Global quark spin correlations in relativistic heavy ion collisions

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The observation of the vector meson's global spin alignment by the STAR Collaboration reveals that strong spin correlations may exist for quarks and antiquarks in relativistic heavy-ion collisions in the normal direction of the reaction plane. We propose a systematic method to describe such correlations in the quark matter. We classify them as local and long range quark spin correlations in the system. We show in particular that the effective quark spin correlations contain the genuine spin correlations originated directly from the dynamical process as well as those induced by averaging over other degrees of freedom. We also show that such correlations can be studied by measuring the vector meson's spin density matrix and hyperon-hyperon and hyperon-anti-hyperon spin correlations. We present the relationships between these measurable quantities and spin correlations of quarks and antiquarks.

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I. INTRODUCTION

The global hyperon polarization has been observed first by the STAR Collaboration at the Relativistic Heavy Ion Collider (RHIC) [1] and later in a series of subsequent experiments [2–6]. This confirms the theoretical predictions made almost two decades ago [7–9]. The experimental results can be described quantitatively using phenomenological transport or hydrodynamical models [10–26], that are now reviewed, e.g., in [27–40]. Such studies open a new avenue in studying properties of the quark gluon plasma (QGP) produced in heavy ion collisions and have attracted much attention in the field.

Recently, the STAR Collaboration has published their measurements on vector mesons' global spin alignment [41]. Their results, together with other measurements [42–45] bring the field of spin physics in heavy-ion collisions to a new climax [46–64]. The STAR results [41], on the one hand, show that the global polarization effects also exist for vector mesons, on the other hand, seem to be inconsistent with the magnitude of hyperon polarization if quark spin correlations due to strong fields are neglected. As was shown in the original papers on the global polarization effect [7–9], if we take quark polarization as a constant and neglect fluctuations and correlations etc., the hyperon polarization should be equal to that of the quark but the vector meson's spin alignment should be proportional to the quark polarization squared. In this case, since the observed hyperon polarization is only a few percent [1–6], the vector meson's spin alignment should be much smaller than those observed in the STAR experiment [41]. Hence, the data clearly reveal that there is strong spin correlation between the quark and antiquark that combine into the vector meson [46–64]. In this sense, this provides the first opportunity to study the spin correlations at the quark level in high energy heavy-ion collisions.

It is clear that the spin correlation of quarks and antiquarks is an important property of QGP. It may contain important information on strong interaction and provide new clue to color confinement in quantum chromodynamics (QCD). So it is crucial to present a unified and systematic description of spin correlations in quark matter and make connection with experimental observables.

The definition of spin correlations in a system of spin-1/2 particles can be found in text books. However,

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to study spin properties of QGP, it is convenient to define the correlations in a way so that one can study them in a given sequence such as number of particles. More precisely, one can start with two particle spin correlations, then continue with two and three particle spin correlations and so on. Such a definition will facilitate the study and reveal the underlying dynamics. The connections between quark spin correlations and experimental observables depend on hadronization mechanism and thus may be model dependent.

The purpose of this paper is to propose a systematic description of spin correlation of quarks and antiquarks in heavy-ion collisions. We show that the spin correlation can be decomposed into the genuine and induced one and also into the local and long range one. We propose that the spin correlation at the quark level can be extracted from the vector meson's diagonal and off diagonal elements of its spin density matrix together with the hyperon-anti-hyperon spin correlation. We present the relationships between the spin correlation at the quark level and measurable quantities in a simple quark recombination model.

The rest of the paper is organized as follows. In Sec. II, we propose the systematic way to describe quark spin correlation in the quark matter system and discuss its properties. In Sec. III A, we present the results of the vector meson's spin alignment and off diagonal elements of the spin density matrix as functions of the quark spin polarization and correlation. In Sec. IV, we present the results for the hyperon polarization and hyperon-hyperon or hyperon-anti-hyperon spin correlation. We also present numerical estimates of spin correlation parameters by fitting the existing data in Sec. VI. Finally, a short summary and an outlook are given in Sec. VII.

II. QUARK SPIN CORRELATIONS IN QUARK MATTER SYSTEM

We consider a quark matter system such as QGP consisting of quarks and antiquarks. The spin properties of the system are described by the spin density matrix. For a single particle, we study the spin polarization while for two or more particles we can study not only the spin polarization but also the spin correlation.

A. The spin density matrix

In a spin-1/2 particle system, the spin state of a particle is described by the spin density matrix that can be expanded in terms of the complete set of the 2×2 Hermitian matrices $\{\mathbb{I}, \hat{\sigma}_i\}$, i.e.,

$$\hat{\rho}^{(q)} = \frac{1}{2}(\mathbb{I} + P_{qi}\hat{\sigma}_i), \tag{1}$$

where $P_{qi} = \langle\hat{\sigma}_i\rangle = \mathrm{Tr}[\hat{\rho}^{(q)}\hat{\sigma}_i]$ with $i = x, y, z$ is the ith component of the quark polarization vector $\boldsymbol{P}_q = (P_{qx}, P_{qy}, P_{qz})$, $\hat{\sigma}_i$ denotes Pauli matrices, and $\mathrm{Tr}\hat{\rho}^{(q)} = 1$ is normalized to one. The symbol $\mathbb{I}$ denotes the unit 2×2 matrix, and in the following of this paper, we simply write it as 1. Also, we use the convention that a sum over repeated indices is implicit through out the paper.

For two particle state in the system, we denote the spin density matrix by $\hat{\rho}^{(12)}$. Conventionally, one expands $\hat{\rho}^{(12)}$ in terms of the complete set of Hermitian matrices $\{1\otimes 1, \hat{\sigma}_{1i}\otimes 1, 1\otimes\hat{\sigma}_{2j}, \hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\}$, i.e.,

$$\hat{\rho}^{(12)} = \frac{1}{4}[1 + P_{1i}\hat{\sigma}_{1i} + P_{2j}\hat{\sigma}_{2j} + t_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}], \tag{2}$$

where $t_{ij}^{(12)}$ is called the spin correlation of particles 1 and 2. Here, as well as in the following of this paper, we take the following scheme of notations: the superscript of the spin density matrix $\hat{\rho}$ or the spin correlation t denotes the type of particle where for hadrons we simply use the symbol while for quarks or antiquarks we put it in a bracket; the subscript denotes the indices of matrix elements or spatial components such as mm' or ij. For polarization vectors, we simply use double subscripts to specify particle type and spatial component, respectively.

There is however a shortcoming in the definition of the spin correlation through $t_{ij}^{(12)}$ in Eq. (2). In case of no spin correlation between particles 1 and 2, we should have $\hat{\rho}^{(12)} = \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)}$ and then $t_{ij}^{(12)} = P_{1i}P_{2j}$ that is non-vanishing. The situation is the same for spin correlations of three or more particles if they are defined in a similar way. This is in particular inconvenient if we study the spin correlations order by order.

To overcome such a shortcoming, we propose to expand $\hat{\rho}^{(12)}$ in the following way:

$$\hat{\rho}^{(12)} = \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)} + \frac{1}{2^2}c_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}, \tag{3}$$

where the spin correlation is described by $c_{ij}^{(12)}$. It is clear that $c_{ij}^{(12)} = 0$ if there is no spin correlation between particles 1 and 2. In the same way, we expand the spin density matrix for a system of three or four particles as

$$\begin{aligned}\hat{\rho}^{(123)} = {}& \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)}\otimes\hat{\rho}^{(3)} + \frac{1}{2^3}c_{ijk}^{(123)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k} \\ & + \frac{1}{2^2}[c_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\rho}^{(3)} + c_{jk}^{(23)}\hat{\rho}^{(1)}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k} \\ & + c_{ik}^{(13)}\hat{\sigma}_{1i}\otimes\hat{\rho}^{(2)}\otimes\hat{\sigma}_{3k}],\end{aligned} \tag{4}$$

