Right-handed slepton bulk region for dark matter in generalized no-scale $\mathcal{F}$-SU(5) with effective supernatural supersymmetry

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We propose generalized no-scale supergravity, the simplest scenario for effective supernatural supersymmetry, naturally solving the supersymmetry electroweak fine-tuning problem and including natural dark matter. A light right-handed slepton bulk region is realized in $\mathcal{F}$-SU(5) and the phenomenological minimal supersymmetric standard model. The bulk may be beyond the LHC reach, although it can be probed at the 1000-day LUX-ZEPLIN, Future Circular Collider at CERN, Circular Electron-Positron Collider, and Hyper-Kamiokande.

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I. INTRODUCTION

A natural solution to the gauge hierarchy problem in the Standard Model (SM) is supersymmetry (SUSY). A few salient solutions to unexplained phenomena provided by supersymmetric SMs (SSMs) with R parity are (i) gauge coupling unification [1], (ii) lightest supersymmetric particle (LSP) neutralino as a dark matter (DM) candidate [2], and (iii) electroweak (EW) gauge symmetry can be broken radiatively due to the large top quark Yukawa coupling. This list is by no means exhaustive, but the first solution above deserves special emphasis, given that gauge coupling unification strongly implies grand unified theories (GUTs) [3–7], which might be constructed from superstring theory. These triumphs are evidence that SUSY builds a bridge between low-energy phenomenology and high-energy fundamental physics, leading to promising new physics beyond the SM.

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As of the present, LHC SUSY searches have been barren of tantalizing signals of new physics, establishing strong constraints on the SSMs. Devoid of observation, SUSY lingers as only a beautiful yet unsubstantiated field theory. The prior ten years of proton-proton collisions without any verifiable "bumps" beyond SM expectations have elevated low mass bounds on the gluino, first-two generation squarks, stop, and sbottom to around 2.3, 1.9, 1.25, and 1.5 TeV, respectively [8–12]. Given these larger than anticipated lower limits on SUSY masses, we must unfortunately face the prospect that we may be encountering the SUSY EW fine-tuning (EWFT) problem. In that event, some encouraging and successful solutions to the EWFT problem have been proposed in the literature [13–27]. In particular, we have proposed supernatural SUSY [28–30] by considering no-scale supergravity (SUGRA) [31] and SUSY-breaking soft terms in M-theory on S^1/Z_2 [32] and demonstrated that the high-energy fine-tuning measure defined by Ellis-Enqvist-Nanopoulos-Zwirner [33] and Barbieri-Giudice [34] (EENZ-BG) is of order one naturally, even if the supersymmetric particle (sparticle) spectra are heavy. Moreover, supernatural SUSY can be generalized to effective supernatural SUSY [24].

Deriving an explanation of the observed dark matter relic density for a bino dominant LSP remains another formidable issue. The methodology to resolve this dilemma typically comes in four distinct approaches: (1) the bulk region where the sfermions (supersymmetric partners of the

SM fermions) are light, (2) the Z/Higgs funnel or Z/Higgs resonance, where the LSP neutralino mass is about half of the masses of the Z boson, SM Higgs, CP-even Higgs H_0, or CP-odd Higgs A_0, (3) coannihilation, where the sfermion masses are close to the LSP neutralino; or (4) mixing scenario or well-tempered scenario, where the LSP neutralino has enough wino or Higgsino components to significantly increase the annihilation cross section. Evaluating these four scenarios, it seems to us that the bulk region may be the most natural. Demanding naturalness in both SUSY and dark matter is therefore a prominent pressing challenge, leading to the compelling question: Is it possible to have a viable bulk region for dark matter in a natural SUSY scenario?

In this paper, we consider $\mathcal{F}$-SU(5), i.e., the flipped $\mathrm{SU}(5)\times\mathrm{U}(1)_X$ GUT model with extra TeV-scale vectorlike particles [35] that have been constructed systematically in local F-theory model building [36,37]. Alternatively, these models can also be realized in free fermionic string constructions [38]. Supernatural SUSY via no-scale SUGRA is not a resolution to the SUSY EWFT problem in the specific case when there is also a light bino LSP, due to a correlation of the bino mass with the wino and gluino masses. Therefore, we propose generalized no-scale SUGRA, where effective supernatural SUSY can be realized. In order to uncover the bulk region for dark matter, we can only consider light right-handed sleptons given that the LHC SUSY searches indicate that all other sfermions must be heavy. First, a determination must be carried out as to whether an interaction between sfermions and the LSP is coannihilation or annihilation. Rendering a judgement involves inspecting the mass difference between the light right-handed sfermions and LSP, though we mention that the ratio of the mass difference $\mathcal{R}_\phi \equiv (m_\phi - m_{\tilde{\chi}_1^0})/m_{\tilde{\chi}_1^0}$ is more important than the absolute mass difference, where ϕ is $\tilde{\tau}_1$ (light stau) or $\tilde{e}_R$ (light selectron). Comprehensive numerical studies that we present in this work show that $\mathcal{R}_\phi \gtrsim 10\%$ is a conservative criterion to formulate the bulk region, i.e., the observed dark matter density is obtained via traditional annihilations, not from coannihilations or resonances, etc. We also investigate outside the tight $\mathcal{F}$-SU(5) constraints by evaluating the phenomenological minimal supersymmetric Standard Model (pMSSM).

