Instabilities of perturbations in some homogeneous color-electric and -magnetic backgrounds in SU(2) gauge theory

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We consider the instabilities of field perturbations around a homogeneous background color-electric and/or -magnetic field in SU(2) pure gauge theory. We investigate a number of distinct cases of background magnetic and electric fields, and we compute the dispersion relations in the linearized theory, identifying stable and unstable momentum modes. In the case of a net homogenous non-Abelian B field, we compute the nonlinear (quadratic and cubic) corrections to the equation of motion and quantify to what extent the instabilities are tempered by these nonlinearities.

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I. INTRODUCTION

The thermalization process of non-Abelian gauge fields has been the subject of intense study—in particular, in the context of heavy ion collisions (HICs; see Ref. [1] for a comprehensive review). In the initial stages of the collision, anisotropic particle distributions along and perpendicular to the beam line may create large anisotropic classical fields [2–6]. In this background, which is not necessarily homogeneous, perturbations may under certain conditions grow (semi)exponentially. These (plasma) instabilities may result in long-wavelength fluctuations with very large occupation numbers that may impact the equilibration process as well [1,7–9]. It turns out that fast hydrodynamization may likely be understood from kinetic theory with input from perturbation theory [10–18], but a thorough understanding of the nonperturbative dynamics as well is still of great interest.

Instabilities in anisotropic backgrounds are well known from both Abelian and non-Abelian gauge theory [19]. While a U(1) theory produces linear field equations of motion for the gauge field instabilities, which could then in principle grow very large, for QCD the equations of motion are nonlinear, and the instabilities could turn out to be short-lived and irrelevant. The Abelianization phenomenon [20] implies that there may be directions in field space where nonlinear contributions vanish, and where the instability could continue unhindered for some time. Ultimately, this will depend on the relative magnitude of the nonlinearities, the range of modes that become unstable, and the self-coupling. Substantial work has been performed analytically and numerically on plasma instabilities in QCD and QCD-like theories (see, for instance, [21–25]), both in a HIC context and for simplified models.

In the present work, we will explore the instabilities of gauge field perturbations around homogeneous and constant E and B fields in pure classical SU(2) gauge theory. For simplicity, we will ignore any fermions coupled to the gauge fields, which may feel and enhance the anisotropy. Our approach is inspired by [26,27], where the authors investigated the dispersion relations of perturbations in the linearized theory, in a variety of gauge field backgrounds. Part of the present work is an extension of their analysis, while later parts attempt to go beyond the linear approximation.

A. The equations of motion order by order

The classical equation of motion for pure SU(2) gauge theory reads

$$D_\mu^{ab} F^{\mu\nu,b} = j^{\nu,a}, \tag{1}$$

where the covariant derivative is

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g\epsilon^{abc} A_\mu^c, \tag{2}$$

and the field strength tensor is

$$F^{\mu\nu,a} = \partial^\mu A^{\nu,a} - \partial^\nu A^{\mu,a} + g\epsilon^{abc} A^{\mu,b} A^{\nu,c}. \tag{3}$$

The gauge field $A_{\mu,a}$ is a 12-component vector, corresponding to color indices $a = 1, 2, 3$ and Lorentz indices $\mu = 0, 1, 2, 3$. Because of gauge invariance, there is some redundancy in these degrees of freedom, and we will make use of this point shortly. The metric signature is taken to be $(+---)$, so that $x_0 = x^0 = t$ and $-x_i = x^i = (x, y, z)$, and $\partial^0 = \partial_0 = \partial/\partial t$, $-\partial^i = \partial_i = \partial/\partial x^i$. Then $E^{i,a} = F^{i0,a}$, while $B^{i,a} = \frac{1}{2}\epsilon^{ijk}F^{kj,a}$.

We will assume that the gauge field may be decomposed into a background field and a fluctuation, with the notation

$$A_a^\mu(x,t) = \bar{A}_a^\mu(x,t) + h_a^\mu(x,t), \tag{4}$$

and with an assumption that the fluctuations are in some sense "small," $h_a^\mu \ll \bar{A}_a^\mu$. It then makes sense to expand order by order in h_a^μ.

1. Background field

The equation of motion for the background field (expanding to zeroth order in the fluctuation h_a^μ) is simply

$$\bar{D}_\mu^{ab}\bar{F}^{\mu\nu,b} = j^{\nu,a}, \tag{5}$$

where the bars on $\bar{D}_\mu^{ab}$ and $\bar{F}^{\mu\nu,b}$ indicate that they are expressed only in terms of the background field $\bar{A}_a^\mu$. We have assumed that the external current enters in this background field equation in its entirety. In practice, we will in the following state the background field $\bar{A}_a^\mu$ and simply infer what the current j_a^ν is required to be to generate such a field. In principle, this current may be taken to arise from some distribution of color-charged particles, but we will not discuss this further.

2. Linear in fluctuations

To linear order in fluctuations, one finds

$$\partial_\mu\partial^\mu h_a^\nu + g\epsilon_{abc}(2\partial^\nu\bar{A}_\mu^b h_c^\mu + 2\partial_\mu\bar{A}_c^\nu h_b^\mu + 2\bar{A}_b^\mu\partial_\mu h_c^\nu) - g^2(2\bar{A}_\mu^c\bar{A}_a^\nu h_c^\mu - 2\bar{A}_c^\nu\bar{A}_\mu^a h_c^\mu + \bar{A}_\mu^c\bar{A}_c^\mu h_a^\nu - \bar{A}_\mu^c\bar{A}_a^\mu h_c^\nu) = 0. \tag{6}$$

We adopt the Lorenz gauge $\partial_\mu\bar{A}_a^\mu = 0$ for the background field, and we have imposed the background gauge fixing condition on the fluctuations, $\bar{D}_\mu^{ab}h^{\mu,b} = 0$, where $\bar{D}_\mu^{ab}$ again denotes the covariant derivative in terms of the background field $\bar{A}_a^\mu$ only. In a more compact notation, this may be rewritten as

$$[g^{\mu\nu}(\bar{D}_\rho\bar{D}^\rho)_{ac} + 2g\epsilon_{abc}\bar{F}_b^{\mu\nu}]h_{\nu,c} = \mathcal{M}_{ac}^{\mu\nu}[12\times 12]h_{\nu,c} = 0. \tag{7}$$

As we will see, in momentum space this turns into a $[12\times 12]$ matrix equation for h_a^μ, which has unstable modes for certain choices of $\bar{A}_a^\mu$. Some of these correspond to nonzero homogeneous color-electric and/or -magnetic fields, and we will in Sec. II consider a fairly broad subset. Given $\bar{A}_a^\mu$, one may compute the dispersion relations for these modes.

3. Quadratic in fluctuations

At early times, when h_a^μ is truly $\ll \bar{A}_a^\mu$, the linear approximation is valid, and unstable modes grow exponentially. However, nonlinearities will eventually impact the evolution. At second order in fluctuations, we have

$$g\epsilon_{abc}(2\partial_\mu h_c^\nu h_b^\mu + \partial^\nu h_\mu^b h_c^\mu) - g^2(2\bar{A}_c^\mu h_\mu^c h_a^\nu - 2\bar{A}_a^\mu h_\mu^c h_c^\nu + \bar{A}_a^\nu h_\mu^c h_c^\mu - \bar{A}_c^\nu h_\mu^c h_a^\mu), \tag{8}$$

which upon taking the expectation values for the fluctuations, and truncating at quadratic order, naturally adds to the background equation of motion (5). Below, we will be investigating correlators of the fluctuations, defined through expectation values of the form

$$C_{ab}^{\mu\nu} = \langle h_a^\mu h_b^\nu\rangle, \qquad \bar{j}_a^\nu = g\epsilon_{abc}\langle 2\partial_\mu h_c^\nu h_b^\mu + \partial^\nu h_\mu^b h_c^\mu\rangle, \tag{9}$$

in a state to be specified in Sec. IV C. Schematically, we may then write

$$\bar{D}_\mu^{ab}\bar{F}^{\mu\nu,b} - g^2\bar{M}_{ab}^{\nu\mu}[C]\bar{A}_{\mu,b} = j_a^\nu - \bar{j}_a^\nu, \tag{10}$$

where $\bar{M}$ is some matrix involving the C correlators in (9). This in effect changes both the background equation of motion and the effective current, and if j_a^ν is constant, it will eventually ruin the assumption of constant background electric/magnetic fields. We will investigate the effect of this nonlinear backreaction below.

4. Cubic in fluctuations

Finally, the equation of motion has cubic contributions in the fluctuations, which read

$$-g^2[h_a^\nu h_\mu^c h_c^\mu - h_a^\mu h_\mu^c h_c^\nu], \tag{11}$$

and which, again upon taking expectation values, naturally add to the linear equation for the modes as

$$[g^{\mu\nu}(\bar{D}_\rho\bar{D}^\rho)_{ac} + 2g\epsilon_{abc}\bar{F}_b^{\mu\nu} - g^2\hat{M}_{ac}^{\mu\nu}(C)]h_{\nu,c} = 0. \tag{12}$$

The matrix $\hat{M}$ is expressed in terms of the C correlators, providing a corrected linear equation for the fluctuations h_a^μ. This gives rise to new time-dependent dispersion relations, altering the pattern of instabilities.

B. This work

In Sec. II, we will consider a fairly general set of homogeneous, anisotropic field backgrounds. For three of these, we will in Sec. III compute the dispersion relations of the fluctuation modes explicitly, and establish for which momentum regions the fluctuations are unstable. The remaining cases are more involved, and we will simply sketch how one would go about finding dispersion relations for them in the Appendix, relying heavily on Refs. [26,27].

In Sec. IV, we choose one of the three cases and compute the eigenmodes explicitly, in order to compute the C correlators. This will in turn allow us to compute the cubic correction matrix $\hat{M}$ and the quadratic corrections $\bar{M}$ and $\bar{J}$, from which we will be able to study the evolution of the instabilities. We provide some discussion, conclusions, and suggestions for future work in Sec. V.

II. ANISOTROPIC BACKGROUNDS WITH CONSTANT, HOMOGENEOUS ELECTRIC/MAGNETIC FIELDS

A. Electric field only

We will first consider a constant, homogeneous color-electric field only, and set up our procedure and notation. First, without loss of generality, we can choose the electric field to be in the $i = 1$, $a = 1$ direction. We hence require that

$$\bar{F}^{i0}_a = E^i_a = \delta^{i1}_{a1} E. \tag{13}$$

Since

$$\bar{F}^{i0}_a = \partial^i \bar{A}^0_a - \partial^0 \bar{A}^i_a + g\epsilon_{abc} \bar{A}^i_b \bar{A}^0_c, \tag{14}$$

such an electric field may correspond to a multitude of choices for $\bar{A}^\mu_a$ (some of which may be gauge equivalent). We choose to restrict ourselves to two cases: one where only the linear-in-A ("Abelian") derivative part of $\bar{F}$ is nonzero, and one where only the quadratic-in-A ("non-Abelian") part is nonzero. A homogeneous, time-independent *Abelian* electric field may then be written (in a compact notation we will use repeatedly for the 12-component objects such as $\bar{A}^\mu_a$, j^μ_a), as

$$\bar{A}^\mu_a = \left[\begin{pmatrix} \bar{A}^0_1 \\ \bar{A}^0_2 \\ \bar{A}^0_3 \end{pmatrix} \begin{pmatrix} \bar{A}^1_1 \\ \bar{A}^1_2 \\ \bar{A}^1_3 \end{pmatrix} \begin{pmatrix} \bar{A}^2_1 \\ \bar{A}^2_2 \\ \bar{A}^2_3 \end{pmatrix} \begin{pmatrix} \bar{A}^3_1 \\ \bar{A}^3_2 \\ \bar{A}^3_3 \end{pmatrix} \right] = \left[\begin{pmatrix} -xC_1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -tC_2 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right], \tag{15}$$

so that

$$E^1_1 \equiv E = C_1 + C_2. \tag{16}$$

This corresponds to a vanishing current $j^{\nu,a} = 0$. Similarly, a homogeneous time-independent *non-Abelian* electric field may be written as

$$\bar{A}^\mu_a = \left[\begin{pmatrix} 0 \\ D_2 \\ D_1 \end{pmatrix} \begin{pmatrix} 0 \\ D_4 \\ D_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right] \tag{17}$$

to find

$$E^1_1 \equiv E = g(D_4 D_1 - D_3 D_2). \tag{18}$$

This corresponds to a current with four nonvanishing components:

$$j^\nu_a = -gE \left[\begin{pmatrix} 0 \\ D_3 \\ -D_4 \end{pmatrix} \begin{pmatrix} 0 \\ D_1 \\ -D_2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right]. \tag{19}$$

B. Magnetic field only

For a homogeneous magnetic field, we can again choose the $\mu = 1$, $a = 1$ direction, and write

$$\frac{1}{2}\epsilon^{ijk} F^{kj}_a = B^i_a = \delta^{i1}_{a1} B. \tag{20}$$

An *Abelian* background field can be parametrized as

$$\bar{A}^\mu_a = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -zC_4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} yC_3 \\ 0 \\ 0 \end{pmatrix} \right], \tag{21}$$

so that

$$B^1_1 = B = C_3 + C_4. \tag{22}$$

This again corresponds to a vanishing current $j^{\nu,a} = 0$. The *non-Abelian* realization is

$$\bar{A}^\mu_a = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ D_6 \\ D_5 \end{pmatrix} \begin{pmatrix} 0 \\ D_8 \\ D_7 \end{pmatrix} \right], \tag{23}$$

with

$$B^1_1 = B = g(D_5 D_8 - D_6 D_7). \tag{24}$$

This again corresponds to a current with four nonvanishing components:

$$j^\nu_a = -gB \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ D_7 \\ -D_8 \end{pmatrix} \begin{pmatrix} 0 \\ -D_5 \\ D_6 \end{pmatrix} \right]. \tag{25}$$

These expressions are minor generalizations to [26], where the authors consider Abelian E with $C_1 = E$, $C_2 = 0$; Abelian B with $C_3 = B$, $C_4 = 0$; non-Abelian E with $D_2 = -D_3 = \sqrt{E/g}$, $D_1 = D_4 = 0$; and non-Abelian B with $D_5 = D_8 = \sqrt{B/g}$, $D_6 = D_7 = 0$. In [27], the same authors in addition considered arbitrary D_2, D_3, and D_5,

D_8, respectively. We see that there are a host of other possibilities for choosing the C_i or the D_i for a given E or B. It will not be possible to keep full generality in the following, but we will further investigate some of the combinations in terms of the resulting dispersion relations.

