Hybrid spectroscopy within the graviton soft-wall model

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(Received 26 February 2024; accepted 16 May 2024; published 21 June 2024)

In this analysis, the so-called holographic graviton soft-wall (GSW) model, first developed to investigate the glueball spectrum, has been adopted to predict the masses of hybrids with different quantum numbers. Results have been compared with other models and lattice calculations. We have extended the GSW model by introducing two modifications based on anomalous dimensions in order to improve our agreement with other calculations and to remove the initial degeneracy not accounted for by lattice predictions. These modifications do not involve new parameters. The next step has been to identify which of our calculated states agree with the PDG data, leading to experimental hybrids. The procedure has been extended to include hybrids made of heavy quarks by incorporating the quark masses into the model.

DOI: 10.1103/PhysRevD.109.114030

I. INTRODUCTION

In the past few years, hadronic models, inspired by the holographic conjecture [1,2], have been vastly used and developed in order to investigate nonperturbative features of glueballs and mesons, thus trying to grasp fundamental features of QCD [3,4]. Recently, we have used the so-called AdS/QCD models to study the scalar glueball spectrum [5,6]. The holographic principle relies on a correspondence between a five-dimensional classical theory with an anti–de Sitter (AdS) metric and a supersymmetric conformal quantum field theory with $N_C \to \infty$. This theory, different from QCD, is taken as a starting point to construct a five-dimensional holographic dual of it. This is the so-called bottom-up approach [7–10]. In this scenario, models are constructed by modifying the five-dimensional classical AdS theory with the aim of resembling QCD as much as possible. The main differences characterizing these models are related to the strategy used to break conformal invariance. Moreover, it must be noted that the relation which these models establish with QCD is at the level of the leading order in the number of colors expansion, and, thus, the mesonic and glueball spectrum and their decay properties are ideal observables to be studied by these models. The starting point for the present investigation is the holographic soft-wall (SW) model scheme, where a dilaton field is introduced to softly break conformal invariance. Within this scheme, we have recently introduced the graviton soft-wall (GSW) model [6,11,12], which has been able to reproduce not only the scalar meson spectrum, but also the lattice QCD scalar glueball masses [13–15], that was not described by the traditional SW models. Moreover, a formalism to study the glueball-meson mixing conditions has been developed, and some predictions, regarding the observably of pure glueball states, have been provided [11,12]. The success of the model in reproducing the scalar QCD spectra has motivated us to extend the GSW model to describe the spectrum of the ρ vector meson, the a_1 axial vector meson, the pseudoscalar meson spectra, and high spin glueballs [16].

For 40 years, the study of quantum chromodynamics (QCD) has served to establish one of the pillars of the Standard Model. Although gluons are now firmly established as the carriers of the strong force, their nonperturbative behavior remains enigmatic. This unfortunate circumstance is chiefly due to two features of QCD: The theory is notoriously difficult to work with in the nonperturbative regime, and experimental manifestations of glue tend to be hidden in the spectrum and dynamics of the conventional hadrons. In particular, experimental manifestations of hadrons that carry valence quark and gluonic degrees of freedom have been postulated since the early days of QCD. These states are called hybrids, and our aim lies in extending our previous experience with conventional hadrons to these states using the very successful GSW model [16].

Let us describe briefly the contents of this work. In Sec. II, we summarize the essence of the GSW model [16]. In Sec. III, we apply the GSW model to calculate the

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spectrum of hybrid states. In Sec. IV, we compare our results with lattice QCD and model calculations. In Sec. V, we present two possible modifications of the GSW model which do not involve new free parameters and allow one to reproduce the essential outcome of lattice data. In Sec. VI, we compare our results with experimental states appearing in the PDG compilation with the same quantum numbers. From Sec. VII to Sec. IX, we repeat the analysis for the heavy particles; i.e., we discuss the GSW model predictions for the spectra and we compare our results with lattice QCD, model calculations, and experimental data. We end by collecting some conclusions of our study.

II. DESCRIPTION OF HADRONS IN THE GSW MODEL

In this section, the essential features of the GSW model are introduced. The development of this approach has been motivated by the impossibility of the conventional SW models to describe the glueball and meson spectra with the same energy scale [6,11,12]. The main difference, which distinguishes the GSW model from the traditional SW, is a deformation of the AdS metric in five dimensions:

$$\begin{aligned} ds^2 &= \frac{R^2}{z^2} e^{\alpha\phi_0(z)}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) \\ &= e^{2A(z)}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) = e^{\alpha\phi_0(z)} g_{MN} dx^M dx^N \\ &= \bar{g}_{MN} dx^M dx^N, \end{aligned} \tag{1}$$

where $A(z) = \log R/z + \alpha\phi_0(z)/2$. The quantities evaluated within this new metric are displayed with an overline. The function $\phi_0(z)$ will be specified later. This kind of modification has been adopted in many studies of the properties of mesons and glueballs within AdS/QCD [17–26]. Let us recall that in Ref. [6] we proposed to study these spectra by considering the equation of motion of a graviton propagating in this deformed metric. Therefore, this metric represents, in this formalism, the gluon dynamics of QCD. Thanks to this choice, the glueball spectrum has been reproduced with just one parameter, hence not increasing the number of free parameters with respect to the original soft-wall model. The metric tensor and its determinant of this new space can be related to the usual AdS_5 metric and its determinant:

$$\bar{g}^{MN} = e^{-\alpha\phi_0(z)} g^{MN}, \tag{2}$$

$$\sqrt{-\bar{g}} = e^{\frac{5}{2}\alpha\phi_0(z)}\sqrt{-g}. \tag{3}$$

Once the gravitational background has been defined by the model, the same strategy used in the SW case is considered in order to obtain the equations of motion (EOMs) for the different fields dual to given hadronic states. The new action, written in terms of the standard AdS metric of the SW model, is given by

$$\begin{aligned} \bar{S} &= \int d^4x dz e^{-\phi_0(z)\beta}\sqrt{-\bar{g}}\mathcal{L}(x_\mu, z) \\ &= \int d^4x dz e^{\phi_0(z)(\frac{5}{2}\alpha - \beta + 1)}\sqrt{-g} e^{-\phi_0(z)}\mathcal{L}(x_\mu, z), \end{aligned} \tag{4}$$

where here the prefactor $\exp[\phi_0(z)(\frac{5}{2}\alpha + \beta + 1)]$ takes into account the dilaton term, as in the SW model, and also the modification of the metric. The parameters α and β parametrize the internal dynamics of the hadrons of QCD described within this holographic framework. In the AdS dynamics, α characterizes the modification of the metric, while β characterizes the SW model dilaton, namely, the breaking of conformal invariance. Since the GSW model has been developed as a modification of the SW model, we propose to fix β to reproduce the kinetic term of the standard SW model action [6,11,12,16]. Thus, in the case of scalar fields, $\beta = \beta_s = 1 + \frac{3}{2}\alpha$, and, in the case of the vector fields, $\beta = \beta_\rho = 1 + \frac{1}{2}\alpha$. The function $\mathcal{L}(x_\mu, z)$ is the Lagrangian density describing the motion of the dual fields in the space described by the metric Eq. (1). The dilaton profile function ϕ_0 adopted in the GSW model is the same of the one usually addressed in SW-based models [6,17,18,24,27–30], i.e., $\phi_0(z) = k^2 z^2$. The action characterizing the fields propagating in the AdS_5 space contains a masslike term whose value is fixed as follows:

