Attempt at constructing a model of grand gauge-Higgs unification with family unification

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We discuss a possibility whether a model of grand gauge-Higgs unification incorporating family unification in higher dimensions can be constructed. We first extend a five-dimensional $SU(6)$ grand gauge-Higgs unification model to a five-dimensional $SU(7)$ grand gauge-Higgs unification model compactified on an orbifold S^1/Z_2 to obtain three generations of quarks and leptons after symmetry breaking of the larger family unified gauge group. A prescription of constructing a six-dimensional $SU(N)$ grand gauge-Higgs unification model including a five-dimensional $SU(7)$ grand gauge-Higgs unification after compactifying the sixth dimension on an orbifold S^1/Z_2 is given. We find a six-dimensional $SU(14)$ grand gauge-Higgs unification model with a set of representations containing three generations of quarks and leptons.

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I. INTRODUCTION

One of the mysteries in the Standard Model (SM) of particle physics is the origin of three generations of quarks and leptons, where quarks and leptons with the same representations and charges have a triple copy structure. Understanding whether three generations are inevitable or accidental is a very nontrivial problem. One of the approaches to solve this problem has been known as “family unification”. In this scenario, quarks, and leptons are embedded into fermions (desirably one fermion) in some nonrepetitive representations of a large group which has the grand unified theory (GUT) gauge group or the SM gauge group as a subgroup and three generations of quarks and leptons appear after symmetry breaking. The study of family unification has a very long history and has been done from various viewpoints [1–23].

A pioneering work was given by Georgi [1]. Three copies of fermions in $\bar{\mathbf{5}}$ and $\mathbf{10}$ of $SU(5)$ representations were obtained from some nonrepetitive and anomaly-free set of totally antisymmetric representations in $SU(11)$ theory. In a process of symmetry breaking, one might worry that there might appear many extra and unwanted massless fermions. If the representations of the fermions are vectorlike or real under the unbroken subgroup, such fermions would have masses of the order of symmetry breaking scale and are decoupled in the low energy theory. Taking into account this point, three generations were obtained.

This line of thought has also applied to the higher-dimensional theory [15]. The advantages of this application are as follows. In odd-dimensional theories, the anomaly constraints for the representations of fermions are relaxed. This greatly helps the possibility of model building extended. In even-dimensional theories, the anomaly constraints certainly exist in general, but it is not so serious if we consider only the Dirac fermions from the beginning. The other advantage is that it is relatively easy to consider symmetry breaking in higher-dimensional theory compactified on an orbifold. In such theories, symmetry breaking can be realized by boundary conditions and we have no need to introduce a complicated and unnatural Higgs potential as in four-dimensional theories.

On the other hand, it has been known that the SM gauge fields and the SM Higgs field can be unified in higher-dimensional gauge theory, which is called “gauge-Higgs unification”. In the scenario, the SM Higgs field is identified with extra spatial components of the higher-dimensional gauge field. Regardless of the nonrenormalizability, the quantum corrections of the physical quantities as Higgs mass and potential are predicted to be finite, which solves the hierarchy problem. The GUT extension of this scenario have been also considered and are very interesting, which has been paid much attention [24–27]. One of authors has proposed a five-dimensional (5D) $SU(6)$ grand gauge-Higgs unification (GGHU). It is remarkable in this model that quarks and leptons of one generation can be embedded into two $\bar{\mathbf{6}}$ and single $\mathbf{20}$ representations of $SU(6)$ without exotic fermions.

In this paper, we attempt to incorporate the family unification into grand gauge-Higgs unification. Employing the above $SU(6)$ model, we immediately notice that we have to extend the $SU(6)$ model since the $\mathbf{20}$ representation is self-conjugate to $\overline{\mathbf{20}}$ and even if three $\mathbf{20}$ are obtained, a pair of them makes a mass term, which implies that three generations are impossible in the approach of [1]. Therefore, we first extend the 5D $SU(6)$ GGHU model to an $SU(7)$ model to circumvent this problem. Then, we search for a six-dimensional (6D) GGHU model with family unification following the approach [1], which includes the 5D $SU(7)$ GGHU model with three generations after compactification to 5D. We find a 6D $SU(14)$ GGHU model with family unification.

The organization of this paper is as follows. In Sec. II, 5D $SU(7)$ GGHU model is constructed as a simple extension of 5D $SU(6)$ GGHU model. In Sec. III, we investigate a GGHU model with family unification in 6D following the approach by [1]. We find an $SU(14)$ GGHU model with family unification. Conclusion is given in the last section.

II. 5D $SU(7)$ THEORY

In this section, we point out some problems of our previous 5D $SU(6)$ GGHU model in the context of family unification and propose an $SU(7)$ model as a simple extension before discussing a 6D GGHU model of family unification.

In [24], a GGHU model of 5D $SU(6)$ gauge theory compactified on S^1/Z_2 was proposed, where the gauge fields and the Higgs fields in GUT are unified in the 5D gauge field. One of the remarkable features of this model is that one generation of quarks and leptons are embedded into a set of representations $2\times\bar{\mathbf{6}}\oplus\mathbf{20}$ of $SU(6)$ and furthermore no massless exotic fermions are left after compactification. However, we immediately notice that three generations of quarks and leptons cannot be obtained from this $SU(6)$ model in the approach of [1] since the $\mathbf{20}$ is self-conjugate to $\overline{\mathbf{20}}$ under $SU(6)$ and therefore a pair of them should be massive.