II. THE $\mathcal{F}$-SU(5) MODEL

For a thorough review of $\mathcal{F}$-SU(5), we invite the reader to explore the wealth of literature on the model [39–41] and references therein, including the no-scale flipped SU(5) string derived inflationary model, cosmology, and gravitational waves [42]. Here we only illuminate a few of the most vital attributes. In the flipped SU(5) models [43], there are three families of SM fermions whose quantum numbers under the $\mathrm{SU}(5)\times\mathrm{U}(1)_X$ gauge group are $F_i = (\mathbf{10},\mathbf{1})$, $\bar{f}_i = (\bar{\mathbf{5}},\mathbf{-3})$, $\bar{l}_i = (\mathbf{1},\mathbf{5})$, where $i = 1$, 2, 3. To break the GUT and electroweak gauge symmetries, we introduce two pairs of Higgs fields $H = (\mathbf{10},\mathbf{1})$, $\bar{H} = (\overline{\mathbf{10}},\mathbf{-1})$, $h = (\mathbf{5},\mathbf{-2})$, $\bar{h} = (\bar{\mathbf{5}},\mathbf{2})$. True string-scale gauge coupling unification [35,36] is achieved by introduction of the vectorlike particles (dubbed "flippons") $XF = (\mathbf{10},\mathbf{1})$, $\overline{XF} = (\overline{\mathbf{10}},\mathbf{1})$, $Xl = (\mathbf{1},\mathbf{-5})$, $\overline{Xl} = (\mathbf{1},\mathbf{5})$, which form complete flipped $\mathrm{SU}(5)\times\mathrm{U}(1)_X$ multiplets. The flipped SU(5) model with the additional vectorlike particles derived from local F-theory model building is referred to as $\mathcal{F}$-SU(5). The renormalization group equation (RGE) β coefficients undergo a shift due to the vectorlike multiplets, lifting the $\mathrm{SU}(5)\times\mathrm{U}(1)_X$ unification to the string scale around 10^{17} GeV (correlating to the mass scales M_5 and M_{1X}), adjacent to the Planck scale. Subsequently, a second stage unification $\mathrm{SU}(3)_C\times\mathrm{SU}(2)_L$ (defined as mass scale M_{32}) occurs near the traditional MSSM GUT scale around 10^{16} GeV. The separated unification structure produces a flat SU(3) RGE running (due to a vanishing $b3$ coefficient enforced by the vectorlike multiplets) from the GUT scale to the TeV scale where all vectorlike multiplets decouple, defined as the scale M_V. The universal vectorlike particle mass scale M_V is treated as a free model parameter. A light M_V mass scale ($M_V \lesssim 10$ TeV) allows for a larger vectorlike particle Yukawa coupling, contributing to the light Higgs boson mass [40].

III. GENERALIZED NO-SCALE SUGRA

In order to generate a light bino, evade the large LHC constraint on the gluino mass, and sustain naturalness conditions, the generalization of no-scale SUGRA is essential. At the $\mathrm{SU}(5)\times\mathrm{U}(1)_X$ unification scale (string scale), we vary the SU(5) gaugino mass M_5 from 1200 to 5000 GeV, yielding a large gluino mass. To produce a light bino, we vary the $\mathrm{U}(1)_X$ gaugino mass M_{1X} from 100 to 600 GeV. Note that no-scale SUGRA is obtained at tree level and can be violated at one loop, so we assume the universal supersymmetry-breaking soft mass M_0 and trilinear soft term A_0 are smaller than about 1% of M_5. Finally, we span $\tan\beta$ from 2 to 65 and the vectorlike particle mass scale M_V from 1 to 10 TeV.

According to effective supernatural SUSY [24], we have shown that a supersymmetry-breaking scenario is natural if all the fundamental parameters that have large EENZ-BG fine-tuning measures are correlated. In our generalized no-scale SUGRA presented here, the fine-tuning measures for the SUSY-breaking soft terms M_{1X}, M_0, and A_0 are all small, and only M_5 might have a large fine-tuning measure. Therefore, our generalized no-scale SUGRA is approximately supernatural SUSY and thus, indeed, natural. More specifically, it is only a small deviation from supernatural SUSY and hence the simplest scenario for effective supernatural SUSY, where only one fundamental parameter may have a large EENZ-BG fine-tuning measure.

IV. NUMERICAL PROCEDURE

To investigate the bulk region in generalized no-scale $\mathcal{F}$-SU(5) and target the requisite M_5 and M_{1X} gaugino mass scales, we need to focus on small M_1. Accomplishing this goal requires we exploit the relationship in $\mathcal{F}$-SU(5) between M_1 for $\mathrm{U}(1)_Y$ and M_{1X} for $\mathrm{U}(1)_X$ at the M_{32} scale, which is [44]

$$\frac{M_1}{\alpha_1} \equiv \frac{24}{25}\frac{M_{1X}}{\alpha_{1X}} + \frac{1}{25}\frac{M_5}{\alpha_5}, \tag{1}$$

where α_i are the gauge couplings at their respective scales. Conjointly, the following experimental constraints are imposed:

(i) require neutralino LSP;
(ii) constraints on the mass of gluino and first and second generation squark masses [45–47] of $m_{\tilde{g}} \gtrsim 2.2$ and $m_{\tilde{q}} \gtrsim 2.0$ TeV;
(iii) rare B-meson decay constraint of [48] $1.6 \times 10^{-9} \leq \mathrm{BR}(B_s^0 \to \mu^+\mu^-) \leq 4.2 \times 10^{-9}$ and branching ratio of rare b-quark decay of $2.99 \times 10^{-4} \leq \mathrm{BR}(b \to s\gamma) \leq 3.87 \times 10^{-4}$ [49];
(iv) attention to both the experimental measurement of the light Higgs boson mass and its theoretical uncertainty [50,51], applying a range $122 \leq m_h \leq 128$ GeV;
(v) constraints on spin-independent DM-nuclei cross sections from XENONnT [52] and LUX-ZEPLIN [53,54]; and
(vi) relic density of cold DM measured by the 5σ Planck 2018 [55] of $0.114 \leq \Omega_{\mathrm{DM}} h^2 \leq 0.126$, where below this range is regarded as undersaturated and above is oversaturated.