$$M_5^2 R^2 = (\Delta - p)(\Delta + p - 4), \tag{5}$$

where Δ is the conformal dimension of the fields and p depends on its p-form. In other analyses, Δ has been corrected by including the contribution of the anomalous conformal dimension that characterizes the chiral symmetry-breaking mechanism [31]. In particular, $\Delta_p = 0$ for scalar, vector, and tensor mesons, and $\Delta_p = -1$ for pseudoscalar and axial vector mesons. Here and in the next sections, we give the GSW predictions for the hybrid masses without taking into account further modifications of the model.

Within these scheme, hybrids and multiquark states, defined as quark and gluon operators in QCD, will be described by the properties of their p-forms. For example, a vector field can be described as [16]

$$V_\mu = \bar{\Psi}\gamma_\mu\Psi, \tag{6}$$

and, thus, $p = 1$, $\Delta = 3$, and, therefore, $M_5^2 R^2 = 0$. Within the present prescription, an axial vector field

$$A_\mu = \bar{\Psi}\gamma_\mu\gamma_5\Psi \tag{7}$$

has also $M_5^2 R^2 = 0$. In the next section, we extend the calculation of the AdS_5 mass to hybrids. In the following, we will refer to vector states as V and axial vector states as A.

III. DESCRIPTION OF THE LIGHT HYBRIDS AND THEIR SPECTRUM

Hybrids are hadrons formed of valence quarks and valence gluons. Because of the nonperturbative nature of these bound states, the use of models is needed to predict and describe possible properties of these systems; see, e.g., Refs. [32–37]. In this scenario, the GSW model, already successfully applied to the studies of glueballs and regular mesons, can be used to calculate the spectra of hybrids with different quantum numbers. In particular, let us start with only nonstrange light hybrids described as a quark-antiquark color octet coupled to a valence gluon leading to a hadronic color singlet.

Let us describe the hybrid fields with the lowest conformal dimensions and characterized by the following quantum numbers J^{PC}, i.e., spin, parity, and charge conjugation. Since these hybrid fields have to be gauge invariant, we will use only gauge-invariant quantities for the quarks and gluon counterparts. In order to construct the fields, we will use for the quarks conventional bilinears and for the gluons their color magnetic field $B_i^a = -\frac{1}{2}\varepsilon^{ijk}F_{ij}^a$, where $F_{\mu\nu}^a$ is the color gauge tensor, $\mu, \nu = 0, 1, 2, 3$ are Lorentz indices, $i, j, k = 1, 2, 3$ are the spatial Lorentz indices, a is the color index, and the color electric field $E_i^a = F_{0i}^a$. To describe the quantum numbers of the hybrids, we recall the transformation properties of the quark bilinears and color magnetic and electric fields under C and P:

$$\begin{aligned}
&\text{scalar} && \bar{\Psi}^P\Psi^P = \Psi\Psi, && \bar{\Psi}^C\Psi^C = \Psi\Psi;\\
&\text{pseudoscalar} && \bar{\Psi}^P\gamma_5\Psi^P = -\bar{\Psi}\gamma_5\Psi, && \bar{\Psi}^C\gamma_5\Psi^C = \bar{\Psi}\gamma_5\Psi;\\
&\text{vector} && \bar{\Psi}^P\gamma_\mu\Psi^P = -\bar{\Psi}\gamma_\mu\Psi, && \bar{\Psi}^C\gamma_\mu\Psi^C = -\bar{\Psi}\gamma_\mu\Psi;\\
&\text{axial vector} && \bar{\Psi}^P\gamma_5\gamma_\mu\Psi^P = -\bar{\Psi}\gamma_5\gamma_\mu\Psi, && \bar{\Psi}^C\gamma_5\gamma_\mu\Psi^C = \bar{\Psi}\gamma_5\gamma_\mu\Psi;\\
&\text{tensor} && \bar{\Psi}^P\sigma_{\mu\nu}\Psi^P = \Psi\sigma_{\mu\nu}\Psi, && \bar{\Psi}^C\sigma_{\mu\nu}\Psi^C = -\bar{\Psi}\sigma_{\mu\nu}\Psi;\\
&\text{color magnetic field} && P^+B_i^aP = B_i^a, && C^+B_i^aC = -B_i^a;\\
&\text{color electric field} && P^+E_i^aP = -E_i^a, && C^+E_i^aC = -E_i^a.
\end{aligned} \tag{8}$$

By using these properties, one can build hybrid field configurations for specific quantum numbers. A crucial role in the solution of the dual field equation of motion is played by the AdS_5 mass Eq. (5), which strongly depends on the value of the p-forms. In order to calculate the hybrid ground states, we consider the configurations corresponding to the minimum AdS_5 mass. Taking these arguments into account, the hybrid field configurations with the lowest AdS_5 masses can be divided in two classes. The first one corresponds to those which have mesonic quantum numbers, and the corresponding hadrons are less wishful phenomenologically. These fields, to lowest order in conformal dimensions, are

$$\begin{aligned}
&0^{-+}\ S = \bar{\Psi}\gamma^i\lambda^a\Psi B_i^a && (\Delta = 5, p = 0, M_5^2R^2 = 5),\\
&0^{++}\ S' = \varepsilon_{ijk}\bar{\Psi}\sigma^{ij}\lambda^a\Psi B^{ka} && (\Delta = 5, p = 0, M_5^2R^2 = 5),\\
&1^{--}\ V_i = \bar{\Psi}\gamma_5\lambda^a\Psi B_i^a && (\Delta = 5, p = 1, M_5^2R^2 = 8),\\
&1^{+-}\ A_i' = \varepsilon_{ijk}\bar{\Psi}\gamma_5\gamma^j\lambda^a\Psi B^{ka} && (\Delta = 5, p = 1, M_5^2R^2 = 8),\\
&1^{++}\ A_i = \varepsilon_{ijk}\bar{\Psi}\gamma^j\lambda^a\Psi E^{ka} && (\Delta = 5, p = 1, M_5^2R^2 = 8).
\end{aligned} \tag{9}$$

