How much strangeness is needed for the axial-vector form factor of the nucleon?

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We consider the axial-vector together with its induced pseudoscalar form factor of the nucleon as computed from the chiral Lagrangian with nucleon and isobar degrees of freedom. The form factors are evaluated at the one-loop level, where particular emphasis is put on the use of on-shell masses in the loop expressions. Our results are presented in terms of a novel set of basis functions that generalize the Passarino-Veltman scheme to the case where power-counting violating structures are to be subtracted. The particularly important role of the isobar degrees of freedom is emphasized. We obtain a significant and simultaneous fit to the available lattice QCD results based on flavor SU(2) ensembles for the baryon masses and form factors up to pion masses of about 500 MeV. Our fit includes sizeable finite volume effects that are implied by using in-box values for the hadron masses entering our one-loop expressions. We conclude that from flavor SU(2) ensembles it appears not possible to predict the empirical form factor at the desired precision. Effects from strange quarks are expected to remedy the situation.

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I. INTRODUCTION

The flavor SU(2) chiral Lagrangian properly formulated with nucleon and isobar degrees of freedom plays an important role in the understanding of lattice QCD results on the form factors of the nucleon [1–7]. Such a framework is designed to be applied to lattice QCD ensembles at fixed physical heavy-quark masses, but unphysical values of the masses of the up and down quarks [8–13].

The use of the flavor SU(2) chiral Lagrangian with nucleon and isobar degrees of freedom has a long history [1–7]. Still, there is some controversy as to what is the most effective framework to tackle such systems. Different assumptions on how to include and consider the isobar field are possible [4,7,14,15]. Here lattice QCD simulations are expected to help identifying the optimal framework.

At this stage most useful are somewhat older lattice QCD data on flavor SU(2) ensembles [11–13] as analyzed already by our group in [7]. This is so since here the convergence properties of the chiral approach are not affected by possible intricate strange quark mass effects. One should keep in mind, however, that one should not expect to obtain from such lattice QCD data an accurate reproduction of the empirical form factor at physical up, down and strange quark masses. It would be quite a surprise if the strange quarks do not play any role. An evaluation of the axial-vector form factor showed that the impact of the pion-isobar loop effects is significant. Therewith a large sensitivity on the used framework was illustrated. The success of the study [7] rests on the novel feature of insisting on the use of on-shell hadron masses inside the loop expressions. Application of more conventional approaches to such SU(2) flavor lattice data have not been documented in the literature so far. Since our approach is not mainstream yet, it is nevertheless useful to scrutinize it against further quantities measurable on lattice QCD ensembles. We note however, that there are some works [16,17] based on flavor SU(3) ensembles that applied more conventional chiral extrapolation techniques like proposed in [18]. In this case, however, an application of flavor SU(2) chiral extrapolation formulas may be questioned since the LEC will have a dependence on the not-so-well known variation of the chiral strange quark masses of such ensembles.

In the present work we will consider the induced pseudoscalar form factor of the nucleon. It offers a unique further test bed of our approach as it is largely determined by the set of low-energy constants (LEC) that enter the axial-vector form factor already. For the first time we derive results in terms of a novel generalization of the

Passarino-Veltman scheme that permits a systematic subtraction of power-counting structures without generating kinematical singularities nor acausal structures [19]. A chiral expansion is implied by using approximated coefficient functions in front of the generalized basis loop functions that are evaluated in terms of on-shell hadron masses rather than bare masses.

II. THE INDUCED PSEUDOSCALAR FORM FACTOR OF THE NUCLEON

The axial current in a nucleon state is parametrized by an axial-vector, $G_A(q^2)$, and an induced pseudoscalar, $G_P(q^2)$, form factor

$$
\begin{aligned}
\langle N(\bar{p})|A_i^\mu(0)|N(p)\rangle &= \bar{u}_N(\bar{p}) F_A^\mu(\bar{p},p)\frac{\tau_i}{2}u_N(p),\\
F_A^\mu(\bar{p},p)|_{\text{on shell}} &= \left(\gamma^\mu G_A(q^2)+\frac{q^\mu}{2\,M_N}G_P(q^2)\right)\gamma_5,\\
G_P(q^2) &= \frac{8\sqrt{2}\,M_N^4}{(d-2)(4\,M_N^2-t)t}\mathrm{Tr}\left[\gamma_\mu\gamma_5\frac{\bar{\not{p}}+M_N}{2\,M_N}F_A^\mu(\bar{p},p)\frac{\not{p}+M_N}{2\,M_N}\right]\\
&\quad-\frac{8\sqrt{2}\,M_N^4(4\,M_N^2(d-1)-t(d-2))}{(d-2)(4\,M_N^2-t)t^2}\mathrm{Tr}\left[\frac{q_\mu}{2\,M_N}\gamma_5\frac{\bar{\not{p}}+M_N}{2\,M_N}F_A^\mu(\bar{p},p)\frac{\not{p}+M_N}{2\,M_N}\right],
\end{aligned}
\tag{1}
$$

with $q^2=(\bar{p}-p)^2$ and $\bar{p}^2=p^2=M_N^2$ (see e.g. [20]). Using exact isospin symmetry the form factors are introduced in terms of the conventional Pauli matrices τ_i. They can be conveniently expressed as a Dirac trace over the amplitude $F_A^\mu(\bar{p},p)$ in space-time dimensions d, where we recall the form relevant for our current work [7].

The chiral Lagrangian with nucleon and isobar degrees of freedom was developed in a series of works [4,6,7,21–29]. Initially the heavy-baryon chiral perturbation theory [21–23] was applied. The relativistic form of the chiral Lagrangian was used in [24–27]. Less well explored is the role of the isobar degrees of freedom [4,6,7,28,29]. We will use the conventions of [7,19,30], in which a renormalization scheme based on a generalized Passarino–Veltman reduction scheme [31] was propagated. Altogether, at the one-loop level we find