To improve this point, we propose a 5D $SU(7)$ model compactified on S^1/Z_2 including the $SU(6)$ model. Z_2 parity matrices are given on each fixed points $y=0,\pi R$ as

$$\begin{aligned} P_0 &= \mathrm{diag}(+1,+1,+1,+1,+1,-1,-1) \quad \text{at } y=0,\\ P_1 &= \mathrm{diag}(+1,+1,-1,-1,-1,-1,+1) \quad \text{at } y=\pi R, \end{aligned} \tag{1}$$

where y, R are the coordinate of the fifth dimension and a radius of the S^1, respectively. For the gauge fields, the Z_2 parity boundary conditions are taken as follows:

$$\begin{aligned} A_\mu(x^\mu,-y) &= P_{0,1}A_\mu(x^\mu,y)P_{0,1}^\dagger,\\ A_5(x^\mu,-y) &= -P_{0,1}A_\mu(x^\mu,y)P_{0,1}^\dagger. \end{aligned} \tag{2}$$

$x^\mu = 0,\ 1,\ 2,\ 3$ denotes the four-dimensional spacetime coordinate.

More explicitly, the parities of $A_{\mu,5}$ can be expressed in a matrix form,

$$A_\mu = \left(\begin{array}{cc|ccc|c|c} (+,+) & (+,+) & (+,-) & (+,-) & (+,-) & (-,-) & (-,+)\\ (+,+) & (+,+) & (+,-) & (+,-) & (+,-) & (-,-) & (-,+)\\ \hline (+,-) & (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,-)\\ (+,-) & (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,-)\\ (+,-) & (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,-)\\ \hline (-,-) & (-,-) & (-,+) & (-,+) & (-,+) & (+,+) & (+,-)\\ \hline (-,+) & (-,+) & (-,-) & (-,-) & (-,-) & (+,-) & (+,+) \end{array}\right), \tag{3}$$

$$A_5 = \left(\begin{array}{cc|ccc|c|c} (-,-) & (-,-) & (-,+) & (-,+) & (-,+) & (+,+) & (+,-)\\ (-,-) & (-,-) & (-,+) & (-,+) & (-,+) & (+,+) & (+,-)\\ \hline (-,+) & (-,+) & (+,-) & (+,-) & (+,-) & (+,-) & (+,+)\\ (-,+) & (-,+) & (+,-) & (+,-) & (+,-) & (+,-) & (+,+)\\ (-,+) & (-,+) & (+,-) & (+,-) & (+,-) & (+,-) & (+,+)\\ \hline (+,+) & (+,+) & (+,-) & (+,-) & (+,-) & (-,-) & (-,+)\\ \hline (+,-) & (+,-) & (+,+) & (+,+) & (+,+) & (-,+) & (-,-) \end{array}\right), \tag{4}$$

where the matrix elements mean a set of Z_2 parities (P_0, P_1).

Noting that Kaluza-Klein (KK) mode expansions of the five-dimensional field Φ with Z_2 parity (P_0, P_1) are given by

$$\begin{aligned}
\Phi^{(+,+)}(x,y) &= \frac{1}{\sqrt{2\pi R}}\phi_0(x) + \frac{1}{\sqrt{\pi R}}\sum_{n=1}^{\infty}\phi_n^{(+,+)}(x)\cos\left(\frac{n}{R}y\right),\\
\Phi^{(+,-)}(x,y) &= \frac{1}{\sqrt{\pi R}}\sum_{n=0}^{\infty}\phi_n^{(+,-)}(x)\cos\left(\frac{n+1/2}{R}y\right),\\
\Phi^{(-,+)}(x,y) &= \frac{1}{\sqrt{\pi R}}\sum_{n=0}^{\infty}\phi_n^{(-,+)}(x)\sin\left(\frac{n+1/2}{R}y\right),\\
\Phi^{(-,-)}(x,y) &= \frac{1}{\sqrt{\pi R}}\sum_{n=0}^{\infty}\phi_n^{(-,-)}(x)\sin\left(\frac{n}{R}y\right),
\end{aligned} \tag{5}$$

$\phi_0(x)$ in $\Phi^{(+,+)}(x,y)$ only remain massless in 4D. From this observation, we find the symmetry breaking pattern $SU(7)\to SU(3)_c\times SU(2)_L\times U(1)_Y\times U(1)_\alpha\times U(1)_\beta$ from A_μ and the SM Higgs doublet is indeed embedded in A_5. Generators of $U(1)_\alpha$ and $U(1)_\beta$ are contained in the $SU(7)$ generators as

$$U(1)_\alpha:\begin{pmatrix}1&&&&&&\\&1&&&&&\\&&1&&&&\\&&&1&&&\\&&&&1&&\\&&&&&-5&\\&&&&&&0\end{pmatrix},\qquad U(1)_\beta:\begin{pmatrix}1&&&&&&\\&1&&&&&\\&&1&&&&\\&&&1&&&\\&&&&1&&\\&&&&&1&\\&&&&&&-6\end{pmatrix}.$$

Looking at A_5, we find the colored Higgs field to be massless at tree level. Taking into account quantum corrections, its mass squared is expected to be $\mathcal{O}(\alpha_s/R^2)$. Since a typical order of the compactification scale is $\mathcal{O}(10\ \mathrm{TeV})$ in GHU, the colored Higgs mass will be at most $\mathcal{O}(10\ \mathrm{TeV})$. For proton stability, the baryon number violating operators by the colored Higgs exchange have to be forbidden or suppressed enough by some mechanism, which will not be discussed in this paper.