The second class are those field configurations with similar properties but with exotic quantum numbers, i.e., quantum numbers that cannot be obtained by mesonic quark-antiquark states. To lowest order in conformal dimensions, these fields are

$$\begin{aligned}
&0^{+-}\ \Sigma = \bar{\Psi}\gamma_5\gamma^i\lambda^a\Psi B_i^a && (\Delta = 5, p = 0, M_5^2R^2 = 5),\\
&0^{--}\ \Sigma' = \bar{\Psi}\gamma_5\lambda^a\Psi E_i^a && (\Delta = 5, p = 0, M_5^2R^2 = 5),\\
&1^{-+}\ W_i = \varepsilon_{ijk}\bar{\Psi}\gamma^j\lambda^a\Psi B^{ka} && (\Delta = 5, p = 1, M_5^2R^2 = 8),\\
&1^{-+}\ W_i' = \bar{\Psi}\gamma_0\lambda^a\Psi E_i^a && (\Delta = 5, p = 1, M_5^2R^2 = 8).
\end{aligned} \tag{10}$$

In the present analysis, we study only $J = 0$ and $J = 1$ light hybrids. The equations of motion for the dual fields of these states are essentially those corresponding to scalar and vector fields, respectively. As shown in Ref. [16], the potential term strongly depends on the AdS_5 masses given in Eqs. (9) and (10). Let us point out that the EOMs can be rearranged as a Schrödinger-like equation. For the scalars, the EOMs read

$$-\frac{d^2\sigma(z)}{dz^2} + \left(k^4z^2 + 2k^2 + \frac{15}{4z^2} + M_5^2R^2\frac{e^{\alpha k^2z^2}}{z^2}\right)\sigma(z) = M^2\sigma(z), \tag{11}$$

and the EOMs for the vectors are

$$-\psi''(z) + \left(k^4z^2 + \frac{3}{4z^2} + M_5^2R^2\frac{e^{\alpha k^2z^2}}{z^2}\right)\psi(z) = M^2\psi(z), \tag{12}$$

TABLE I. The hybrid masses obtained from Eqs. (11) and (12) with the values of the conformal masses shown in Eqs. (9) and (10).

Mesonic hybrids				
	$n=0$	$n=1$	$n=2$	$n=3$
0^{-+}	2074	2536	2986	3429
0^{++}	2074	2536	2986	3429
1^{--}	2149	2647	3125	3592
1^{+-}	2149	2647	3125	3592
1^{++}	2149	2647	3125	3592
Exotic hybrids				
	$n=0$	$n=1$	$n=2$	$n=3$
0^{+-}	2074	2536	2986	3429
0^{--}	2074	2536	2986	3429
1^{-+}	2149	2647	3125	3592

where M^2 is related to the mode energies. We must recall that the parameters of the model have been fixed in our previous works, being $\alpha = 0.55 \pm 0.04$ and $k = \frac{370}{\sqrt{\alpha}}$ MeV [16]. In particular, we remark here that the value of $\alpha k^2 = 370^2$ MeV^2 has been fixed from the glueball spectrum in Ref. [6]. Later, when we studied the light scalar spectrum, in the corresponding EOM, the value of α and k appeared separated. We then fixed α to the experimental data for the light scalar meson masses; see Refs. [12,16]. The fitting procedure led to an uncertainty in α which represents the error in the fit. For the sake of simplicity, we display results corresponding to $\alpha = 0.55$.

Equations (11) and (12) have been solved numerically, and the corresponding results are displayed in Table I.

Since the GSW is supposed to be a faithful $1/N_C$ leading-order representation of QCD, we cannot consider, e.g., the gluon a mass as addressed for other conventional model calculations. Before concluding this section, it is worth noticing that mode functions can be derived from the solution to the above EOMs. These quantities can be linked to the light-front wave function [38,39] and used to compute different observables, including the decay constants of the hybrids. In the future, we will also study the hybrid wave functions to explore potential mixing effects with regular mesons that have the same quantum numbers. Such a strategy has been already used to evaluate the mixing between light scalar mesons and glueballs; see Ref. [11].

IV. LATTICE QCD AND MODEL CALCULATIONS FOR LIGHT HYBRIDS

Because of the lack of experimental data, in order to test the present model, we will compare the GSW predictions with those of other approaches. In particular, the spectra of these hybrids have been studied by lattice QCD (LQCD) [40–42], sum rule methods [43–46], and different models [32–37]. Following the lines of Refs. [40,47,48], we report in Table II the main results of Refs. [32–37,40–46]

TABLE II. We show the masses obtained for the hybrid hadrons, whose quantum numbers are defined above in Eqs. (9) and (10) and in Refs. [32–37,40,41,43–48]. The theoretical errors in the GSW predictions are due to the uncertainty on the α parameter. Masses are given in GeV unity.

Hybrid masses as in Ref. [47]					
	Bag [32,33]	Flux tube [34,35]	Constituent gluon [36,37]	LQCD ($m_\pi = 396$ MeV) [41]	GSW
0^{-+}	1.3	1.7–1.9	1.8–2.2	2.1	2.074 ± 0.028
0^{++}	>1.9	⋯	1.3–2.2	>2.4	2.074 ± 0.028
1^{--}	1.7	1.7–1.9	1.8–2.2	2.3	2.149 ± 0.017
1^{+-}	>1.9	1.7–1.9	1.8–2.1	>2.4	2.149 ± 0.017
1^{++}	>1.9	1.7–1.9	1.3–2.2	>2.4	2.149 ± 0.017
0^{+-}	⋯	1.7–1.9	⋯	>2.4	2.074 ± 0.028
0^{--}	⋯	⋯	1.8–2.3	⋯	2.074 ± 0.028
1^{-+}	1.5	1.7–1.9	1.8–2.2	2.0	2.149 ± 0.017

Hybrid masses as in Refs. [40,48]					
	Sum rule [43,44]	Bethe-Salpeter [45,46]	LQCD (renormalon) [42]	LQCD (anisotropic) [40]	GSW
0^{++}	⋯	⋯	1.98	⋯	2.074 ± 0.028
1^{--}	⋯	⋯	0.87	⋯	2.149 ± 0.017
1^{+-}	⋯	⋯	1.25	⋯	2.149 ± 0.017
0^{+-}	⋯	1.082	⋯	⋯	2.074 ± 0.028
0^{--}	⋯	1.319	⋯	⋯	2.074 ± 0.028
1^{-+}	1.3–1.8	1.439	2.15	2.013	2.149 ± 0.017

compared with the predictions of the GSW model already shown in Table I. We will specify the name of the model in the table as used in the cited references. The masses will be given in GeV.