$$\begin{aligned}\hat{\rho}^{(1234)} = {} & \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)} \otimes \hat{\rho}^{(3)} \otimes \hat{\rho}^{(4)} + \frac{1}{2^4} c_{ijkl}^{(1234)} \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2j} \otimes \hat{\sigma}_{3k} \otimes \hat{\sigma}_{4l} \\ & + \frac{1}{2^2} [c_{ij}^{(12)} \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2j} \otimes \hat{\rho}^{(3)} \otimes \hat{\rho}^{(4)} + c_{kl}^{(34)} \hat{\rho}^{(1)} \otimes \hat{\rho}^{(2)} \otimes \hat{\sigma}_{3k} \otimes \hat{\sigma}_{4l} + c_{ik}^{(13)} \hat{\sigma}_{1i} \otimes \hat{\rho}^{(2)} \otimes \hat{\sigma}_{3k} \otimes \hat{\rho}^{(4)} \\ & + c_{jl}^{(24)} \hat{\rho}^{(1)} \otimes \hat{\sigma}_{2j} \otimes \hat{\rho}^{(3)} \otimes \hat{\sigma}_{4l} + c_{il}^{(14)} \hat{\sigma}_{1i} \otimes \hat{\rho}^{(2)} \otimes \hat{\rho}^{(3)} \otimes \hat{\sigma}_{4l} + c_{jk}^{(23)} \hat{\rho}^{(1)} \otimes \hat{\sigma}_{2j} \otimes \hat{\sigma}_{3k} \otimes \hat{\rho}^{(4)}] \\ & + \frac{1}{2^3} [c_{ijk}^{(123)} \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2j} \otimes \hat{\sigma}_{3k} \otimes \hat{\rho}^{(4)} + c_{ijl}^{(124)} \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2j} \otimes \hat{\rho}^{(3)} \otimes \hat{\sigma}_{4l} + c_{ikl}^{(134)} \hat{\sigma}_{1i} \otimes \hat{\rho}^{(2)} \otimes \hat{\sigma}_{3k} \otimes \hat{\sigma}_{4l} \\ & + c_{jkl}^{(234)} \hat{\rho}^{(1)} \otimes \hat{\sigma}_{2j} \otimes \hat{\sigma}_{3k} \otimes \hat{\sigma}_{4l}]. \end{aligned} \tag{5}$$

The polarizations and spin correlations can be extracted by taking expectation values of a direct product of Pauli matrices on spin density matrices. The results are

$$P_{1i} = \langle \hat{\sigma}_{1i} \rangle, \tag{6}$$

$$c_{ij}^{(12)} = \langle \hat{\sigma}_{1i} \hat{\sigma}_{2j} \rangle - P_{1i} P_{2j}, \tag{7}$$

$$\begin{aligned} c_{ijk}^{(123)} = {} & \langle \hat{\sigma}_{1i} \hat{\sigma}_{2j} \hat{\sigma}_{3k} \rangle - c_{ij}^{(12)} P_{3k} - c_{jk}^{(23)} P_{1i} - c_{ik}^{(13)} P_{2j} \\ & - P_{1i} P_{2j} P_{3k}, \end{aligned} \tag{8}$$

$$\begin{aligned} c_{ijkl}^{(1234)} = {} & \langle \hat{\sigma}_{1i} \hat{\sigma}_{2j} \hat{\sigma}_{3k} \hat{\sigma}_{4l} \rangle - c_{ijk}^{(123)} P_{4l} - c_{ijl}^{(124)} P_{3k} - c_{ikl}^{(134)} P_{2j} \\ & - c_{jkl}^{(234)} P_{1i} - c_{ij}^{(12)} P_{3k} P_{4l} - c_{ik}^{(13)} P_{2j} P_{4l} \\ & - c_{il}^{(14)} P_{2j} P_{3k} - c_{jk}^{(23)} P_{1i} P_{4l} - c_{jl}^{(24)} P_{1i} P_{3k} \\ & - c_{kl}^{(34)} P_{1i} P_{2j} - P_{1i} P_{2j} P_{3k} P_{4l}. \end{aligned} \tag{9}$$

For a four-particle system, according to Eq. (5), if the system does not have any spin correlations, i.e., the spin density matrix of the system is the direct product of spin density matrices of single particles, we have

$$\begin{aligned} & c_{ij}^{(12)} = c_{jk}^{(23)} = c_{ik}^{(13)} = c_{kl}^{(34)} = c_{jl}^{(24)} = c_{il}^{(14)} = 0, \\ & c_{ijk}^{(123)} = c_{jkl}^{(234)} = c_{ijl}^{(124)} = c_{ikl}^{(134)} = 0, \\ & c_{ijkl}^{(1234)} = 0. \end{aligned} \tag{10}$$

If there are only two-particle spin correlations, we have

$$c_{ijk}^{(123)} = c_{jkl}^{(234)} = c_{ijl}^{(124)} = c_{ikl}^{(134)} = c_{ijkl}^{(1234)} = 0. \tag{11}$$

If there are only two-particle and three-particle spin correlations, we have $c_{ijkl}^{(1234)} = 0$. In this way, we can include spin correlations order by order.

We note that if we define the spin correlation for two spin-1/2 particles $h_1 h_2$ in the conventional way, i.e.,

$$c_{nn} = \frac{f_{++} + f_{--} - f_{+-} - f_{-+}}{f_{++} + f_{--} + f_{+-} + f_{-+}}, \tag{12}$$

where n stands for the spin quantization direction $\hat{\boldsymbol{n}}$, $f_{m_1 m_2} = \langle m_1 m_2 | \hat{\rho}^{(12)} | m_1 m_2 \rangle$ (with $m_1, m_2 = \pm$ denoting spin states) is the fraction of the particle pair in the spin state $|m_1 m_2\rangle$. We then obtain the relationship between c_{nn} and $c_{ij}^{(12)}$ defined above as

$$c_{nn} = c_{nn}^{(12)} + P_{1n} P_{2n}. \tag{13}$$

B. With other degrees of freedom

We suppose particles in the system have other degrees of freedom that are denoted in general by α. We consider here a very simple case that the polarization and spin correlation have α dependence so that spin density matrices are given by

$$\hat{\rho}^{(q)}(\alpha) = \frac{1}{2} [1 + P_{qi}(\alpha) \hat{\sigma}_i], \tag{14}$$

$$\begin{aligned} \hat{\rho}^{(12)}(\alpha_1, \alpha_2) = {} & \hat{\rho}^{(1)}(\alpha_1) \otimes \hat{\rho}^{(2)}(\alpha_2) \\ & + \frac{1}{2^2} c_{ij}^{(12)}(\alpha_1, \alpha_2) \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2j}. \end{aligned} \tag{15}$$

Now suppose we have a system (12) composed of 1 and 2. We assume that the system is at the state $|\alpha_{12}\rangle$ so the probability to find particle 1 and 2 at α_1 and α_2, respectively, is determined by the amplitude $\langle \alpha_1, \alpha_2 | \alpha_{12} \rangle$. We obtain the effective spin density matrix for the system (12) at α_{12} as

$$\begin{aligned} \hat{\bar{\rho}}^{(12)}(\alpha_{12}) & = \langle \alpha_{12} | \hat{\rho}^{(12)} | \alpha_{12} \rangle \\ & = \sum_{\alpha_1 \alpha_2} |\langle \alpha_1, \alpha_2 | \alpha_{12} \rangle|^2 \hat{\rho}^{(12)}(\alpha_1, \alpha_2). \end{aligned} \tag{16}$$

We decompose this effective spin density matrix $\hat{\bar{\rho}}^{(12)}(\alpha_{12})$ in the same way as that in Eq. (3) or (15), i.e.,

$$\begin{aligned} \hat{\bar{\rho}}^{(12)}(\alpha_{12}) = {} & \hat{\bar{\rho}}^{(1)}(\alpha_{12}) \otimes \hat{\bar{\rho}}^{(2)}(\alpha_{12}) \\ & + \frac{1}{2^2} \bar{c}_{ij}^{(12)}(\alpha_{12}) \hat{\sigma}_{1i} \otimes \hat{\sigma}_{2j}, \end{aligned} \tag{17}$$

where $\hat{\bar{\rho}}^{(1)}(\alpha_{12})$ is the average of $\hat{\rho}^{(1)}(\alpha_1)$ weighted by the wave function $\langle \alpha_1, \alpha_2 | \alpha_{12} \rangle$ squared and is decomposed as

$$\hat{\bar{\rho}}^{(1)}(\alpha_{12}) = \langle \hat{\rho}^{(1)}(\alpha_1)\rangle \equiv \sum_{\alpha_1\alpha_2} |\langle \alpha_1,\alpha_2|\alpha_{12}\rangle|^2 \hat{\rho}^{(1)}(\alpha_1) = \frac{1}{2}[1+\bar{P}_{1i}(\alpha_{12})\hat{\sigma}_{1i}], \quad (18)$$

and similar for $\hat{\bar{\rho}}^{(2)}(\alpha_{12})$. Here, as well as in the following of this paper, we use $\langle\cdots\rangle$ to denote such an average on the state of the system. In this way, by reversing Eq. (18), we obtain that

$$\bar{P}_{1i}(\alpha_{12}) = \langle P_{1i}(\alpha_1)\rangle \equiv \sum_{\alpha_1\alpha_2} |\langle \alpha_1,\alpha_2|\alpha_{12}\rangle|^2 P_{1i}(\alpha_1). \quad (19)$$