In the $SU(7)$ theory, $2\times\bar{\mathbf{6}}\oplus\mathbf{20}$ are embedded in $2\times\bar{\mathbf{7}}\oplus\overline{\mathbf{35}}$,

$$\begin{aligned}
\bar{\mathbf{7}} &= \bar{\mathbf{6}}\oplus\mathbf{1},\\
\overline{\mathbf{35}} &= \mathbf{20}\oplus\overline{\mathbf{15}}.
\end{aligned} \tag{6}$$

More explicitly, one generation of quarks and leptons are contained as follows:

$$\bar{\mathbf{7}} = \begin{cases}
\bar{\mathbf{7}}_L = \underbrace{(\bar{\mathbf{3}},\mathbf{1})^{(+,-)}_{(\frac{1}{3},-1,-1)}\oplus l_L(\mathbf{1},\mathbf{2})^{(+,+)}_{(-\frac{1}{2},-1,-1)}\oplus(\mathbf{1},\mathbf{1})^{(-,-)}_{(0,5,-1)}}_{\bar{\mathbf{6}}}\oplus\underbrace{(\mathbf{1},\mathbf{1})^{(-,+)}_{(0,0,6)}}_{\mathbf{1}}\\
\bar{\mathbf{7}}_R = \underbrace{(\bar{\mathbf{3}},\mathbf{1})^{(-,+)}_{(\frac{1}{3},-1,-1)}\oplus(\mathbf{1},\mathbf{2})^{(-,-)}_{(-\frac{1}{2},-1,-1)}\oplus\nu_R(\mathbf{1},\mathbf{1})^{(+,+)}_{(0,5,-1)}}_{\bar{\mathbf{6}}}\oplus\underbrace{(\mathbf{1},\mathbf{1})^{(+,-)}_{(0,0,6)}}_{\mathbf{1}}
\end{cases}$$

$$\bar{\mathbf{7}} = \begin{cases}
\bar{\mathbf{7}}_L = \underbrace{(\bar{\mathbf{3}},\mathbf{1})^{(-,-)}_{(\frac{1}{3},-1,-1)}\oplus l_L(\mathbf{1},\mathbf{2})^{(-,+)}_{(-\frac{1}{2},-1,-1)}\oplus(\mathbf{1},\mathbf{1})^{(+,-)}_{(0,5,-1)}}_{\bar{\mathbf{6}}}\oplus\underbrace{\chi_1(\mathbf{1},\mathbf{1})^{(+,+)}_{(0,0,6)}}_{\mathbf{1}}\\
\bar{\mathbf{7}}_R = \underbrace{\bar{d}_R(\bar{\mathbf{3}},\mathbf{1})^{(+,+)}_{(\frac{1}{3},-1,-1)}\oplus(\mathbf{1},\mathbf{2})^{(+,-)}_{(-\frac{1}{2},-1,-1)}\oplus(\mathbf{1},\mathbf{1})^{(-,+)}_{(0,5,-1)}}_{\bar{\mathbf{6}}}\oplus\underbrace{(\mathbf{1},\mathbf{1})^{(-,-)}_{(0,0,6)}}_{\mathbf{1}}
\end{cases}$$

$$
\overline{\mathbf{35}} = \begin{cases}
\overline{\mathbf{35}}_L = \underbrace{q_L(\mathbf{3},\mathbf{2})^{(+,+)}_{(\frac{1}{6},-3,-3)} \oplus (\bar{\mathbf{3}},\mathbf{1})^{(+,-)}_{(-\frac{2}{3},-3,-3)} \oplus (\mathbf{1},\mathbf{1})^{(+,-)}_{(1,-3,-3)}}_{\mathbf{20}} \\
\underbrace{\oplus (\bar{\mathbf{3}},\mathbf{2})^{(-,+)}_{(-\frac{1}{6},3,-3)} \oplus (\mathbf{3},\mathbf{1})^{(-,-)}_{(\frac{2}{3},3.,-3)} \oplus (\mathbf{1},\mathbf{1})^{(-,-)}_{(-1,3,-3)}}_{\mathbf{20}} \\
\oplus \underbrace{(\bar{\mathbf{3}},\mathbf{2})^{(-,-)}_{(-\frac{1}{6},-2,4)} \oplus (\mathbf{3},\mathbf{1})^{(-,+)}_{(\frac{2}{3},-2.,4)} \oplus \chi_2(\bar{\mathbf{3}},\mathbf{1})^{(+,+)}_{(\frac{1}{3},4,4)} \oplus (\mathbf{1},\mathbf{2})^{(+,-)}_{(-\frac{1}{2},4,4)} \oplus (\mathbf{1},\mathbf{1})^{(-,+)}_{(-1,-2,4)}}_{\overline{\mathbf{15}}} \\
\overline{\mathbf{35}}_R = \underbrace{(\mathbf{3},\mathbf{2})^{(-,-)}_{(\frac{1}{6},-3,-3)} \oplus (\bar{\mathbf{3}},\mathbf{1})^{(-,+)}_{(-\frac{2}{3},-3,-3)} \oplus (\mathbf{1},\mathbf{1})^{(-,+)}_{(1,-3)}}_{\mathbf{20}} \\
\underbrace{\oplus (\bar{\mathbf{3}},\mathbf{2})^{(+,-)}_{(-\frac{1}{6},3,-3)} \oplus u_R(\mathbf{3},\mathbf{1})^{(+,+)}_{(\frac{2}{3},3,-3)} \oplus e_R(\mathbf{1},\mathbf{1})^{(+,+)}_{(-1,3,-3)}}_{\mathbf{20}} \\
\oplus \underbrace{\chi_3(\bar{\mathbf{3}},\mathbf{2})^{(+,+)}_{(-\frac{1}{6},-2,4)} \oplus (\mathbf{3},\mathbf{1})^{(+,-)}_{(\frac{2}{3},-2.,4)} \oplus (\bar{\mathbf{3}},\mathbf{1})^{(-,-)}_{(\frac{1}{3},4,4)} \oplus (\mathbf{1},\mathbf{2})^{(-,+)}_{(-\frac{1}{2},4,4)} \oplus (\mathbf{1},\mathbf{1})^{(+,-)}_{(-1,-2,4)}}_{\overline{\mathbf{15}}}
\end{cases},
$$