All SUSY RGE numerical calculations are executed with micrOMEGAs 2.1 [56] incorporating a proprietary revision to the SuSpect 2.34 [57] code base to evolve vectorlike particle and $\mathcal{F}$-SU(5) enhanced RGEs. We deploy micrOMEGAs to compute the relic density in the instances of both annihilation and coannihilation, plus, to ensure we are wielding the correct number for annihilation only, a parallel computation valid when coannihilation is absent is performed, given by [58]

$$\Omega_{\tilde{B}} h^2 = 1.3 \times 10^{-2} \left(\frac{m_{\tilde{e}_R}}{100\ \mathrm{GeV}}\right)^2 \frac{(1+r)^4}{r(1+r^2)} \times \left(1 + 0.07 \log \frac{\sqrt{r}\,100\ \mathrm{GeV}}{m_{\tilde{e}_R}}\right), \tag{2}$$

where $m_{\tilde{e}_R}$ is the mass of right-handed sleptons, $r \equiv M_1^2/m_{\tilde{e}_R}^2$, and M_1 is the bino mass.

Investigation of the bulk region for a pure binolike neutralino necessitates avoiding resonance annihilation and coannihilation. Prevention demands the following conditions be implemented:

(i) 99.9% binolike LSPs are selected to prohibit large annihilation cross sections induced by Higgsino or wino components.
(ii) $2m_{\tilde{\chi}_1^0} \ll m_{H^0}, m_{A^0}$ and $2m_{\tilde{\chi}_1^0} \gg m_h$ are enforced to avoid the "Higgs funnel."
(iii) First and second right-handed sleptons, $\tilde{\tau}_1$ and $\tilde{e}_R$, are naturally light, so coannihilation processes are negligible when $\mathcal{R}_{\tilde{\tau}_1} \equiv \frac{m_{\tilde{\tau}_1} - m_{\tilde{\chi}_1^0}}{m_{\tilde{\chi}_1^0}} \gtrsim 10\%$ and $\mathcal{R}_{\tilde{e}_R} \equiv \frac{m_{\tilde{e}_R} - m_{\tilde{\chi}_1^0}}{m_{\tilde{\chi}_1^0}} \gtrsim 10\%$.
(iv) Require the SM Higgs resonance to vanish, which transpires when $|\mu|^2 \gg M_Z^2$, via the coupling $g_{h\tilde{\chi}_1^0\tilde{\chi}_1^0} \propto \frac{M_Z(2\mu\cos\beta + M_1)}{\mu^2 - M_1^2}$ [59].

V. NUMERICAL RESULTS

Constructing the bulk region requires we consider only traditional annihilations. In order to assure that our calculation using only annihilations is accurate, a consistency check was performed between two disparate methods, the relic density calculated by micrOMEGAs and the formula in Eq. (2). We found the two distinct computations were consistent. With the validity of our annihilation calculation method now established, the expression $\mathcal{R}_\phi < 10\%$ is tested to authenticate that it does, in fact, deliver a large coannihilation percentage. Indeed, for $\mathcal{R}_\phi = 5\%$–8%, the relic density computed using only annihilations deviates from the coannihilation computation by $\geq 50\%$. The large deviation between the two results verifies the use of $\mathcal{R}_\phi < 10\%$ for identifying a large coannihilation factor. On the other hand, numerical results reveal that for $\mathcal{R}_\phi = 10\%$–12%, the annihilation and coannihilation calculations only deviate by 20%–30%, a negligible amount of coannihilation compared to 70%–80% annihilation.

The bulk region in generalized no-scale $\mathcal{F}$-SU(5) is illustrated in Fig. 1, with $\mathcal{R}_{\tilde{\tau}_1}$ plotted as a function of the

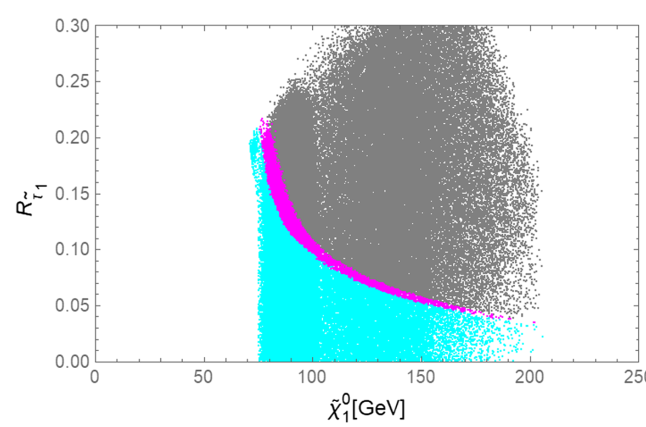

FIG. 1. Bulk region in generalized no-scale $\mathcal{F}$-SU(5). Cyan, magenta, gray points correspond to undersaturated, saturated, oversaturated DM relic density.

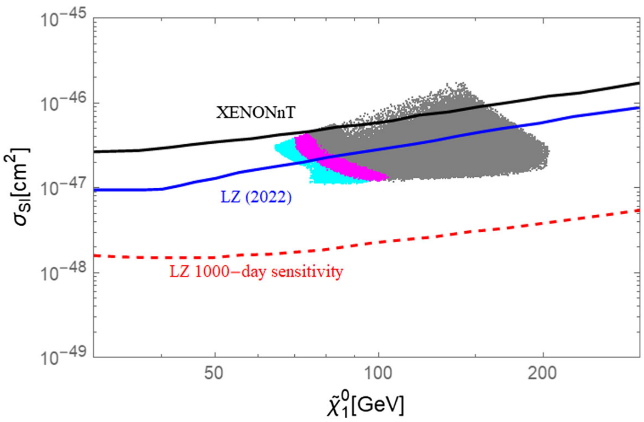

FIG. 2. Generalized no-scale $\mathcal{F}$-SU(5) bulk region LSPs plotted in reference to spin-independent DM-nuclei cross sections from XENONnT [52] and LUX-ZEPLIN [53,54]. Cyan, magenta, gray points correspond to undersaturated, saturated, oversaturated DM relic density with $\mathcal{R}_{\tilde{\tau}_1} \gtrsim 10\%$. We underscore the significance of the 1000-day LUX-ZEPLIN run that should fully probe the $\mathcal{F}$-SU(5) bulk and about 50% of the pMSSM bulk (not shown).