C. Electric and magnetic fields, combined

The natural generalization is to consider the combination of an electric field E_1^1 together with a magnetic field that is/is not aligned in space and/or color. Allowing for all combinations of Abelian/non-Abelian $\bar{A}_a^\mu$, one might expect 16 distinct cases. Some combinations are, however, not possible, since additional components of magnetic and electric field are sourced. The task is therefore to identify vector potentials with the property that they generate only E_1^1 and then some combination of B_a^i, enforcing that they are either of Abelian or non-Abelian type, and that E and B are constant in space and time. One then finds a limited set of options, which we will briefly review in the following.

1. Abelian E and Abelian B

Choosing a vector potential of the form

$$\bar{A}_a^\mu = \left[\begin{pmatrix} -xC_1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -tC_2 - yC_{16} + zC_9 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} -zC_4 + xC_{15} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -xC_{10} + yC_3 \\ 0 \\ 0 \end{pmatrix}\right] \tag{26}$$

gives Abelian E and B fields aligned in color, with

$$E_1^1 = C_1 + C_2, \quad B_a^i = g\left[\begin{pmatrix} C_3 + C_4 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} C_9 + C_{10} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} C_{15} + C_{16} \\ 0 \\ 0 \end{pmatrix}\right]. \tag{27}$$

Explicitly setting $\bar{A}_1^0 = 0$ ($C_1 = 0$), this opens up a few more options:

$$\bar{A}_a^\mu = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -tC_2 - yC_{16} + zC_9 \\ -yC_{18} + zC_{11} \\ -yC_{20} + zC_{13} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right], \tag{28}$$

which gives Abelian E and B fields orthogonal in space:

$$E_1^1 = C_2, \quad B_a^i = g\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} C_9 \\ C_{11} \\ C_{13} \end{pmatrix} \begin{pmatrix} C_{16} \\ C_{18} \\ C_{20} \end{pmatrix}\right]. \tag{29}$$

2. Abelian E and non-Abelian B

A vector potential of the form

$$\bar{A}_a^\mu = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -tC_2 \\ D_4 \\ D_3 \end{pmatrix} \begin{pmatrix} D_{11} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} D_{12} \\ 0 \\ 0 \end{pmatrix}\right] \tag{30}$$

gives Abelian E and non-Abelian B fields orthogonal in space and color:

$$E_1^1 = C_2, \quad B_a^i = g\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ D_3 D_{12} \\ -D_4 D_{12} \end{pmatrix} \begin{pmatrix} 0 \\ -D_3 D_{11} \\ D_4 D_{11} \end{pmatrix}\right]. \tag{31}$$

3. Non-Abelian E and non-Abelian B

Finally, we may choose

$$\bar{A}_a^\mu = \left[\begin{pmatrix} 0 \\ 0 \\ D_1 \end{pmatrix} \begin{pmatrix} 0 \\ D_4 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ D_5 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ D_7 \end{pmatrix}\right], \tag{32}$$

which gives non-Abelian E and B fields orthogonal in space, but aligned in color.

$$E_1^1 = gD_1 D_4, \quad B_a^i = g\left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} D_4 D_7 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -D_4 D_5 \\ 0 \\ 0 \end{pmatrix}\right]. \tag{33}$$

The combination

$$\bar{A}_a^\mu = \left[\begin{pmatrix} 0 \\ D_2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ D_3 \end{pmatrix} \begin{pmatrix} 0 \\ D_6 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ D_8 \\ 0 \end{pmatrix}\right] \tag{34}$$

is equivalent.

In summary, we have seen that we can find

(1) Abelian E or B, or non-Abelian E or B, with a somewhat generalized parametrization of what was reported in [26].
(2) Space-aligned, color-aligned E and B: Possible as an Abelian/Abelian combination. $C_1 \neq 0$, $C_3 \neq 0$ is the special case reported in [26].
(3) Space-aligned, color-orthogonal E and B: Cannot be realized in our setup.
(4) Space-orthogonal, color-aligned E and B: Possible as Abelian/Abelian and non-Abelian/non-Abelian combinations.

(5) Space-orthogonal, color-orthogonal E and B: Possible as Abelian/Abelian and Abelian/non-Abelian combinations.

We also see that there are several computationally distinct cases, depending on whether the vector potential is itself space- and/or time-dependent. We will press on with the three purely non-Abelian cases (E only, B only, E and B), where $\bar{A}_a^\mu$ is homogeneous and constant. We refer to the Appendix for a brief discussion of the issues involved for the Abelian vector potentials.

III. DISPERSION RELATIONS IN THE LINEARIZED THEORY

A. Non-Abelian $B, E = 0$

Let us return to the case of a non-Abelian B field and zero E field. In four-dimensional momentum space, the linear equation for the modes of the fluctuations reduces to a set of 12 coupled linear algebraic equations for h_a^μ, which can then be split into three independent sections:

$$\mathcal{M}_{ac}^{\mu\nu}[12\times 12]h_\nu^c = \begin{pmatrix} M_{00} & 0 & 0 & 0 \\ 0 & M_{11} & 0 & 0 \\ 0 & 0 & M_{22} & M_{23} \\ 0 & 0 & M_{32} & M_{33} \end{pmatrix}\begin{pmatrix} h^0_{1,2,3} \\ h^1_{1,2,3} \\ h^2_{1,2,3} \\ h^3_{1,2,3} \end{pmatrix}, \tag{35}$$

where in addition $M_{00} = M_{11} = M_{22} = M_{33}$ are all Hermitian, and $M_{23} = M_{32}^\dagger$, so that the whole matrix M is Hermitian. Hence, the equations for $h^0_{1,2,3}$ (M_{00}) and $h^1_{1,2,3}$ (M_{11}) are identical:

$$\begin{pmatrix} \Box_k + D_5^2 + D_6^2 + D_7^2 + D_8^2 & 2i(D_5k_y + D_7k_z) & -2i(D_6k_y + D_8k_z) \\ -2i(D_5k_y + D_7k_z) & \Box_k + D_5^2 + D_7^2 & -(D_5D_6 + D_7D_8) \\ 2i(D_6k_y + D_8k_z) & -(D_5D_6 + D_7D_8) & \Box_k + D_6^2 + D_8^2 \end{pmatrix}\begin{pmatrix} h_1^{0,1} \\ h_2^{0,1} \\ h_3^{0,1} \end{pmatrix} = 0, \tag{36}$$

while the $h^{2,3}_{1,2,3}$ sections mix ($M_{22,23,32,33}$):

$$\begin{pmatrix} \Box_k + D_5^2 + D_6^2 + D_7^2 + D_8^2 & 2i(D_5k_y + D_7k_z) & -2i(D_6k_y + D_8k_z) \\ -2i(D_5k_y + D_7k_z) & \Box_k + D_5^2 + D_7^2 & -(D_5D_6 + D_7D_8) \\ 2i(D_6k_y + D_8k_z) & -(D_5D_6 + D_7D_8) & \Box_k + D_6^2 + D_8^2 \end{pmatrix}\begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2B \\ 0 & 2B & 0 \end{pmatrix}\begin{pmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \end{pmatrix} = 0, \tag{37}$$

and

$$\begin{pmatrix} \Box_k + D_5^2 + D_6^2 + D_7^2 + D_8^2 & 2i(D_5k_y + D_7k_z) & -2i(D_6k_y + D_8k_z) \\ -2i(D_5k_y + D_7k_z) & \Box_k + D_5^2 + D_7^2 & -(D_5D_6 + D_7D_8) \\ 2i(D_6k_y + D_8k_z) & -(D_5D_6 + D_7D_8) & \Box_k + D_6^2 + D_8^2 \end{pmatrix}\begin{pmatrix} h_1^3 \\ h_2^3 \\ h_3^3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2B \\ 0 & -2B & 0 \end{pmatrix}\begin{pmatrix} h_1^2 \\ h_2^2 \\ h_3^2 \end{pmatrix} = 0. \tag{38}$$

To somewhat mitigate notational clutter, we have absorbed g into $\bar{A}$, $gD_i \to D_i$, and defined the box operator in momentum space $\Box_k = -\omega^2 + k_x^2 + k_T^2$, with $k_T^2 = k_y^2 + k_z^2$. We note that $D_{5,6,7,8}$ not only appear in the combination $B = D_5D_8 - D_6D_7$, but also depend on D_i individually.

1. Non-Abelian B: 0 and 1 sectors

The condition on ω to satisfy the equation is that $\mathrm{Det}[M_{00}] = 0 = \mathrm{Det}[M_{11}]$, which reduces to the cubic equation

$$\Box_k^3 + 2\alpha_B\Box_k^2 + \beta_B\Box_k + \gamma_B = 0, \tag{39}$$

with

$$\alpha_B = (D_5^2 + D_6^2 + D_7^2 + D_8^2) \geq 2B, \tag{40}$$

$$\beta_B = \alpha_B^2 + B^2 - 4((D_5k_y + D_7k_z)^2 + (D_6k_y + D_8k_z)^2), \tag{41}$$

$$\gamma_B = B^2(\alpha_B - 4k_T^2). \tag{42}$$

Let us first consider the zero-momentum case $k_T = k_x = 0$, so that $\omega^2 = -\Box_k$, and

$$-(\omega^2)^3 + 2\alpha_B(\omega^2)^2 - (\alpha_B^2 + B^2)\omega^2 + B^2\alpha_B = 0. \tag{43}$$

The solutions for ω^2 depend only on the combination α_B/B:

$$\frac{\omega^2}{B} = \frac{\alpha_B}{B}, \qquad \frac{\omega^2}{B} = \frac{1}{2}\left(\frac{\alpha_B}{B} \pm \sqrt{\frac{\alpha_B^2}{B^2} - 4}\right), \tag{44}$$

and they are all real and positive (see Fig. 1, top). For instance, in the case $D_5 = D_8 = 0$, $D_6 = -D_7 = \sqrt{B}$, we find $\alpha_B/B = 2$ so that the eigenvalues reduce to $\omega^2 = (2B, B, B)$. For the more general case $D_6 = \sqrt{B}\lambda$, $-D_7 = \sqrt{B}/\lambda$ for some λ [26], $\alpha_B/B = \lambda^2 + \lambda^{-2} \geq 2$, and the solutions are again real and positive (see Fig. 1, bottom). As a sanity check, we note that when $B = 0$, the eigenvalues are $\omega^2 = (0, \alpha_B, \alpha_B)$. These are real and positive, and they are only zero when all the D_i's are zero.

For nonzero momentum k_T, k_x, the explicit realization of $D_{5,\ldots,8}$ becomes more important. Again, for the case $D_6 = -D_7 = \sqrt{B}$, one finds ($\alpha_B/B = 2$, $\beta_B/B^2 = 5 - 4k_T^2/B$, $\gamma_B/B^3 = 2 - 4k_T^2/B$)

$$\left(-\frac{\omega^2}{B} + \frac{k_T^2}{B} + \frac{k_x^2}{B}\right)^3 + 4\left(-\frac{\omega^2}{B} + \frac{k_T^2}{B} + \frac{k_x^2}{B}\right)^2 + \left(5 - 4\frac{k_T^2}{B}\right)\left(-\frac{\omega^2}{B} + \frac{k_T^2}{B} + \frac{k_x^2}{B}\right) + 2 - 4\frac{k_T^2}{B} = 0, \tag{45}$$

which has the solutions (see Fig. 2)

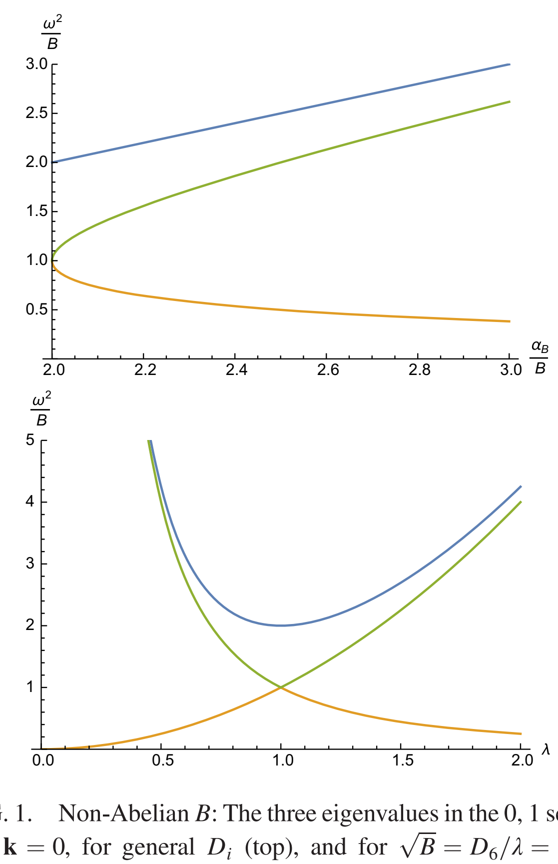

FIG. 1. Non-Abelian B: The three eigenvalues in the 0, 1 sectors for $\mathbf{k} = 0$, for general D_i (top), and for $\sqrt{B} = D_6/\lambda = -D_7\lambda$ (bottom).

$$\frac{\omega^2}{B} = 1 + \frac{k_x^2}{B} + \frac{k_T^2}{B}, \quad \frac{\omega^2}{B} = \frac{3}{2} + \frac{k_x^2}{B} + \frac{k_T^2}{B} \pm \frac{1}{2}\sqrt{1 + 16\frac{k_T^2}{B}}. \tag{46}$$

These are real and positive for all B, k_x, k_T, and hence the 0 and 1 components of the fluctuations are stable for this choice of $D_{5,\ldots,8}$.