The GSW model calculations lead to values of the hybrid masses within the range (2.07–2.15 GeV), similar to other model calculations [32–37,45,46]. However, as one can notice, the masses of the lightest particles differ between the predictions of mentioned collaborations. For the much sought exotic 1^{-+}, the corresponding mass, obtained from the GSW model, is similar to that of the three LQCD calculations shown [40–42] but too high if compared with the Bethe-Salpeter approach [45,46]. We also agree quite well with the LQCD [41] and LQCD (renormalon) calculation of Ref. [42] in three states. However, our 0^{+-} and 0^{--} appear to be very high compared to the Bethe-Salpeter approach [45,46] but reasonable compared with the constituent gluon [36,37] and low compared to LQCD [41]. In summary, we report that the 0^{-+} state is well reproduced, while the other states are underestimated except the 1^{-+} case. Remarkably, these calculations, which do not involve any free parameter, are in line with those of other models or LQCD. In particular, we predict that scalar hybrids are lighter than the vector ones, as addressed by lattice data, except for the 1^{-+}. However, as one might notice, the lattice calculations clearly indicate that only a few states can be degenerate; for example, the mass of the 0^{-+} is lower than that of the 0^{++}. In order to remove this degeneracy, in the next section, we propose different possible modifications of the GSW model. Let us remark that, in the next section, only some possible simple extensions of the GSW model will be discussed in order to highlight how the present approach can be improved without the inclusion of further free parameters. Such a choice is essential to preserve the relevant predictive power of the present model. It is also worth stressing that the holographic approach represents a $1/N_c$ leading-order calculation, which is good to determine hybrids with masses around 2 GeV.

V. BEYOND THE GSW MODEL

As previously shown, the calculation of the hybrid spectra within the GSW, without invoking any new free parameter, is comparable with other analyses; however, as one can notice, the degeneracy for different parity states, predicted by the model itself, seems to be unphysical according to the lattice results. Therefore, the comparison with the lattice analyses suggests that an improvement of the model is necessary. To this aim, we propose the following philosophy: (i) Since there are no experimental data for the hybrid spectra to guide us, we adopt some simple modifications for example purposes just to show that the model, properly modified, could grasp the basic dynamics underlying the hybrid structure; (ii) in order to keep the predicting power of the model, we avoid any approach that directly involves new parameters to be fitted. Summarizing, our aim here is to propose an extended scenario to reproduce the hierarchy of the hybrid masses predicted by lattice QCD and model calculations. The improvement that we present is based on approaches already discussed in several investigations involving holographic models [49–51]. All these scenarios do not require any additional free parameter. As one might notice, since the present experimental and lattice scenarios are not well constrained, the main purpose of this section is to show that the GSW can be considered as a solid baseline for any future calculations or comparisons. In fact, it has been proved that the present approach is able to reproduce the almost linear trajectory of the glueball masses [6,11,12], the spectra of light and heavy mesons (scalars, pseudoscalars, vectors, and axial vectors) [12,16] with only two parameters. Let us also point out that the spirit of the modifications we are proposing is to parametrize the possible peculiar dynamics underlying the hybrid structure. In fact, we propose to keep the same wrap factor in the metric in Eq. (1), the profile function of the dilaton Eq. (4), and the parameters characterizing the model. Therefore, in order to describe hadrons with different dynamics from that of glueballs and regular mesons, some adjustments are needed. In particular, to improve the model we consider that the field interpolators might lead to anomalous dimensions that might affect the conformal mass [49–51]:

$$M_5^2 R^2 = (\Delta + \Delta_p - p)(\Delta + \Delta_p + p - 4). \tag{13}$$

However, there is no direct correspondence between the anomalous dimensions in QCD and the corresponding holomorphic anomalous dimensions. Let us discuss in what follows two modifications of the GSW associated with the proposal Δ_p. Let us mention that, in principle, there are several ways to introduce this quantity. For example, as shown in Ref. [51], one could consider Δ_p to be dependent on z [51]. However, the shape function is not trivially fixed, and further phenomenological constraints will be needed. Here, as an example, we consider the approach proposed in Ref. [31], where Δ_p is introduced to parametrize twist effects that lead to modifications of the orbital angular momentum L, thus allowing one to distinguish the spectra of states with different parity. In general, there could be several ways to achieve the goal. Nevertheless, let us remark that the present holographic approach relies on $1/N_C$ calculations, and, thus, the different channels are parametrized only by the fifth-dimensional mass which depends exclusively on the conformal dimension Δ and spin p. In this scenario, therefore, it is natural to modify this quantity in order to distinguish the different channels. The introduction of Δ_p has precisely this purpose.

A. First modification

The first modification consists in introducing the anomalous dimension Δ_p that leads one to distinguish scalar and vector fields from the pseudoscalar and the axial vector ones. Such a strategy was quite successful in the study of regular mesons. In this case, one assigns $\Delta_p = -1$ for states whose field operator definition involves the γ_5 matrix [49]. Using this input and Eq. (13), one obtains the following conformal masses for each field interpolator of the mesonic type:

$$\begin{aligned}
&0^{-+}\ S = \bar{\Psi}\gamma^i\lambda^a\Psi B_i^a && (\Delta = 5, \Delta_p = 0, p = 0, M_5^2R^2 = 5),\\
&0^{++}\ S' = \varepsilon_{ijk}\bar{\Psi}\sigma^{ij}\lambda^a\Psi B^{ka} && (\Delta = 5, \Delta_p = 0, p = 0, M_5^2R^2 = 5),\\
&1^{--}\ V_i = \bar{\Psi}\gamma_5\lambda^a\Psi B_i^a && (\Delta = 5, \Delta_p = -1, p = 1, M_5^2R^2 = 3),\\
&1^{+-}\ A_i' = \varepsilon_{ijk}\bar{\Psi}\gamma_5\gamma^j\lambda^a\Psi B^{ka} && (\Delta = 5, \Delta_p = -1, p = 1, M_5^2R^2 = 3),\\
&1^{++}\ A_i = \varepsilon_{ijk}\bar{\Psi}\gamma^j\lambda^a\Psi E^{ka} && (\Delta = 5, \Delta_p = 0, p = 1, M_5^2R^2 = 8).
\end{aligned} \tag{14}$$

For the nonmesonic type, we get

$$\begin{aligned}
&0^{+-}\ \Sigma = \bar{\Psi}\gamma_5\gamma^i\lambda^a\Psi B_i^a && (\Delta = 5, \Delta_p = -1, p = 0, M_5^2R^2 = 0),\\
&0^{--}\ \Sigma' = \bar{\Psi}\gamma_5\lambda^a\Psi E_i^a && (\Delta = 5, \Delta_p = -1, p = 0, M_5^2R^2 = 0),\\
&1^{-+}\ W_i = \varepsilon_{ijk}\bar{\Psi}\gamma^j\lambda^a\Psi B^{ka} && (\Delta = 5, \Delta_p = 0, p = 1, M_5^2R^2 = 8),\\
&1^{-+}\ W_i' = \bar{\Psi}\gamma_0\lambda^a\Psi E_i^a && (\Delta = 5, \Delta_p = 0, p = 1, M_5^2R^2 = 8).
\end{aligned} \tag{15}$$

The resulting spectrum is displayed in Table III in the columns labeled GSWm1. As one can see, now the degeneracy between the first scalars (0^{-+} and 0^{++}) and the second ones (0^{+-} and 0^{--}) is removed, as that for the two vectors 1^{--} and 1^{+-} compared to 1^{++} and 1^{-+}. However, in this case, except for the 1^{-+} we predict vector hybrids lighter than the scalar ones. Moreover, the masses of the 1^{--} and the 1^{+-} states largely underestimate the lattice predictions. We conclude that this simple modification based on the phenomenology of regular mesons does not lead to a significant improvement to the GSW model. On the contrary, the hierarchy between the scalar and vector mesons is not reproduced.