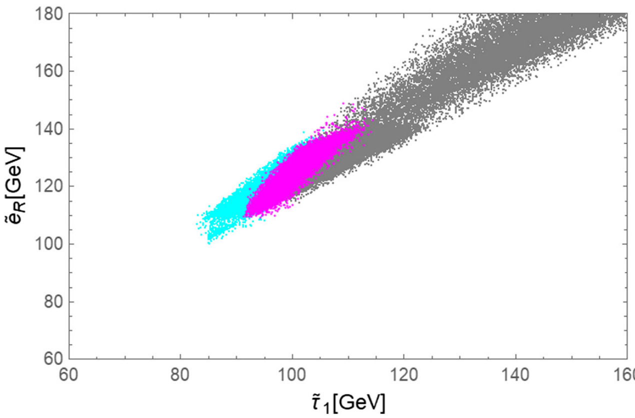

FIG. 3. Light right-handed slepton masses in the generalized no-scale $\mathcal{F}$-SU(5) bulk region. Cyan, magenta, gray points correspond to undersaturated, saturated, oversaturated DM relic density with $\mathcal{R}_{\tilde{\tau}_1} \gtrsim 10\%$.

binolike neutralino $m_{\tilde{\chi}_1^0}$. The mass hierarchy in $\mathcal{F}$-SU(5) is $m_{\tilde{\chi}_1^0} < m_{\tilde{\tau}_1} < m_{\tilde{e}_R} = m_{\tilde{\mu}_R}$, hence $\mathcal{R}_{\tilde{e}_R}$ always exceeds $\mathcal{R}_{\tilde{\tau}_1}$. All points in Fig. 1 satisfy the itemized experimental constraints specified in the prior section. Note that no-scale $\mathcal{F}$-SU(5) is in tension with the recent muon anomalous magnetic moment measurements [60], though this can be fully remedied with the addition of a SM singlet and chirality flip [41]. Cyan, magenta, and gray points in Fig. 1 correspond to the undersaturated, saturated, and oversaturated DM relic density, respectively. If the bino contributes all the DM abundance, the ratio $\mathcal{R}_{\tilde{\tau}_1} \gtrsim 10\%$ implies $m_{\tilde{\chi}_1^0} \leq 103.0$ GeV. The light bino LSP of the bulk region could be fully probed within the next few years, where Fig. 2 shows the compelling expected sensitivity of the 1000-day LUX-ZEPLIN experiment [53] and broad coverage of the no-scale $\mathcal{F}$-SU(5) bulk region. The $\tilde{\tau}_1 - \tilde{e}_R$ plane is delineated in Fig. 3. Within the generalized no-scale $\mathcal{F}$-SU(5) bulk region, Fig. 3 shows the upper limit of $\tilde{\tau}_1$ and $\tilde{e}_R$ are around 115 and 150 GeV, respectively. Recognize that these right-handed sleptons and bino LSP are *naturally* light, thus the LSP has not been fine-tuned to fortuitously conform to Planck satellite 5σ relic density observations. Two benchmark points are provided in Table I highlighting all the parameters and constraints discussed here.

The August 2023 ATLAS summary plot [61] highlighting observed exclusion limits from ATLAS SUSY searches for electroweak production of sleptons [62–66] is shown in Fig. 4 with several generalized no-scale $\mathcal{F}$-SU(5) benchmark points in the bulk region superimposed. Though not shown in this paper, the situation is similar with regard to CMS SUSY searches for electroweak production of sleptons [67–69]. All points plot in Fig. 4 are primarily traditional annihilation only and adhere to our requirement $\mathcal{R}_\phi \gtrsim 10\%$. The blue points correspond to $\tilde{e}_R = \tilde{\mu}_R$ in $\mathcal{F}$-SU(5), whereas the green points correspond to $\tilde{\tau}_1$ in $\mathcal{F}$-SU(5). It is important to point out that the ATLAS thin orange shaded region in Fig. 4 applies only to $\tilde{e}_R = \tilde{\mu}_R$ and not the $\tilde{\tau}_1$. The ATLAS green shaded region in Fig. 4 depicts the $\tilde{\tau}_1$ constraints, which $\mathcal{F}$-SU(5) is well beyond. A corollary picture of the bulk at the LHC is depicted in Fig. 5, with $\mathcal{F}$-SU(5) points superimposed over an August

TABLE I. Two benchmark points for generalized no-scale $\mathcal{F}$-SU(5). All masses are in GeV.