$$
\begin{aligned}
F_A^\mu(\bar{p},p) &= \left(\frac{1}{\sqrt{2}}g_A\,\gamma^\nu\gamma_5\right)\left(Z_N\,g_\nu^\mu-\frac{q_\nu q^\mu}{q^2-m_\pi^2}[Z_N+f_\pi/f+Z_\pi-2]\right)\\
&\quad+\frac{1}{\sqrt{2}}(g_R\,q^2+4\,g_\chi^+(2\,B_0\,m))\gamma^\mu\gamma_5-\frac{2\,M}{\sqrt{2}}(4\,g_\chi^++g_\chi^-)\gamma_5\frac{q^\mu(2\,B_0\,m)}{q^2-m_\pi^2}-\frac{1}{\sqrt{2}}g_R\,\not{q}\gamma_5 q^\mu\\
&\quad+\frac{g_A}{f^2}\left\{J_\pi^\mu(\bar{p},p)+J_{\pi N}^\mu(\bar{p},p)+J_{N\pi}^\mu(\bar{p},p)\right\}+\frac{g_A^3}{4f^2}J_{N\pi N}^\mu(\bar{p},p)+\frac{5\,h_A\,f_S^2}{9f^2}J_{\Delta\pi\Delta}^\mu(\bar{p},p)\\
&\quad+\frac{f_S}{3f^2}\left\{J_{\pi\Delta}^\mu(\bar{p},p)+J_{\Delta\pi}^\mu(\bar{p},p)\right\}+\frac{2\,g_A\,f_S^2}{3f^2}\left\{J_{N\pi\Delta}^\mu(\bar{p},p)+J_{\Delta\pi N}^\mu(\bar{p},p)\right\}+\mathcal{O}(Q^4),
\end{aligned}
\tag{2}
$$

where we encounter some low-energy constants (LEC) and loop integrals $J_{\dots}^\mu(\bar{p},p)$. Contributions from two-loop diagrams are relevant at chiral order Q^4. Unlike in our previous works on $G_A(q^2)$ our focus here is on the induced pseudoscalar form factor $G_P(q^2)$. In turn additional contributions are required that were not considered by us before. The wave-function factor, not only for the nucleon, Z_N, but also for pion Z_π together with the LEC l_4 is encountered. We use the one-loop expression as given by [24] in Eqs. (33)–(35) with

$$
\begin{aligned}
Z_\pi &= 1+\frac{2}{3f^2}\bar{I}_\pi-2\frac{m_\pi^2}{f^2}l_4\,, \qquad f_\pi = f-\frac{1}{f}\bar{I}_\pi+\frac{m_\pi^2}{f}l_4\,,\\
\bar{I}_\pi &= \frac{m_\pi^2}{(4\pi)^2}\log\frac{m_\pi^2}{\mu^2}\,, \qquad m_\pi^2 = 2\,B_0\,m+\frac{m_\pi^2}{2f^2}\bar{I}_\pi-\frac{m_\pi^4}{2f^2}l_3\,,
\end{aligned}
\tag{3}
$$

in terms of the renormalization scale μ of dimensional regularization. Explicit expressions for the form of the nucleon wave function factor Z_N are given in [7].

Like in a computation of $G_A(q^2)$ the leading order LEC g_A, h_A, f_S but also some subleading order LEC g_R, g_χ^+, but now in addition g_χ^-, are needed in (2). Given the specific form of the form factor in (2) it is straight forward to match our convention for the LEC to other choices in the literature. Further two-body LEC g_S, g_V, g_T, g_R and $f_A^\pm, f_M$ are involved in the loop functions, which we will specify in terms of their integrands

$$J^\mu_{\cdots}(\bar p, p) = i\,\mu^{4-d}\int \frac{d^d l}{(2\pi)^d} K^\nu_{\cdots}(\bar p, p; l)(g^\mu_\nu - q_\nu q^\mu/(q^2-m_\pi^2)). \tag{4}$$

From our previous works [7,30] one can find

$$\begin{aligned}
K^\mu_\pi &= -\frac{1}{\sqrt{2}}\frac{\gamma^\mu\gamma_5}{l^2-m_\pi^2} + \frac{1}{\sqrt{2}}\frac{q^\mu}{3\,m_\pi^2}\frac{4\,M_N}{l^2-m_\pi^2}\gamma_5,\\
K^\mu_{\pi N} &= -\frac{1}{\sqrt{2}}\Big(\gamma^\mu + 2\,g_S\,l^\mu + 2\,g_T\,i\,\sigma^{\mu\nu}l_\nu - 16\,\sqrt{3}\,g_F\,i\,\sigma^{\mu\nu}q_\nu\\
&\quad + \frac{1}{2}g_V\Big[(\gamma^\mu\,\bar p\cdot l + \not{l}\,\bar p^\mu) + (\gamma^\mu(p-l)\cdot l + \not{l}\,(p-l)^\mu)\Big]\Big)S_N(p-l)\,\frac{\not{l}\gamma_5}{l^2-m_\pi^2},\\
K^\mu_{N\pi} &= -\frac{1}{\sqrt{2}}\frac{\not{l}\gamma_5}{l^2-m_\pi^2}S_N(\bar p - l)\Big(\gamma^\mu + 2\,g_S\,l^\mu - 2\,g_T\,i\,\sigma^{\mu\nu}l_\nu\\
&\quad + \frac{1}{2}g_V\Big[(\gamma^\mu\,p\cdot l + \not{l}\,p^\mu) + (\gamma^\mu(\bar p-l)\cdot l + \not{l}\,(\bar p-l)^\mu)\Big] + 16\,\sqrt{3}\,g_F\,i\,\sigma^{\mu\nu}q_\nu\Big),\\
K^\mu_{N\pi N} &= -\frac{1}{\sqrt{2}}\not{l}\gamma_5\,S_N(\bar p - l)\,\frac{\gamma^\mu\gamma_5}{l^2-m_\pi^2}\,S_N(p-l)\,\not{l}\gamma_5,\\
K^\mu_{\pi\Delta} &= -\frac{1}{\sqrt{2}}\frac{\gamma_5 l_\nu}{l^2-m_\pi^2}\Big((f_A^- - 5f_A^+)\not{l}\,S_\Delta^{\mu\nu}(p-l) - (f_A^- + 5f_A^+)\,\gamma^\mu\,l_\alpha\,S_\Delta^{\alpha\nu}(p-l)\\
&\quad - 4f_M\,q_\alpha\Big[\gamma^\alpha\,S_\Delta^{\mu\nu}(p-l) - \gamma^\mu S_\Delta^{\alpha\nu}(p-l)\Big]\Big),\\
K^\mu_{\Delta\pi} &= \frac{1}{\sqrt{2}}\Big((f_A^- - 5f_A^+)\,S_\Delta^{\nu\mu}(\bar p - l)\not{l} - (f_A^- + 5f_A^+)\,S_\Delta^{\nu\alpha}(\bar p - l)l_\alpha\gamma^\mu\\
&\quad - 4f_M\Big[S_\Delta^{\nu\alpha}(\bar p - l)\gamma^\mu - S_\Delta^{\nu\mu}(\bar p - l)\gamma^\alpha\Big]q_\alpha\Big)\,\frac{l_\nu\gamma_5}{l^2-m_\pi^2},\\
K^\mu_{N\pi\Delta} &= \frac{1}{\sqrt{2}}\frac{\not{l}\gamma_5\,l_\tau}{l^2-m_\pi^2}S_N(\bar p - l)\Big(S_\Delta^{\mu\tau}(p-l) + 2\frac{f_E}{f_S}\Big[\gamma_\mu S_\Delta^{\alpha\tau}(p-l) - \gamma^\alpha S_\Delta^{\mu\tau}(p-l)\Big]q_\alpha\Big),\\
K^\mu_{\Delta\pi N} &= \frac{1}{\sqrt{2}}\Big(S_\Delta^{\sigma\mu}(\bar p - l) + 2\frac{f_E}{f_S}\Big[S_\Delta^{\sigma\mu}(\bar p - l)\gamma^\alpha - S_\Delta^{\sigma\alpha}(\bar p - l)\gamma^\mu\Big]q_\alpha\Big)S_N(p-l)\,\frac{l_\sigma\not{l}\gamma_5}{l^2-m_\pi^2},\\
K^\mu_{\Delta\pi\Delta} &= -\frac{1}{\sqrt{2}}l_\sigma S_\Delta^{\sigma\alpha}(\bar p - l)\,\frac{\gamma^\mu\gamma_5}{l^2-m_\pi^2}\,S^\Delta_{\alpha\beta}(p-l)l^\beta,
\end{aligned} \tag{5}$$