However, the situation for $\bar{c}_{ij}^{(12)}(\alpha_{12})$ is different. By reversing Eq. (17) and using Eq. (15), we obtain that

$$\bar{c}_{ij}^{(12)}(\alpha_{12}) = \langle c_{ij}^{(12)}(\alpha_1,\alpha_2) + P_{1i}(\alpha_1)P_{2j}(\alpha_2)\rangle - \bar{P}_{1i}(\alpha_{12})\bar{P}_{2j}(\alpha_{12}). \quad (20)$$

We see that $\bar{c}_{ij}^{(12)}(\alpha_{12})$ is not simply the average of $c_{ij}^{(12)}(\alpha_1,\alpha_2)$ weighted by the wave function $\langle\alpha_1,\alpha_2|\alpha_{12}\rangle$ squared. In particular, in the case of $c_{ij}^{(12)}(\alpha_1,\alpha_2)=0$, we have

$$\bar{c}_{ij}^{(12;0)}(\alpha_{12}) = \langle P_{1i}(\alpha_1)P_{2j}(\alpha_2)\rangle - \langle P_{1i}(\alpha_1)\rangle\langle P_{2j}(\alpha_2)\rangle. \quad (21)$$

We see clearly that $\bar{c}_{ij}^{(12;0)}(\alpha_{12})$ is in general nonzero if P_{qi} ($q=1,2$) have α dependences even if $c_{ij}^{(12)}(\alpha_1,\alpha_2)=0$ so that $\hat{\rho}^{(12)}(\alpha_1,\alpha_2)=\hat{\rho}^{(1)}(\alpha_1)\otimes\hat{\rho}^{(2)}(\alpha_2)$. To distinguish them from each other, we propose to call $c_{ij}^{(12)}(\alpha_1,\alpha_2)$, the genuine spin correlation, but the corresponding $\bar{c}_{ij}^{(12)}(\alpha_{12})$, the effective spin correlation, and $\bar{c}_{ij}^{(12;0)}(\alpha_{12})$, the induced spin correlation. To be consistent, we will also call $\bar{P}_{qi}(\alpha_{12})$, the effective, and $P_{qi}(\alpha_q)$, the genuine polarization.

Also, we suggest to distinguish the induced spin correlations into local and long range correlations depending on whether they are short or long ranged in the α space. An example that leads to such induced spin correlations was given in Refs. [51–55]. The spin correlation between s and $\bar{s}$ was shown to be strong and local in phase space due to strong interaction with the ϕ-meson field.

Similarly, for a three-particle system (123), the α-dependent spin density matrix reads

$$\hat{\rho}^{(123)}(\alpha_1,\alpha_2,\alpha_3) = \hat{\rho}^{(1)}(\alpha_1)\otimes\hat{\rho}^{(2)}(\alpha_2)\otimes\hat{\rho}^{(3)}(\alpha_3) + \frac{1}{2^2}[c_{ij}^{(12)}(\alpha_1,\alpha_2)\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\rho}^{(3)}(\alpha_3) + c_{jk}^{(23)}(\alpha_2,\alpha_3)\hat{\rho}^{(1)}(\alpha_1)\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k} + c_{ik}^{(13)}(\alpha_1,\alpha_3)\hat{\sigma}_{1i}\otimes\hat{\rho}^{(2)}(\alpha_2)\otimes\hat{\sigma}_{3k}] + \frac{1}{2^3}c_{ijk}^{(123)}(\alpha_1,\alpha_2,\alpha_3)\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}. \quad (22)$$

If the system (123) is in the state $|\alpha_{123}\rangle$, the effective spin density matrix is given by

$$\hat{\bar{\rho}}^{(123)}(\alpha_{123}) = \hat{\bar{\rho}}^{(1)}(\alpha_{123})\otimes\hat{\bar{\rho}}^{(2)}(\alpha_{123})\hat{\bar{\rho}}^{(3)}(\alpha_{123}) + \frac{1}{2^2}[\bar{c}_{ij}^{(12)}(\alpha_{123})\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\bar{\rho}}^{(3)}(\alpha_{123}) + \bar{c}_{jk}^{(23)}(\alpha_{123})\hat{\bar{\rho}}^{(1)}(\alpha_{123})\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k} + \bar{c}_{ik}^{(13)}(\alpha_{123})\hat{\sigma}_{1i}\otimes\hat{\bar{\rho}}^{(2)}(\alpha_{123})\otimes\hat{\sigma}_{3k}] + \frac{1}{2^3}\bar{c}_{ijk}^{(123)}(\alpha_{123})\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}, \quad (23)$$

where the effective polarizations such as $\bar{P}_{1i}(\alpha_{123})$ and effective two-particle correlations such as $\bar{c}_{ij}^{(12)}(\alpha_{123})$ have similar expressions as those in the two-particle system given by Eqs. (19) and (20), and the effective three-particle correlation $\bar{c}_{ijk}^{(123)}(\alpha_{123})$ is given by

$$\bar{c}_{ijk}^{(123)}(\alpha_{123}) = \langle c_{ijk}^{(123)}(\alpha_1,\alpha_2,\alpha_3) + P_{1i}(\alpha_1)P_{2j}(\alpha_2)P_{3k}(\alpha_3) + c_{ij}^{(12)}(\alpha_1,\alpha_2)P_{3k}(\alpha_3) + c_{ik}^{(13)}(\alpha_1,\alpha_3)P_{2j}(\alpha_2) + c_{jk}^{(23)}(\alpha_2,\alpha_3)P_{1i}(\alpha_1)\rangle - \bar{c}_{ij}^{(12)}(\alpha_{123})\bar{P}_{3k}(\alpha_{123}) - \bar{c}_{ik}^{(13)}(\alpha_{123})\bar{P}_{2j}(\alpha_{123}) - \bar{c}_{jk}^{(23)}(\alpha_{123})\bar{P}_{1i}(\alpha_{123}) - \bar{P}_{1i}(\alpha_{123})\bar{P}_{2j}(\alpha_{123})\bar{P}_{3k}(\alpha_{123}). \quad (24)$$

If $c_{ij}^{(12)}(\alpha_1,\alpha_2,\alpha_3)=0$ and $c_{ijk}^{(123)}(\alpha_1,\alpha_2,\alpha_3)=0$, the spin density matrix of the system in Eq. (22) is the direct product of single-particle spin density matrices. In this case, we have a similar result for the induced two-particle spin correlations to Eq. (21), and the induced three-particle spin correlation becomes

$$\bar{c}_{ijk}^{(123;0)}(\alpha_{123}) = \langle P_{1i}(\alpha_1)P_{2j}(\alpha_2)P_{3k}(\alpha_3)\rangle + 2\langle P_{1i}(\alpha_1)\rangle\langle P_{2j}(\alpha_2)\rangle\langle P_{3k}(\alpha_3)\rangle - \langle P_{1i}(\alpha_1)P_{2j}(\alpha_2)\rangle\langle P_{3k}(\alpha_3)\rangle - \langle P_{1i}(\alpha_1)P_{3k}(\alpha_3)\rangle\langle P_{2j}(\alpha_2)\rangle - \langle P_{2j}(\alpha_2)P_{3k}(\alpha_3)\rangle\langle P_{1i}(\alpha_1)\rangle. \quad (25)$$

If $c_{ijk}^{(123)}(\alpha_1,\alpha_2,\alpha_3)=0$ but there are two-particle spin correlations, the induced three-particle spin correlation has the form,

$$\begin{aligned}\bar{c}_{ijk}^{(123;1)}(\alpha_{123}) &= \langle P_{1i}(\alpha_1)P_{2j}(\alpha_2)P_{3k}(\alpha_3)+c_{ij}^{(12)}(\alpha_1,\alpha_2)P_{3k}(\alpha_3)+c_{ik}^{(13)}(\alpha_1,\alpha_3)P_{2j}(\alpha_2)+c_{jk}^{(23)}(\alpha_2,\alpha_3)P_{1i}(\alpha_1)\rangle\\ &\quad-\bar{c}_{ij}^{(12)}(\alpha_{123})\bar{P}_{3k}(\alpha_{123})-\bar{c}_{ik}^{(13)}(\alpha_{123})\bar{P}_{2j}(\alpha_{123})-\bar{c}_{jk}^{(23)}(\alpha_{123})\bar{P}_{1i}(\alpha_{123})\\ &\quad-\bar{P}_{1i}(\alpha_{123})\bar{P}_{2j}(\alpha_{123})\bar{P}_{3k}(\alpha_{123}). \end{aligned} \tag{26}$$

We see that if the system only has two-particle spin correlations but has α-dependence, three-particle spin correlations are not vanishing due to averages over α in a given region and/or given α-dependent weight. We call $\bar{c}_{ijk}^{(123;1)}(\alpha_{123})$ in Eq. (26) the first order induced spin correlation and $\bar{c}_{ijk}^{(123;0)}(\alpha_{123})$ in Eq. (25) the zeroth order induced spin correlation.