where the bold face numbers in the right-hand side are the representations under $SU(3)_c \times SU(2)_L$ and the numbers in the subscript are the charges of $U(1)_Y \times U(1)_\alpha \times U(1)_\beta$. $L(R)$ means 4D left(right)-handed chiralities.

Furthermore, we must note that these representations have exotic fermions χ_i ($i = 1, 2, 3$) absent in the Standard Model as the price of extending the gauge group $SU(6)$ to $SU(7)$. These exotic fermions can be massive and removed by introducing the 4D fermions $\bar{\chi}_i$ with conjugate representations and opposite chirality to χ_i and Dirac mass terms $m_i\bar{\chi}_i\chi_i$ on the fixed point.

III. 6D $SU(N)$ THEORY

In this section, we attempt constructing a six-dimensional (6D) $SU(N)$ theory on S^1/Z_2 to realize a family unification of the previous 5D $SU(7)$ theory. Our strategy is based on an approach by Georgi [1] and its applications to higher-dimensional theory by Kawamura *et al.* [15,16,18,20,21]. In [1], the 4D theory of the gauge group $SU(N)$ including $SU(5)$ as a subgroup with fermions in only antisymmetric tensor representation of $SU(N)$ was considered and a possibility that three generations of quarks and leptons are obtained after symmetry breaking was explored. This is a very nontrivial requirement since all of antisymmetric tensor representations are not replicated and the fermion content has to be anomaly-free. Furthermore, if some fermions are vectorlike or real representations under $SU(5)$ after symmetry breaking, they will have a mass of order of the symmetry breaking scale. In [15,16,18,20,21], the argument by Georgi was applied to the higher-dimensional theory. The advantages of this application are as follows. There is no anomaly-free conditions in the case of odd dimensions. If the Dirac fermions are considered, the anomaly is canceled even in even dimensions. Moreover, the symmetry breaking can be easily realized by boundary conditions in extra spatial dimensions. Following approaches by [1] and [15,16,18,20,21], we would like to attempt constructing a 6D model where both gauge-Higgs unification and family unification are incorporated.

In 6D theory, gamma matrices Γ^M $(M = 0, 1, 2, 3, 5, 6)$ are 8×8 matrices satisfying Clifford algebra,

$$\{\Gamma^M, \Gamma^N\} = 2\eta^{MN} \quad (M, N = 0, 1, 2, 3, 5, 6), \qquad (7)$$

where η^{MN} is metric and $\eta = \mathrm{diag}(+1, -1, -1, -1, -1, -1)$. In addition, Γ^7 is defined as

$$\Gamma^7 \coloneqq i\Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^5\Gamma^6. \qquad (8)$$

In our notation, the 6D Weyl fermions $\Psi_\pm$ are eigenstates of Γ^7 with eigenvalue ±1, respectively and are decomposed into 4D Weyl fermions ψ_L and ψ_R,

$$\Psi = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} \psi_{+L} \\ \psi_{+R} \\ \psi_{-L} \\ \psi_{-R} \end{pmatrix}. \qquad (9)$$

By compactifying the sixth dimension z to S^1/Z_2 orbifold with a radius R' of S^1, we consider a symmetry breaking $SU(N) \to SU(7) \times SU(p) \times SU(q) \times U(1)^2$ realized by the following Z_2 parities P'_0 and P'_1,

$$P'_0 = \mathrm{diag}\overbrace{(+1,\ldots,+1,+1,\ldots,+1,-1,\ldots,-1)}^{N} \quad \text{at } z=0,$$
$$P'_1 = \mathrm{diag}(\underbrace{+1,\ldots,+1}_{7},\underbrace{-1,\ldots,-1}_{p},\underbrace{+1,\ldots,+1}_{q}) \quad \text{at } z=\pi R', \tag{10}$$

where the rank of the gauge symmetries are unchanged by the symmetry breaking, $7+p+q=N$. We introduce here a symbol $[k]_N$, which means the rank k totally antisymmetric tensor representation of $SU(N)$. This is decomposed into multiplets of $SU(7)\times SU(p)\times SU(q)$ as

$$[k]_N = \sum_{l=0}^{k}\sum_{m=0}^{k-l}\sum_{n=0}^{k-l-m}([l]_7,[m]_p,[n]_q). \tag{11}$$