with the baryon propagators

$$\begin{aligned}
S_N(k) &= \frac{1}{\not{k} - M_N},\\
S_\Delta^{\mu\nu}(k) &= \frac{-1}{\not{k} - M_\Delta}\Big(g^{\mu\nu} - \frac{\gamma^\mu\gamma^\nu}{d-1} + \frac{(k^\mu\gamma^\nu - k^\nu\gamma^\mu)}{(d-1)M_\Delta} - \frac{(d-2)k^\mu k^\nu}{(d-1)M_\Delta^2}\Big).
\end{aligned} \tag{6}$$

The main target of our current work is the derivation of the induced pseudoscalar form factor $G_P(q^2)$ as implied by (5) and properly expanded into its chiral moments. We will apply our counting rules formulated in terms of on-shell hadron masses [7,30].

III. CHIRAL EXPANSION OF THE FORM FACTOR

In this section we will express the loop functions as introduced with (5) in terms of a generalized Passarino-Veltman reduction scheme [7,19,31]. In a first step we apply the projection scheme (1) that avoids the need to consider tensor-type loop integrals. Such a projection is always possible. We write

$$G_A(t) = g_A\, Z_N + 4\, g_\chi^+\, m_\pi^2 + g_R\, t + \frac{g_A}{f^2}\Big\{J^A_\pi(t) + J^A_{\pi N}(t) + J^A_{N\pi}(t)\Big\} + \frac{g_A^3}{4f^2} J^A_{N\pi N}(t) + \frac{5\, h_A\, f_S^2}{9f^2} J^A_{\Delta\pi\Delta}(t)$$
$$+ \frac{f_S}{3f^2}\Big\{J^A_{\pi\Delta}(t) + J^A_{\Delta\pi}(t)\Big\} + \frac{2\, g_A\, f_S^2}{3f^2}\Big\{J^A_{N\pi\Delta}(t) + J^A_{\Delta\pi N}(t)\Big\} + \mathcal{O}(Q^4), \tag{7}$$

and

$$\frac{t-m_\pi^2}{4\,M_N^2} G_P(t) = -g_A(Z_N + Z_\pi + f_\pi/f - 2) - m_\pi^2(4\, g_\chi^+ + g_\chi^-) - g_R(t - m_\pi^2)$$
$$+ \frac{g_A}{f^2}\Big\{J^P_\pi(t) + J^P_{\pi N}(t) + J^P_{N\pi}(t)\Big\} + \frac{g_A^3}{4f^2} J^P_{N\pi N}(t) + \frac{5\, h_A\, f_S^2}{9f^2} J^P_{\Delta\pi\Delta}(t)$$
$$+ \frac{f_S}{3f^2}\Big\{J^P_{\pi\Delta}(t) + J^P_{\Delta\pi}(t)\Big\} + \frac{2\, g_A\, f_S^2}{3f^2}\Big\{J^P_{N\pi\Delta}(t) + J^P_{\Delta\pi N}(t)\Big\} + \mathcal{O}(Q^4), \tag{8}$$

where for any loop function $J^\mu_{\cdots}(\bar p, p) \to 4\, M_N^2\, J^P_{\cdots}(t)/(t - m_\pi^2)$ and $J^\mu_{\cdots}(\bar p, p) \to J^A_{\cdots}(t)$ we introduced their suitably projected forms. It is convenient to monitor the consequences of the chiral Ward identities, which imply in particular a correlation of the pseudoscalar and axial-vector loop functions. In the chiral limit it should hold

$$\lim_{m\to 0}\left[G_A(t) + \frac{t}{4\,M_N^2} G_P(t)\right] = 0 \to \qquad \lim_{m\to 0}[J^A_{\ldots} + J^P_{\ldots}] = 0, \tag{9}$$

which we verified by explicit computations in our renormalization scheme.

The contributions to the form factor $G_A(t)$ and $G_P(t)$ are computed in application of a novel reduction scheme [19], in terms of scalar loop functions only. Recently the Passarino–Veltman scheme was supplemented systematically by a set of additional basis functions that leads to expressions free of kinematical constraints and that comply with the expectation of dimensional counting rules [19]. Such an extension is required once triangle loop contributions are considered. While in our previous work [7] the Passarino-Veltman set was extended by one specific loop function in an evaluation of $G_A(t)$ only, it turned out that the more general scheme as proposed in [19] is required for $G_P(t)$. Our truncated expressions are implied by an expansion of the coefficient functions in chiral moments

$$t \sim m_\pi^2 \sim Q^2, \qquad \delta = M_\Delta - M_N\left(1 + \frac{\Delta}{M}\right) \sim Q^2, \tag{10}$$

where we keep the on-shell masses unexpanded. The chiral limit value of the nucleon and isobar masses are M and $M + \Delta$ respectively.