III. SPIN DENSITY MATRIX FOR VECTOR MESONS

We take a simple case that quarks and antiquarks in the system combine with each other to form hadrons. We use this as an illustrating example to show the relationship between the quark-quark spin correlations and the polarization of hadrons as well as other measurable quantities.

In this section, we consider the combination process $q_1+\bar{q}_2\to V$ and present the results for the spin density matrix of the vector meson V. We use $\hat{\mathcal{M}}$ to denote the transition matrix for a $q_1\bar{q}_2$ to form V in the combination process so that the spin density matrix of V is given by

$$\hat{\rho}^V=\hat{\mathcal{M}}\hat{\rho}^{(q_1\bar{q}_2)}\hat{\mathcal{M}}^\dagger. \tag{27}$$

Using this, we will calculate elements of $\hat{\rho}^V$ in various cases in this section.

A. With only spin degree of freedom

If we only consider the spin degree of freedom, the matrix element of $\hat{\rho}^V$ is given by

$$\begin{aligned}\rho_{mm'}^V&=\langle jm|\hat{\mathcal{M}}\hat{\rho}^{(q_1\bar{q}_2)}\hat{\mathcal{M}}^\dagger|jm'\rangle\\ &=\sum_{m_n,m_n'}\langle jm|\hat{\mathcal{M}}|m_n\rangle\langle m_n|\hat{\rho}^{(q_1\bar{q}_2)}|m_n'\rangle\langle m_n'|\hat{\mathcal{M}}^\dagger|jm'\rangle,\end{aligned} \tag{28}$$

where $j=1$ is the spin of V. Hereafter, we will use shorthand notations $|m_n\rangle\equiv|j_1m_1,j_2m_2\rangle$ and $|m_n'\rangle\equiv|j_1m_1',j_2m_2'\rangle$ for spin states of the quark-antiquark system in case of no ambiguity.

The transition matrix element $\langle jm|\hat{\mathcal{M}}|m_n\rangle$ can be further written as

$$\langle jm|\hat{\mathcal{M}}|m_n\rangle=\sum_{j'm'}\langle jm|\hat{\mathcal{M}}|j'm'\rangle\langle j'm'|m_n\rangle, \tag{29}$$

where $\langle m_n|jm\rangle$ is the well-known Clebsch-Gordan coefficient. The space rotation invariance demands that $j=j'$ and $m=m'$ and that $\langle jm|\hat{\mathcal{M}}|jm\rangle$ be independent of m. We therefore obtain that

$$\rho_{mm'}^V=N_V\sum_{m_n;m_n'}\langle jm|m_n\rangle\langle m_n|\hat{\rho}^{(q_1\bar{q}_2)}|m_n'\rangle\langle m_n'|jm'\rangle, \tag{30}$$

where N_V is a constant that can be absorbed into the normalization constant.

We insert $\hat{\rho}^{(q_1\bar{q}_2)}$ by Eq. (3) into Eq. (30) and obtain the element of the vector meson's spin density matrix,

$$\begin{aligned}\rho_{00}^V=\frac{1}{C_V}\{1&+c_{xx}^{(q_1\bar{q}_2)}+c_{yy}^{(q_1\bar{q}_2)}-c_{zz}^{(q_1\bar{q}_2)}+P_{q_1x}P_{\bar{q}_2x}\\ &+P_{q_1y}P_{\bar{q}_2y}-P_{q_1z}P_{\bar{q}_2z}\},\end{aligned} \tag{31}$$

$$\begin{aligned}\rho_{1-1}^V=\frac{1}{C_V}\{&c_{xx}^{(q_1\bar{q}_2)}-c_{yy}^{(q_1\bar{q}_2)}+P_{q_1x}P_{\bar{q}_2x}-P_{q_1y}P_{\bar{q}_2y}\\ &-i[c_{xy}^{(q_1\bar{q}_2)}+c_{yx}^{(q_1\bar{q}_2)}+P_{q_1x}P_{\bar{q}_2y}+P_{q_1y}P_{\bar{q}_2x}]\},\end{aligned} \tag{32}$$

$$\begin{aligned}\rho_{10}^V=\frac{1}{\sqrt{2}C_V}\{&c_{xz}^{(q_1\bar{q}_2)}+c_{zx}^{(q_1\bar{q}_2)}+P_{q_1x}(1+P_{\bar{q}_2z})\\ &+(1+P_{q_1z})P_{\bar{q}_2x}-i[c_{yz}^{(q_1\bar{q}_2)}+c_{zy}^{(q_1\bar{q}_2)}\\ &+P_{q_1y}(1+P_{\bar{q}_2z})+(1+P_{q_1z})P_{\bar{q}_2y}]\},\end{aligned} \tag{33}$$

$$\begin{aligned}\rho_{0-1}^V=\frac{1}{\sqrt{2}C_V}\{&-c_{xz}^{(q_1\bar{q}_2)}-c_{zx}^{(q_1\bar{q}_2)}+P_{q_1x}(1-P_{\bar{q}_2z})\\ &+(1-P_{q_2z})P_{\bar{q}_2x}+i[c_{yz}^{(q_1\bar{q}_2)}+c_{zy}^{(q_1\bar{q}_2)}\\ &-P_{q_1y}(1-P_{\bar{q}_2z})-P_{\bar{q}_2y}(1-P_{q_1z})]\},\end{aligned} \tag{34}$$

where $C_V=\mathrm{Tr}\hat{\rho}^V=3+c_{ii}^{(q_1\bar{q}_2)}+P_{q_1i}P_{\bar{q}_2i}$ is the normalization constant.

From Eqs. (31)–(34), we see clearly that we have contributions from quark-anti-quark spin correlations in all elements of the spin density matrix of the vector meson.

B. With other degrees of freedom

If there are other degrees of freedom, we have

$$\rho^V_{mm'}(\alpha_V) = \langle jm,\alpha_V|\hat{\mathcal{M}}\hat\rho^{(q_1\bar q_2)}\hat{\mathcal{M}}^\dagger|jm',\alpha_V\rangle = \sum_{m_n;m'_n}\sum_{\alpha_n}\langle jm,\alpha_V|\hat{\mathcal{M}}|m_n,\alpha_n\rangle \times \langle m_n|\hat\rho^{(q_1\bar q_2)}(\alpha_n)|m'_n\rangle\langle m'_n,\alpha_n|\hat{\mathcal{M}}^\dagger|jm',\alpha_V\rangle, \tag{35}$$

where similar to m_n, we use α_n to denote (α_1,α_2); and we considered only the case discussed in Sec. II B, i.e., we considered only the α-dependence of $\hat\rho^{(q_1\bar q_2)}$ but neglected its off diagonal elements with respect to α.