B. Second modification

By following the line of the procedure described in the previous section but trying to take into account differences between regular and hybrid mesons, we propose to introduce the anomalous dimension to eliminate the degeneracy between states with different parity, as proposed in Refs. [49,50]. However, we assume that the lowest states, the 0^{-+} and the 1^{--}, correspond to $\Delta_p = 0$, while those with opposite parity are associated to $\Delta_p = 1$. This strategy is almost equivalent to that discussed in Refs. [49,50], where there is an exchange of one unit between states with different parity. Let us stress that in this way Δ_p is not a "standard" free parameter; indeed, only its sign is chosen to reproduce the lattice hierarchy. Hence, the parameters of the model are still those already fixed (α and k).

This modification is necessary just to reproduce the hierarchy of the masses displayed in Table III. Now, the conformal masses for the mesonic hybrids read

TABLE III. We show the masses obtained for the hybrid hadrons, whose quantum numbers are defined above in Eqs. (9) and (10) and in Refs. [40–42]. We compare the lattice calculations with the GSW predictions together with its modifications. The theoretical errors in the GSW predictions are due to the uncertainty on the α parameter.

	LQCD (renormalon) [42]	QCD (anisotropic) [40]	LQCD ($m_\pi = 396$ MeV) [41]	GSW	GSWm1	GSWm2
0^{-+}	...	...	2.1	2.074 ± 0.028	2.074 ± 0.028	2.074 ± 0.028
0^{++}	1.98	...	>2.4	2.074 ± 0.028	2.074 ± 0.028	2.694 ± 0.021
1^{--}	0.87	...	2.3	2.149 ± 0.017	1.562 ± 0.023	2.149 ± 0.017
1^{+-}	1.25	...	>2.4	2.149 ± 0.017	1.562 ± 0.023	2.747 ± 0.013
1^{++}	...	...	>2.4	2.149 ± 0.017	2.149 ± 0.017	2.747 ± 0.013
0^{+-}	...	...	>2.4	2.074 ± 0.028	1.411 ± 0.052	2.694 ± 0.021
0^{--}	...	...	...	2.074 ± 0.028	1.411 ± 0.052	2.074 ± 0.028
1^{-+}	2.15	2.013	2.0	2.149 ± 0.017	2.149 ± 0.017	2.149 ± 0.017

$$\begin{aligned}
&0^{-+}\ S = \bar{\Psi}\gamma^i\lambda^a\Psi B_i^a && (\Delta = 5, \Delta_p = 0, p = 0, M_5^2R^2 = 5),\\
&0^{++}\ S' = \varepsilon_{ijk}\bar{\Psi}\sigma^{ij}\lambda^a\Psi B^{ka} && (\Delta = 5, \Delta_p = 1, p = 0, M_5^2R^2 = 12),\\
&1^{--}\ V_i = \bar{\Psi}\gamma_5\lambda^a\Psi B_i^a && (\Delta = 5, \Delta_p = 0, p = 1, M_5^2R^2 = 8),\\
&1^{+-}\ A_i' = \varepsilon_{ijk}\bar{\Psi}\gamma_5\gamma^j\lambda^a\Psi B^{ka} && (\Delta = 5, \Delta_p = 1, p = 1, M_5^2R^2 = 15),\\
&1^{++}\ A_i = \varepsilon_{ijk}\bar{\Psi}\gamma^j\lambda^a\Psi E^{ka} && (\Delta = 5, \Delta_p = 1, p = 1, M_5^2R^2 = 15).
\end{aligned} \tag{16}$$

For the nonmesonic hybrids, they become

$$\begin{aligned}
&0^{+-}\ \Sigma = \bar{\Psi}\gamma_5\gamma^i\lambda^a\Psi B_i^a && (\Delta = 5, \Delta_p = 1, p = 0, M_5^2R^2 = 12),\\
&0^{--}\ \Sigma' = \bar{\Psi}\gamma_5\lambda^a\Psi E_i^a && (\Delta = 5, \Delta_p = 0, p = 0, M_5^2R^2 = 5),\\
&1^{-+}\ W_i = \varepsilon_{ijk}\bar{\Psi}\gamma^j\lambda^a\Psi B^{ka} && (\Delta = 5, \Delta_p = 0, p = 1, M_5^2R^2 = 8),\\
&1^{-+}\ W_i' = \bar{\Psi}\gamma_0\lambda^a\Psi E_i^a && (\Delta = 5, \Delta_p = 0, p = 1, M_5^2R^2 = 8).
\end{aligned} \tag{17}$$

Let us refer to this modification as GSWm2. As one can see in Table III, the present free parameter approach is in agreement with the predictions of the recent lattice calculations [41] except for the state 1^{--}, which is slightly underestimated. Of course, this is just an example of how a simple modification could lead to a good description of the present knowledge of hybrid states. Therefore, let us stress again that the GSW model, in general, can also be used to provide useful predictions for the physics of exotic hadrons. For example, we might use the GSW model calculations to try to identify which states, among those addressed in the Particle Data Group, could be considered as hybrid candidates.

C. Third modification

In this final part of the section, we consider the strategy discussed in the previous section, making now Δ_p dependent on z. To this aim, we follow the strategies presented in Refs. [51–53]. Because of the current lack of constraints on the lattice, model calculations, and experimental scenarios, we will consider as an example the SW model for tetraquarks in Ref. [51]. In this model, the author proposed the following Δ_p [51]:

$$\Delta_p = \gamma(z) = -az^\eta + bz^k, \tag{18}$$

for the spectrum of tetraquarks, in comparison with the conventional one for regular mesons. The author obtained the following set of parameters: $a = 4$, $\eta = 0.001$, $b = 0.05$, and $k = 2$ for his fit. In the present study, where $\Delta = 5$, if the above parameters are adopted to evaluate Δ_p, one gets complex masses. For the sake of simplicity and as an example, we assume the following shape for Δ_p:

$$\Delta_p = bz^k \tag{19}$$

with $b = 0.05$ and $k = 2$. The results with this fit are displayed in Table IV and addressed as GSWm3. Keeping the parameters b and k in Eq. (19) of Ref. [51], the predicted masses follow the order determined by lattice calculations. Given the current lack of well-established values for hybrid masses in the lattice scenario, distinguishing between the second and third modifications is not physically relevant. For the discussions that follow, we will consider the second modification as a reference due to its simplicity.

TABLE IV. The same description of Table III for the mass predictions of the GSW including in this case the third modification for comparison.

