^2-1))\phi^T - 2\sqrt{\zeta} r\phi^a]]E_n(t_g)h_g(x_B,x,x_3,b,b_B) \\
& + (r^2-1)[\epsilon_T^{D^*}\cdot\epsilon_T^{\rho}[r^2(x_B-x_3)+\zeta(r^2(x_3-1)+x-1)+\zeta^2(1-x)] - i\epsilon^{nv\epsilon_T^{D^*}\epsilon_T^{\rho}} \\
& \times [r^2(x_B-x_3)+\zeta(r^2(x_3-1)-\bar{\zeta}(x-1))]]\phi^T E_n(t_h)h_h(x_B,x,x_3,b,b_B)\}. \qquad \text{(A18)}
\end{aligned}
$$

The hard functions h_i, the hard scales t_i with $i\in\{a,b,c,d,e,f,g,h,m,n,o,p\}$, and the evolution factors $E_{e,a,n}$ have their explicit expression in the Appendix of Ref. [99].

APPENDIX B: DECAY AMPLITUDE FOR $B^+ \to \bar{D}^0[a_0(980,1450)^+ \to]K^+\bar{K}^0$

The decay constants of the pseudoscalar meson P and the scalar meson S are defined by

$$
\begin{aligned}
\langle P(p)|\bar{q}_2\gamma_\mu\gamma_5 q_1|0\rangle &= -if_P p_\mu, \\
\langle S(p)|\bar{q}_2\gamma_\mu q_1|0\rangle &= f_S p_\mu, \\
\langle S|\bar{q}_2 q_1|0\rangle &= m_S\bar{f}_S.
\end{aligned}
$$

When the kaon pair $K^+\bar{K}^0$ originated from the intermediate $a_0(980,1450)^+$, we have [96]

$$
\langle K^+\bar{K}^0|\bar{d}u|0\rangle = \langle K^+\bar{K}^0|a_0\rangle \frac{1}{\mathcal{D}_{a_0}}\langle a_0|\bar{d}u|0\rangle \frac{g_{a_0KK}}{\mathcal{D}_{a_0}}\langle a_0|\bar{d}u|0\rangle, \qquad \text{(B1)}
$$

with the denominator [157,158]

$$
\mathcal{D}_{a_0} = m_{a_0}^2 - s - i(g_{\pi\eta}^2\rho_{\pi\eta} + g_{\bar{K}^0K}^2\rho_{\bar{K}^0K} + g_{\pi\eta'}^2\rho_{\pi\eta'}). \qquad \text{(B2)}
$$

The $B\to D^{(*)}$ matrix element is described by the transition form factors [159]

$$
\langle D(p')|\bar{c}\gamma^\mu b|B(p)\rangle = F_0^{BD}(q^2)\frac{m_B^2-m_D^2}{q^2}q^\mu + F_1^{BD}(q^2)\left[(p+p')^\mu - \frac{m_B^2-m_D^2}{q^2}q^\mu\right], \qquad \text{(B3)}
$$

where $q=p-p'$. We parametrize the matrix element for the $B\to a_0$ transition in terms of form factors F_0^{Ba} and F_1^{Ba} as

$$
\langle a_0(p')|\bar{q}\gamma^\mu\gamma_5 b|B(p)\rangle = iF_0^{Ba}(q^2)\frac{m_B^2-m_{a_0}^2}{q^2}q^\mu + iF_1^{Ba}(q^2)\left[(p+p')^\mu - \frac{m_B^2-m_{a_0}^2}{q^2}q^\mu\right]. \qquad \text{(B4)}
$$

With Eqs. (2) and (3) and related transition form factors above, we have the decay amplitude

$$
\mathcal{M}(B^+\to\bar{D}^0[a_0^+\to]K^+\bar{K}^0) = \frac{G_F}{\sqrt{2}}V_{cb}^*V_{ud}[a_1(m_B^2-m_{a_0}^2)f_D F_0^{Ba}(m_D^2) + a_2(m_B^2-m_D^2)f_{a_0}F_0^{BD}(s)]\frac{g_{a_0KK}}{\mathcal{D}_{a_0}}. \qquad \text{(B5)}
$$

The expressions and related parameters for F_0^{Ba} and $F_0^{BD}(s)$ are found in Refs. [68,144,146,160].

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