The transition matrix element $\langle jm,\alpha_V|\hat{\mathcal{M}}|m_n,\alpha_n\rangle$ is further simplified as

$$\langle jm,\alpha_V|\hat{\mathcal{M}}|m_n,\alpha_n\rangle = \sum_{\alpha'_V,j'm'}\langle jm,\alpha_V|\hat{\mathcal{M}}|j'm',\alpha'_V\rangle \times \langle j'm',\alpha'_V|m_n,\alpha_n\rangle. \tag{36}$$

If we consider only the case where all j, m and α are conserved, we obtain

$$\langle jm,\alpha_V|\hat{\mathcal{M}}|m_n,\alpha_n\rangle = \langle jm,\alpha_V|\hat{\mathcal{M}}|jm,\alpha_V\rangle\langle jm,\alpha_V|m_n,\alpha_n\rangle. \tag{37}$$

where the rotation invariance of $\hat{\mathcal{M}}$ leads to that $\langle jm,\alpha_V|\hat{\mathcal{M}}|jm,\alpha_V\rangle$ is independent of m. In this case, the matrix element $\rho^V_{mm'}(\alpha_V)$ is obtained as

$$\rho^V_{mm'}(\alpha_V) = N(\alpha_V)\sum_{m_n;m'_n}\langle jm|m_n\rangle\langle m'_n|jm'\rangle \times \langle m_n|\hat{\bar\rho}^{(q_1\bar q_2)}(\alpha_V)|m'_n\rangle, \tag{38}$$

where $N_V(\alpha_V)$ is a constant for given α_V and $\hat{\bar\rho}^{(q_1\bar q_2)}(\alpha_V)$ is given by

$$\hat{\bar\rho}^{(q_1\bar q_2)}(\alpha_V) = \sum_{\alpha_n}\hat\rho^{(q_1\bar q_2)}(\alpha_n)\psi^*(\alpha_n;\alpha_V;m,m_n) \times \psi(\alpha_n;\alpha_V;m',m'_n), \tag{39}$$

that in general depends on m, m', m_n, and m'_n through the function,

$$\psi(\alpha_n;\alpha_V;m,m_n) = \frac{\langle m_n,\alpha_n|jm,\alpha_V\rangle}{\langle m_n|jm\rangle}. \tag{40}$$

Note that ψ does not depend on m and m_n if the wave function is factorized,

$$\langle m_n,\alpha_n|jm,\alpha_V\rangle = \langle\alpha_n|\alpha_V\rangle\langle m_n|jm\rangle. \tag{41}$$

In this case, we have $\psi(\alpha_n;\alpha_V;m',m'_n) = \langle\alpha_n|\alpha_V\rangle$ that depends on (α_n,α_V) only and

$$\hat{\bar\rho}^{(q_1\bar q_2)}(\alpha_V) = \langle\hat\rho^{(q_1\bar q_2)}(\alpha_n)\rangle_V \equiv \sum_{\alpha_n}\hat\rho^{(q_1\bar q_2)}(\alpha_n)|\langle\alpha_n|\alpha_V\rangle|^2. \tag{42}$$

We have a similar formula to Eq. (17) for $\hat{\bar\rho}^{(q_1\bar q_2)}(\alpha_V)$,

$$\hat{\bar\rho}^{(q_1\bar q_2)}(\alpha_V) = \hat{\bar\rho}^{(q_1)}(\alpha_V)\otimes\hat{\bar\rho}^{(\bar q_2)}(\alpha_V) + \frac{1}{2^2}\bar c^{(q_1\bar q_2)}_{ij}(\alpha_V)\hat\sigma_{1i}\otimes\hat\sigma_{2j}, \tag{43}$$

where the effective spin density of q_1 and the effective spin correlation are given by

$$\hat{\bar\rho}^{(q_1)}(\alpha_V) = \langle\hat\rho^{(q_1)}(\alpha_1)\rangle_V = \sum_{\alpha_n}|\langle\alpha_n|\alpha_V\rangle|^2\hat\rho^{(q_1)}(\alpha_1) = \frac{1}{2}[1+\bar P_{q_1i}(\alpha_V)\hat\sigma_{1i}],$$
$$\bar P_{q_1i}(\alpha_V) = \langle P_{q_1i}(\alpha_1)\rangle_V = \sum_{\alpha_n}|\langle\alpha_n|\alpha_V\rangle|^2P_{q_1i}(\alpha_1),$$
$$\bar c^{(q_1\bar q_2)}_{ij}(\alpha_V) = \langle c^{(q_1\bar q_2)}_{ij}(\alpha_n)+P_{qi}(\alpha_1)P_{\bar qj}(\alpha_2)\rangle_V - \langle P_{q_1i}(\alpha_1)\rangle_V\langle P_{\bar q_2j}(\alpha_2)\rangle_V, \tag{44}$$

and similar for $\hat{\bar\rho}^{(\bar q_2)}(\alpha_V)$ and $\bar P_{\bar q_2i}(\alpha_V)$. We see that the above results are similar to Eqs. (19) and (20).

Equation (42) just corresponds to the case discussed in Sec. II B. We note that the factorization in Eq. (41) is true in nonrelativistic quark models. In relativistic quantum systems, spin is not an independent degree of freedom so the wave function is not factorizable as in Eq. (41). In the following of this paper, we limit ourselves to the factorizable case and leave the general case for future studies.

Using Eqs. (38) and (43), we can obtain the results for $\rho^V_{mm'}(\alpha_V)$ similar to $\rho^V_{mm'}$ in Eqs. (31)–(34). The results can be obtained from Eqs. (31)–(34) by the replacement of spin polarization and correlation quantities, $P_{q_1i}\to\bar P_{q_1i}(\alpha_V)$, $P_{\bar q_2j}\to\bar P_{\bar q_2j}(\alpha_V)$ and $c^{(q_1\bar q_2)}_{ij}\to\bar c^{(q_1\bar q_2)}_{ij}(\alpha_V)$, where $\bar P_{q_1i}(\alpha_V)$ and $\bar c^{(q_1\bar q_2)}_{ij}(\alpha_V)$ are defined in Eq. (44).

By applying Eq. (20), the above results can be put in similar forms as those given by Eqs. (31)–(34) but with the average taken over each numerator and each denominator separately weighted by $|\langle\alpha_n|\alpha_V\rangle|^2$. Here, we show the result of ρ^V_{00} corresponding to Eq. (31),

$$\rho^V_{00}(\alpha_V) = \frac{1}{\langle C_V\rangle_V}[1+\langle P_{q_1x}P_{\bar q_2x}+P_{q_1y}P_{\bar q_2y}-P_{q_1z}P_{\bar q_2z} + c^{(q_1\bar q_2)}_{xx}+c^{(q_1\bar q_2)}_{yy}-c^{(q_1\bar q_2)}_{zz}\rangle_V]. \tag{45}$$

In practice, in particular in experiments, we often study $\hat{\rho}^V(\alpha_V)$ averaged over α_V in a given kinematic region. In this case, we obtain e.g., for ρ_{00}^V,

$$\langle\rho_{00}^V\rangle = \frac{1}{\langle\langle C_V\rangle_V\rangle_S}[1 + \langle\bar{P}_{q_1x}\bar{P}_{\bar{q}_2x} + \bar{P}_{q_1y}\bar{P}_{\bar{q}_2y} - \bar{P}_{q_1z}\bar{P}_{\bar{q}_2z} + \bar{c}_{xx}^{(q_1\bar{q}_2)} + \bar{c}_{yy}^{(q_1\bar{q}_2)} - \bar{c}_{zz}^{(q_1\bar{q}_2)}\rangle_S]$$
$$= \frac{1}{\langle\langle C_V\rangle_V\rangle_S}[1 + \langle\langle P_{q_1x}P_{\bar{q}_2x} + P_{q_1y}P_{\bar{q}_2y} - P_{q_1z}P_{\bar{q}_2z} + c_{xx}^{(q_1\bar{q}_2)} + c_{yy}^{(q_1\bar{q}_2)} - c_{zz}^{(q_1\bar{q}_2)}\rangle_V\rangle_S], \quad (46)$$

where S denotes the kinematic region of α_V or the subsystem over that we average.

We emphasized [48] in particular that the average now is twofold. For example, for $c_{ij}^{(q_1\bar{q}_2)}$, it is

$$\langle c_{ij}^{(q_1\bar{q}_2)}\rangle = \sum_{\alpha_V} f_V(\alpha_V)\langle c_{ij}^{(q_1\bar{q}_2)}(\alpha_n)\rangle_V = \sum_{\alpha_V} f_V(\alpha_V)\sum_{\alpha_n}|\langle\alpha_n|\alpha_V\rangle|^2 c_{ij}^{(q_1\bar{q}_2)}(\alpha_n), \quad (47)$$

where $f_V(\alpha_V)$ is the α_V distribution of V. We see that for the genuine quark spin correlation $c_{ij}^{(q_1\bar{q}_2)}(\alpha_{q_1},\alpha_{\bar{q}_2})$ and polarizations $P_{q_1}(\alpha_{q_1})$ and $P_{\bar{q}_2}(\alpha_{\bar{q}_2})$ of q_1 and $\bar{q}_2$, we first average over $(\alpha_{q_1},\alpha_{\bar{q}_2})$ inside the vector meson V. In this step, we obtain $\langle c_{ij}^{(q_1\bar{q}_2)}(\alpha_{q_1},\alpha_{\bar{q}_2})\rangle_V$ and the induced correlation $c_{ij}^{(q_1\bar{q}_2,0)}(\alpha_V) = \langle P_{q_1}(\alpha_{q_1})P_{\bar{q}_2}(\alpha_{\bar{q}_2})\rangle_V - \langle P_{q_1}(\alpha_{q_1})\rangle_V\langle P_{\bar{q}_2}(\alpha_{\bar{q}_2})\rangle_V$. It is clear that in this step only local quark-anti-quark spin correlations contribute. In the second step, we average over V with different α_V. For both the genuine and induced correlations, we average the results obtained in the first step at different α_V weighted by the α_V distribution $f_V(\alpha_V)$. We never consider a q_1 and $\bar{q}_2$ separated by a large distance in the α space. Hence, we do not have any contribution from long range correlation. This implies that by studying vector meson spin alignment and off diagonal elements of the spin density matrix, we study only local quark-anti-quark spin correlations.