M_5	3996	2591
M_{1X}	473	268
M_0	23.15	21.49
A_0	0	0
$\tan\beta$	3.04	2.53
M_V	9063	4603
m_h	125.43	123.06
m_A	7325	5395
$m_{\tilde{\chi}_1^0}$	161.3	92.3
$m_{\tilde{\tau}_1}$	169.6	103.4
$m_{\tilde{e}_R}$	216.1	130.1
$m_{\tilde{t}_1}$	4273	2747
$m_{\tilde{g}}$	4986	3259
$m_{\tilde{u}_R}$	6798	4606
$\mathrm{BR}(B_S^0 \to \mu^+\mu^-) \times 10^{-9}$	3.03	3.05
$\mathrm{BR}(b \to s\gamma) \times 10^{-4}$	3.61	3.61
$\sigma_{SI} \times 10^{-12}$ pb	6.28	17.10
$\tau_p \times 10^{34}$ yr	5.01	3.95
$\mathcal{R}_{\tilde{\tau}_1}$	5%	12%
$\Omega_{\tilde{\chi}} h^2$	0.1256	0.118
$\Omega_{\tilde{\chi}} h^2$ (no coannihilation)	0.386	0.147
Coannihilation rate	>50%	~20%

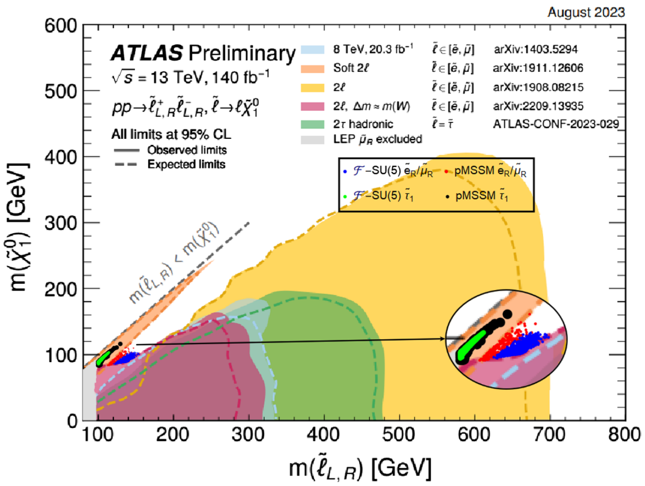

FIG. 4. Generalized no-scale $\mathcal{F}$-SU(5) and pMSSM bulk regions superimposed over the August 2023 ATLAS summary plot [61] of SUSY searches for electroweak production of sleptons [62–66]. The blue points correspond to $\tilde{e}_R = \tilde{\mu}_R$ in $\mathcal{F}$-SU(5), whereas the red points are $\tilde{e}_R = \tilde{\mu}_R$ in the pMSSM. The green points correspond to $\tilde{\tau}_1$ in $\mathcal{F}$-SU(5), while the black points are $\tilde{\tau}_1$ in the pMSSM. Inset: enlargement of the bulk. All points plotted here are annihilation only per our requirement $\mathcal{R}_\phi \gtrsim 10\%$. Note the ATLAS orange shaded sliver applies to $\tilde{e}_R = \tilde{\mu}_R$ only and not the $\tilde{\tau}_1$. The $\tilde{\tau}_1$ constraints are the ATLAS green shaded region, which both $\mathcal{F}$-SU(5) and pMSSM points are comfortably beyond. The evasion of the bulk region at the LHC thus far is evident.

2023 ATLAS summary plot for smuon SUSY searches [62–65]. In this particular instance, the smuon mass is plotted as a function of $\Delta m = (\tilde{\mu}_{L,R}, \tilde{\chi}^0_1)$ for a bino LSP in order to emphasize probing of those regions consistent with

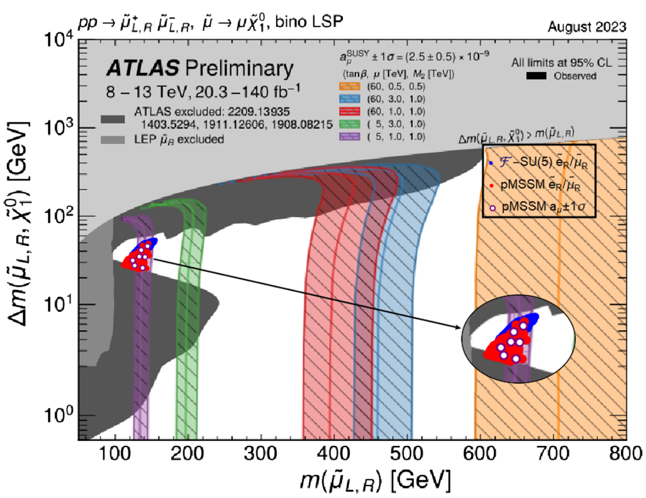

FIG. 5. Generalized no-scale $\mathcal{F}$-SU(5) and pMSSM bulk regions superimposed over the August 2023 ATLAS summary plot [61] of SUSY searches for electroweak production of smuons [62–65], plotted here in terms of $\Delta m = (\tilde{\mu}_{L,R}, \tilde{\chi}^0_1)$ for a bino LSP, emphasizing consistency of the bulk with recent muon anomalous magnetic moment measurements [60]. Inset: enlargement of the bulk.

recent muon anomalous magnetic moment measurements [60]. The predicament depicted in Figs. 4 and 5 is that, given the compressed nature of these spectra, this bulk region may not be probed at the LHC, though these light sleptons could conceivably be observed when the forthcoming circular colliders power up their beams, namely, the Future Circular Collider (FCC-ee) [70,71] at CERN and the Circular Electron-Positron Collider (CEPC) [72] with its sensitivity specified in Ref. [73]. Likewise, the proton lifetime via dimension-six proton decay is near 3–4 × 10^{34} yr, so this "fast" dimension-six proton decay is within reach of the future Hyper-Kamiokande experiment [74].