While we detail all axial-vector loop functions in the Appendix, the so-far unknown pseudoscalar ones are given here

$$\begin{aligned}
\bar J^P_\pi(t) &= \frac{1}{3}\bar I_\pi + \mathcal{O}(Q^4),\\
\bar J^P_{\pi N}(t) + \bar J^P_{N\pi}(t) &= m_\pi^2\left(1 - \frac{8}{3}M_N(g_S - 2g_T)\right)\bar I_{\pi N} + \mathcal{O}(Q^4),\\
\bar J^P_{N\pi N}(t) &= -\bar I_\pi - m_\pi^2\,\bar I_{\pi N} - 4\,m_\pi^2 M_N^2\left(\bar I^{(2,0)}_{N\pi N}(t) + \bar I^{(0,2)}_{N\pi N}(t)\right) + \mathcal{O}(Q^4),\\
\bar J^P_{\pi\Delta}(t) + \bar J^P_{\Delta\pi}(t) &= \frac{8}{9}\left(25 f_A^+\left[m_\pi^2\alpha^P_{12} - 2\delta M_N\, r\,\alpha^P_{13}\right] + f_A^-\left[m_\pi^2\alpha^P_{22} - 2\delta M_N\, r\,\alpha^P_{23}\right] - 8 f_M\, r^2\left[t\,\alpha^P_{31} - m_\pi^2\alpha^P_{32}\right]\right)M_N\bar I_{\pi\Delta} + \mathcal{O}(Q^4),\\
\bar J^P_{N\pi\Delta}(t) + \bar J^P_{\Delta\pi N}(t) &= \frac{16}{9r}m_\pi^2\left(\bar I_{\pi N} - \alpha^P_{42}\bar I_{\pi\Delta}\right) + \frac{2}{9}\left(r\,t\,\alpha^P_{41} + 8\delta M_N\alpha^P_{43} + 12\frac{f_E M_N\, r}{f_S}\left[t\,\alpha^P_{51} - m_\pi^2\alpha^P_{52}\right]\right)\bar I_{\pi\Delta}\\
&\quad - \frac{8}{3}\frac{f_E M_N\, r}{f_S}\left[t\,\alpha^P_{61} - m_\pi^2\alpha^P_{62}\right]M_N^2\left(\bar I^{(1,0)}_{\Delta\pi N}(t) + \bar I^{(0,1)}_{N\pi\Delta}(t)\right) - \frac{4}{9}\left[t\,\alpha^P_{71} - 3\,m_\pi^2\alpha^P_{72}\right]\\
&\quad \times M_N^2\left(\bar I^{(2,0)}_{\Delta\pi N}(t) + \bar I^{(0,2)}_{N\pi\Delta}(t)\right) + \mathcal{O}(Q^4),\\
\bar J^P_{\Delta\pi\Delta}(t) &= -\frac{2}{3}\left(2\,r\,t\,\alpha^P_{81} + \frac{5}{9}m_\pi^2\alpha^P_{82} - \frac{10}{3}\delta M_N\alpha^P_{83}\right)\bar I_{\pi\Delta} - \frac{4}{3}\left[t\,\alpha^P_{91} + \frac{1}{3}m_\pi^2\alpha^P_{92}\right]M_N^2\left(\bar I^{(2,0)}_{\Delta\pi\Delta}(t) + \bar I^{(0,2)}_{\Delta\pi\Delta}(t)\right) + \mathcal{O}(Q^4),
\end{aligned} \tag{11}$$

with $r = \Delta/M$ and $\alpha_{ab}^{A,P} \to 1$ at $r \to 0$. In Table I we provide specific values for such coefficients as needed in our global fit scenario. The coefficients α_{ab}^{A} characterize the corresponding loop functions $\bar{J}^{A}_{...}(t)$ as they are detailed in the Appendix. Particularly important are α_{82}^{P} and α_{82}^{A} that differ significantly from their limit value.

The set of basis loop functions are

$$
\begin{aligned}
\bar{I}_{\pi} &= \frac{m_\pi^2}{16\,\pi^2}\log\frac{m_\pi^2}{\mu^2} \sim Q^2, \\
F_{L\pi R}(u,v) &= m_\pi^2 + u(M_L^2 - m_\pi^2 - (1-u)M_N^2) + v(M_R^2 - m_\pi^2 - (1-v)M_N^2) + uv(2\,M_N^2 - t), \\
\bar{I}_{\pi R} &= \frac{\gamma_N^R - 2}{16\,\pi^2} - \int_0^1 \frac{dv}{16\,\pi^2}\log\frac{F_{L\pi R}(0,v)}{M_R^2} \sim Q, \\
\bar{I}^{(m,n)}_{L\pi R}(t) &= -\frac{\gamma^{(m,n)}_{L\pi R}}{16\,\pi^2 M^2} + \int_0^1\int_0^{1-u} \frac{dv\,du\,u^m\,v^n}{16\,\pi^2 F_{L\pi R}(u,v)} \sim Q^0,
\end{aligned} \tag{12}
$$

where we introduced some subtraction terms for later convenience. We recall from [19] that it suffices to consider $\bar{I}^{(0,n)}_{L\pi R}(t)$ and $\bar{I}^{(n,0)}_{L\pi R}(t)$ as additional basis functions. All remaining terms can be decomposed in terms of those without running into kinematical constraints or power-counting violating structures.

At $\gamma_N^R \to 0$ and $\gamma^{(m,n)}_{L\pi R} \to 0$ our renormalized loop functions follow the expectation of dimensional counting with $\Delta/M = r \sim Q$ as indicated in (12). In this case all bubble and triangle contributions in (8) start at order Q^3. The only Q^2 term is implied by the pion tadpole contribution. While such a counting is well justified for somewhat larger pion masses with $m_\pi \sim \Delta$, it loses its efficiency in the chiral domain with $m_\pi \ll \Delta$. Here one may integrate out the isobar degrees of freedom and expand in m_π/Δ. In order to properly treat both domains in our scheme we use the subtraction terms

$$
\begin{aligned}
&a = r\,(2+r), \qquad \gamma_N^\Delta = a\log\frac{1+a}{a}, \qquad \gamma_N^N = 0\,, \\
&\gamma^{(0,0)}_{\Delta\pi N} = \gamma^{(0,0)}_{N\pi\Delta} = \frac{1}{a}\log(1+a) + \log\frac{1+a}{a}, \\
&\gamma^{(1,0)}_{\Delta\pi N} = \gamma^{(0,1)}_{N\pi\Delta} = \frac{1+a}{3a} - \frac{1}{3a^2}\log(1+a) - \frac{a}{3}\log\frac{1+a}{a}, \\
&\gamma^{(0,1)}_{\Delta\pi N} = \gamma^{(1,0)}_{N\pi\Delta} = \frac{-2+a}{6\,a} + \frac{2+3a}{6\,a^2}\log(1+a) - \frac{a}{6}\log\frac{1+a}{a}, \\
&\gamma^{(2,0)}_{\Delta\pi N} = \gamma^{(0,2)}_{N\pi\Delta} = \frac{-2+a+a^2-2\,a^3}{10\,a^2} + \frac{1}{5a^3}\log(1+a) + \frac{a^2}{5}\log\frac{1+a}{a}, \\
&\gamma^{(0,2)}_{\Delta\pi N} = \gamma^{(2,0)}_{N\pi\Delta} = \frac{-12-24\,a+a^2-2\,a^3}{60\,a^2} + \frac{6+5a(3+2\,a)}{30\,a^3}\log(1+a) + \frac{a^2}{30}\log\frac{1+a}{a}, \\
&\gamma^{(0,0)}_{\Delta\pi\Delta} = \log\frac{1+a}{a}, \\
&\gamma^{(1,0)}_{\Delta\pi\Delta} = \gamma^{(0,1)}_{\Delta\pi\Delta} = \frac{1}{2}\left(1 - a\log\frac{1+a}{a}\right), \\
&\gamma^{(2,0)}_{\Delta\pi\Delta} = \gamma^{(0,2)}_{\Delta\pi\Delta} = \frac{1}{6}\left(1 - 2\,a + 2\,a^2\log\frac{1+a}{a}\right),
\end{aligned} \tag{13}
$$