IV. GLOBAL HYPERON POLARIZATION

For hyperon formation from three quarks, $q_1 + q_2 + q_3 \to H$, similar to the vector meson discussed in Sec. III A, the spin density matrix for the hyperon is given by

$$\hat{\rho}^H = \hat{\mathcal{M}}\hat{\rho}^{(q_1q_2q_3)}\hat{\mathcal{M}}^\dagger. \quad (48)$$

The calculation of the hyperon's polarization is also similar to the vector meson which we will present in this section.

A. With only spin degrees of freedom

In this case, the matrix element of $\rho_{mm'}^H$ is written as

$$\rho_{mm'}^H = \langle jm|\hat{\mathcal{M}}\rho^{(q_1q_2q_3)}\hat{\mathcal{M}}^\dagger|jm'\rangle = \sum_{m_n;m_n'}\langle jm|\hat{\mathcal{M}}|m_n\rangle\langle m_n|\hat{\rho}^{(q_1q_2q_3)}|m_n'\rangle\langle m_n'|\hat{\mathcal{M}}^\dagger|jm'\rangle, \quad (49)$$

where $j = 1/2$ is the spin of the hyperon H. For the quark spin state, we also omit j_n in $|j_n m_n\rangle$ since they are all $1/2$ and use the shorthand notation $|m_n\rangle$ to stand for $|m_1, m_2, m_3\rangle$. Similar to Eq. (29), the transition matrix element $\langle jm|\hat{\mathcal{M}}|m_n\rangle$ is given by

$$\langle jm|\hat{\mathcal{M}}|m_n\rangle = \sum_{j'm'}\langle jm|\hat{\mathcal{M}}|j'm'\rangle\langle j'm'|m_n\rangle. \quad (50)$$

Similar to the vector meson case discussed in Sec. III A, we use again the space rotation invariance that demands that $j = j'$ and $m = m'$ and that $\langle jm|\hat{\mathcal{M}}|jm\rangle$ is independent of m. We therefore obtain that

$$\rho_{mm'}^H = N_H\sum_{m_n;m_n'}\langle jm|m_n\rangle\langle m_n|\hat{\rho}^{(q_1q_2q_3)}|m_n'\rangle\langle m_n'|jm'\rangle. \quad (51)$$

Note in particular that the spin density matrix in Eq. (51) for the hyperon has the same form as that in Eq. (30) for the vector meson.

By inserting Eq. (4) into Eq. (51), we obtain the spin density matrix $\hat{\rho}^H$ and the hyperon polarization $P_H = \Delta\rho_H/C_H$, where $\Delta\rho_H = \rho_{1/2,1/2}^H - \rho_{-1/2,-1/2}^H$ and $C_H = \mathrm{Tr}\hat{\rho}^H$ is the normalization constant. The result for Λ is the simplest one since according to the wave function $|\Lambda^\uparrow\rangle = |uds\rangle\frac{1}{\sqrt{2}}|(\uparrow\downarrow - \downarrow\uparrow)\uparrow\rangle$ (where $\uparrow$ or $\downarrow$ denotes spin) spin of Λ is completely carried by the s quark. The result is given by

$$P_\Lambda = P_{sz} - \frac{\delta\rho_\Lambda}{C_\Lambda}, \quad (52)$$

$$\delta\rho_\Lambda = c_{iz}^{(us)}P_{di} + c_{iz}^{(ds)}P_{ui} + c_{iiz}^{(uds)}, \quad (53)$$

$$C_\Lambda = 1 - c_{ii}^{(ud)} - P_{ui}P_{di}. \quad (54)$$

From Eqs. (52)–(54), we see in particular that the Λ polarization is not simply equal to that of the s quark when quark spin correlations are considered. This is because to produce a spin-up or spin-down Λ, we need not only a spin-up or spin-down s quark but also a spin zero ud-di-quark. If the spin of s and those of u and d are correlated, the probability to have a spin zero ud-di-quark in the case that s is spin-up can be different from that in the case that s is spin-down. This leads to influences on the final polarization of Λ.

For other $J^P = (1/2)^+$ hyperons, we obtain

$$P_{H_{112}} = \frac{1}{3}(4P_{q_1 z} - P_{q_2 z}) + \frac{\delta\rho_{H_{112}}}{C_{H_{112}}}, \tag{55}$$

$$P_{\Sigma^0} = \frac{1}{3}(2P_{uz} + 2P_{dz} - P_{sz}) + \frac{\delta\rho_{\Sigma^0}}{C_{\Sigma^0}}, \tag{56}$$

where $\delta\rho_{H_{112}}$, $C_{H_{112}}$, $\delta\rho_{\Sigma^0}$ and C_{Σ^0} are given by

$$\delta\rho_{H_{112}} = -\frac{4}{3}(P_{q_1 i}P_{q_1 i} - P_{q_1 i}P_{q_2 i} + c_{ii}^{(q_1 q_1)} - c_{ii}^{(q_1 q_2)})(P_{q_1 z} - P_{q_2 z}) - 4c_{iz}^{(q_1 q_1)}P_{q_2 i} + 2(c_{iz}^{(q_1 q_2)} - 2c_{zi}^{(q_1 q_2)})P_{q_1 i} + c_{iiz}^{(q_1 q_1 q_2)} - 4c_{zii}^{(q_1 q_1 q_2)}, \tag{57}$$

$$C_{H_{112}} = 3 + P_{q_1 i}P_{q_1 i} - 4P_{q_1 i}P_{q_2 i} + c_{ii}^{(q_1 q_1)} - 4c_{ii}^{(q_1 q_2)}, \tag{58}$$

$$\begin{aligned}\delta\rho_{\Sigma^0} = {} & -\frac{2}{3}(P_{ui}P_{di} + c_{ii}^{(ud)})(P_{uz} + P_{dz} - 2P_{sz}) + \frac{2}{3}(P_{ui}P_{si} + c_{ii}^{(us)})(2P_{uz} - P_{dz} - P_{sz}) \\ & + \frac{2}{3}(P_{di}P_{si} + c_{ii}^{(ds)})(2P_{dz} - P_{uz} - P_{sz}) + (c_{zi}^{(us)} - 2c_{iz}^{(us)})P_{di} \\ & + (c_{zi}^{(ds)} - 2c_{iz}^{(ds)})P_{ui} - 2(c_{iz}^{(ud)} + c_{zi}^{(ud)})P_{si} + c_{iiz}^{(uds)} - 2c_{izi}^{(uds)} - 2c_{zii}^{(uds)},\end{aligned} \tag{59}$$

$$C_{\Sigma^0} = 3 + P_{ui}P_{di} - 2P_{di}P_{si} - 2P_{ui}P_{si} + c_{ii}^{(ud)} - 2c_{ii}^{(us)} - 2c_{ii}^{(ds)}, \tag{60}$$

where H_{112} denotes $J^P = (1/2)^+$ hyperon with quark flavor $q_1 q_1 q_2$ and we have used $c_{xz}^{(q_1 q_1)} = c_{zx}^{(q_1 q_1)}$ and $c_{ijk}^{(q_1 q_1 q_2)} = c_{jik}^{(q_1 q_1 q_2)}$. We see again that there are contributions from spin correlations in hyperons' polarizations.