For fixed B, a three-dimensional space of realizations of $D_{5,\ldots,8}$ is available, which we will not fully explore here. We will, however, briefly consider again the case $D_6 = -D_7 = \sqrt{B}$, fixing $D_8 = 0$, which allows us to choose any value for D_5 without changing B. We choose $D_5 = \sqrt{qB}$ and vary q. In Fig. 3, we show the eigenvalues for $q = 1$ and their dependence on $k_{x,y,z}$ (respectively, keeping the other two constant). We note how the $k_T = \sqrt{k_y^2 + k_z^2}$ dependence in Fig. 2 is resolved into different dependences on k_y and k_z. We can further examine the q dependence of the lowest eigenvalue as a function of k_y and k_z, shown in Fig. 4. We see a nontrivial dependence, but the eigenvalue remains real and positive.

2. Non-Abelian *B*: 2 and 3 sectors

For the 2 and 3 components, the requirement $\mathrm{Det}[M_{22,23,32,33}] = 0$ amounts to [compare to (39)]

$$[\Box_k^3 + 2\alpha_B\Box_k^2 + (\beta_B - 4B^2)\Box_k + \gamma_B - 4B^2\alpha_B.]^2 = 0. \tag{47}$$

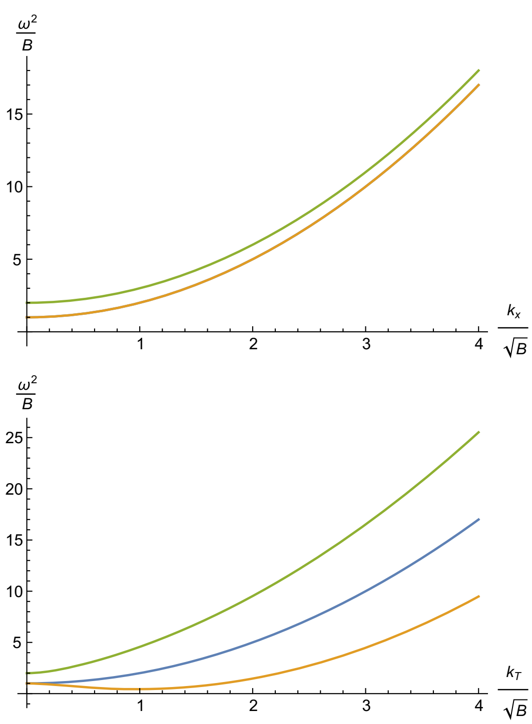

FIG. 2. Non-Abelian B: The three eigenvalues in the 0, 1 sectors for $\sqrt{B} = D_6 = -D_7$, with $k_T = 0$ (top) and $k_x = 0$ (bottom).

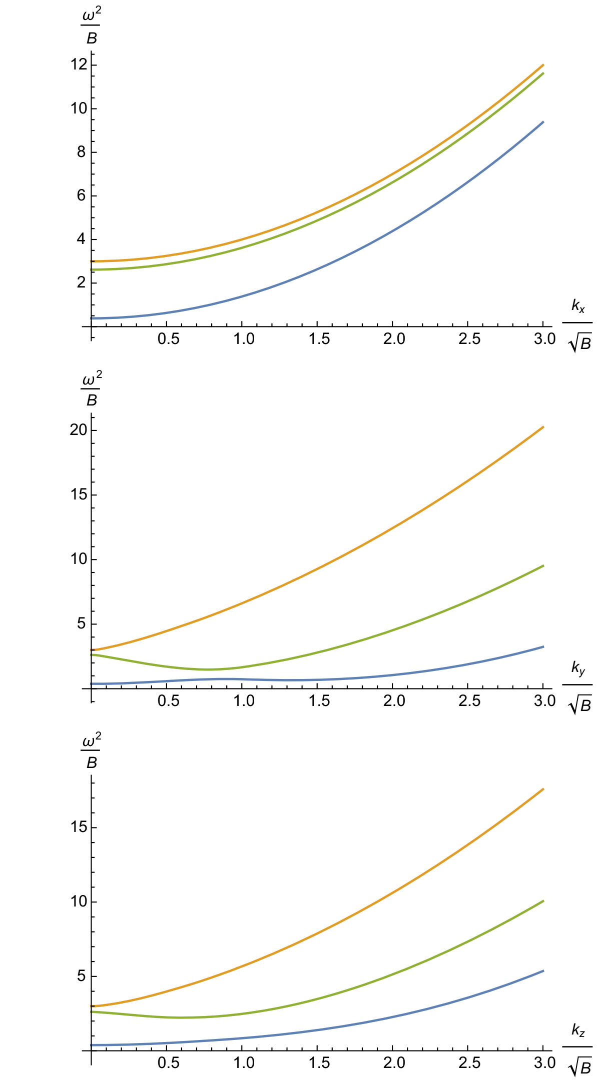

FIG. 3. Non-Abelian B: The three eigenvalues in the 0, 1 sectors for $\sqrt{B} = D_6 = -D_7$, with nonzero $D_5 = \sqrt{qB}$, $q = 1$; for nonzero k_x (top), k_y (middle), and k_z (bottom).

We may focus on the expression inside the square bracket and infer that any solution is a double solution. For $k_T, k_x = 0$, this becomes

$$-(\omega^2)^3 + 2\alpha_B(\omega^2)^2 - (\alpha_B^2 - 3B^2)\omega^2 - 3B^2\alpha_B = 0, \quad (48)$$

which has the solutions

$$\frac{\omega^2}{B} = \frac{\alpha_B}{B}, \qquad \omega^2 = \frac{1}{2}\left(\frac{\alpha_B}{B} \pm \sqrt{\frac{\alpha_B^2}{B^2} + 12}\right). \quad (49)$$

One of these eigenvalues is negative for all α_B/B (see Fig. 5, top). In particular, for $D_5 = D_8 = 0$, $D_6 = -D_7 = \sqrt{B}$, we find $\omega^2 = (3B, 2B, -B)$, while the parametrization $D_6 = \sqrt{B}\lambda$, $-D_7 = \sqrt{B}/\lambda$ allows the lowest ω^2/B to take values in the interval $[-1, 0]$ (see Fig. 5, bottom). Again, when $B = 0$, $\omega^2 = (0, \alpha_B, \alpha_B)$.

For nonzero momentum, we consider again $D_6 = -D_7 = \sqrt{B}$, for which we need to solve

$$\left(-\frac{\omega^2}{B} + \frac{k_T^2}{B} + \frac{k_x^2}{B}\right)^3 + 4\left(-\frac{\omega^2}{B} + \frac{k_T^2}{B} + \frac{k_x^2}{B}\right)^2 + \left(1 - 4\frac{k_T^2}{B}\right)\left(-\frac{\omega^2}{B} + \frac{k_T^2}{B} + \frac{k_x^2}{B}\right) - 6 - 4\frac{k_T^2}{B} = 0. \quad (50)$$

This is a general cubic equation, which may be solved straightforwardly, to reveal (Fig. 6) that the (double) unstable mode becomes stable as k_x and/or k_T increase. Explicitly, for $k_T = 0$, the unstable region is $|k_x|/\sqrt{B} < 1$, while for $k_x = 0$, it is $k_T/\sqrt{B} < \sqrt{(3-\sqrt{8})^{1/3} + (3+\sqrt{8})^{1/3}} \simeq 1.53$.

We can again extend our analysis somewhat, by allowing $D_5 = \sqrt{qB}$ while keeping $D_8 = 0$. In Fig. 7, we show the negative eigenvalue and its momentum dependence (nonzero k_x, k_y, or k_z) for different values of $q = 0, 1, 2, 3, 4$. We see that larger n reduces the region of instability, and we find that for $n \to \infty$, $\omega^2(k = 0) \to 0$.

Hence, in the presence of a constant, homogeneous non-Abelian magnetic field, for this choice of $D_{5,\ldots,8}$, of the 12

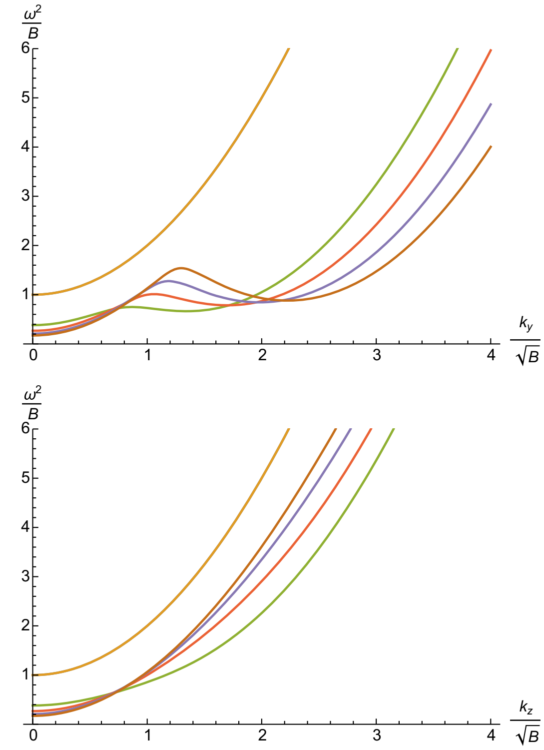

FIG. 4. Non-Abelian B: The lowest eigenvalues in the 0, 1 sectors for $\sqrt{B} = D_6 = -D_7$, with nonzero $D_5 = \sqrt{qB}$, $q = 0, 1, 2, 3, 4$; For nonzero k_y (top) and k_z (bottom).

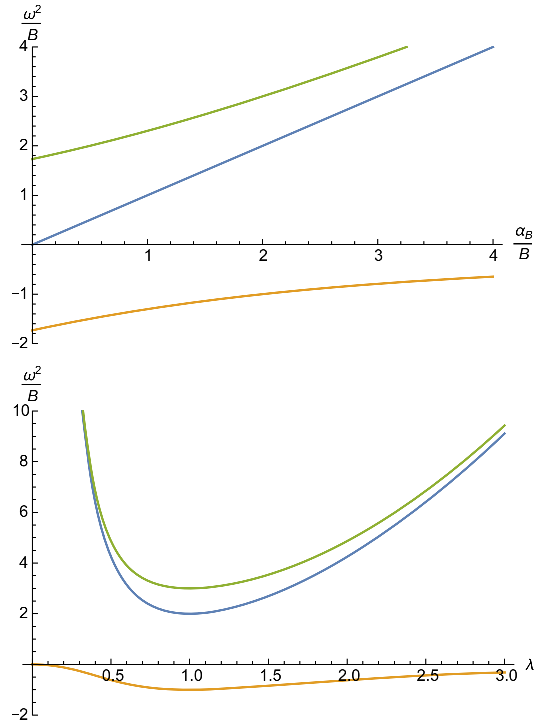

FIG. 5. Non-Abelian B: The three double eigenvalues in the 2, 3 sectors; for $\mathbf{k}=0$, general D_i (top), and for $\sqrt{B}=D_6/\lambda=-D_7\lambda$ (bottom).

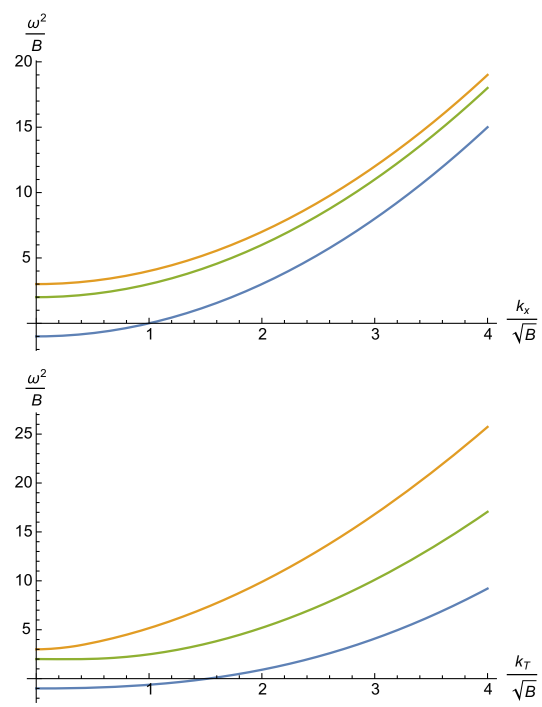

FIG. 6. Non-Abelian B: The three eigenvalues in the 2, 3 sectors for $\sqrt{B}=D_6=-D_7$, with $k_T=0$ (top) and $k_x=0$ (bottom).