We note that $[k]_N$ does not exist for the case of $k>N$. $[k]_N$ can be expressed as an antisymmetric part of the tensor product of k fundamental representations $\boldsymbol{N}$,

$$[k]_N = (\boldsymbol{N}\times\cdots\times\boldsymbol{N})_a, \tag{12}$$

where a means an antisymmetric part. The Z_2 transformations at $z=0,\pi R$ of the 6D Dirac fermion in the $[k]_N$ representation are given as follows:

$$(\boldsymbol{N}\times\cdots\times\boldsymbol{N})_a \to \eta_k(P'_0\boldsymbol{N}\times\cdots\times P'_0\boldsymbol{N})_a \quad \text{at } z=0,$$
$$(\boldsymbol{N}\times\cdots\times\boldsymbol{N})_a \to \eta'_k(P'_1\boldsymbol{N}\times\cdots\times P'_1\boldsymbol{N})_a \quad \text{at } z=\pi R', \tag{13}$$

where $\eta_k,\eta'_k=\pm1$. For example, $[1]_N=\boldsymbol{N}$ is decomposed as

$$\boldsymbol{N} = (\boldsymbol{7},\boldsymbol{1},\boldsymbol{1})\oplus(\boldsymbol{1},\boldsymbol{p},\boldsymbol{1})\oplus(\boldsymbol{1},\boldsymbol{1},\boldsymbol{q}), \tag{14}$$

and the Z_2 transformation at $z=0$ can be read as

$$\boldsymbol{N}\to\eta_k P'_0\boldsymbol{N} = \eta_k(\boldsymbol{7},\boldsymbol{1},\boldsymbol{1})\oplus+\eta_k(\boldsymbol{1},\boldsymbol{p},\boldsymbol{1})\oplus -\eta_k(\boldsymbol{1},\boldsymbol{1},\boldsymbol{q}). \tag{15}$$

In this way, we obtain the Z_2 parities $\mathcal{P}'_0$ and $\mathcal{P}'_1$ of the representation $([l]_7,[m]_p,[n]_q)$,

$$\mathcal{P}'_0 = (-1)^n\eta_k = (-1)^{l+m}(-1)^k\eta_k,$$
$$\mathcal{P}'_1 = (-1)^m\eta'_k = (-1)^{l+n}(-1)^k\eta'_k, \tag{16}$$

where a condition $l+m+n=k$ is taken into account in the second quality.

Here we can define that "the left-handed (LH)" fermion Ψ_- has η_k,η'_k factors, and "the right-handed (RH)" fermion Ψ_+ has $-\eta_k$, $-\eta'_k$ factors, more explicitly,

$$\Psi_-:\ (-1)^{l+m}(-1)^k\eta_k, \qquad (-1)^{l+m}(-1)^k\eta'_k, \tag{17}$$

$$\Psi_+:\ -(-1)^{l+m}(-1)^k\eta_k, \qquad -(-1)^{l+m}(-1)^k\eta'_k. \tag{18}$$

This relative sign is due to that of the chiral operator in 6D.

TABLE I. Z_2 parity assignment of **7**, **21**, and **35** (and their conjugate) representations of $SU(7)$. $\pm$ are given by the corresponding Γ_7 eigenvalues.

	$\mathcal{P}'_0$	$\mathcal{P}'_1$
$\mathbf{7}_\pm$	$\pm(-1)^m$	$\pm(-1)^n$
$\mathbf{21}_\pm$	$\mp(-1)^m$	$\mp(-1)^n$
$\mathbf{35}_\pm$	$\pm(-1)^m$	$\pm(-1)^n$
$\overline{\mathbf{35}}_\pm$	$\mp(-1)^m$	$\mp(-1)^n$
$\overline{\mathbf{21}}_\pm$	$\pm(-1)^m$	$\pm(-1)^n$
$\bar{\mathbf{7}}_\pm$	$\mp(-1)^m$	$\mp(-1)^n$

Now choosing as $((-1)^k\eta_k,(-1)^k\eta'_k)=(+1,+1)$ for simplicity, we obtain the Z_2 parity assignment for **7**, **21**, and **35** representations of $SU(7)$ by putting $l=1,2,3$ as in Table I. From the information of Z_2 parities listed in Table I, we can obtain the numbers of massless fermions with Z_2-even parity in the representations of $\bar{\mathbf{7}}_-$, $\mathbf{21}_-$, and $\overline{\mathbf{35}}_-$

$$n^{(+,+)}_{\bar{\mathbf{7}}_-,k} = \sum_{m,n=\text{even}} {}_pC_m\cdot{}_qC_n\cdot(\delta_{k,1+m+n}+\delta_{k,6+m+n}) - \sum_{m,n=\text{odd}} {}_pC_m\cdot{}_qC_n\cdot(\delta_{k,1+m+n}+\delta_{k,6+m+n}),$$
$$n^{(+,+)}_{\mathbf{21}_-,k} = \sum_{m,n=\text{even}} {}_pC_m\cdot{}_qC_n\cdot(\delta_{k,2+m+n}+\delta_{k,5+m+n}) - \sum_{m,n=\text{odd}} {}_pC_m\cdot{}_qC_n\cdot(\delta_{k,2+m+n}+\delta_{k,5+m+n}),$$
$$n^{(+,+)}_{\overline{\mathbf{35}}_-,k} = \sum_{m,n=\text{even}} {}_pC_m\cdot{}_qC_n\cdot(\delta_{k,3+m+n}+\delta_{k,4+m+n}) - \sum_{m,n=\text{odd}} {}_pC_m\cdot{}_qC_n\cdot(\delta_{k,3+m+n}+\delta_{k,4+m+n}) \tag{19}$$