TABLE I. Values of the α_{ab}^{P} and α_{ab}^{A} in our global fit with $r = 0.343$.

$\alpha_{12}^{A}/\alpha_{12}^{P}$	1.164/1.185	$\alpha_{51}^{A}, \alpha_{51}^{P}, \alpha_{52}^{P}$	1.239
$\alpha_{13}^{A}, \alpha_{13}^{P}$	1.194	$\alpha_{61}^{A}, \alpha_{61}^{P}, \alpha_{62}^{P}$	0.761
$\alpha_{22}^{A}/\alpha_{22}^{P}$	0.715/0.612	$\alpha_{71}^{A}, \alpha_{71}^{P}, \alpha_{72}^{P}$	0.949
$\alpha_{23}^{A}, \alpha_{23}^{P}$	0.533	$\alpha_{81}^{A}, \alpha_{81}^{P}$	1.554
$\alpha_{31}^{A}, \alpha_{31}^{P}, \alpha_{32}^{P}$	1.044	$\alpha_{82}^{A}/\alpha_{82}^{P}$	2.783/3.481
$\alpha_{41}^{A}, \alpha_{41}^{P}$	0.932	$\alpha_{83}^{A}, \alpha_{83}^{P}$	1.399
$\alpha_{42}^{A}/\alpha_{42}^{P}$	1.276/1.250	$\alpha_{91}^{A}, \alpha_{91}^{P}$	1.034
$\alpha_{43}^{A}, \alpha_{43}^{P}$	1.494	α_{92}^{P}	0.977

which are crucial in the chiral domain, since if not included an explicit evaluation of a class of two-loop diagrams would be needed [32]. Now we are in line with dimensional counting rules

$$\bar{I}_{\pi\Delta}(M_N^2) \sim \frac{m_\pi^2}{\Delta\, M} \sim Q^2, \qquad \bar{I}^{(m,n)}_{N\pi\Delta}(t) \sim \frac{m_\pi}{\Delta\, M^2} \sim Q,$$
$$\bar{I}^{(m,n)}_{\Delta\pi\Delta}(t) \sim \frac{m_\pi^2}{\Delta^2\, M^2} \sim Q^2, \tag{14}$$

that arise in the chiral domain with $m_\pi/\Delta \sim Q$.

IV. LEC FROM LATTICE QCD DATA

We consider QCD lattice data on two-flavor ensembles from ETMC [33,34], CLS [11,35], and RQCD [10] QCD lattice collaborations. While all such references provide the nucleon pion and nucleon masses together with the form factor $G_A(t)$ on their ensembles, only the ETMC provides in addition the isobar mass. This is unfortunate since the latter play a crucial role in the understanding of such form factors. Moreover, results on $G_P(t)$ are provided on CLS and RQCD ensembles only. After a comparison of their ratios

$$\frac{t - m_\pi^2}{4\, M_N^2} \frac{G_P(t)}{G_A(t)}, \tag{15}$$

it is evident that their results are largely incompatible for $G_P(t)$. This follows, since in our previous work we found that their results on $G_A(t)$ appear quite compatible. As explained in [11,35] the form factor $G_P(t)$ is suffering from sizeable and difficult-to-control excited state contamination at small pion masses. While we made an attempt to fit the results of [10] with our scheme we badly failed to recover their results for $G_P(t)$.

In the following we discuss the results of global fits to the available dataset, where we excluded all results for $G_P(t)$ from [10]. We use the evolutionary fit algorithm GENEVA [36] with the recently implemented mpi consumer [37]. With this significant update it is possible now to call the evolutionary algorithm as a heterogeneous mpi job. As compared to previous versions a much better scaling behavior in the number of involved cores is observed. Our results are typically obtained on 2830 cores from reserved nodes on the Green Cube at GSI.

Our fit strategy was detailed already in our previous work [7]. We adjust the LEC to our expressions for the nucleon and isobar masses, M_N and M_Δ (with finite-volume effects included following Ref. [38]), and for the nucleon axial-vector form factor, $G_A(t)$ and $G_P(t)$ from (7), (8) (without explicit finite-volume effects), to the lattice data. As explained, we use on-shell in-box meson and baryon masses in the loop expressions throughout this work. We coin such volume effects as *implicit* volume effects mostly driven by the volume dependence of the in-box isobar mass.

TABLE II. Lattice scales as determined in our global fit.

Group	Scale (fm)	This work	Lattice group
ETMC	$a_{\beta=3.8}$	$0.1095(^{+0.0005}_{-0.0004})$	0.0995(7) [33]
	$a_{\beta=3.9}$	$0.0941(^{+0.0002}_{-0.0005})$	0.089(5) [34]
	$a_{\beta=4.05}$	$0.0736(^{+0.0002}_{-0.0002})$	0.070(4) [34]
	$a_{\beta=4.2}$	$0.0590(^{+0.0002}_{-0.0002})$	0.056(4) [34]
CLS	$a_{\beta=5.2}$	$0.0850(^{+0.0001}_{-0.0002})$	0.079 [11]
	$a_{\beta=5.3}$	$0.0705(^{+0.0003}_{-0.0002})$	0.063 [11]
	$a_{\beta=5.5}$	$0.0525(^{+0.0001}_{-0.0001})$	0.050 [11]
RQCD	$a_{\beta=5.2}$	$0.0834(^{+0.0002}_{-0.0003})$	0.081 [10]
	$a_{\beta=5.29}$	$0.0704(^{+0.0002}_{-0.0003})$	0.071 [10]
	$a_{\beta=5.4}$	$0.0598(^{+0.0003}_{-0.0005})$	0.060 [10]

The results depend on the pion mass and box size of a given ensemble. It is important to have accurate values for the QCD lattice scales available on all considered lattice ensembles. Distinct QCD β values are associated with distinct lattice scale parameters. We consider the various lattice scales as free parameters in our global fit where we isospin averaged empirical baryon masses as the scale setting condition. Such a strategy was successfully used in various global fits to lattice data [7,38–40]. Our results for all lattice scales are shown in Table II. The scales given by the ETMC and CLS collaborations differ significantly from our values. There is, however, a clear trend that the smaller the lattice scale, the closer our fitted scales get to the ones given by the lattice collaborations. The RQCD scales, on the other hand, can be reproduced quite accurately. Since the lattice set up of the CLS and RQCD groups coincide, one would expect identical lattice scales on the $\beta = 5.2$ ensembles in Table II. Within uncertainties this is the case for our results.