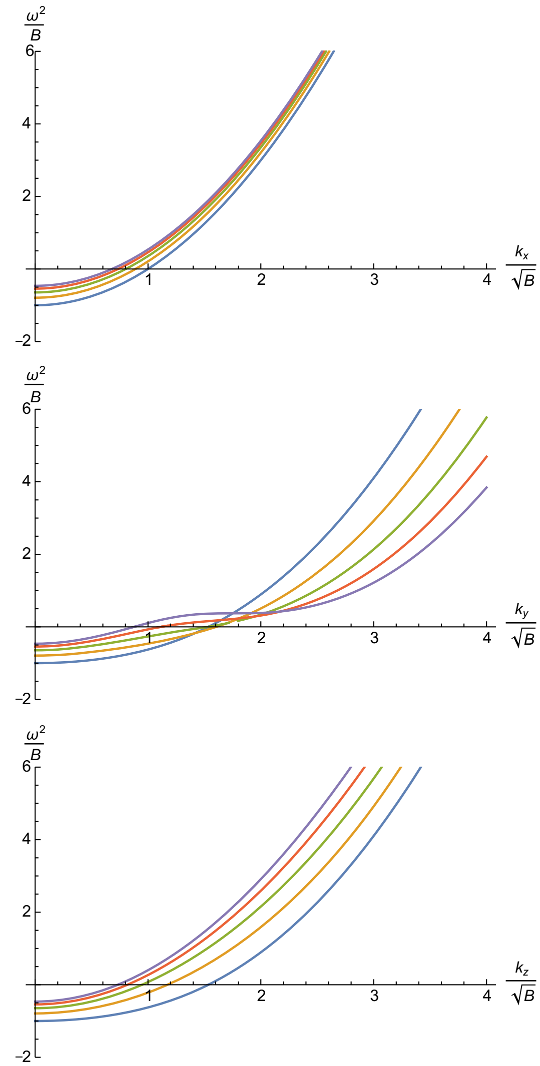

FIG. 7. Non-Abelian B: The lowest (negative) eigenvalue in the 2, 3 sectors for $\sqrt{B}=D_6=-D_7$, with nonzero $D_5=\sqrt{qB}$, $q=0,1,2,3,4$; for nonzero k_x (top), k_y (middle), and k_z (bottom).

sets of momentum modes present, two are potentially unstable, growing as

$$\propto \exp(\pm|\omega|t), \tag{51}$$

with an ω given by the two lowest, degenerate solutions to (47) [exemplified by the $-$ solution in the special case of (49)]. Furthermore, the region of instability is centered around the origin ($\mathbf{k}=0$), and for large enough $\mathbf{k}$, these modes are again stable. Hence, only a finite IR region of momentum modes are unstable, and this region is dependent not only on B, but also on the concrete choice of $D_{5,\ldots,8}$, exemplified here by varying D_5.

In Sec. IV, we will further find the eigenmodes of the system for non-Abelian B only, with $\sqrt{B} = D_6 = -D_7$, $D_5 = D_8 = 0$, and we will compute the quadratic and cubic corrections to the evolution equations and solve them including these corrections.

B. Non-Abelian $E, B = 0$

For a non-Abelian background E field and zero B field, the linear equation for the Fourier modes again splits into three independent sections. The sections for $h^2_{1,2,3}$ (M_2) and $h^3_{1,2,3}$ (M_3) are now identical and decoupled from each other, and they read (again absorbing g as $gD_i \to D_i$)

$$\begin{pmatrix} \Box_k - D_1^2 - D_2^2 + D_3^2 + D_4^2 & -2i(D_1\omega - D_3k_x) & 2i(D_2\omega - D_4k_x) \\ +2i(D_1\omega - D_3k_x) & \Box_k - D_1^2 + D_3^2 & D_1D_2 - D_3D_4 \\ -2i(D_2\omega - D_4k_x) & D_1D_2 - D_3D_4 & \Box_k - D_2^2 + D_4^2 \end{pmatrix} \begin{pmatrix} h_1^{2,3} \\ h_2^{2,3} \\ h_3^{2,3} \end{pmatrix} = 0, \tag{52}$$

while the $h^{0,1}_{1,2,3}$-sections mix ($M_{00}, M_{01}, M_{10}, M_{11}$)

$$\begin{pmatrix} \Box_k - D_1^2 - D_2^2 + D_3^2 + D_4^2 & -2i(D_1\omega - D_3k_x) & 2i(D_2\omega - D_4k_x) \\ 2i(D_1\omega - D_3k_x) & \Box_k - D_1^2 + D_3^2 & D_1D_2 - D_3D_4 \\ -2i(D_2\omega - D_4k_x) & D_1D_2 - D_3D_4 & \Box_k - D_2^2 + D_4^2 \end{pmatrix} \begin{pmatrix} h_1^0 \\ h_2^0 \\ h_3^0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2E \\ 0 & 2E & 0 \end{pmatrix} \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \end{pmatrix} = 0 \tag{53}$$

and

$$\begin{pmatrix} \Box_k - D_1^2 - D_2^2 + D_3^2 + D_4^2 & -2i(D_1\omega - D_3k_x) & 2i(D_2\omega - D_4k_x) \\ 2i(D_1\omega - D_3k_x) & \Box_k - D_1^2 + D_3^2 & D_1D_2 - D_3D_4 \\ -2i(D_2\omega - D_4k_x) & D_1D_2 - D_3D_4 & \Box_k - D_2^2 + D_4^2 \end{pmatrix} \begin{pmatrix} h_1^1 \\ h_2^1 \\ h_3^1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2E \\ 0 & 2E & 0 \end{pmatrix} \begin{pmatrix} h_1^0 \\ h_2^0 \\ h_3^0 \end{pmatrix} = 0. \tag{54}$$

1. Non-Abelian E: 2 and 3 sectors

The condition on ω is now that $\mathrm{Det}[M_{22}] = 0 = \mathrm{Det}[M_{33}]$, which is again a cubic equation of the form

$$\Box_k^3 + 2\alpha_-\Box_k^2 + \beta_E\Box_k + \gamma_E = 0, \tag{55}$$

where still $\Box_k = -\omega^2 + k_x^2 + k_T^2$, $k_T^2 = k_y^2 + k_z^2$, and

$$\alpha_- = -D_1^2 - D_2^2 + D_3^2 + D_4^2, \tag{56}$$

$$\alpha_+ = D_1^2 + D_2^2 + D_3^2 + D_4^2, \tag{57}$$

$$\beta_E = \alpha_-^2 - E^2 - 4((D_1\omega - D_3k_x)^2 + (D_2\omega - D_4k_x)^2), \tag{58}$$

$$\gamma_E = -E^2(\alpha_- - 4(k_x^2 - \omega^2)), \tag{59}$$

keeping in mind that $E = D_1D_4 - D_2D_3$. For $k_T = k_x = 0$, we obtain

$$-(\omega^2)^3 + 2\alpha_+(\omega^2)^2 - (\alpha_-^2 + 3E^2)\omega^2 - E^2\alpha_- = 0. \tag{60}$$

There is now one real solution for ω^2, while the other two solutions are in general nonreal. Hence, already in the 2 and 3 sectors, there are unstable modes in the presence of finite E. However, for the special case $D_1 = D_4 = \sqrt{E}$, $D_2 = D_3 = 0$, the eigenvalues are $\omega^2 = (0, E, 3E)$, while for the generalization $D_1 = \sqrt{E}\lambda$, $D_4 = \sqrt{E}/\lambda$, one of the eigenvalues is negative in the interval $\lambda \in [0, 1]$, and the other two are real and positive (see Fig. 8). It appears that the curve traced out by $D_1D_4 = E$ in the space of $D_{1,\ldots,4}$ is quite special, avoiding the regions where the eigenvalues are nonreal. In Fig. 9, we show the α_+/α_- plane and the contours of nonzero imaginary values of the eigenvalues. Overlaid in red, the curve traced out at λ is varied, which precisely avoids those regions. Overlaid in green, the

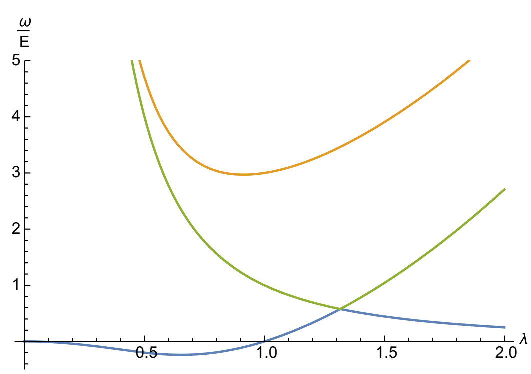

FIG. 8. Non-Abelian E: The three eigenvalues in the 2, 3 sectors for $k_T = k_x = 0$, $\sqrt{E} = D_1/\lambda = D_4\lambda$.

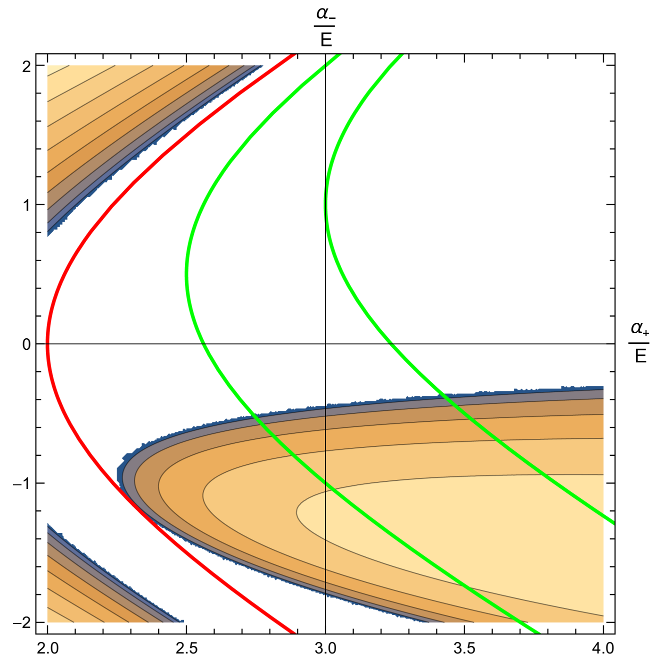

FIG. 9. Non-Abelian E: Regions of the nonzero imaginary part of the eigenvalues in α_+/α_- space. Overlaid are the curve $\sqrt{E} = D_1/\lambda = D_4\lambda$ (red), and curves when further adding nonzero $D_3 = \sqrt{qB}$, $q = 0.5, 1$ (green). In this and later contour plots, the change in colors denotes equisurfaces, but the details are not important here.

curves are traced out by adding nonzero $D_3 = \sqrt{qE}$, $q = 0.5, 1$. As another sanity check, we note that for $E = 0$, $\omega^2 = (0, \alpha_+ \pm \sqrt{\alpha_+^2 - \alpha_-^2})$, which is real and positive, but it is nonzero whenever the D_i's are.

For nonzero momentum, we immediately specialize to $D_1 = D_4 = \sqrt{E}$, $D_2 = D_3 = 0$, so that we need to solve

$$\left(-\frac{\omega^2}{E}+\frac{k_T^2}{E}+\frac{k_x^2}{E}\right)\left[\left(-\frac{\omega^2}{E}+\frac{k_T^2}{E}+\frac{k_x^2}{E}\right)^2-4\left(\frac{\omega^2}{E}+\frac{k_x^2}{E}\right)-1\right] -4\left(\frac{\omega^2}{E}-\frac{k_x^2}{E}\right)=0. \quad (61)$$

The solutions are real, and one of them is negative for some regions of (k_x, k_T^2), as shown in Fig. 10.

2. Non-Abelian E: 0 and 1 sectors

We compute $\mathrm{Det}[M_{00,01,10,11}] = 0$ to find [compare (55)]

$$[\Box_k^3 + 2\alpha_-\Box_k^2 + (\beta_E + 4E^2)\Box_k + \gamma_e + 4E^2\alpha_-]^2 = 0, \quad (62)$$

and so we may consider the expression inside the square, and treat each solution as a double root. We again first consider the case $k_x = k_T = 0$, for which we have

$$-(\omega^2)^3 + 2\alpha_+(\omega^2)^2 - (\alpha_-^2 + 7E^2)\omega^2 + 3E^2\alpha_- = 0. \quad (63)$$

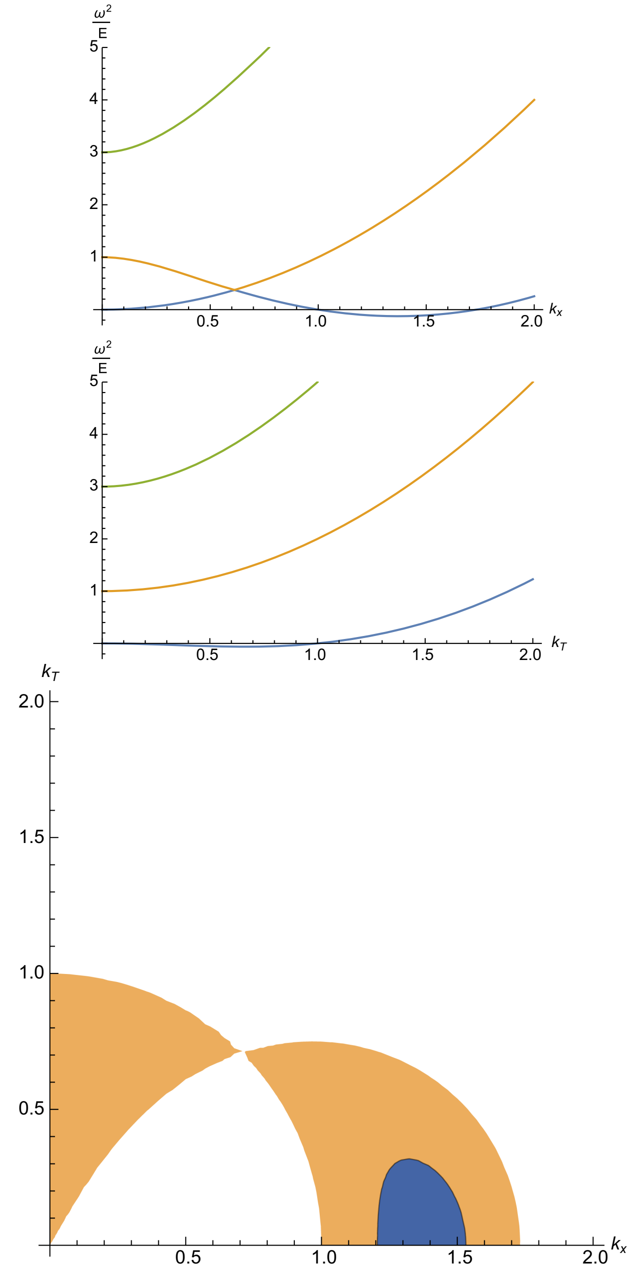

FIG. 10. Non-Abelian E: The three eigenvalues in the 2, 3 sectors for $\sqrt{E} = D_1 = D_4$, as a function of k_x (top) and k_T (middle), and the region in the k_x/k_T plane, where the lowest eigenvalue is negative (bottom).