B. With other degrees of freedom

If there are other degrees of freedom besides spin, the matrix elements of $\rho^H_{mm'}(\alpha_H)$ is given by

$$\begin{aligned}\rho^H_{mm'}(\alpha_H) &= \langle jm, \alpha_H|\hat{\mathcal{M}}\hat{\rho}^{(q_1 q_2 q_3)}(\alpha_n)\hat{\mathcal{M}}^\dagger|jm', \alpha_H\rangle \\ &= \sum_{m_n, m'_n}\sum_{\alpha_n}\langle jm, \alpha_H|\hat{\mathcal{M}}|m_n, \alpha_n\rangle \\ &\quad \times \langle m'_n, \alpha_n|\hat{\mathcal{M}}^\dagger|jm', \alpha_H\rangle\langle m_n|\hat{\rho}^{(q_1 q_2 q_3)}(\alpha_n)|m'_n\rangle.\end{aligned} \tag{61}$$

The transition matrix element $\langle jm, \alpha_H|\mathcal{M}|m_n, \alpha_n\rangle$ can be further written as

$$\langle jm, \alpha_H|\hat{\mathcal{M}}|m_n, \alpha_n\rangle = \sum_{\alpha'_H, j'm'}\langle jm, \alpha_H|\hat{\mathcal{M}}|j'm', \alpha'_H\rangle\langle j'm', \alpha'_H|m_n, \alpha_n\rangle. \tag{62}$$

In the case that j, m and α are conserved, we obtain

$$\langle jm, \alpha_H|\hat{\mathcal{M}}|m_n, \alpha_n\rangle = \langle jm, \alpha_H|\hat{\mathcal{M}}|jm, \alpha_H\rangle\langle jm, \alpha_H|m_n, \alpha_n\rangle, \tag{63}$$

so that the matrix element $\rho^H_{mm'}(\alpha_H)$ is given by

$$\rho^H_{mm'}(\alpha_H) = N_H(\alpha_H)\sum_{m_n, m'_n}\langle jm|m_n\rangle\langle m'_n|jm'\rangle \times \langle m_n|\hat{\bar{\rho}}^{(q_1 q_2 q_3)}(\alpha_H)|m'_n\rangle, \tag{64}$$

where $\hat{\bar{\rho}}^{(q_1 q_2 q_3)}(\alpha_H)$ in general depends on m, m_n, m', m'_n and is given by

$$\hat{\bar{\rho}}^{(q_1 q_2 q_3)}(\alpha_H) = \sum_{\alpha_n}\hat{\rho}^{(q_1 q_2 q_3)}(\alpha_n)\psi^*(\alpha_n, \alpha_H; m, m_n) \times \psi(\alpha_n, \alpha_H; m', m'_n), \tag{65}$$

where $\psi(\alpha_n, \alpha_H; m, m_n)$ is defined as

$$\psi(\alpha_n, \alpha_H; m, m_n) = \frac{\langle m_n, m, \alpha_n|jm, \alpha_H\rangle}{\langle m_n|jm\rangle}. \tag{66}$$

If the wave function is factorizable, $\langle m_n, \alpha_n|jm, \alpha_H\rangle = \langle\alpha_n|\alpha_H\rangle\langle m_n|jm\rangle$, we have $\psi(\alpha_n, \alpha_H; m, m_n) = \langle\alpha_n|\alpha_H\rangle$. So Eq. (65) is simplified as

$$\hat{\bar{\rho}}^{(q_1q_2q_3)}(\alpha_H) = \sum_{\alpha_n} \hat{\rho}^{(q_1q_2q_3)}(\alpha_n)|\langle\alpha_n|\alpha_H\rangle|^2. \quad (67)$$

Inserting Eq. (67) into Eq. (64), we can calculate the hyperon polarization and results take exactly the same form as Eqs. (52)–(60) except that all quantities are replaced by effective ones, e.g., $P_{qi} \to \bar{P}_{qi}$, $c_{ij}^{(12)} \to \bar{c}_{ij}^{(12)}$ and so on. For example, the polarization of Λ is in the form,

$$P_\Lambda(\alpha_\Lambda) = \bar{P}_{sz} - \frac{\bar{c}_{iiz}^{(uds)} + \bar{c}_{iz}^{(us)}\bar{P}_{di} + \bar{c}_{iz}^{(ds)}\bar{P}_{ui}}{1 - \bar{c}_{ii}^{(ud)} - \bar{P}_{ui}\bar{P}_{di}}. \quad (68)$$

By applying Eqs. (17) and (24), we can rewrite Eq. (68) as

$$P_\Lambda(\alpha_\Lambda) = \frac{\langle P_{sz}(1-c_{ii}^{(ud)} - P_{ui}P_{di}) - c_{iiz}^{(uds)} - c_{iz}^{(us)}P_{di} - c_{iz}^{(ds)}P_{ui}\rangle_\Lambda}{\langle 1 - c_{ii}^{(ud)} - P_{ui}P_{di}\rangle_\Lambda}, \quad (69)$$

where the averages are taken with the weight $|\langle\alpha_n|\alpha_\Lambda\rangle|^2$. We see that the situation is similar to vector mesons. If all the genuine quark correlations $c_{ij}^{(q_1q_2)} = 0$ and $c_{ijk}^{(q_1q_2q_3)} = 0$, we still have induced correlations,

$$P_\Lambda(\alpha_\Lambda) = \frac{\langle P_{sz}(1 - P_{ui}P_{di})\rangle_\Lambda}{\langle 1 - P_{ui}P_{di}\rangle_\Lambda}. \quad (70)$$

Similar to the vector meson's spin alignment, we often run into $P_\Lambda(\alpha_\Lambda)$ averaged over α_Λ in a given kinematic region. Then, we will have an additional average over the distribution $f_\Lambda(\alpha_\Lambda)$. It is also obvious that we have only local quark-quark spin correlations in this case.

Together with the results obtained in Sec. III, we see that by the spin polarization of one hadron, we can only probe local quark spin correlations.

V. GLOBAL SPIN CORRELATIONS OF HYPERONS

The calculations can be extended in a straightforward manner to hyperon-hyperon and hyperon-anti-hyperon spin correlations. In this section, we take hyperon-anti-hyperon as an example to show the calculation and results.

For a spin-1/2 hyperon pair $H_1\bar{H}_2$, the spin correlation is usually defined in the conventional way as given by Eq. (12),

$$c_{nn}^{H_1\bar{H}_2} = \frac{f_{++}^{H_1\bar{H}_2} + f_{--}^{H_1\bar{H}_2} - f_{+-}^{H_1\bar{H}_2} - f_{-+}^{H_1\bar{H}_2}}{f_{++}^{H_1\bar{H}_2} + f_{--}^{H_1\bar{H}_2} + f_{+-}^{H_1\bar{H}_2} + f_{-+}^{H_1\bar{H}_2}}, \quad (71)$$

where $f_{m_{H_1}m_{\bar{H}_2}}^{H_1\bar{H}_2} = \langle m_{H_1}m_{\bar{H}_2}|\hat{\rho}^{H_1\bar{H}_2}|m_{H_1}m_{\bar{H}_2}\rangle$ and m_{H_1}, $m_{\bar{H}_2} = \pm$ denoting the spin states parallel or antiparallel to the $\hat{\mathbf{n}}$ direction, respectively. We simply adopt this definition and calculate its relationship to those quantities defined at the quark level using quark combination mechanism. In the calculation, the most convenient way is to rotate the Cartesian system so that $\hat{\mathbf{n}}$ direction becomes z direction in the new system. We choose this case as an example to do the calculation and denote $c_{nn}^{H_1\bar{H}_2}$ by $c_{zz}^{H_1\bar{H}_2}$ in the following of this paper.

Now the task is to compute the spin density matrix element $\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m_{H_1}m_{\bar{H}_2}} = \langle m_{H_1}m_{\bar{H}_2}|\hat{\rho}^{H_1\bar{H}_2}|m_{H_1}m_{\bar{H}_2}\rangle$. Similar to spin density operators for vector mesons and hyperons given by Eqs. (27) and (48), $\hat{\rho}^{H_1\bar{H}_2}$ is related to that of the six body system $q_1q_2q_3\bar{q}_4\bar{q}_5\bar{q}_6$ by

$$\hat{\rho}^{H_1\bar{H}_2} = \hat{\mathcal{M}}\hat{\rho}^{(1\ldots6)}\hat{\mathcal{M}}^\dagger, \quad (72)$$

where we simply used "1…6" to label $q_1q_2q_3\bar{q}_4\bar{q}_5\bar{q}_6$. The matrix element of $\hat{\rho}_{H_1\bar{H}_2}$ is given by

$$\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m'_{H_1}m'_{\bar{H}_2}} = \langle j_{H_1}m_{H_1}, j_{\bar{H}_2}m_{\bar{H}_2}|\hat{\mathcal{M}}\hat{\rho}^{(1\ldots6)}\hat{\mathcal{M}}^\dagger|j_{H_1}m'_{H_1}, j_{\bar{H}_2}m'_{\bar{H}_2}\rangle. \quad (73)$$