	LQCD [42]	QCD [40]	LQCD [41]	GSW	GSWm1	GSWm2	GSWm3
0^{-+}	⋯	⋯	2.1	2.074 ± 0.028	2.074 ± 0.028	2.074 ± 0.028	2.074 ± 0.028
0^{++}	1.98	⋯	>2.4	2.074 ± 0.028	2.074 ± 0.028	2.694 ± 0.021	2.317 ± 0.021
1^{--}	0.87	⋯	2.3	2.149 ± 0.017	1.562 ± 0.023	2.149 ± 0.017	2.149 ± 0.017
1^{+-}	1.25	⋯	>2.4	2.149 ± 0.017	1.562 ± 0.023	2.747 ± 0.013	2.360 ± 0.012
1^{++}	⋯	⋯	>2.4	2.149 ± 0.017	2.149 ± 0.017	2.747 ± 0.013	2.360 ± 0.012
0^{+-}	⋯	⋯	>2.4	2.074 ± 0.028	1.411 ± 0.052	2.694 ± 0.021	2.317 ± 0.021
0^{--}	⋯	⋯	⋯	2.074 ± 0.028	1.411 ± 0.052	2.074 ± 0.028	2.074 ± 0.028
1^{-+}	2.15	2.013	2.0	2.149 ± 0.017	2.149 ± 0.017	2.149 ± 0.017	2.149 ± 0.017

TABLE V. We show the masses of the particles in the PDG whose quantum numbers correspond to mesons [54] and compare them with our calculated values for the hybrids for the first and the second modes. It must be noted that those particles marked with $*$ are presented in PDG outside the summary table.

	$\pi(1800)$	$\eta(2010)^*$	$\pi(2070)^*$	$\eta(2100)^*$	$\eta(2190)^*$	$\eta(2320)^*$	$\pi(2360)^*$	$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
0^{-+}	1800^{+9}_{-10}	2010^{+35}_{-60}	2070 ± 35	2050^{+105}_{-50}	2190 ± 50	2320 ± 15	2360 ± 25	2074	2536	2074	2536
	$f_0(1710)$	$a_0(2020)^*$	$f_0(2060)^*$	$X(2540)^*$				$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
0^{++}	1740^{+8}_{-7}	2025 ± 30	2060 ± 10	2540^{+52}_{-28}				2074	2536	2694	3179
	$\omega(1420)$	$\rho(1450)$	$\omega(1650)$	$\phi(1680)$	$\rho(1700)$			$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
1^{--}	1410 ± 60	1465 ± 20	1670 ± 30	1680 ± 20	1720 ± 20			2149	2647	2149	2647
	$\omega(1960)^*$	$\phi(2170)$	$\omega(2205)^*$	$\rho(2270)^*$	$\omega(2290)^*$	$\omega(2330)^*$		$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
1^{--}	1960 ± 25	2162 ± 7	2205 ± 30	2270 ± 45	2290 ± 20	2330 ± 30		2149	2647	2149	2647
	$h_1(1170)$	$h_1(1415)$	$h_1(1595)$	$b_1(1960)^*$	$h_1(1965)^*$	$h_1(2215)^*$	$b_1(2240)^*$	$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
1^{+-}	1166 ± 8	1416 ± 8	1594^{+25}_{-75}	1960 ± 35	1965 ± 45	2215 ± 40	2240 ± 35	2149	2647	2746	3251
	$a_1(1930)^*$	$f_1(1970)^*$	$a_1(2095)^*$	$a_1(2270)^*$	$f_1(2310)^*$			$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
1^{++}	1930^{+30}_{-70}	1971 ± 15	2096 ± 138	2270^{+55}_{-40}	2310 ± 60			2149	2647	2746	3251

VI. PHENOMENOLOGICAL ANALYSIS FOR LIGHT HYBRIDS

Let us discuss our results in light of the experimental data for the spectra [54]. We recall that the hybrid states 0^{-+}, 0^{++}, 1^{--}, and 1^{++} have the same quantum numbers as regular mesons; hence, one cannot directly consider the latter as hybrid states. However, we will proceed to compare the spectra of heavy mesons, reported in Ref. [54], with our predictions for hybrids assuming that the model will guide us to identify which kind of these states might be hybrids. The corresponding masses are displayed in Table V. In the table, we also report the results for both the ground ($\mathrm{GSW}^{n=0}$) and the first excited ($\mathrm{GSW}^{n=1}$) states obtained from the GSW model. Since lattice QCD calculations indicate the necessity of an improvement of the present approach, we also include the masses predicted by the simple modification GSWm2 discussed in the previous section. From the comparisons between the theoretical predictions and the experimental data, one might conclude that, from the point of view of the model, the following states could be hybrids

0^{-+}: The $\pi(2070)$ and $\eta(2100)$ could be consistent with the ground state of a hybrid.

0^{++}: The $f_0(2060)$ could be the ground state, while the $X(2540)$ could be a first excited state. If future improvements of lattice calculations will confirm the need for the modification of the GSW model, the $X(2540)$ could be also consistent with the ground state of the corresponding hybrid.

1^{--}: The $\phi(2170)$ and $\omega(2205)^*$ both have masses consistent with the hybrid ground state.

1^{+-}: The $h_1(1965)^*$ could be a hybrid ground state.

1^{++}: The $a_1(2095)$ could be a ground state hybrid.

Let us remark that such a strategy is motivated by the predictive power of this model, as reflected in Refs. [6,16,55]. Hence, we might infer that the model provides a reasonable depiction of certain aspects of QCD, and the utilization of its predictions to identify states potentially attributed to hybrids is well justified.

In Table V, we observe that the masses of the candidate particles, many of which have not been considered or discovered, fall within the range of the lowest mode of our calculation in all cases, and some even fall within the range of the second mode. Upon this analysis, it is clear that finding a pure hybrid state will be challenging, as they are likely to be mixed with mesons. In fact, it is worth to notice that the widths of these states are very large, and, therefore, one might suspect that mixing of states could occur. In future investigations, we will consider applying the same strategy adopted in Ref. [11], where the GSW model has been used to establish the mixing condition between glueballs and meson states.

TABLE VI. We show the masses of the particles in the PDG for the quantum numbers that do not correspond to mesons. The $\pi_1(2015)$ has been omitted from the particle table [54].

				$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
0^{+-}				2074	2536	2694	3179
				$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
0^{--}				2074	2536	2074	2536
	$\pi_1(1400)$	$\pi_1(1600)$	$\pi_1(2015)$	$\mathrm{GSW}^{n=0}$	$\mathrm{GSW}^{n=1}$	$\mathrm{GSW}^{n=0}_{m2}$	$\mathrm{GSW}^{n=1}_{m2}$
1^{-+}	1354 ± 25	1661^{+15}_{-11}	2001 ± 122	2149	2647	2149	2647

More interesting are the other quantum numbers which do not correspond to known mesons 0^{+-}, 0^{--}, and 1^{-+}. No particle appears in the PDG tables for the first two, but for the 1^{-+} we have probably one candidate, the $\pi_1(2015)$ (see Table VI). In this case, as one can see, the reported mass is in agreement with the ground state predicted by the GSW model within the experimental error.