where $\delta_{k,l+m+n}$ expresses $k=l+m+n$. For the case of $\bar{\mathbf{7}}_-$, the net number of massless 5D fermions is given by the difference between the sum with respect to even m, n come from $\bar{\mathbf{7}}_-$, $\mathbf{7}_+$ and that with respect to odd m, n from $\mathbf{7}_-$, $\bar{\mathbf{7}}_+$,

$$n_{\bar{\mathbf{7}}_-} = \#\bar{\mathbf{7}}_- + \#\mathbf{7}_+ - \#\bar{\mathbf{7}}_+ - \#\mathbf{7}_- \tag{20}$$

where # expresses the number of each multiplet. Similar arguments for $\mathbf{21}_-$ and $\overline{\mathbf{35}}_-$ are also applied.

Now, we would like to find a 6D family unified model of 5D SU(7) grand gauge-Higgs unification with three replicated $2\times\bar{\mathbf{7}}\oplus\overline{\mathbf{35}}$ representations including three families of quarks and leptons.

Taking $N=14$, $p=5$, and $q=2$ of our interest, (19) is rewritten as

$$
\begin{aligned}
n^{(+,+)}_{\bar{\mathbf{7}}_-,k} &= \sum_{m=0,2,4}\sum_{n=0,2} {}_5C_m \cdot {}_2C_n \cdot (\delta_{k,1+m+n} + \delta_{k,6+m+n}) \\
&\quad - \sum_{m=1,3,5}\sum_{n=1} {}_5C_m \cdot {}_2C_n \cdot (\delta_{k,1+m+n} + \delta_{k,6+m+n}), \\
n^{(+,+)}_{\mathbf{21}_-,k} &= \sum_{m=0,2,4}\sum_{n=0,2} {}_5C_m \cdot {}_2C_n \cdot (\delta_{k,2+m+n} + \delta_{k,5+m+n}) \\
&\quad - \sum_{m=1,3,5}\sum_{n=1} {}_5C_m \cdot {}_2C_n \cdot (\delta_{k,2+m+n} + \delta_{k,5+m+n}), \\
n^{(+,+)}_{\overline{\mathbf{35}}_-,k} &= \sum_{m=0,2,4}\sum_{n=0,2} {}_5C_m \cdot {}_2C_n \cdot (\delta_{k,3+m+n} + \delta_{k,4+m+n}) \\
&\quad - \sum_{m=1,3,5}\sum_{n=1} {}_5C_m \cdot {}_2C_n \cdot (\delta_{k,3+m+n} + \delta_{k,4+m+n}). \quad (21)
\end{aligned}
$$

From these results, the list of $n^{(+,+)}_{\bar{\mathbf{7}}_-,k}$, $n^{(+,+)}_{\mathbf{21}_-,k}$, and $n^{(+,+)}_{\overline{\mathbf{35}}_-,k}$ for various values of k in Table II are obtained. We note that $n^{(+,+)}_{R_-,k} = -5$ means $n^{(+,+)}_{R_+,k} = 5$ because of $n^{(+,+)}_{R_-,k} = -n^{(+,+)}_{R_+,k}$ ($R = \bar{\mathbf{7}}, \mathbf{21}, \overline{\mathbf{35}}$). Consider a set of representations $[1]_{14} \oplus [2]_{14} \oplus [3]_{14} \oplus [4]_{14} \oplus [6]_{14} \oplus [11]_{14} \oplus [12]_{14}$, all numbers of $\bar{\mathbf{7}}_-$, $\mathbf{21}_-$ and $\overline{\mathbf{35}}_-$ are given by

$$
\begin{aligned}
n_{\bar{\mathbf{7}}_-} &:= \sum_{k=1,2,3,4,6,11,12} n^{(+,+)}_{\bar{\mathbf{7}}_-,k} = 1+0+1+0+1+0+3 = 6, \\
n_{\mathbf{21}_-} &:= \sum_{k=1,2,3,4,6,11,12} n^{(+,+)}_{\mathbf{21}_+,k} = 0+1+0+1-5+3+0 = 0, \\
n_{\overline{\mathbf{35}}_-} &:= \sum_{k=1,2,3,4,6,11,12} n^{(+,+)}_{\overline{\mathbf{35}}_-,k} = 0+0+1+1+1+0+0 = 3.
\end{aligned} \quad (22)
$$

Remarkably, we have succeeded in obtaining a desirable set of massless fermions in the representations $3\times(2\times\bar{\mathbf{7}}\oplus\overline{\mathbf{35}})$, which contain three families of quarks and leptons. We think this result to be very nontrivial since the number of massless **21** representations has to vanish in addition to $3 \times (2 \times \bar{\mathbf{7}} \oplus \overline{\mathbf{35}})$ to realize three generations. We have investigated various models up to $SU(14)$, the above $SU(14)$ model was an only satisfactory solution. One might think that other models can obtained if larger gauge groups are considered. However, it seems to be difficult to find other GGHU models with family unification since the representations and the number of massless fermions become more complicated in such cases. It is therefore plausible to conclude that our 6D $SU(14)$ theory obtained in this paper is a simplest GGHU model with family unification in a class of SU theories although we cannot prove it formally.