The methodology just discussed is extended to include the much less constrained generic pMSSM. The pMSSM contains 22 free parameters, and we input M_A and μ in lieu of $m^2_{H_u}$ and $m^2_{H_d}$. The scanning ranges of the pMSSM parameters are as follows:

$$\begin{aligned}
&20 \le M_1 \le 1000\ \text{GeV}, \qquad 2 \le \tan\beta \le 65,\\
&1000 \le M_2 \le 5000\ \text{GeV}, \qquad 1000 \le M_A, \mu \le 6000\ \text{GeV},\\
&1200 \le M_3 \le 5000\ \text{GeV}, \qquad M_1 \le m_{\tilde{e}_R}, m_{\tilde{\tau}_R} \le 2M_1,\\
&2500 \le m_{\tilde{q}}, m_{\tilde{Q}}, m_{\tilde{u}_R}, m_{\tilde{t}_R}, m_{\tilde{d}_R}, m_{\tilde{b}_R} \le 5000\ \text{GeV},\\
&700 \le m_{\tilde{l}} \le 2000\ \text{GeV}, \qquad 1200 \le m_{\tilde{L}} \le 5000\ \text{GeV},\\
&-5000 \le A_u, A_d, A_e, A_t, A_b, A_\tau \le 5000\ \text{GeV},
\end{aligned} \tag{3}$$

where $m_{\tilde{q}}$, $m_{\tilde{u}_R}$, $m_{\tilde{d}_R}$, $m_{\tilde{l}}$, and $m_{\tilde{e}_R}$ are the first/second generation sfermion mass parameters and $m_{\tilde{Q}}$, $m_{\tilde{t}_R}$, $m_{\tilde{b}_R}$, $m_{\tilde{L}}$, and $m_{\tilde{\tau}_R}$ are third generation sfermions. In contrast with $\mathcal{F}$-SU(5), there is no rigid mass ordering among $\tilde{e}_R$ and $\tilde{\tau}_1$ in the pMSSM. Our search imposes $\mathcal{R}_{\tilde{e}_R} > \mathcal{R}_{\tilde{\tau}_1}$ while floating the ratio $\mathcal{R}_{\tilde{\tau}_1}$ and vice versa. Numerical findings disclose the ratio $\mathcal{R}_{\tilde{\tau}_1} \gtrsim 10\%$ implies $m_{\tilde{\chi}^0_1} \le 117.7$ GeV. In the latter case where we imposed $\mathcal{R}_{\tilde{\tau}_1} > \mathcal{R}_{\tilde{e}_R}$, all pMSSM points with an $\tilde{e}_R$ next to lightest supersymmetric particle (NLSP) are excluded by the ATLAS soft lepton SUSY search [63]. Therefore, like generalized no-scale $\mathcal{F}$-SU(5), the only viable pMSSM region in the bulk is for the case $m_{\tilde{\chi}^0_1} < m_{\tilde{\tau}_1} < m_{\tilde{e}_R} = m_{\tilde{\mu}_R}$. We only scanned for small $\tilde{e}_R$ in the pMSSM, but, in principle, $\tilde{e}_R$ can be much heavier than $\tilde{\tau}_1$ in the pMSSM. The pMSSM bulk is not included in Fig. 2, though the 1000-day LUX-ZEPLIN experiment [53] is anticipated to probe about 50% of the pMSSM bulk. The ATLAS SUSY search exhibited in Fig. 4 also superimposes points from the bulk region in the pMSSM. The red points are $\tilde{e}_R = \tilde{\mu}_R$ in the pMSSM, while the black points are $\tilde{\tau}_1$ in the pMSSM. In the pMSSM, the bulk alone can explain recent muon anomalous magnetic moment measurements [60]. Indeed, pMSSM bulk region points are plotted in Fig. 5 along with $\mathcal{F}$-SU(5), and consistency with the anomalous magnetic moment results for small $\tan\beta$ is clear (ATLAS generated purple band). The conclusions leaping from Figs. 4 and 5 for both $\mathcal{F}$-SU(5) and the pMSSM are uniform; the bulk region has so far stealthily eluded the reach of the LHC, potentially leaving the prospect of discovery residing with dark matter direct detection experiments and next generation circular colliders.

VI. CONCLUSION

The void of a substantiated SUSY signal at the LHC has promoted naturalness to an indispensable foundation of SUSY GUT models. It is not sufficient for a model to only attack and then solve the electroweak fine-tuning problem, the model should also accommodate natural dark matter. The challenge resounds as to whether such broad naturalness is even possible to formulate, and if so, could such a sweeping natural model even be probed at the LHC?

The EWFT problem can be elegantly solved with effective supernatural SUSY, but this technique alone cannot naturally generate light sfermions and thus a light bino LSP, a scenario we regard as the most natural dark matter. In pursuit of the ambitious objective to construct a fully natural model, we proposed in this work a new perspective on no-scale supergravity, defining it as generalized no-scale SUGRA. Arising from our generalization of no-scale SUGRA is a marriage of natural SUSY with natural dark matter, a very favorable and welcome merger. The blissful union of natural SUSY with natural dark matter was fulfilled in the GUT model $\mathcal{F}$-SU(5), deriving a region of the model space that naturally supports light right-handed sleptons and a light LSP, known as the bulk region, where $m_{\tilde{\chi}_1^0} \leq 103.0$ GeV with negligible coannihilation and upper limits on $m_{\tilde{\tau}_1}$ and $m_{\tilde{e}_R}$ about 115 and 150 GeV, respectively. The $\mathcal{F}$-SU(5) bulk region LSPs will receive full coverage during the presently running 1000-day LUX-ZEPLIN experiment. Our analytical results unveiled that the light right-handed sleptons in the bulk could be beyond the LHC reach, although they may be probed amid the imminent era of advanced circular colliders, for example, the FCC-ee at CERN and CEPC. Equally meaningful, the fast proton decay could be observed at the future Hyper-Kamiokande experiment.

We extended the analysis beyond $\mathcal{F}$-SU(5) and scrutinized the pMSSM, exposing comparable results. We are partial to the beautiful simplicity of the stringy physical model no-scale $\mathcal{F}$-SU(5) and its unprecedented cosmology, yet we also recognize the advantages of studying the larger and much less constrained generic pMSSM. Our expanded investigation revealed that a bulk region can subsist in the pMSSM also, though only with a light stau NLSP, where $m_{\tilde{\chi}_1^0} \leq 117.7$ GeV with negligible coannihilation.