In Ref. [56], the authors studied the 1^{-+} mesons in the soft-wall model, finding that the lightest state has mass 1.1 GeV (if the mass scale is fixed from the rho mass). Thus, the GSW model predicts higher masses for these exotic states, as well as it does for scalar glueballs. The difference in the outcomes of the two models is important, since the SW model could support the hypothesis that $\pi_1(1400)$ is a candidate for such states, while according to the GSW model another structure should be assumed for $\pi_1(1400)$.

Let us conclude by noting that, when considering the masses in our calculations of the 0^{+-} and 0^{--} states, they should be investigated. However, in this case, special decay properties need to be examined for a distinctive characterization.

VII. THE SPECTRUM OF THE NONLIGHT HYBRIDS

Since the holographic approach here adopted relies on conformal symmetry, predictions can be realistic once the chiral symmetry of QCD is restored. Hence, the proposed model does not contain any dependence on the flavor of the constituent quarks of the hadrons. Nevertheless, further modifications of the approach can be taken into account to reproduce the masses of heavy hadrons [12,16]. In particular, we apply the approach introduced in Sec. III to s, c, and b quark-antiquark pairs. For this purpose, we follow the prescription addressed in Refs. [12,16], namely, to add a constant to the mass of the light mesons:

$$M_{\mathrm{heavy},n} = M_{\mathrm{light},n} + C. \tag{20}$$

Let us report the values of the constant C corresponding to the considered quark flavors [12,16]: $C_c = 2400$ MeV and $C_b = 8700$ MeV. We add here also C_s associated to strangeness which we have not studied before, $C_s = 300$ MeV. We then add the above constants to the mass spectra predicted by the GSW model previously calculated, for ground and excited states. We show in Tables VII and VIII the results obtained by performing this operation. We also apply the present procedure to the predictions obtained also for $\Delta_p = 1$, i.e., the second modification. In the next sections, we compare the results of this analysis with the predictions of other quark models and lattice QCD calculations.

TABLE VII. We show the masses obtained for the heavy hybrid interpolating fields defined above, Eqs. (9) and (10) having fixed all our parameters with the scalar hadrons in Ref. [16].

Mesonic hybrids ($\Delta_p = 0$)								
	$\bar{s}s$	$n=0$	$n=1$	$\bar{c}c$	$n=0$	$n=1$	$\bar{b}b$	$n=0$	$n=1$
0^{-+}		2374	2836		4474	4936		10774	11236
0^{++}		2374	2836		4474	4936		10774	11236
1^{--}		2449	2947		4549	5074		10849	11347
1^{+-}		2449	2947		4549	5074		10849	11347
1^{++}		2449	2947		4549	5074		10849	11347

Exotic hybrids ($\Delta_p = 0$)									
	$\bar{s}s$	$n=0$	$n=1$	$\bar{c}c$	$n=0$	$n=1$	$\bar{b}b$	$n=0$	$n=1$
0^{+-}		2374	2836		4474	4936		10774	11236
0^{--}		2374	2836		4474	4936		10774	11236
1^{-+}		2449	2947		4549	5074		10849	11347

TABLE VIII. We show the masses obtained for the heavy hybrid interpolating fields defined above, Eqs. (16) and (17) having fixed all our parameters with the scalar hadrons in Ref. [16].

Mesonic hybrids ($\Delta_p = 1$)									
	$\bar{s}s$	$n=0$	$n=1$	$\bar{c}c$	$n=0$	$n=1$	$\bar{b}b$	$n=0$	$n=1$
0^{-+}		2374	2836		4474	4936		10774	11236
0^{++}		2994	3479		5094	5579		11394	11879
1^{--}		2449	2947		4549	5074		10849	11347
1^{+-}		3047	3551		5147	5651		11447	11951
1^{++}		3047	3551		5147	5651		11447	11951

Exotic hybrids ($\Delta_p = 1$)									
	$\bar{s}s$	$n=0$	$n=1$	$\bar{c}c$	$n=0$	$n=1$	$\bar{b}b$	$n=0$	$n=1$
0^{+-}		2994	3479		5094	5579		11394	11879
0^{--}		2374	2836		4474	4936		10774	11236
1^{-+}		2449	2947		4549	5074		10849	11347

VIII. LATTICE QCD AND MODEL CALCULATIONS FOR HEAVY HYBRIDS

There have been many calculations interested in the study of heavy hybrid hadrons. In Table IX, we show some of their results [40,57–59].

Let us proceed by comparing these spectra with the outcomes of the GSW model (Table VII) and its modification (Table VIII). Let us start from the lattice evaluation from Refs. [40,57]—the predictions of the model overestimate the $s\bar{s}$ ground states 0^{-+} and 0^{++}. However, our calculations almost agree with the 1^{-+} ground and excited states. A good agreement is also found for the ground state of the 1^{-+} for the $c\bar{c}$ hadron [40,57]. We also compare our results with the model of Ref. [58] (SR). The main

TABLE IX. We show the masses obtained for the heavy hybrid hadrons, whose quantum numbers are defined by Eqs. (9) and (10), in Refs. [40,57–59].

Heavy hybrid masses									
	LQCD	LQCD	LQCD	SR	SR	NRQCD	NRQCD	NRQCD	NRQCD
	$(\bar{s}s)$ [57] $n=0$	$(\bar{s}s)$ [57] $n=1$	$(\bar{c}c)$ [40] $n=0$	$(\bar{c}c)$ [58] $n=0$	SR $(\bar{b}b)$ [58] $n=0$	$(\bar{c}c)$ [59] $n=0$	$(\bar{c}c)$ [59] $n=1$	$(\bar{b}b)$ [59] $n=0$	$(\bar{b}b$ [59]) $n=1$
0^{-+}	1.7	…	…	3.61	9.68	4.011	4.355	10.690	10.885
0^{++}	…	…	…	5.34	11.20	4.486	4.920	11.011	11.299
1^{--}	1.7	…	…	3.36	9.70	4.011	4.355	10.690	10.885
1^{+-}	…	…	…	4.53	10.70	4.145	4.511	10.761	10.970
1^{++}	…	…	…	5.06	11.09	4.145	4.511	10.761	10.970
0^{+-}	…	…	…	4.09	10.17	4.145	4.511	10.761	10.970
0^{--}	…	…		5.51	11.48	…	…	…	…
1^{-+}	2.1–2.2	3.6	4.369	3.70	9.79	4.011	4.355	10690	10885

differences can be found in the 0^{-+}, the 1^{--}, and the 1^{-+} states. Let us conclude with the comparison of the GSW model calculations, with those of Ref. [59] (NRQCD). Here, one can notice that our predictions are in line with the outcome of Ref. [59] for the $c\bar{c}$ and $b\bar{b}$ states for $n=0$. However, the excited states are overestimated.