Again, we have one real solution and two solutions that are nonreal in part of the parameter space, shown in Fig. 11 as a function of α_+ and α_-. Overlaid is the now-familiar curve parametrized by $D_1 = \sqrt{E}\lambda$, $D_4 = \sqrt{E}/\lambda$, $D_2 = D_3 = 0$ which lies fully in the region of nonreal eigenvalues. For $\lambda = 1$, the eigenvalues are $\omega^2 = (0, (2 \pm i\sqrt{3})E)$. The real part of two eigenvalues is always positive, while the third has a negative real part whenever α_- is negative.

For nonzero momentum and $D_1 = D_4 = \sqrt{E}$, we need to solve

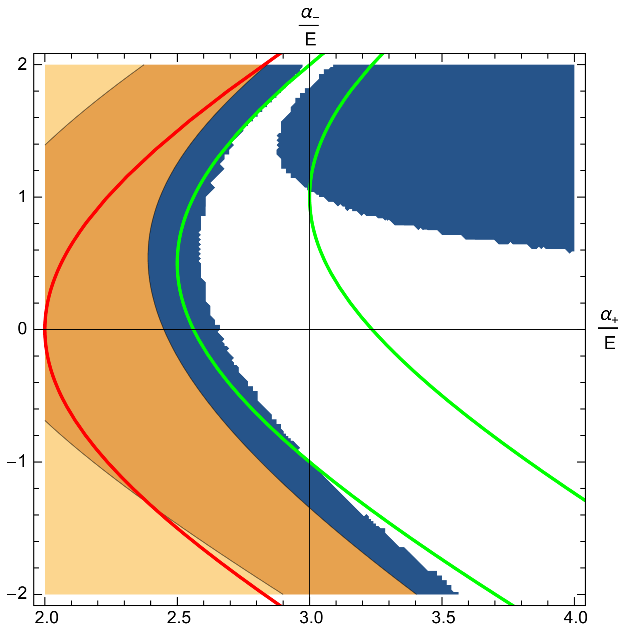

FIG. 11. Non-Abelian E: Regions of the nonzero imaginary part of the eigenvalues in the 0, 1 sectors in α_+/α_- space (shaded). Overlaid is the curve $\sqrt{E} = D_1/\lambda = D_4\lambda$ (red), as well as curves that result when further adding nonzero D_3 (green).

$$\left(-\frac{\omega^2}{E}+\frac{k_T^2}{E}+\frac{k_x^2}{E}\right)\left[\left(-\frac{\omega^2}{E}+\frac{k_T^2}{E}+\frac{k_x^2}{E}\right)^2-4\left(\frac{\omega^2}{E}+\frac{k_x^2}{E}\right)+3\right] -4\left(\frac{\omega^2}{E}-\frac{k_x^2}{E}\right)=0, \tag{64}$$

which provides one real solution and two solutions that are in general nonreal. In Fig. 12, we show the real and imaginary parts of these eigenvalues as functions of k_x (top) and k_T (bottom), respectively. We see that the eigenvalues are real and positive everywhere, except for a finite region near the origin (an ellipse in the $k_x - k_T$ plane).

And so, for the non-Abelian electric field only, of the set of 12 momentum modes present, several are potentially unstable, depending on the precise choice of $D_{1,\ldots,4}$. For the special case of $D_1 = D_4 = \sqrt{E}$, the 0, 1 sectors have a finite-momentum region with two unstable modes near the origin with nonreal eigenvalues, while in the 2, 3 sectors, there are two unstable modes in certain momentum regions with real, negative eigenvalues (imaginary ω).

C. Non-Abelian E and B

Including both an electric and a magnetic field allows us to specify two directions in space, and hence completely break isotropy. We recall the non-Abelian/non-Abelian configuration

$$\bar{A}_3^0 = D_1, \qquad \bar{A}_2^1 = D_4, \qquad \bar{A}_3^3 = D_7, \tag{65}$$

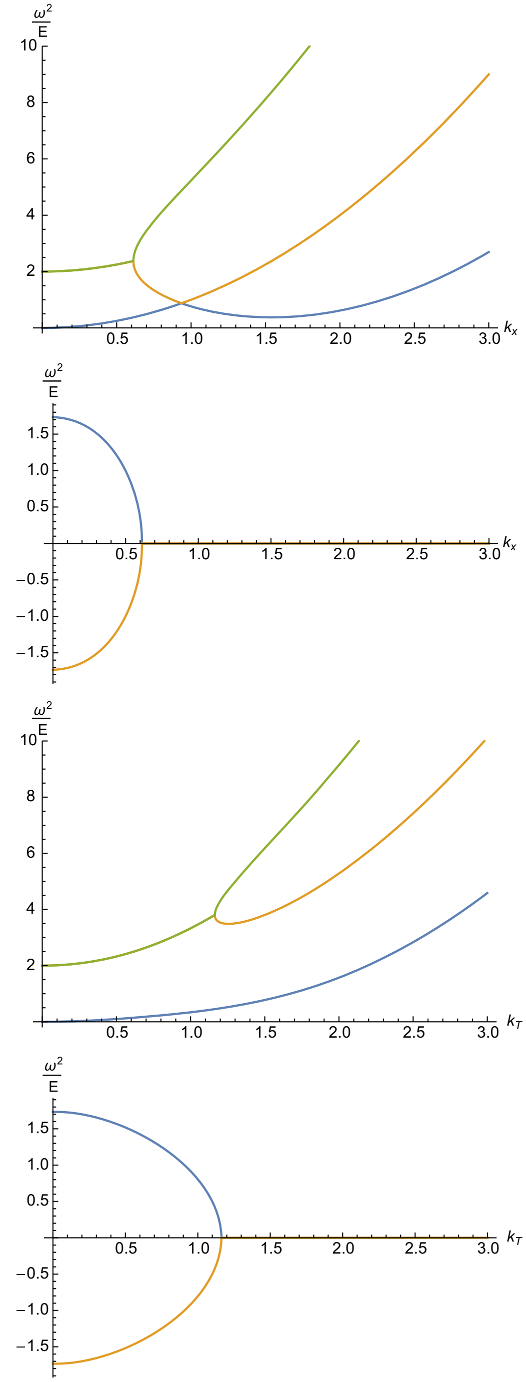

FIG. 12. Non-Abelian E: Real and imaginary parts for the three (double) eigenvalues in the 0, 1 sectors, as functions of k_x (top two) and k_T (bottom two), for $k_{T,x} = 0$, respectively.

which gives

$$E_1^1 = gD_1D_4, \qquad B_1^2 = gD_4D_7. \tag{66}$$

The linear equation in momentum space is again a 12-by-12 matrix, but now the sectors are coupled. Introducing a shorthand, where we absorb g as before and define

$$
M_{\rm diag} = \begin{pmatrix} \Box_k - D_1^2 + D_4^2 + D_7^2 & 2i(D_7 k_z - D_1\omega) & -2iD_4 k_x \\ -2i(D_7 k_z - D_1\omega) & \Box_k - D_1^2 + D_7^2 & 0 \\ 2iD_4 k_x & 0 & \Box_k + D_4^2 \end{pmatrix} \tag{67}
$$

and

$$
M_E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2E \\ 0 & 2E & 0 \end{pmatrix}, \qquad M_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2B \\ 0 & -2B & 0 \end{pmatrix}, \tag{68}
$$

we have schematically

$$
\begin{pmatrix} M_{\rm diag} & M_E & 0 & 0 \\ -M_E^T & M_{\rm diag} & 0 & M_B \\ 0 & 0 & M_{\rm diag} & 0 \\ 0 & M_B^T & 0 & M_{\rm diag} \end{pmatrix} \begin{pmatrix} h^0_{1,2,3} \\ h^1_{1,2,3} \\ h^2_{1,2,3} \\ h^3_{1,2,3} \end{pmatrix} = 0. \tag{69}
$$

This couples the 0, 1, and 3 sectors, while the 2 sector is decoupled. Note that this is not simply combining the non-Abelian E and the non-Abelian B that we considered above.

We write $D_1 = E/D_4$ and $D_7 = B/D_4$, in which case the determinant of the entire matrix may be conveniently written as

$$
\begin{aligned} &\frac{1}{D_4^{16}}[D_4^4\Box_k^3 + 2D_4^2\alpha_3\Box_k^2 + \beta_3\Box_k + \gamma_3]^2 \\ &\quad \times [D_4^4\Box_k^3 + 2D_4^2\alpha_3\Box_k^2 + (\beta_3 - 4D_4^4(\alpha_3 - D_4^4))\Box_k \\ &\quad + \gamma_3 - 4D_4^2\alpha_3(\alpha_3 - D_4^4)]^2 = 0, \end{aligned} \tag{70}
$$

with

$$
\alpha_3 = B^2 - E^2 + D_4^4, \tag{71}
$$

$$
\beta_3 = \alpha_3(\alpha_3 + D_4^4) - D_4^8 - 4D_4^6 k_x^2 - 4D_4^2(Bk_z - E\omega)^2, \tag{72}
$$

$$
\begin{aligned} \gamma_3 = D_4^2[&\alpha_3(\alpha_3 - D_4^4) - 4D_4^2(\alpha_3 - D_4^4)k_x^2 \\ &- 4D_4^2(Bk_z - E\omega)^2], \end{aligned} \tag{73}
$$

which reduces to (56) in the limit $B = 0$ ($D_7 = 0$). We see that the 12 roots split up into six double roots, with three for each term.

We can again specialize to zero momentum, and we will rescale all quantities with the appropriate power of D_4 (ω/D_4, B/D_4^2, E/D_4^2, α_3/D_4^4, and so on, which amounts to setting $D_4 = 1$) to find the requirements

$$
\begin{aligned} &-(\omega^2)^3 + 2(B^2 + E^2 + 1)(\omega^2)^2 \\ &\quad - ((B^2 - E^2)^2 + 3B^2 + E^2 + 1)\omega^2 \\ &\quad + (B^2 - E^2)(B^2 - E^2 + 1) = 0 \end{aligned} \tag{74}
$$

and

$$
\begin{aligned} &-(\omega^2)^3 + 2(B^2 + E^2 + 1)(\omega^2)^2 \\ &\quad - ((B^2 - E^2)^2 - B^2 + 5E^2 + 1)\omega^2 \\ &\quad - 3(B^2 - E^2)(B^2 - E^2 + 1) = 0. \end{aligned} \tag{75}
$$

These provide $3+3$ distinct (double) eigenvalues for ω^2/D_4^2. The first three can be found straightforwardly:

$$
\omega^2 = 1, \tag{76}
$$

$$
\omega^2 = \frac{1}{2}\left(1 + 2B^2 + 2E^2 \pm \sqrt{1 + 8E^2 + 16B^2E^2}\right).
$$

These are all real, and one is negative in a region of the $E-B$ plane, as shown in Fig. 13. The remaining three eigenvalues are in general complex, and Fig. 14 shows the regions in the $E-B$ plane, where the real part is negative (top) or the imaginary part is non-zero (bottom).

The solutions for nonzero momentum now involve k_x, k_y, and k_z independently, in addition to E and B, and it is perhaps not so helpful attempting to display the eigenvalues in all generality. But, for instance, for $k_y \neq 0$, we simply get

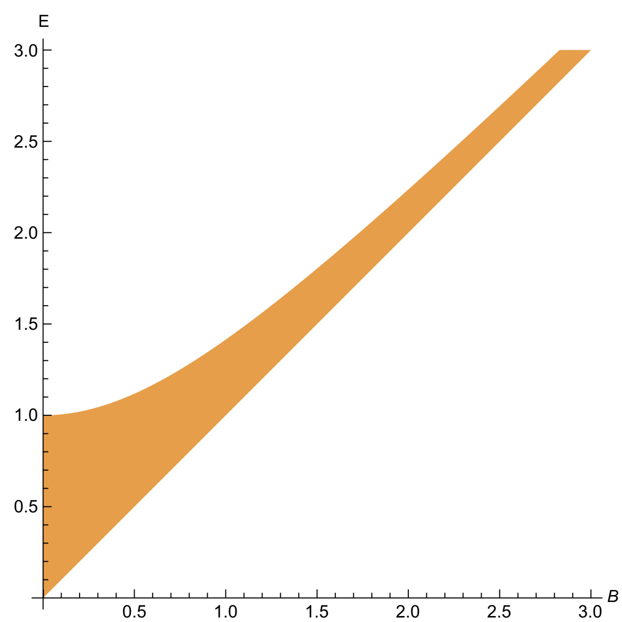

FIG. 13. Non-Abelian E and B: The region in the $E-B$ plane where one of the first three (double) eigenvalues is negative.