Should the LHC fail in the next few years to mount another substantial discovery beyond the light Higgs boson, a strategic plan forged by the high-energy physics community will undoubtably emerge. We suggest the tactics presented here that can formulate a model supporting both natural SUSY and natural dark matter be given explicit deliberation as to why discovery has been delayed and where to focus the forthcoming circular colliders.

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[1] S. Dimopoulos, S. Raby, and F. Wilczek, Phys. Rev. D **24**, 1681 (1981); J. R. Ellis, S. Kelley, and D. V. Nanopoulos, Phys. Lett. B **260**, 131 (1991); U. Amaldi, W. de Boer, and H. Furstenau, Phys. Lett. B **260**, 447 (1991); P. Langacker and M. X. Luo, Phys. Rev. D **44**, 817 (1991).
[2] H. Goldberg, Phys. Rev. Lett. **50**, 1419 (1983); **103**, 099905(E) (2009); J. R. Ellis, J. S. Hagelin, D. V. Nanopoulos, K. A. Olive, and M. Srednicki, Nucl. Phys. **B238**, 453 (1984).
[3] H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 438 (1974).
[4] J. C. Pati and A. Salam, Phys. Rev. D **10**, 275 (1974); **11**, 703(E) (1975).
[5] R. N. Mohapatra and J. C. Pati, Phys. Rev. D **11**, 566 (1975).
[6] H. Fritzsch and P. Minkowski, Ann. Phys. (N.Y.) **93**, 193 (1975).
[7] H. Georgi, AIP Conf. Proc. **23**, 575 (1975).
[8] ATLAS Collaboration, Report No. ATLAS-CONF-2019-040.
[9] G. Aad *et al.* (ATLAS Collaboration), Eur. Phys. J. C **80**, 737 (2020).

[10] G. Aad *et al.* (ATLAS Collaboration), J. High Energy Phys. 12 (2019) 060.

[11] CMS Collaboration, Report No. CMS PAS SUS-19-005.

[12] CMS Collaboration, Report No. CMS PAS SUS-19-006.

[13] S. Dimopoulos and G. F. Giudice, Phys. Lett. B **357**, 573 (1995).

[14] A. G. Cohen, D. B. Kaplan, and A. E. Nelson, Phys. Lett. B **388**, 588 (1996).

[15] R. Kitano and Y. Nomura, Phys. Lett. B **631**, 58 (2005).

[16] R. Kitano and Y. Nomura, Phys. Rev. D **73**, 095004 (2006).

[17] T. J. LeCompte and S. P. Martin, Phys. Rev. D **84**, 015004 (2011).

[18] T. J. LeCompte and S. P. Martin, Phys. Rev. D **85**, 035023 (2012).

[19] J. Fan, M. Reece, and J. T. Ruderman, J. High Energy Phys. 11 (2011) 012.

[20] G. D. Kribs and A. Martin, Phys. Rev. D **85**, 115014 (2012).

[21] H. Baer, V. Barger, P. Huang, D. Mickelson, A. Mustafayev, and X. Tata, Phys. Rev. D **87**, 035017 (2013).

[22] H. Baer, V. Barger, P. Huang, D. Mickelson, A. Mustafayev, and X. Tata, Phys. Rev. D **87**, 115028 (2013).

[23] M. Drees and J. S. Kim, Phys. Rev. D **93**, 095005 (2016).

[24] R. Ding, T. Li, F. Staub, and B. Zhu, Phys. Rev. D **93**, 095028 (2016).

[25] H. Baer, V. Barger, and M. Savoy, Phys. Rev. D **93**, 035016 (2016).

[26] B. Batell, G. F. Giudice, and M. McCullough, J. High Energy Phys. 12 (2015) 162.

[27] J. Fan, M. Reece, and L. T. Wang, J. High Energy Phys. 08 (2015) 152.

[28] T. Leggett, T. Li, J. A. Maxin, D. V. Nanopoulos, and J. W. Walker, arXiv:1403.3099; Phys. Lett. B **740**, 66 (2015).

[29] G. Du, T. Li, D. V. Nanopoulos, and S. Raza, Phys. Rev. D **92**, 025038 (2015).

[30] T. Li, S. Raza, and X. C. Wang, Phys. Rev. D **93**, 115014 (2016).

[31] E. Cremmer, S. Ferrara, C. Kounnas, and D. V. Nanopoulos, Phys. Lett. **133B**, 61 (1983); J. R. Ellis, A. B. Lahanas, D. V. Nanopoulos, and K. Tamvakis, Phys. Lett. **134B**, 429 (1984); J. R. Ellis, C. Kounnas, and D. V. Nanopoulos, Nucl. Phys. **B241**, 406 (1984); **B247**, 373 (1984); A. B. Lahanas and D. V. Nanopoulos, Phys. Rep. **145**, 1 (1987).

[32] T. j. Li, Phys. Rev. D **59**, 107902 (1999).

[33] J. R. Ellis, K. Enqvist, D. V. Nanopoulos, and F. Zwirner, Mod. Phys. Lett. A **01**, 57 (1986).

[34] R. Barbieri and G. F. Giudice, Nucl. Phys. **B306**, 63 (1988).

[35] J. Jiang, T. Li, and D. V. Nanopoulos, Nucl. Phys. **B772**, 49 (2007).

[36] J. Jiang, T. Li, D. V. Nanopoulos, and D. Xie, Phys. Lett. B **677**, 322 (2009).

[37] J. Jiang, T. Li, D. V. Nanopoulos, and D. Xie, Nucl. Phys. **B830**, 195 (2010).