IX. PHENOMENOLOGICAL ANALYSIS FOR HEAVY HYBRIDS

In this section, we will proceed as before. We will use the GSW model to examine the data and determine if they correspond to any states that could potentially be hybrids. In Table X, we present particle masses from the PDG [54] with mesonic quantum numbers also shared with possible heavy hybrids, and we compare them with the first two modes of our calculation.

In the $s\bar{s}$ case, the prediction of the GSW model overestimates, for example, the $\phi(2170)$ state. For the $c\bar{c}$ hadrons, the $\chi_{c0}(4500)$, the $\Psi(4660)$, and the $Z_c(4430)$ states are possible candidates. Finally, for $b\bar{b}$ hadrons, the $\chi_{b0}(2P)$, the $\Upsilon(10860)$, and the $\Upsilon(11020)$ are close to the spectra predicted by the GSW model. The $Z_b(10650)$ and the $\chi_{b1}(3P)$ are

TABLE X. We show the masses of the particles in the PDG whose quantum numbers addressed in the present analysis but corresponding to mesons [54] and compare them with the results of the calculations of the heavy hybrid spectra with the GSW model and its modification (GSW_{m2}).

$\bar{s}s$ hybrids	$\phi(1020)$	$\phi(1680)$	$\phi(2170)$					GSW	GSW_{m2}
1^{--}	1019.46 ± 0.02	1680 ± 20	2162 ± 7					2449	2449
$\bar{c}c$ hybrids	η_c $(1S)$	η_c $(2S)$						GSW	GSW_{m2}
0^{-+}	2983.9 ± 0.4	3637.5 ± 1.1						4474	4474
	χ_{c0} $(1P)$	$\chi_{c0}(3860)$	$\chi_{c0}(3915)$	$\chi_{c0}(4500)$	$\chi_{c0}(4700)$			GSW	GSW_{m2}
0^{++}	3862^{+66}_{-55}	3414.7 ± 0.3	3921.7 ± 1.8	4474 ± 6	4694^{+20}_{-7}			4474	5094
	$\Psi(3770)$	$\Psi(4040)$	$\Psi(4160)$	$\Psi(4230)$	$\Psi(4360)$	$\Psi(4415)$	$\Psi(4660)$	GSW	GSW_{m2}
1^{--}	3773.7 ± 0.4	4039 ± 1	4191 ± 5	4222.6 ± 2.6	4372 ± 9	4421 ± 4	4630 ± 6	4549	4549
	h_c $(1P)$	$Z_c(3900)$	$Z_c(4200)$	$Z_c(4430)$				GSW	GSW_{m2}
1^{+-}	3525.38 ± 0.11	3887.1 ± 2.6	4196^{+48}_{-42}	4478^{+15}_{-18}				4549	5147
	χ_{c1} $(1P)$	$\chi_{c1}(3872)$	$\chi_{c1}(4140)$	$\chi_{c1}(4274)$				GSW	GSW_{m2}
1^{++}	3510.67 ± 0.05	3871.65 ± 0.06	4146.5 ± 3.0	$4286^{+8}_{.9}$				4549	5147
$\bar{b}b$ hybrids	η_b $(1S)$							GSW	GSW_{m2}
0^{-+}	9398.7 ± 2.0							10774	10774
	χ_{b0} $(1P)$	χ_{b0} $(2P)$						GSW	GSW_{m2}
0^{++}	9859.44 ± 0.73	10232.5 ± 0.9						10774	11394
	Υ $(1S)$	Υ $(2S)$	Υ $(3S)$	Υ $(4S)$	$\Upsilon(10860)$	$\Upsilon(11020)$		GSW	GSW_{m2}
1^{--}	9460.30 ± 0.26	10023.26 ± 0.31	10355.2 ± 0.31	10579.4 ± 1.2	$10885.2^{+2.6}_{1.6}$	11000 ± 4		10849	10849
	h_b $(1P)$	h_b $(2P)$	$Z_b(10610)$	$Z_b(10650)$				GSW	GSW_{m2}
1^{+-}	9899.3 ± 0.8	10259.8 ± 1.6	10607.2 ± 2.0	10652.2 ± 1.5				10849	11447
	χ_{b1} $(1P)$	χ_{b1} $(2P)$	χ_{b1} $(3P)$					GSW	GSW_{m2}
1^{++}	9892.78 ± 0.57	10255.46 ± 0.72	10513.42 ± 0.94					10849	11447

overestimated, but the disagreement is not particularly big.

X. CONCLUSIONS

We have used the GSW model, previously developed for conventional hadrons, to analyze hybrid hadrons. To this aim, we have computed the conformal masses by examining the potential minimal p-form field configuration that can be obtained from quark and gluon fields. These configurations define the spins and parities of the hybrids and determine the corresponding conformal masses, which, in turn, characterize the corresponding bound state equations in the fifth dimension. It is important to highlight that our calculation is parameter-free, as the two parameters employed in the approach have been determined by the scalar glueballs and mesons [12,16]. For the heavy hybrids, we have included the same additional parameters as those used for heavy mesons, which essentially represent the heavy quark masses. The results obtained in our parameter-free calculation has been compared with other model calculations. We have analyzed in detail the similarities and differences. We tend to agree with the lattice results and the NRQCD better than with the SR approach. Moreover, we found out that the GSW model, due to the simplicity lowest-order p-forms, leads to the same conformal mass for different states and, therefore, to mass degeneracies. Looking back at Table I, we see degeneracies between the 0^{-+}, the 0^{++}, the 0^{+-}, and the 0^{--} hybrids and the 1^{--}, the 1^{+-}, the 1^{++}, and the 1^{-+} hybrids. In order to remove such a degeneracy, not predicted by recent lattice calculations, we took into account the effects of anomalous dimensions for some of these states, and, hence, the corresponding degeneracies are eliminated, as can be seen in Table III for the two modifications studied. The remaining degeneracies correspond to underlying symmetries not accounted for in QCD. Therefore, additional adjustments to the naive GSW model should be pursued for more accurate predictions. Therefore, we can assert that the model has the potential to be modified to accurately parametrize the dynamics underlying the spectra of hybrid mesons once the experimental scenario is clarified. Finally, we have proposed a different approach to analyze the PDG spectra. We assume our model's results are accurate mass values for hybrids and attempt to identify potential states in the data using these mass values, indicating the possible presence of hybrids in the spectra. In this way, we have predicted several possible hybrid states, but, in many cases given the closeness to conventional mesons states, one expects strong mixing between mesons and hybrids.

ACKNOWLEDGMENTS

M. R. thanks for the financial support received under the STRONG-2020 project of the European Union's Horizon 2020 research and innovation program: Fund No. 824093. V. V. was supported by MCIN/AEI/10.13039/501100011033, European Regional Development Fund Grant No. PID2019–105439 GB-C21, and GVA PROMETEO/2021/083.

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