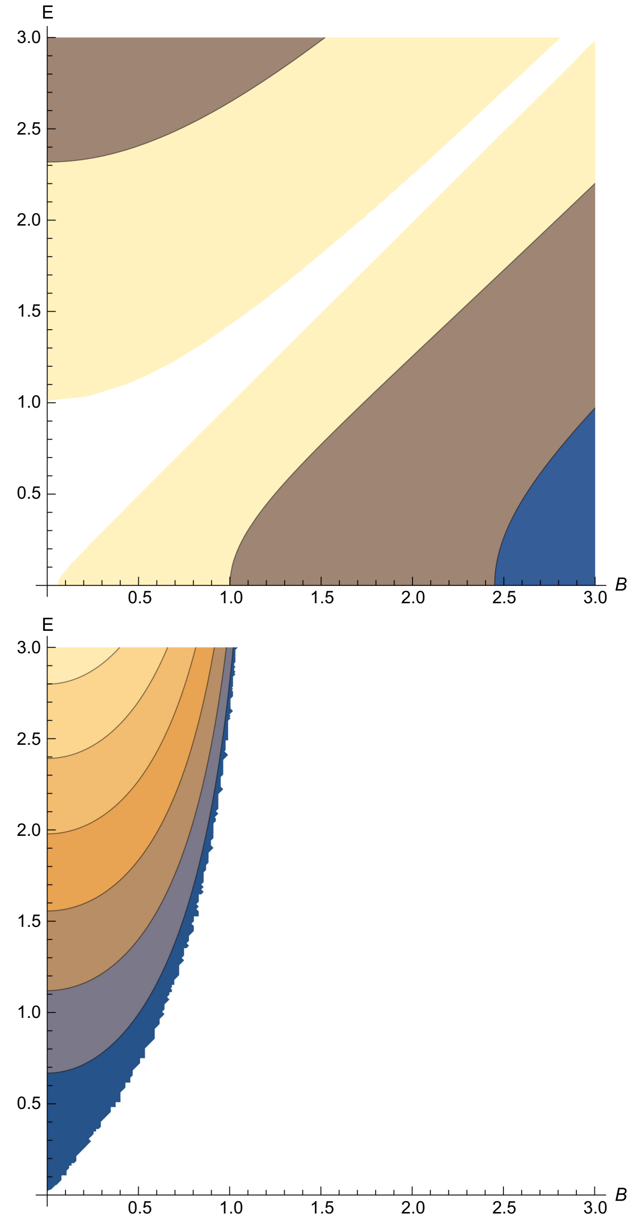

FIG. 14. Non-Abelian E and B: Regions in the $E-B$ plane, where one or more of the second set of three (double) eigenvalues has a negative real part (top) or nonzero imaginary part (bottom).

the same relations, but with $\omega^2 \to \omega^2 - k_y^2$. Hence, all solutions for ω^2 are shifted by $+k_y^2$, so that unstable but real ω solutions become stable for large k_y. For $k_x \neq 0$, a similar shift appears, but there are additional terms involving k_x^2 as well in both equations:

$$+4k_x^2(\omega^2 - k_x^2 - B^2 + E^2). \tag{77}$$

Something new happens for $k_z \neq 0$, where in addition to the shift $\omega^2 \to \omega^2 - k_z^2$, we pick up terms linear in ω, k_z, and proportional to BE,

$$+4(B^2k_z^2 - 2BEk_z\omega)(\omega^2 - k_z^2 - 1), \tag{78}$$

so that the (relative) signs of E, B, k_z matter for determining ω.

We will not pursue this further, but simply note that there is much fun to be had in this large parameter space.

IV. COMPUTING THE QUADRATIC AND CUBIC CONTRIBUTIONS FOR NON-ABELIAN B

A. Setting up the matrix structures: Mode equation

Having completed our survey of realizations of instabilities, we now return to Eq. (12), which implies that the equation of motion for the fluctuations h_a^μ including self-interaction at leading order takes the form

$$[g^{\mu\nu}(\bar{D}_\rho\bar{D}^\rho)_{ac} + 2g\epsilon_{abc}\bar{F}_b^{\mu\nu}]h_{\nu,c} - g^2\hat{M}_{ab}^{\mu\nu}(C)h_{\nu,b} = 0, \tag{79}$$

where the matrix $\hat{M}_{ab}^{\mu\nu}$ is expressed in terms of the expectation values of the fluctuation fields themselves:

$$C_{ab}^{\mu\nu} = \langle h_a^\mu(\mathbf{x},t)h_b^\nu(\mathbf{x},t)\rangle = \langle h_a^\mu(\mathbf{0},t)h_b^\nu(\mathbf{0},t)\rangle \tag{80}$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\langle h_a^\mu(\mathbf{k},t)(h_b^\nu(\mathbf{k},t))^\dagger\rangle, \tag{81}$$

where we have assumed evaluation in a homogeneous state. For the case of a non-Abelian B field, the matrix structure is as follows (in the 2 and 3 sectors):

$$\hat{M}^{ij}ab = \begin{bmatrix} -\hat{M}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\hat{M}_{22} & -\hat{M}_{23} & 0 & 0 & -\hat{M}_{26} \\ 0 & -\hat{M}_{32} & -\hat{M}_{33} & 0 & -\hat{M}_{35} & 0 \\ 0 & 0 & 0 & -\hat{M}_{44} & 0 & 0 \\ 0 & 0 & -\hat{M}_{53} & 0 & -\hat{M}_{55} & -\hat{M}_{56} \\ 0 & -\hat{M}_{62} & 0 & 0 & -\hat{M}_{65} & -\hat{M}_{66} \end{bmatrix}, \tag{82}$$

where

$$\hat{M}_{11} = C_{22}^{33} + C_{33}^{33}, \qquad \hat{M}_{22} = C_{33}^{33} + C_{11}^{33}, \tag{83}$$

$$\hat{M}_{33} = C_{11}^{33} + C_{22}^{33}, \qquad \hat{M}_{44} = C_{22}^{22} + C_{33}^{22}, \tag{84}$$

$$\hat{M}_{55} = C_{33}^{22} + C_{11}^{22}, \qquad \hat{M}_{66} = C_{11}^{22} + C_{22}^{22}, \tag{85}$$

and

$$\hat{M}_{23} = \hat{M}_{32} = -C_{32}^{33} = -C_{23}^{33}, \tag{86}$$

$$\hat{M}_{56} = \hat{M}_{65} = -C_{32}^{22} = -C_{23}^{22}, \tag{87}$$

$$\hat{M}_{26} = \hat{M}_{62} = 2C_{23}^{23} - C_{32}^{23}, \tag{88}$$

$$\hat{M}_{35} = \hat{M}_{53} = 2C^{23}_{32} - C^{23}_{23}. \quad (89)$$

We have at this point already implemented that several correlators vanish identically, and we have for simplicity by hand set to zero all correlators involving the modes in the 0 and 1 sectors, since they are stable. The 0 and 1 sectors of the matrix have the same structure as the 2 and 3 sectors, but without the sector-mixing components $\hat{M}_{26,62,35,53}$. In our particular setup, we in addition find that $\hat{M}_{11} = \hat{M}_{44}$, $\hat{M}_{22} = \hat{M}_{33} = \hat{M}_{55} = \hat{M}_{66}$, $\hat{M}_{23} = \hat{M}_{32} = \hat{M}_{56} = \hat{M}_{65}$, and $\hat{M}_{35} = \hat{M}_{53} = -\hat{M}_{26} = -\hat{M}_{62}$, as there are only four distinct correlators to take into account: $C^{22}_{11}, C^{22}_{22}, C^{23}_{23}, C^{22}_{23}$.

Our task is then to compute these time-dependent correlators, and again solve the eigenvalue equations ($\mathrm{Det}[\mathcal{M} - g^2\hat{M}] = 0$) to find corrected, time-dependent values of ω^2. We will do this below, but already at this stage, we note that in the limit where $\hat{M} \gg \mathcal{M}$, the corrected system now has three double eigenvalues, $\hat{M}_{11}, \hat{M}_{22} \pm \sqrt{\hat{M}^2_{23} + \hat{M}^2_{26}}$, of which one is negative whenever $\hat{M}^2_{23} + \hat{M}^2_{26} > \hat{M}^2_{22}$. This means that (some of) the instabilities remain even at late times, provided the background field remains constant. In the 0 and 1 sectors, all modes remain stable, since $\hat{M}_{26} \to 0$, and as we will see, $\hat{M}_{22} > \hat{M}_{23}$.

B. Setting up the matrix structures: Background equation

Similarly, we can express the quadratic contribution (10) in terms of correlators:

$$\bar{D}^{ab}_{\mu}\bar{F}^{\mu\nu,b} = j^{\nu}_{a} - \bar{M}^{\nu\mu}_{ab}[C]\bar{A}_{\mu,b}. \quad (90)$$

Again, in our specific setup, the space-derivative terms in the correction to the current $\bar{j}^{\nu}_{a}$ vanish because the correlators are space-independent. The time-derivative terms also vanish, because they either act on a stable correlator (in the $0-\mu$ sector, which we neglect) or because of the antisymmetrization of ϵ_{abc}. We are therefore left with the matrix $\bar{M}(C)$, which multiplies $\bar{A}$, and we find that this matrix has the same structure as $\hat{M}$. Since there is no original current in the 0 and 1 sectors, it is consistent to set $\bar{A}^{0,1}_{a} = 0$ throughout, and we will consider it no further. In the 2 and 3 sectors, however, we would have to solve the full Yang-Mills equation with a time-dependent current.

We will not attempt this here, but simply estimate at which time the assumption of constant $\bar{A}$ is likely to break down. We quantify this as when the correction is larger than the original current:

$$\bar{M}^{\nu\mu}_{ab}[C]\bar{A}_{\mu,b} > j^{\nu}_{a}. \quad (91)$$

We recall that

$$j^{\nu}_{a} = -gB\left[\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\D_7\\-D_8\end{pmatrix}\begin{pmatrix}0\\-D_5\\D_6\end{pmatrix}\right], \quad (92)$$

with $D_6 = -D_7 = \sqrt{B/g}$, and $D_5 = D_8 = 0$. The correction simply becomes

$$(\bar{M}\,\bar{A})^{\nu}_{a} = -gB\left[\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\J_7\\-J_8\end{pmatrix}\begin{pmatrix}0\\-J_5\\J_6\end{pmatrix}\right], \quad (93)$$

with

$$J_6 = -J_7 = \sqrt{\frac{B}{g}}\frac{g(C^{33}_{11} + C^{33}_{33} - 3C^{23}_{23})}{B},$$
$$J_5 = -J_8 = \sqrt{\frac{B}{g}}\frac{gC^{22}_{23}}{B}, \quad (94)$$

so that the form of the total current is unchanged, and it will still generate a B field in the $\mu = 1$, $a = 1$ direction. This B field is, however, now time-dependent, and the time-dependence of $\bar{A}^{\mu}_{a}$ could also potentially source an Abelian E field. This means that as soon as either of the correlators is larger than B/g, our approximation breaks down. We will now estimate when that happens.

C. Eigenvectors, quantization, and correlators

The 2 and 3 sectors of the linear system have six eigenvalues for each set of momenta $k_{x,y,z}$, $\omega^i_{\mathbf{k}}$, corresponding to six eigenvectors $\phi_i(\mathbf{k}, t)$ which are solutions to

$$\partial^2_t\phi_i(\mathbf{k}, t) = -\omega^2_i(\mathbf{k})\phi_i(\mathbf{k}, t), \quad i = 1, \ldots, 6, \quad (95)$$

$$\to \phi_i(\mathbf{k}, t) = \alpha^i_k e^{i\omega^i_{\mathbf{k}}t} + \beta^i_k e^{-i\omega^i_{\mathbf{k}}t}. \quad (96)$$

In order to define the state in which to compute the expectation values, we will assume that the system is prepared in a vanishing background $\bar{A}^{\mu}_{a} = 0$ for $t < 0$, and that $\bar{A}^{\mu}_{a}$ is instantaneously turned on at $t = 0$, remaining constant for $t > 0$. Then, we may write for $t < 0$

$$\phi_i(\mathbf{k}, t) = \frac{1}{\sqrt{2\omega^0_k}}(a^i_{\mathbf{k}}e^{i\omega^0_k t} + (a^i_{-\mathbf{k}})^{\dagger}e^{-i\omega^0_k t}), \quad (97)$$

$$\partial_t\phi_i(\mathbf{k}, t) = \frac{i\omega^0_k}{\sqrt{2\omega^0_k}}(a^i_{\mathbf{k}}e^{i\omega^0_k t} - (a^i_{-\mathbf{k}})^{\dagger}e^{-i\omega^0_k t}), \quad (98)$$

where we note that for $\bar{A}^{\mu}_{a} = 0$, all the eigenvalues coincide: $\omega_i(\mathbf{k}) \to \omega^0_k = |\mathbf{k}|$.

For $t > 0$, we may write in general

$$\phi_i(\mathbf{k}, t) = \alpha_k^i e^{i\omega_k^i t} + \beta_{\mathbf{k}}^i e^{-i\omega_k^i t}, \tag{99}$$

$$\partial_t \phi_i(\mathbf{k}, t) = i\omega_k^i (\alpha_{\mathbf{k}}^i e^{i\omega_k^i t} - \beta_{\mathbf{k}}^i e^{-i\omega_k^i t}). \tag{100}$$

Matching the expressions at $t = 0$, we can express $\alpha_{\mathbf{k}}^i$ and $\beta_{\mathbf{k}}^i$ in terms of $a_{\mathbf{k}}^i$ and $a_{-\mathbf{k}}^i$ as

$$\alpha_{\mathbf{k}}^i = \frac{1}{\sqrt{2\omega_k^0}} \left[a_{\mathbf{k}}^i \left(1 + \frac{\omega_k^0}{\omega_{\mathbf{k}}^i} \right) + (a_{-\mathbf{k}}^i)^\dagger \left(1 - \frac{\omega_k^0}{\omega_{\mathbf{k}}^i} \right) \right], \tag{101}$$

$$\beta_{\mathbf{k}}^i = \frac{1}{\sqrt{2\omega_k^0}} \left[a_{\mathbf{k}}^i \left(1 - \frac{\omega_k^0}{\omega_{\mathbf{k}}^i} \right) + (a_{-\mathbf{k}}^i)^\dagger \left(1 + \frac{\omega_k^0}{\omega_{\mathbf{k}}^i} \right) \right]. \tag{102}$$

Choosing our initial state as the vacuum at $t < 0$, $a_{\mathbf{k}}^i|0\rangle = 0$ and using the standard commutation relations $[a_{\mathbf{k}}^i, (a_{\mathbf{k}'}^j)^\dagger] = (2\pi)^3 \delta^{ij} \delta^3(\mathbf{k} - \mathbf{k}')$, we can straightforwardly compute expectation values of any combination of $\alpha_{\mathbf{k}}^i$, $\beta_{\mathbf{k}}^i$.