TABLE II. Numbers of massless fermions in $\bar{\mathbf{7}}_-$, $\mathbf{21}_-$, $\overline{\mathbf{35}}_-$ representations for various k-rank totally antisymmetric tensors of $SU(14)$.

k	$n^{(+,+)}_{\bar{\mathbf{7}}_-,k}$	$n^{(+,+)}_{\mathbf{21}_-,k}$	$n^{(+,+)}_{\overline{\mathbf{35}}_-,k}$
1	1	0	0
2	0	1	0
3	1	0	1
4	0	1	1
5	−5	1	1
6	1	−5	1
7	3	1	−5
8	1	3	−5
9	0	−5	3
10	−5	0	3
11	0	3	0
12	3	0	0
13	0	0	0

In our model of GGHU with family unification, it is a very nontrivial issue to obtain Yukawa coupling. We note that the left-handed quark doublet and the right-handed up quark are included in $\overline{35}$ representation of the 5D fermion, which come from $[3]_{14}$, $[4]_{14}$, $[6]_{14}$ representations of 6D SU(14) theory. This means that up-type Yukawa coupling can be obtained from the gauge interaction in extra spatial components of SU(14). However, the right-handed down quark are included in $\bar{7}$ represention different from $\overline{35}$ representation where the left-handed quark doubled is present. Similarly, the representations where the left-handed lepton doublet and the right-handed electron are included are different. Therefore, the down-type quark and the charged lepton Yukawa couplings cannot be obtained from the SU(14) gauge interactions such as the up-type quark Yukawa couplings. One of the setups to realize the down-type and the charged Yukawa couplings has been known that 4D quarks and leptons are introduced on the brane and they couple to the bulk fermions through the localized Dirac mass terms. Then, integrating out the bulk fermions provides nonlocal Yukawa coupling, which was proposed in [28] and has been extensively studied in SU(6) GGHU model [29]. We also have to construct such a setup in the present model along the above idea, but it seems to be very nontrivial and complicated, which is beyond the scope of this paper.

Some comments on anomaly cancellation are given. A chiral gauge theory in even dimensions, in general, is possible to have gauge anomalies and global anomaly. Fermions in the representations $[k]_{14}$ ($k = 1, 2, 3, 4, 6, 11, 12$) we introduced are massless 6D Dirac fermions, that is, both of $[k]_{14+}$ and $[k]_{14-}$ are introduced. Therefore, the gauge anomalies are canceled. As for the global anomaly, the global anomaly is absent since the sixth homotopy group of $SU(14)$, $\Pi_6(SU(14))$, is known to be vanished [30].

IV. CONCLUSION

We have attempted to construct a grand gauge-Higgs unification model with family unification in this paper. A 6D $SU(14)$ GGHU theory as a family unified 5D $SU(7)$

GGHU theory was found, where the set of representations $[1]_{14} \oplus [2]_{14} \oplus [3]_{14} \oplus [4]_{14} \oplus [6]_{14} \oplus [11]_{14} \oplus [12]_{14}$ are decomposed into $3 \times (2 \times \bar{\mathbf{7}} \oplus \overline{\mathbf{35}})$ after symmetry breaking $SU(14) \to SU(7) \times SU(5) \times SU(2) \times U(1)^2$. Three generations of quarks and leptons are obtained from $3 \times (2 \times \bar{\mathbf{7}} \oplus \overline{\mathbf{35}})$ after compactification to 4D. As far as we know, this is the first GGHU model unifying three generations of quarks and leptons, which is expected to be a guideline of a model building along this line.

We have many issues to be explored since our work done in this paper is still a first step towards a construction of realistic model of GGHU with family unification.

In our model, we have unfortunately some massless exotic fermions. It is of course desirable to find a model without them. In order to realize it, it would be interesting to study the theories with other gauge groups and fermion matter content. In this paper, we have set $((-1)^k \eta_k, (-1)^k \eta'_k) = (+1, +1)$ for all k for simplicity. Considering other patterns for $((-1)^k \eta_k, (-1)^k \eta'_k)$ might help reducing the number of massless exotic fermions.

In our family unified 6D $SU(14)$ model, two step compactification is considered. It would be interesting to investigate a possibility that three generations of quarks and leptons are directly obtained from the compactification of 6D GGHU theory to 4D one such as [20].

As one of the other interesting directions, it would be interesting to consider models of SO gauge theory. The family unification via the spinor representation in SO theory has been much studied since the work [6]. Although several irreducible representations are necessary to realize the family unification in SU theory, the family unification in SO theory is expected to be realized by fewer irreducible representations than SU theory since the dimension of spinor representation is exponentially large with respect to the size of the gauge group.

There are many phenomenological issues to be studied, generating the realistic Yukawa hierarchy, the study of electroweak symmetry breaking, the flavor physics, the gauge coupling unification and the proton decay analysis and so on. All of them are left for our future study.