[38] J. L. Lopez, D. V. Nanopoulos, and K. J. Yuan, Nucl. Phys. **B399**, 654 (1993).

[39] T. Li, J. A. Maxin, D. V. Nanopoulos, and J. W. Walker, Phys. Rev. D **83**, 056015 (2011); T. Li, J. A. Maxin, and D. V. Nanopoulos, Phys. Lett. B **764**, 167 (2017).

[40] T. Li, J. A. Maxin, D. V. Nanopoulos, and J. W. Walker, Phys. Lett. B **710**, 207 (2012).

[41] T. Li, J. A. Maxin, and D. V. Nanopoulos, Eur. Phys. J. C **81**, 1059 (2021).

[42] J. Ellis, M. A. G. Garcia, N. Nagata, D. V. Nanopoulos, and K. A. Olive, Phys. Lett. B **797**, 134864 (2019); J. Ellis, M. A. G. Garcia, N. Nagata, D. V. Nanopoulos, and K. A. Olive, J. Cosmol. Astropart. Phys. 01 (2020) 035; I. Antoniadis, D. V. Nanopoulos, and J. Rizos, J. Cosmol. Astropart. Phys. 03 (2021) 017; S. Basilakos, D. V. Nanopoulos, T. Papanikolaou, E. N. Saridakis, and C. Tzerefos, Phys. Lett. B **849**, 138446 (2024).

[43] S. M. Barr, Phys. Lett. **112B**, 219 (1982); J. P. Derendinger, J. E. Kim, and D. V. Nanopoulos, Phys. Lett. **139B**, 170 (1984); I. Antoniadis, J. R. Ellis, J. S. Hagelin, and D. V. Nanopoulos, Phys. Lett. B **194**, 231 (1987).

[44] T. Li, J. A. Maxin, D. V. Nanopoulos, and J. W. Walker, Nucl. Phys. **B848**, 314 (2011).

[45] M. Aaboud *et al.* (ATLAS Collaboration), Phys. Rev. D **97**, 112001 (2018).

[46] T. A. Vami (ATLAS and CMS Collaborations), Proc. Sci. LHCP2019 (**2019**) 168 [arXiv:1909.11753].

[47] A. M. Sirunyan *et al.* (CMS Collaboration), Eur. Phys. J. C **77**, 710 (2017).

[48] CMS and LHCb Collaborations, Nature (London) **522**, 68 (2015).

[49] Y. Amhis *et al.* (HFLAV Collaboration), arXiv:1412.7515.

[50] P. Slavich, S. Heinemeyer, E. Bagnaschi, H. Bahl, M. Goodsell, H. E. Haber, T. Hahn, R. Harlander, W. Hollik, G. Lee *et al.*, Eur. Phys. J. C **81**, 450 (2021).

[51] B. C. Allanach, A. Djouadi, J. L. Kneur, W. Porod, and P. Slavich, J. High Energy Phys. 09 (2004) 044.

[52] E. Aprile *et al.* (XENON Collaboration), Phys. Rev. Lett. **131**, 041003 (2023).

[53] D. S. Akerib *et al.* (LZ Collaboration), Phys. Rev. D **101**, 052002 (2020).

[54] J. Aalbers *et al.* (LZ Collaboration), Phys. Rev. Lett. **131**, 041002 (2023).

[55] N. Aghanim *et al.* (Planck Collaboration), Astron. Astrophys. **641**, A1 (2020).

[56] G. Belanger, F. Boudjema, A. Pukhov, and A. Semenov, Comput. Phys. Commun. **149**, 103 (2002).

[57] A. Djouadi, J. L. Kneur, and G. Moultaka, Comput. Phys. Commun. **176**, 426 (2007).

[58] N. Arkani-Hamed, A. Delgado, and G. F. Giudice, Nucl. Phys. **B741**, 108 (2006).

[59] A. Djouadi, M. Drees, and J. L. Kneur, Phys. Lett. B **624**, 60 (2005).

[60] D. P. Aguillard *et al.* (Muon $g-2$ Collaboration), Phys. Rev. Lett. **131**, 161802 (2023).

[61] ATLAS Collaboration, Report No. ATL-PHYS-PUB-2023-025.

[62] G. Aad *et al.* (ATLAS Collaboration), J. High Energy Phys. 05 (2014) 071.

[63] G. Aad *et al.* (ATLAS Collaboration), Phys. Rev. D **101**, 052005 (2020).

[64] G. Aad *et al.* (ATLAS Collaboration), Eur. Phys. J. C **80**, 123 (2020).

[65] G. Aad *et al.* (ATLAS Collaboration), J. High Energy Phys. 06 (2023) 031.

[66] ATLAS Collaboration, Report No. ATLAS-CONF-2023-029.

[67] A. M. Sirunyan *et al.* (CMS Collaboration), J. High Energy Phys. 04 (2021) 123.

[68] CMS Collaboration, Report No. CMS-PAS-SUS-21-008.

[69] A. Tumasyan *et al.* (CMS Collaboration), Phys. Rev. D **108**, 012011 (2023).

[70] A. Abada *et al.* (FCC Collaboration), Eur. Phys. J. C **79**, 474 (2019).

[71] A. Abada *et al.* (FCC Collaboration), Eur. Phys. J. Special Topics **228**, 261 (2019).

[72] J. B. Guimarães da Costa *et al.* (CEPC Study Group), arXiv:1811.10545.

[73] J. Yuan, H. Cheng, and X. Zhuang, arXiv:2203.10580.

[74] K. Abe *et al.* (Hyper-Kamiokande Collaboration), arXiv:1805.04163; P. N. Bhattiprolu, S. P. Martin, and J. D. Wells, Phys. Rev. D **107**, 055016 (2023).