The fluctuation fields $h_a^\mu(\mathbf{k})$ in the 2 and 3 sectors are linear combinations of the eigenmodes $\phi_i(\mathbf{k})$, written in terms of a unitary matrix $U^{ij}(\mathbf{k})$. With ($\mu = 2, 3$, $a = 1, 2, 3 \to 1, \ldots, 6$), so

$$\begin{aligned} \langle h^i(\mathbf{k})(h^j(\mathbf{k}))^\dagger \rangle &= \langle U_{ir} \phi^r(\mathbf{k}) [U_{js} \phi^s(\mathbf{k})]^\dagger \rangle \\ &= U^{ir} [U^\dagger]_{sj} \langle \phi^r(\mathbf{k}) [\phi^s(\mathbf{k})]^\dagger \rangle, \end{aligned} \tag{103}$$

keeping in mind that $\langle \phi^r(\mathbf{k}) [\phi^s(\mathbf{k})]^\dagger \rangle \propto \delta^{rs}$. We wish to compute the correlators

$$\begin{aligned} C_{ab}^{\mu\nu} &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \langle h_a^\mu(\mathbf{k}) h_b^\nu(-\mathbf{k}) \rangle \\ &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\langle h^i(\mathbf{k}) [h^j(\mathbf{k})]^\dagger + [h^i(\mathbf{k})]^\dagger h^j(\mathbf{k}) \rangle}{2}, \end{aligned} \tag{104}$$

which follow from combining Eqs. (96), (102), and (103), and using these, we can construct the nonlinear correction to the linear mode equation (79).

One final complication is that the diagonal correlators are (IR and UV) divergent, and so in order to gauge the effect of the instability, we will in practice compute

$$C_{ab;\mathrm{ren}}^{\mu\nu} = \int \frac{d^3\mathbf{k}}{(2\pi)^3} [\langle h_a^\mu(\mathbf{k}, t) h_b^\nu(-\mathbf{k}, t) \rangle - \langle h_a^\mu(\mathbf{k}, 0) h_b^\nu(-\mathbf{k}, 0) \rangle]. \tag{105}$$

In Fig. 15, we show the four distinct (renormalized) correlators in momentum space at times $t = 1, 2, 3$

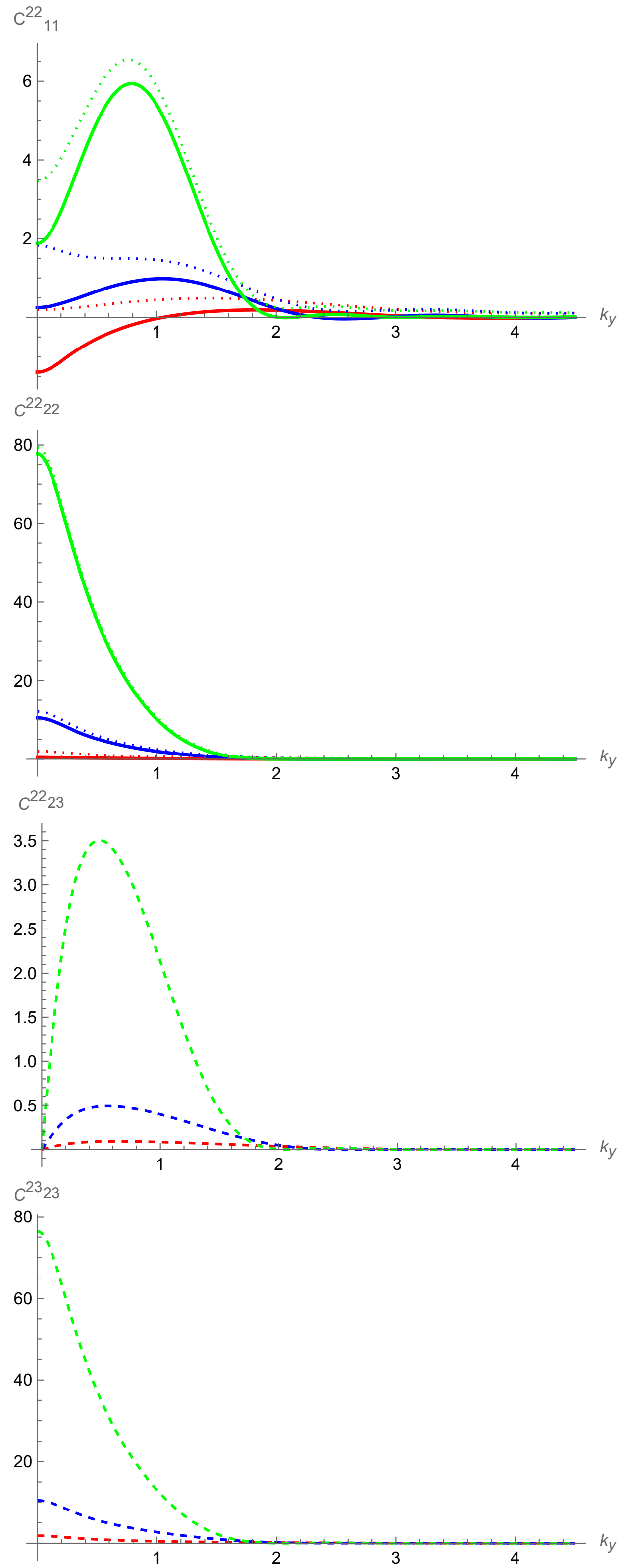

FIG. 15. Non-Abelian B: The correlators C_{11}^{22}, C_{22}^{22}, C_{23}^{22}, and C_{23}^{23} (top to bottom) as functions of k_y for times $gBt = 1, 2, 3$ (red, blue, green). Renormalized (full lines) and unrenormalized correlators (dotted lines) are shown.

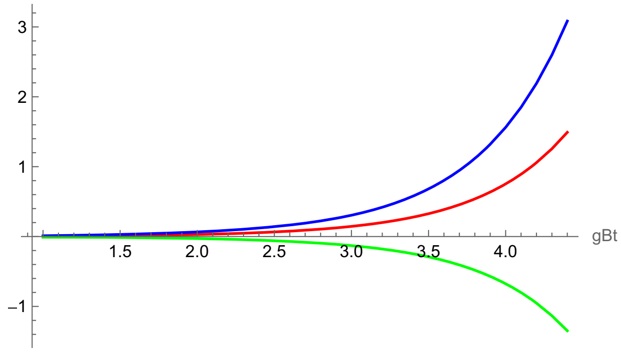

FIG. 16. Non-Abelian B: The three eigenvalues M_{11}, $M_{22} \pm \sqrt{M_{23}^2 - M_{26}^2}$.

(in units where $gB = 1$). We see that the C_{22}^{22} and C_{23}^{23} correlators are substantially larger than the other two.

D. Corrected dispersion relations

In Fig. 16, we show M_{11}, $M_{22} \pm \sqrt{M_{23}^2 + M_{26}^2}$, which was our late-time estimate for the eigenvalues in Sec. IV A. The lowest eigenvalue is indeed negative throughout, and so for any momentum, for late enough time, instabilities remain if they are not switched off for other reasons.

We can do better, and simply compute the eigenvalues at $k = 0$ of the whole matrix $\mathcal{M} - g^2\hat{M}$, as a function of time. Figure 17 shows the three distinct eigenvalues in the 2 and 3 sectors as a function of time. The initially unstable mode is briefly stabilized around $gBt = 3.5$, but by 4.7 the late-time behavior takes over, and the eigenvalue becomes negative again.

E. Corrected current

In a similar way, we can insert the correlators into the effective current. Figure 18 shows the quantities $J_{5,6,7,8}$ (94) in time. We see that $J_6 = -J_7$ grows the fastest, and it becomes order 1 (our criterion for $g = B = 1$) around $gBt = 4-5$. At this time, we can then no longer assume a

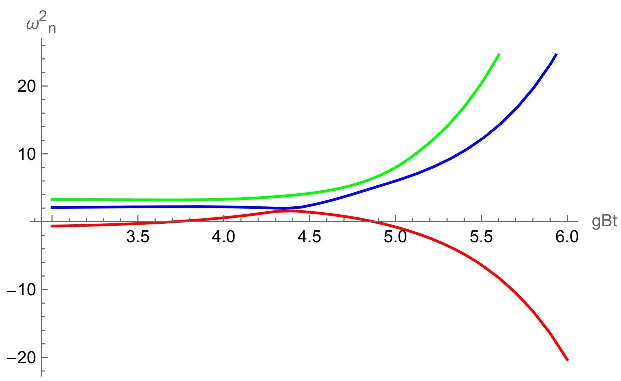

FIG. 17. Non-Abelian B: The corrected eigenvalues ω_i as a function of time.

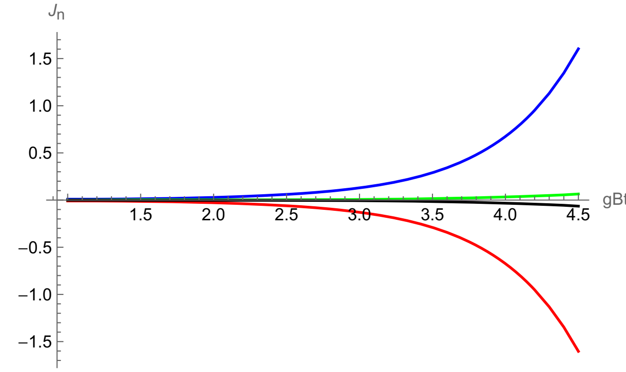

FIG. 18. Non-Abelian B: The quantities $J_{5,6,7,8}/\sqrt{B/g}$ in time.

constant background field. Whether this leads to stability is beyond our approach here, but it may be readily studied using numerical simulations.

V. CONCLUSIONS

To summarize, we have investigated the instabilities of field fluctuations in pure SU(2) classical gauge theory with a constant homogeneous background E and/or B field. We first, in Sec. II, presented a fairly general set of background vector potentials, generalizing the setup of [26,27]. We then, in Sec. III, focused on non-Abelian E and B field backgrounds, and we derived the dispersion relation that the fluctuations obey. Nonreal instances of frequencies $\omega(k)$ signal unstable modes in the linear theory, and we confirmed that these arise for homogeneous E fields, B fields, and when combining the two. As an aside, we also found that the instability does not simply depend on the B or E field, but on the vector field components $\bar{A}_a^\mu$ separately.

Our focus was primarily to identify the computationally distinct possibilities (space-/time-dependent/independent, Abelian/non-Abelian), and we have not investigated in great detail whether some of these cases are gauge equivalent. But as an example, one can convince oneself that there is a gauge transformation connecting the $C_1 = 0$ and $C_2 = 0$ instances of the Abelian E-only case in Eq. (15), but that there is no gauge transformation connecting the non-Abelian B-only case, $D_6D_7 = B$, $D_5 = D_8 = 0$, to $D_5 \neq 0$ $D_8 = 0$ (Fig. 9), under the constraint that it does not generate other nonzero components of A_a^μ.

Of particular interest is understanding to what extent these instabilities shut themselves off due to the nonlinearities of the theory. To that end, we computed the nonlinear (Gaussian) corrections to both the background field equation and the linear fluctuation equation. We found that the corrections certainly influence the instabilities of the fluctuations, but that rather than turning them off, they can make them stronger. This, however, assumes that the background field stays constant and homogeneous throughout, which is spoiled by the nonlinearities in the background field equation. We estimated the time when the background field is no longer a constant non-Abelian B field to be $gBt \simeq 4$, but further details of how this comes about, and the explicit effect on the instabilities is likely better addressed numerically.

Although the context of this type of instabilities is the thermalization of the plasma in HICs, anisotropic gauge fields are not unique to QCD. Anisotropic conditions also appear in cosmology—for instance, in the context of first-order phase transitions. In that case, the walls of nucleated bubbles sweep through the ambient plasma, so that immediately in front of and behind the advancing wall, the plasma is anisotropic (see, for instance, [28–30]). Although it is not clear to what extent net E and B fields are produced, it would be worth investigating whether an anisotropic out-of-equilibrium background could potentially source similar instabilities.

The obvious continuation of this work is to make a full investigation of the other realizations of background E and B fields presented in the Appendix. This could also readily be compared to direct numerical simulations of the pure gauge SU(2) system, going beyond the Gaussian approximation to the nonlinearities. Further generalizing to SU(3) gauge theory involves writing down a $\mathcal{M}(32\times 32)$ matrix equation and solving for the dispersion relations and then unstable modes there. This is ongoing work, but we note that the results for SU(2) presented here naturally arise as a subsector in the SU(3) theory.

ACKNOWLEDGMENTS

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APPENDIX: SPACE- OR TIME-DEPENDENT VECTOR POTENTIALS

For completeness, we now briefly present a few cases of Abelian vector potentials, and how one might go about solving the linearized equations of motion.