We include in the fit ensembles with pion masses up to 500 MeV and for lattice sizes with $m_\pi L \geq 4.0$. MeV. For the form factor we include the data points up to momentum transfer $t = -0.36\ \mathrm{GeV}^2$. The fit minimizes the least-squares differences χ^2 of our expressions with respect to the lattice data points. In this χ^2 determination, all available lattice points that meet our requirements contribute with equal weight. The χ^2 per lattice point reached is $\chi^2_{\mathrm{min}}/N_{\mathrm{data}} = 0.799$. With 124 used lattice data points and 32 degrees of freedom (22 LEC and 10 lattice scales), we reach for the total χ^2 per degree of freedom

$$\chi^2_{\mathrm{min}}/N_{\mathrm{df}} = 99.12/(124 - 32) = 1.077, \tag{16}$$

which signals a fair description of the available lattice data. As compared to [7] the improvement of the QCD lattice data reproduction is a consequence of using an updated

TABLE III. Low-energy constants as determined in our fit. The $*$ parameters are not fitted to the lattice data. While b_χ and d_χ are adjusted to the isospin-averaged masses of the nucleon and the isobar at the physical point, the value of $h_A = 9\,g_A - 6\,f_S$ is implied by its large-N_c sum rule. The renormalization scale μ is set to a conventional value.

LEC	Fit result	LEC	Fit result	LEC	Fit result
f [MeV]	$83.43(^{+0.30}_{-0.81})$	b^*_χ [GeV^{-1}]	$-0.6805(^{+0.0085}_{-0.0020})$	g_S [GeV^{-1}]	$0.9163(^{+0.0060}_{-0.0072})$
M [MeV]	$893.79(^{+0.55}_{-0.16})$	d^*_χ [GeV^{-1}]	$-0.3224(^{+0.0103}_{-0.0118})$	g_V [GeV^{-2}]	$-0.8096(^{+0.0792}_{-0.1784})$
$M+\Delta$ [MeV]	$1200.42(^{+0.72}_{-0.39})$	c_χ [GeV^{-3}]	$1.4627(^{+0.0256}_{-0.0452})$	g_T [GeV^{-1}]	$1.5035(^{+0.0762}_{-0.0200})$
g_A	$1.1449(^{+0.0019}_{-0.0049})$	e_χ [GeV^{-3}]	$1.2609(^{+0.0170}_{-0.0452})$	g_R [GeV^{-2}]	$1.1233(^{+0.0118}_{-0.0318})$
f_S	$1.5857(^{+0.0040}_{-0.0176})$	g^+_χ [GeV^{-2}]	$-3.9827(^{+0.0226}_{-0.0863})$	h_S [GeV^{-1}]	$0.7748(^{+0.0205}_{-0.0103})$
h^*_A	$0.7893(^{+0.1123}_{-0.0229})$	g^-_χ [GeV^{-2}]	$-0.2752(^{+0.034}_{-0.083})$	h_V [GeV^{-2}]	$1.9760(^{+0.152}_{-0.149})$
f_E [GeV^{-1}]	$0.6949(^{+0.3075}_{-0.3812})$	f_M [GeV^{-1}]	$-0.7335(^{+0.0543}_{-0.4320})$	f^+_A [GeV^{-1}]	$-0.0617(^{+0.0222}_{-0.0652})$
l_3	$0.0193(^{+0.0003}_{-0.0003})$	l_4	$-0.0151(^{+0.0003}_{-0.0011})$	μ^* [MeV]	770
ζ_N	$0.2277(^{+0.0207}_{-0.0475})$	ζ_Δ	$-0.0180(^{+0.0022}_{-0.0044})$		

evaluation of the loop functions but also by considering a large set of data, that includes the form factor $G_A(t)$ *and* $G_P(t)$. We search for local minima of our chi-square function where we are interested only in those minima, which have naturally sized LEC. This is easily possible within our evolutionary algorithm.

In Table III we collect the values for the LEC from our global fit. We give asymmetric one-sigma error bars. They are based on a one standard deviation (σ) change for the value of $\chi^2_{\rm min}$ (i.e., an increase by 1). We determined the region for the LEC meeting this range, from which follow the errors for the LEC. While our results are in qualitative agreement with previous studies there are important differences. For instance it shows an expected value for $\bar{l}_3$ [41,42]. Also our values for f, M, $M+\Delta$, and g_A are within range of [42]. On the other hand we find values for g_S and g_V that disagree significantly from previous SU(2) works like Refs. [41,43]. However, in most papers the constants g_S and g_V are determined in a theory without isobars.

A comparison of our l_3 and l_4 in Table III with values listed in [44] reveals a striking tension. While our l_3 translates into an $\bar{l}_3$ value of about ~ 3.07 our value for l_4 has a sign opposite to typical values from [44]. We scrutinized our results for l_4 by performing further fits by selecting ensembles with $m_\pi < 450$ MeV but also $m_\pi < 550$ MeV. In both cases the fit quality is changed slightly only. While the LEC change somewhat outside the tiny 1-sigma error band none of the striking features of our best fit scenario in Table III with $m_\pi < 500$ MeV is altered. In particular our l_4 remains negative always.

It is interesting to observe that in our current fit the sign of h_A changed as compared to [7]. A negative sign was also obtained previously in [14] from a study of loop corrections in pion-nucleon scattering. While at leading order in a $1/N_c$ expansion one may favor a large and positive $h_A \sim 9\,g_A/5$ value, at subleading order its sign is not fixed with $h_A = 9\,g_A - 6\,f_S$. Depending on the values of $g_A > 0$ and $f_S > 0$ it may turn positive or negative. In the axial-vector form factor the contribution $\sim h_A\, f_S^2\, J_{\pi\Delta\pi}(t)$ probes the sign of h_A. A possible more direct strategy to determine such a phase was suggested recently in [45]. The fact that we find a strong sensitivity of h_A on the detailed form of how to incorporate the isobar degrees of freedom, we may speculate that the specifics of loop corrections in pion-nucleon scattering may be subject to similar effects. Here the novel development [19] may turn out instrumental.