[1] H. Georgi, Towards a grand unified theory of flavor, Nucl. Phys. **B156**, 126 (1979).
[2] P. H. Frampton, SU(N) grand unification with several quark—lepton generations, Phys. Lett. **88B**, 299 (1979).
[3] P. Frampton and S. Nandi, SU(9) grand unification of flavor with three generations, Phys. Rev. Lett. **43**, 1460 (1979).
[4] P. H. Frampton, Unification of flavor, Phys. Lett. **89B**, 352 (1980).
[5] M. Ida, Y. Kayama, and T. Kitazoe, Inclusion of generations in SO(14), Prog. Theor. Phys. **64**, 1745 (1980).
[6] F. Wilczek and A. Zee, Families from spinors, Phys. Rev. D **25**, 553 (1982).
[7] S. Nandi, A. Stern, and E. C. G. Sudarshan, Structure of proton decay in SO(n) family unification, Phys. Rev. D **26**, 1653 (1982).
[8] Y. Fujimoto, SO(18) unification, Phys. Rev. D **26**, 3183 (1982).
[9] J. Bagger, S. Dimopoulos, and E. Masso, Proton lifetime in orthogonal theories of family unification, Phys. Lett. **145B**, 211 (1984).
[10] J. A. Bagger, S. Dimopoulos, E. Masso, and M. H. Reno, A realistic theory of family unification, Nucl. Phys. **B258**, 565 (1985).
[11] S. M. Barr, E_8 family unification, mirror fermions, and new low-energy physics, Phys. Rev. D **37**, 204 (1988).
[12] P. H. Frampton and T. W. Kephart, Minimal family unification, Phys. Rev. D **51**, R1 (1995).
[13] K. S. Babu, S. M. Barr, and B. Kyae, Family unification in five-dimensions and six-dimensions, Phys. Rev. D **65**, 115008 (2002).
[14] Z. y. Han and W. Skiba, Family unification on an orbifold, Phys. Rev. D **70**, 035013 (2004).
[15] Y. Kawamura, T. Kinami, and K. y. Oda, Orbifold family unification, Phys. Rev. D **76**, 035001 (2007).
[16] Y. Kawamura and T. Kinami, Sfermion mass relations in orbifold family unification, Prog. Theor. Phys. **119**, 285 (2008).
[17] P. H. Frampton and T. W. Kephart, Fermion mixings in SU(9) family unification, Phys. Lett. B **681**, 343 (2009).
[18] Y. Kawamura and T. Miura, Orbifold family unification in $SO(2N)$ gauge theory, Phys. Rev. D **81**, 075011 (2010).
[19] C. H. Albright, Robert P. Feger, and Thomas W. Kephart, An explicit SU(12) family and flavor unification model with natural fermion masses and mixings, Phys. Rev. D **86**, 015012 (2012).
[20] Y. Goto, Y. Kawamura, and T. Miura, Orbifold family unification on six dimensions, Phys. Rev. D **88**, 055016 (2013).
[21] Y. Goto and Y. Kawamura, Predictions of SU (9) orbifold family unification, Phys. Lett. B **752**, 252 (2016).
[22] M. Reig, J. W. F. Valle, C. A. Vaquera-Araujo, and F. Wilczek, A model of comprehensive unification, Phys. Lett. B **774**, 667 (2017).
[23] N. Yamatsu, Family unification in special grand unification, Prog. Theor. Exp. Phys. **2018**, 091B01 (2018).
[24] C. S. Lim and N. Maru, Towards a realistic grand gauge-Higgs unification, Phys. Lett. B **653**, 320 (2007).
[25] K. Kojima, K. Takenaga, and T. Yamashita, Grand gauge-Higgs unification, Phys. Rev. D **84**, 051701 (2011); Gauge symmetry breaking patterns in an SU(5) grand gauge-Higgs unification model, Phys. Rev. D **95**, 015021 (2017);

The standard model gauge symmetry from higher-rank unified groups in grand gauge-Higgs unification models, J. High Energy Phys. 06 (2017) 018; Grand gauge-Higgs unification on T^2/Z_3 via the diagonal embedding method, Phys. Rev. D **108**, 035031 (2023).

[26] Y. Hosotani and N. Yamatsu, Gauge-Higgs grand unification, Prog. Theor. Exp. Phys. **2015**, 111B01 (2015); A. Furui, Y. Hosotani, and N. Yamatsu, Toward realistic gauge-Higgs grand unification, Prog. Theor. Exp. Phys. **2016**, 093B01 (2016); Y. Hosotani and N. Yamatsu, Gauge-Higgs seesaw mechanism in 6-dimensional grand unification, Prog. Theor. Exp. Phys. **2017**, 091B01 (2017); Electroweak symmetry breaking and mass spectra in six-dimensional gauge–Higgs grand unification, Prog. Theor. Exp. Phys. **2018**, 023B05 (2018).

[27] A. Angelescu, A. Bally, S. Blasi, and F. Goertz, Minimal SU(6) gauge-Higgs grand unification, Phys. Rev. D **105**, 035026 (2022); A. Angelescu, A. Bally, F. Goertz, and S. Weber, SU(6) gauge-Higgs grand unification: Minimal viable models and flavor, J. High Energy Phys. 04 (2023) 012.

[28] C. A. Scrucca, M. Serone, and L. Silvestrini, Electroweak symmetry breaking and fermion masses from extra dimensions, Nucl. Phys. **B669**, 128 (2003).

[29] N. Maru and Y. Yatagai, Fermion mass hierarchy in grand gauge-Higgs unification, Prog. Theor. Exp. Phys. **2019**, 083B03 (2019); Improving fermion mass hierarchy in grand gauge–Higgs unification with localized gauge kinetic terms, Eur. Phys. J. C **80**, 933 (2020); N. Maru, H. Takahashi, and Y. Yatagai, Gauge coupling unification in simplified grand gauge-Higgs unification, Phys. Rev. D **106**, 055033 (2022).

[30] B. A. Dobrescu and E. Poppitz, Number of fermion generations derived from anomaly cancellation, Phys. Rev. Lett. **87**, 031801 (2001).