If the background field $\bar{A}^\mu_a$ has no space dependence, we can simply Fourier-transform in three dimensions and write

$$h^\mu_a(x,y,z,t) = \int \frac{dk_x dk_y dk_z}{(2\pi)^3} e^{-ik_x x - ik_y y - ik_z z} h^\mu_a(k_x,k_y,k_z,t). \tag{A1}$$

Then, the equation of motion in momentum space reads

$$\mathcal{M}(\partial_t^2, \partial_t, k_x, k_y, k_z, \bar{A}) h^\mu_a = 0, \tag{A2}$$

which can be solved as a set of coupled linear differential equations. One may further Fourier-transform in time, in the sense that

$$h^\mu_a(k_x,k_y,k_z,t) = \int \frac{d\omega}{2\pi} h^\mu_a(k_x,k_y,k_z,\omega) e^{i\omega t}, \tag{A3}$$

in which case one may substitute $\partial_t \to i\omega$ in (A2). The expectation is that for some of the modes (for certain values of $k_x, k_y, k_z, \bar{A}$), ω_k has a nonzero imaginary part, and the mode grows (or decays) exponentially in time.

If the background field depends on a spatial coordinate, we can only Fourier-transform in the remaining coordinates (say y, z), to find

$$h^\mu_a(x,y,z,t) = \int \frac{dk_y dk_z}{(2\pi)^2} e^{-ik_y y - ik_z z} h^\mu_a(x,k_y,k_z,t), \tag{A4}$$

and we are led to a relation of the form

$$\mathcal{M}(\partial_t^2, \partial_t, \partial_x^2, \partial_x, k_y, k_z, \bar{A}) h^\mu_a = 0. \tag{A5}$$

This implies that the field profile is not a product of plane waves in x, y, z, but that the x dependence is some other function to be solved for explicitly. Similarly, if the background field depends on two spatial coordinates, we can only Fourier-transform in the remaining coordinates (say, z), to find

$$h^\mu_a(x,y,z,t) = \int \frac{dk_z}{(2\pi)} e^{-ik_z z} h^\mu_a(x,y,k_z,t). \tag{A6}$$

Neither the x nor y dependence can then be expressed as plane waves. Finally, if $\bar{A}^\mu_a$ is a function of t, the Fourier transform in time also fails, and the field has a nontrivial time evolution. We again refer to [26] and the Appendix for a discussion on this.

1. Abelian configuration of B field ($C_3 = B \neq 0$)

Consider a potential $\bar{A}^\mu_a$ in the $\mu = 3$ direction with a constant homogeneous magnetic field in the $i = 2$ direction and in the $a = 1$ color direction:

$$\bar{A}^\mu_a = \left[\begin{pmatrix}0\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\end{pmatrix} \begin{pmatrix}By\\0\\0\end{pmatrix} \right]. \tag{A7}$$

The linearized Yang-Mills equation becomes

$$\Box h^\mu_a - 2gBy\epsilon^{ab1}\partial_z h^\mu_b - 2gB\epsilon^{ab1}(\delta^{\mu y}h^z_b - \delta^{\mu z}h^y_b) - g^2B^2y^2\epsilon^{ac1}\epsilon^{cb1}h^\mu_b = 0. \tag{A8}$$

Let us define the functions

$$T^\pm = h^0_2 \pm ih^0_3, \qquad X^\pm = h^x_2 \pm ih^x_3,$$
$$Y^\pm = h^y_2 \pm ih^y_3, \qquad Z^\pm = h^z_2 \pm ih^z_3.$$

Modifying Eq. (A8) and then solving for each mode in Fourier space, we obtain the dispersion relations

$$\left(-\omega^2 + k_x^2 + (k_z \mp gBy)^2 - \frac{d^2}{dy^2}\right)T^{\pm}(y) = 0, \quad \text{(A9)}$$

$$\left(-\omega^2 + k_x^2 + (k_z \mp gBy)^2 - \frac{d^2}{dy^2}\right)X^{\pm}(y) = 0, \quad \text{(A10)}$$

$$\left(-\omega^2 \pm 2gB + k_x^2 + (k_z - gBy)^2 - \frac{d^2}{dy^2}\right)U^{\pm}(y) = 0, \quad \text{(A11)}$$

$$\left(-\omega^2 \mp 2gB + k_x^2 + (k_z + gBy)^2 - \frac{d^2}{dy^2}\right)W^{\pm}(y) = 0, \quad \text{(A12)}$$

where

$$U^{\pm} \equiv Y^+ \pm iZ^+, \qquad W^{\pm} \equiv Y^- \pm iZ^-.$$

The above equations resemble a Schrödinger equation for a harmonic oscillator and have a discrete spectrum of frequencies. We can solve for the frequencies for W^+ and U^- to find

$$\omega_0^2 = 2gB\left(n + \frac{1}{2}\right) + k_x^2 \quad \text{(A13)}$$

and

$$\omega_{\pm}^2 = 2gB\left(n + \frac{1}{2}\right) \pm 2gB + k_x^2 \quad \text{(A14)}$$

for discrete $n = 0, 1, 2, \ldots$ and continuous k_x.

It is clear that ω_0^2 and ω_+^2 are always positive (≥ 0) for any value of n, but ω_-^2 is negative whenever $n < \frac{1}{2} - \frac{k_x^2}{2gB}$. That means there are unstable modes of U^- and W^+ which grow exponentially. The remaining modes $T^{\pm}$ and $X^{\pm}$, U^+ and W^- are stable for all k_x. We see that the dependence of k_z is absorbed in a redefinition of the y coordinate.

2. Abelian configuration of E field ($C_1 = E \neq 0$)

Consider instead a potential $\bar{A}_a^{\mu}$ in the $\mu = 0$ direction with a constant homogeneous electric field in the $i = 1$ direction and in the $a = 1$ color direction:

$$\bar{A}_a^{\mu} = \left[\begin{pmatrix} -Ex \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right]. \quad \text{(A15)}$$

The linearized Yang-Mills equation becomes

$$g^{\mu\nu}[(\partial_\rho \delta^{ad} - gf^{ade}\bar{A}_\rho^e)(\partial^\rho \delta_{dc} - gf_{dcf}\bar{A}_f^\rho)h_\nu^c] + 2gf^{abc}\bar{F}_b^{\mu\nu}h_\nu^c = 0. \quad \text{(A16)}$$

Solving Eq. (A16) for the modes in Fourier space gives us dispersion relations such as

$$\left(k_y^2 + k_z^2 \pm 2igE - (\omega + gEx)^2 - \frac{d^2}{dx^2}\right)G^{\pm}(x) = 0, \quad \text{(A17)}$$

$$\left(k_y^2 + k_z^2 \mp 2igE - (\omega - gEx)^2 - \frac{d^2}{dx^2}\right)H^{\pm}(x) = 0, \quad \text{(A18)}$$

$$\left(k_y^2 + k_z^2 - (\omega \pm gEx)^2 - \frac{d^2}{dx^2}\right)Y^{\pm}(x) = 0, \quad \text{(A19)}$$

$$\left(k_y^2 + k_z^2 - (\omega \mp gEx)^2 - \frac{d^2}{dx^2}\right)Z^{\pm}(x) = 0, \quad \text{(A20)}$$

where the functions $G^{\pm}$ and $H^{\pm}$ are defined as

$$G^{\pm} \equiv T^+ \pm X^+, \qquad H^{\pm} \equiv T^- \pm X^-.$$

The above equations again resemble an inverted Schrödinger equation, an "upside-down" harmonic oscillator. This does not have any normalizable solutions.

3. Abelian E and Abelian B ($C_1 \neq 0 \neq C_3$)

Consider a potential $\bar{A}_a^{\mu}$ corresponding to a constant homogeneous electric field in the x direction and a constant homogeneous magnetic field also in the x direction:

$$\bar{A}_a^{\mu} = \left[\begin{pmatrix} -Ex \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} By \\ 0 \\ 0 \end{pmatrix}\right]. \quad \text{(A21)}$$

The linearized Yang-Mills equation in this case becomes

$$\left((k_z - gBy)^2 - (\omega + gEx)^2 \pm 2igE - \frac{d^2}{dx^2} - \frac{d^2}{dy^2}\right) \times G^{\pm}(x, y) = 0, \quad \text{(A22)}$$

$$\left((k_z + gBy)^2 - (\omega - gEx)^2 \mp 2igE - \frac{d^2}{dx^2} - \frac{d^2}{dy^2}\right) \times H^{\pm}(x, y) = 0, \quad \text{(A23)}$$

$$\left((k_z - gBy)^2 - (\omega + gEx)^2 \pm 2gB - \frac{d^2}{dx^2} - \frac{d^2}{dy^2}\right) \times U^{\pm}(x, y) = 0, \quad \text{(A24)}$$

$$\left((k_z + gBy)^2 - (\omega - gEx)^2 \mp 2gB - \frac{d^2}{dx^2} - \frac{d^2}{dy^2}\right) \times W^{\pm}(x,y) = 0. \tag{A25}$$

Since there is explicit dependence on x and y, we are left with a partial differential equation in both coordinates. The equations (A22)–(A25) can be further solved using variable separation. For example, considering Eq. (A24), we can write

$$U^{\pm}(x,y) = U_E^{\pm}(x)U_B^{\pm}(y). \tag{A26}$$

We can split them into two equations:

$$\left[Q_U^{\pm} - \frac{d^2}{dx^2} - g^2E^2\left(\frac{\omega}{gE} + x\right)^2\right]U_E^{\pm}(x) = 0, \tag{A27}$$

$$\left[-Q_U^{\pm} \pm 2gB - \frac{d^2}{dy^2} + g^2B^2\left(\frac{k_z}{gB} - y\right)^2\right]U_B^{\pm}(y) = 0, \tag{A28}$$

where $Q_U^{\pm}$ is the separation constant. Equation (A28) is another Schrödinger equation with the following replacements: $Q_U^{\pm} \mp 2gB \to 2m\varepsilon$, $gB \to m\bar{\omega}$, $\frac{k_z}{gB} \to y_0$. Since the oscillator energy $\varepsilon_n = (n + \frac{1}{2})\bar{\omega}$, $n = 0, 1, 2, 3, \ldots$,

$$Q_U^{\pm} = gB(2n+1) \pm 2gB. \tag{A29}$$

Equation (A27) is an inverted Schrödinger equation with the following replacements: $Q_U^{\pm} \to -2m\varepsilon$, $gE \to m\bar{\omega}$, $\frac{\omega}{gE} \to x_0$. It has no normalizable solutions.

Similarly, by separating variables, one can write

$$W^{\pm}(x,y) = W_E^{\pm}(x)W_B^{\pm}(y). \tag{A30}$$

This can also be split into two equations:

$$\left[Q_W^{\pm} - \frac{d^2}{dx^2} - g^2E^2\left(\frac{\omega}{gE} - x\right)^2\right]W_E^{\pm}(x) = 0, \tag{A31}$$

$$\left[-Q_W^{\pm} \mp 2gB - \frac{d^2}{dy^2} + g^2B^2\left(\frac{k_z}{gB} + y\right)^2\right]W_B^{\pm}(y) = 0, \tag{A32}$$

where $Q_W^{\pm}$ is the separation constant. Similarly, one can solve the remaining two equations.

4. Abelian E and non-Abelian B ($C_2 = E$, $D_3 = \lambda\sqrt{B/g}$, and $D_{12} = \frac{1}{\lambda}\sqrt{B/g}$)

Consider a potential $\bar{A}_a^{\mu}$ corresponding to a constant homogeneous electric field in the x direction and constant homogenous magnetic field components in the x and z directions:

$$\bar{A}_a^{\mu} = \left[\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}-tC_2\\0\\D_3\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}D_{12}\\0\\0\end{pmatrix}\right]. \tag{A33}$$

In this case, the linearized Yang-Mills equation reduces to

$$\begin{aligned}&\Box h_a^{\mu} + 2g\epsilon^{a1b}(D_{12}\partial_z - tC_2\partial_x)h_b^{\mu}\\&\quad + 2g\epsilon^{a3b}D_3\partial_x h_b^{\mu} - g^2\epsilon^{a1d}\epsilon^{d1b}(t^2C_2^2 + D_{12}^2)h_b^{\mu}\\&\quad + g^2tC_2D_3h_b^{\mu}(\epsilon^{a1d}\epsilon^{d3b} + \epsilon^{a3d}\epsilon^{d1b}) - g^2\epsilon^{a3d}\epsilon^{d3b}D_3^2h_b^{\mu}\\&\quad + 2g^2\epsilon^{a2b}(\delta^{\mu3}D_{12}D_3h_b^x - \delta^{\mu1}D_3D_{12}h_b^z)\\&\quad + 2g\epsilon^{a1b}(C_2\delta^{\mu0}a_b^x + C_2\delta^{\mu1}a_b^0) = 0.\end{aligned} \tag{A34}$$

We may now Fourier-transform in space, but not in time, and the linear equation for the modes of the fluctuations reduce to a set of 12 coupled linear differential equations, which we can split into four 3×3 matrices corresponding to the $\mu = 0, 1, 2, 3$ sectors introduced earlier, with cross terms coupling the 0, 1, and 3 sectors. The diagonal 3×3 matrix has the form

$$M_{ANA}^t = \begin{bmatrix}(\Box + g^2D_3^2) & 2igk_xD_3 & g^2tD_3C_2\\ -2igk_xD_3 & \Box + g^2(t^2C_2^2 + D_{12}^2 + D_3^2) & 2ig(D_{12}k_z - tC_2k_x)\\ g^2tD_3C_2 & -2ig(D_{12}k_z - tC_2k_x) & \Box + g^2(t^2C_2^2 + D_{12}^2),\end{bmatrix} \tag{A35}$$

where $\Box = \frac{\partial^2}{\partial t^2} + \mathbf{k}^2$, $\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2$. Diagonalizing this set of equations is nontrivial, but it would result in a set of 12 nonlinear differential equations in t, which would most likely require numerical evaluation.

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