In Table IV we list additional observable quantities as they are implied by our set of LEC. Most interesting is our prediction for the axial-vector coupling constant, which is significantly below its empirical value $G_A(0) = 1.2732(23)$ [46]. We interpret this discrepancy as the effect from the neglected strange-quark mass effects in our approach.

The axial radius may be compared with its empirical value $\langle r_A^2\rangle = 0.46(24)$ fm^2 [47,48], where we find our value to be roughly consistent with its empirical expectation. Previous lattice values show quite some spread [6,11,12,49] but tend to prefer also a small radius with for instance $\langle r_A^2\rangle = 0.213(6)(13)(3)$ fm^2 from [49]. Similarly the empirical value for $g_P = 10.6(2.7)$ from [50]

TABLE IV. Observables as determined in our fit.

Observable	Fit results
$G_A(0)$	$1.2284(^{+0.0021}_{-0.0059})$
$\langle r_A^2\rangle$ [fm^2]	$0.20137(^{+0.0032}_{-0.0035})$
g_P	$8.2521(^{+0.039}_{-0.039})$
$\sigma_{\pi N}$ [MeV]	$42.22(^{+0.02}_{-0.05})$
$\sigma_{\pi\Delta}$ [MeV]	$35.27(^{+0.01}_{-0.06})$
f_π [MeV]	$84.96(^{+0.29}_{-0.82})$

$$g_P = \frac{m_\mu}{2\,M_N}\,G_P(-0.877 m_\mu^2), \tag{17}$$

is surprisingly close to our range in Table IV. So one may expect that the effect of strange quarks is less visible here.

Most striking we find our value for the pion-nucleon sigma term

$$\sigma_{\pi N} = m\frac{\partial}{\partial m}M_N, \tag{18}$$

and the pion decay constant which both show a significant conflict with the empirical value $\sigma_{\pi N} = 58(5)$ MeV from Ref. [51–53] and the PDG value $f_\pi = (92.21 \pm 0.14)$ MeV [54]. Our sigma term also differs significantly from our previous value obtained [7], which we take as strong hint that details how to incorporate the isobar degrees of freedom are crucial for such observables. Our current value is quite consistent with previous analysis [55,56] that obtained $\sigma_{\pi N} = 41(5)(4)$ MeV based on a flavor-SU(2) extrapolation of an older set of lattice data for the nucleon mass [34,57–59]. While some recent fits of flavor SU(3) lattice QCD data in [60,61] appear compatible with the empirical value, there is a large spread in values in the literature based on different assumptions and datasets (see, e.g., [39,40,62,63]).

This raises the question on the role of strange quarks in the sigma term but also in the pion decay constant. The reason for our discovery in the decay constants stems to a large extent in the sign of l_4. In contrast to previous works based on flavor SU(2) ensembles as listed in [44], in our work we determined l_4 from the pseudoscalar induced form factor of the nucleon for the first time. Setting the lattice scale for SU(2) ensembles by the empirical decay constant, a rather popular scheme, may masks the potential discovery of such an effect. Lattice studies with a scale set to the nucleon mass would be much preferred in this case.

V. SUMMARY AND OUTLOOK

In our work we presented a chiral extrapolation study of lattice QCD data based on flavor SU(2) ensembles of CLS, RQCD, and ETMC. Our emphasis is the role of the isobar in a computation of the axial-vector form factor of the nucleon. Here the evaluation of one-loop effects involving the isobar was worked out in a novel chiral framework, that permits a systematic subtraction of power-counting terms, but still use on-shell hadron masses. For the first time we achieved a simultaneous reproduction on QCD lattice data for both components of that form factor on ensembles with pion masses up to 500 MeV.

Our results indicate the crucial importance of strange quark effects on the axial-vector form factor. Lattice data in the absence of active strange degrees of freedom do not seem to be able to reproduce the empirical form factors at the physical point. The axial-vector coupling constant of the nucleon is underestimated, but also the pion-nucleon sigma term turns out to be much below its empirical value. While our current fit to the available dataset is already excellent it cannot be ruled out that our conclusions are affected upon a full consideration of finite-box effects or a further improved dataset.

It is an open challenge to generate and analyse further lattice QCD data at fixed strange quark masses, which can be extrapolated faithfully to the physical pion masses and volumes. Only then a significant comparison with empirical results is convincing. Given this challenge it is of utmost importance to generate high statistic data on the isobar masses for such systems on ensembles with varying volumes and pion masses. On our side we plan to provide a full computation of finite box effects for such form factors. We expect that with those it may be possible to also consider further discretization effects by the spurion field approach.

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APPENDIX: LOOP FUNCTIONS FOR G_A

We confirm the form of the axial-vector loop functions as derived in [7]. A revision of such results is required as to arrive at results that comply with (9) and (11). The following form is found

$$\begin{aligned}
\bar J^A_\pi(t) &= -\bar I_\pi + \mathcal{O}(Q^4),\\
\bar J^A_{\pi N}(t) + \bar J^A_{N\pi}(t) &= 2\,m_\pi^2\left(-1 + \frac{4}{3}M_N(g_S - 2\,g_T)\right)\bar I_{\pi N} + \mathcal{O}(Q^4),\\
\bar J^A_{N\pi N}(t) &= \bar I_\pi + m_\pi^2\,\bar I_{\pi N} + \mathcal{O}(Q^4),\\
\bar J^A_{\pi\Delta}(t) + \bar J^A_{\Delta\pi}(t) &= \frac{8}{9}\left(25 f_A^+\left[-m_\pi^2\,\alpha^A_{12} + 2\,\delta M_N\,r\,\alpha^A_{13}\right] + f_A^-\left[-m_\pi^2\,\alpha^A_{22} + 2\,\delta M_N\,r\,\alpha^A_{23}\right] + 8 f_M\,r^2 t\,\alpha^A_{31}\right)M_N\,\bar I_{\pi\Delta} + \mathcal{O}(Q^4),
\end{aligned}$$

$$
\begin{aligned}
\bar{J}^A_{N\pi\Delta}(t)+\bar{J}^A_{\Delta\pi N}(t) &= -\frac{16}{9r}m_\pi^2\left(\bar{I}_{\pi N}-\alpha^A_{42}\,\bar{I}_{\pi\Delta}\right)-\frac{2}{9}\left(r\,t\,\alpha^A_{41}+8\,\delta M_N\,\alpha^A_{43}+12\frac{f_E\,M_N\,r}{f_S}t\,\alpha^A_{51}\right)\bar{I}_{\pi\Delta}\\
&\quad+\frac{8}{3}\frac{f_E\,M_N\,r}{f_S}t\,\alpha^A_{61}M_N^2\left(\bar{I}^{(1,0)}_{\Delta\pi N}(t)+\bar{I}^{(0,1)}_{N\pi\Delta}(t)\right)+\frac{4}{9}t\,\alpha^A_{71}M_N^2\left(\bar{I}^{(2,0)}_{\Delta\pi N}(t)+\bar{I}^{(0,2)}_{N\pi\Delta}(t)\right)+\mathcal{O}(Q^4),\\
\bar{J}^A_{\Delta\pi\Delta}(t) &= \frac{2}{3}\left(2\,r\,t\,\alpha^A_{81}+\frac{5}{9}m_\pi^2\,\alpha^A_{82}-\frac{10}{3}\delta M_N\,\alpha^A_{83}\right)\bar{I}_{\pi\Delta}+\frac{4}{3}t\,\alpha^A_{91}M_N^2\left(\bar{I}^{(2,0)}_{\Delta\pi\Delta}(t)+\bar{I}^{(0,2)}_{\Delta\pi\Delta}(t)\right)+\mathcal{O}(Q^4), \qquad \text{(A1)}
\end{aligned}
$$

together with the rational functions $\alpha^A_{\ldots}$ and $\alpha^P_{\ldots}$ which take the form

$$
\begin{aligned}
\alpha^A_{12} &= \frac{(2+r)^2(20+24r+19r^2+3r^3)}{80(1+r)^2},\\
\alpha^A_{13} &= \frac{(2+r)^3(20+36r+29r^2+5r^3)}{160(1+r)^3},\\
\alpha^A_{22} &= \frac{(2+r)^2(4-r^2-3r^3)}{16(1+r)^2},\\
\alpha^A_{23} &= \frac{(2+r)^3(4-5r^2-5r^3)}{32(1+r)^3},\\
\alpha^A_{31} &= \frac{(2+r)^4}{16(1+r)^2},\\
\alpha^A_{41} &= \frac{(2+r)^2(1+r-r^2)}{4(1+r)^2},\\
\alpha^A_{42} &= \frac{(2+r)^3(4+4r+3r^2)}{32(1+r)^2},\\
\alpha^A_{43} &= \frac{(2+r)^4(1+2r+2r^2)}{16(1+r)^3},\\
\alpha^A_{51} &= \frac{(2+r)^2(3+4r+4r^2+r^3)}{12(1+r)^2},\\
\alpha^A_{61} &= \frac{(2+r)^2}{4(1+r)^2},\\
\alpha^A_{71} &= \frac{(2+r)(1+r+r^2)}{2(1+r)^2},\\
\alpha^A_{81} &= \frac{(2+r)^3(18+60r+29r^2+4r^3)}{144(1+r)^3},\\
\alpha^A_{82} &= \frac{(2+r)(20+212r+398r^2+325r^3+130r^4+19r^5)}{40(1+r)^4},\\
\alpha^A_{83} &= \frac{(2+r)^2(30+130r+319r^2+418r^3+296r^4+107r^5+14r^6)}{120(1+r)^5},\\
\alpha^A_{91} &= \frac{(2+r)^2(6+6r+r^2)}{24(1+r)^2}.\\
\alpha^P_{12} &= \frac{(2+r)^2(40+48r+42r^2+18r^3+9r^4+2r^5)}{160(1+r)^2},\\
\alpha^P_{13} &= \frac{(2+r)^3(20+36r+29r^2+5r^3)}{160(1+r)^3},
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{22}^P &= \frac{(2+r)^2(8-6r^2-18r^3-9r^4-2r^5)}{32(1+r)^2}, \\
\alpha_{23}^P &= \frac{(2+r)^3(4-5r^2-5r^3)}{32(1+r)^3}, \\
\alpha_{31}^P &= \frac{(2+r)^4}{16(1+r)^2}, \qquad \alpha_{32}^P = \alpha_{31}^P, \\
\alpha_{41}^P &= \frac{(2+r)^2(1+r-r^2)}{4(1+r)^2}, \\
\alpha_{42}^P &= \frac{(2+r)^2(8+12r+7r^2+5r^3+r^5)}{32(1+r)^2}, \\
\alpha_{43}^P &= \frac{(2+r)^4(1+2r+2r^2)}{16(1+r)^3}, \\
\alpha_{51}^P &= \frac{(2+r)^2(3+4r+4r^2+r^3)}{12(1+r)^2}, \qquad \alpha_{52}^P = \alpha_{51}^P, \\
\alpha_{61}^P &= \frac{(2+r)^2}{4(1+r)^2}, \qquad \alpha_{62}^P = \alpha_{61}^P, \\
\alpha_{71}^P &= \frac{(2+r)(1+r+r^2)}{2(1+r)^2}, \qquad \alpha_{72}^P = \alpha_{71}^P, \\
\alpha_{81}^P &= \frac{(2+r)^3(18+60r+29r^2+4r^3)}{144(1+r)^3}, \\
\alpha_{82}^P &= \frac{(2+r)(20+236r+518r^2+603r^3+442r^4+187r^5+40r^6+3r^7)}{40(1+r)^4}, \\
\alpha_{83}^P &= \frac{(2+r)^2(30+130r+319r^2+418r^3+296r^4+107r^5+14r^6)}{120(1+r)^5}, \\
\alpha_{91}^P &= \frac{(2+r)^2(6+6r+r^2)}{24(1+r)^2}, \\
\alpha_{92}^P &= \frac{(2+r)^2(2+2r-r^2)}{8(1+r)^2}.
\end{aligned} \tag{A2}
$$

The expectation (9) implies specific relations among the coefficients

$$
\begin{aligned}
&\alpha_{n1}^A = \alpha_{n1}^P \quad \text{for } n = 1, \ldots, 9, \quad \text{and} \quad \alpha_{32}^P = \alpha_{31}^P,\ \alpha_{52}^P = \alpha_{51}^P,\ \alpha_{62}^P = \alpha_{61}^P, \\
&\alpha_{n3}^A = \alpha_{n3}^P \quad \text{for } n = 1, \ldots, 9,
\end{aligned} \tag{A3}
$$

which is indeed verified by our explicit computations. Note that the identities in the second line (A3) are accidental and not related to the chiral Ward identities.

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