The complete expansion of $\hat{\rho}^{(1\ldots6)}$ is

$$\begin{aligned}\hat{\rho}^{(1\ldots6)} = {}& \hat{\rho}^{(1)}\otimes\hat{\rho}^{(2)}\otimes\hat{\rho}^{(3)}\otimes\hat{\rho}^{(4)}\otimes\hat{\rho}^{(5)}\otimes\hat{\rho}^{(6)} + \frac{1}{2^2}[c_{ij}^{(12)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\rho}^{(3)}\otimes\hat{\rho}^{(4)}\otimes\hat{\rho}^{(5)}\otimes\hat{\rho}^{(6)} + 14\,\text{exchange terms}] \\ & + \frac{1}{2^3}[c_{ijk}^{(123)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}\otimes\hat{\rho}^{(4)}\otimes\hat{\rho}^{(5)}\otimes\hat{\rho}^{(6)} + 19\,\text{exchange terms}] \\ & + \frac{1}{2^4}[c_{ijkl}^{(1234)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}\otimes\hat{\sigma}_{4l}\otimes\hat{\rho}^{(5)}\otimes\hat{\rho}^{(6)} + 14\,\text{exchange terms}] \\ & + \frac{1}{2^5}[c_{ijklm}^{(12345)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}\otimes\hat{\sigma}_{4l}\otimes\hat{\sigma}_{5m}\otimes\hat{\rho}^{(6)} + 5\,\text{exchange terms}] \\ & + \frac{1}{2^6}c_{ijklmn}^{(123456)}\hat{\sigma}_{1i}\otimes\hat{\sigma}_{2j}\otimes\hat{\sigma}_{3k}\otimes\hat{\sigma}_{4l}\otimes\hat{\sigma}_{5m}\otimes\hat{\sigma}_{6n}.\end{aligned} \quad (74)$$

In the rest part of this section, we take $\Lambda\bar{\Lambda}$ as an example to show the calculation of the hyperon-anti-hyperon spin correlation. For simplicity, we only consider two-particle spin correlations and set all other correlations with more than two particles as zero. As before, we consider two cases, the one with only spin degree of freedom and the one with other degrees of freedom denoted by α. The calculations can be extended to other hyperons and/or include spin correlations of more than two particles in a straightforward way.

A. With spin degree of freedom

As in previous sections, we insert the completeness identity $\sum_{m_n} |m_n\rangle\langle m_n| = 1$ into Eq. (73) and obtain

$$\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m'_{H_1}m'_{\bar{H}_2}} = \sum_{m_n,m'_n} \langle m_{H_1}m_{\bar{H}_2}|\hat{\mathcal{M}}|m_n\rangle\langle m_n|\hat{\rho}^{(1\ldots6)}|m'_n\rangle \times \langle m'_n|\hat{\mathcal{M}}^\dagger|m'_{H_1}m'_{\bar{H}_2}\rangle, \tag{75}$$

where we have suppressed $j_{H_1} = j_{\bar{H}_2} = 1/2$ for $H_1 = \Lambda$ and $\bar{H}_2 = \bar{\Lambda}$. The transition matrix element can be written as

$$\begin{aligned}\langle m_{H_1}m_{\bar{H}_2}|\hat{\mathcal{M}}|m_n\rangle &= \langle m_{H_1}m_{\bar{H}_2}|\hat{\mathcal{M}}|m_{H_1}m_{\bar{H}_2}\rangle\langle m_{H_1}m_{\bar{H}_2}|m_n\rangle \\ &= \langle m_{H_1}|\hat{\mathcal{M}}_H|m_{H_1}\rangle\langle m_{\bar{H}_2}|\hat{\mathcal{M}}_{\bar{H}}|m_{\bar{H}_2}\rangle \\ &\quad\times \langle m_{H_1}m_{\bar{H}_2}|m_n\rangle,\end{aligned} \tag{76}$$

where we have assumed a factorization form for $\langle m_{H_1}m_{\bar{H}_2}|\hat{\mathcal{M}}|m_{H_1}m_{\bar{H}_2}\rangle$ with $\hat{\mathcal{M}} = \hat{\mathcal{M}}_H\hat{\mathcal{M}}_{\bar{H}}$, so that the transition matrix contributes only to the normalization constant and has no effect on the spin part. Then, we obtain

$$\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m'_{H_1}m'_{\bar{H}_2}} = N_{H_1\bar{H}_2}\sum_{m_n,m'_n} \langle m_{H_1}m_{\bar{H}_2}|m_n\rangle\langle m_n|\hat{\rho}^{(1\ldots6)}|m'_n\rangle \times \langle m'_n|m'_{H_1}m'_{\bar{H}_2}\rangle. \tag{77}$$

We note that in production processes of H_1 and $\bar{H}_2$, $q_1 + q_2 + q_3 \to H_1$ and $\bar{q}_4 + \bar{q}_5 + \bar{q}_6 \to \bar{H}_2$, the Clebsch-Gordan coefficient $\langle m_{H_1}m_{\bar{H}_2}|m_n\rangle$ is just the product of $\langle m_{H_1}|m_1m_2m_3\rangle$ and $\langle m_{\bar{H}_2}|m_4m_5m_6\rangle$.

When all two-particle spin correlations are considered, the result for the spin correlation of $\Lambda\bar{\Lambda}$ is

$$\begin{aligned}c^{\Lambda\bar{\Lambda}}_{zz} = P_{sz}P_{\bar{s}z} + \frac{1}{C_{\Lambda\bar{\Lambda}}}\{&c^{(s\bar{s})}_{zz}(1 - P_{ui}P_{di})(1 - P_{\bar{u}i}P_{\bar{d}i}) \\ &- P_{sz}[(c^{(d\bar{s})}_{iz}P_{ui} + c^{(u\bar{s})}_{iz}P_{di})(1 - P_{\bar{u}i}P_{\bar{d}i}) \\ &+ (c^{(\bar{d}\bar{s})}_{iz}P_{\bar{u}i} + c^{(\bar{u}\bar{s})}_{iz}P_{\bar{d}i})(1 - P_{ui}P_{di})] \\ &- P_{\bar{s}z}[(c^{(ds)}_{iz}P_{ui} + c^{(us)}_{iz}P_{di})(1 - P_{\bar{u}i}P_{\bar{d}i}) \\ &+ (c^{(\bar{d}s)}_{iz}P_{\bar{u}i} + c^{(\bar{u}s)}_{iz}P_{\bar{d}i})(1 - P_{ui}P_{di})]\},\end{aligned} \tag{78}$$

where the normalization constant $C_{\Lambda\bar{\Lambda}}$ is given by

$$\begin{aligned}C_{\Lambda\bar{\Lambda}} = C_\Lambda C_{\bar{\Lambda}} - c^{(ud)}_{ii}c^{(\bar{u}\bar{d})}_{jj} + c^{(u\bar{u})}_{ij}P_{di}P_{\bar{d}j} + c^{(d\bar{d})}_{ij}P_{ui}P_{\bar{u}j} \\ + c^{(d\bar{u})}_{ij}P_{ui}P_{\bar{d}j} + c^{(u\bar{d})}_{ij}P_{di}P_{\bar{u}j},\end{aligned} \tag{79}$$

where C_Λ is given by Eq. (54) and $C_{\bar{\Lambda}}$ is obtained from C_Λ with the replacement of all quantities for quarks by those for corresponding antiquarks.

We compare the results given by Eqs. (78)–(79) with those given by Eqs. (52)–(54) for Λ polarization. We note that we need to put all spin correlations of more than two quarks and/or antiquarks and products of two particle spin correlations as zero since we consider only two particle spin correlations. In this way, we obtain

$$\begin{aligned}c^{\Lambda\bar{\Lambda}}_{zz} \approx P_{\Lambda z}P_{\bar{\Lambda}z} + c^{(s\bar{s})}_{zz} - \frac{P_{sz}}{C_\Lambda}[c^{(d\bar{s})}_{iz}P_{ui} + c^{(u\bar{s})}_{iz}P_{di}] \\ - \frac{P_{\bar{s}z}}{C_{\bar{\Lambda}}}[c^{(s\bar{d})}_{zi}P_{\bar{u}i} + c^{(s\bar{u})}_{zi}P_{\bar{d}i}].\end{aligned} \tag{80}$$

From Eq. (80), we see clearly that the spin correlation of Λ and $\bar{\Lambda}$ comes from those of quarks and antiquarks. We also see clearly that $c^{\Lambda\bar{\Lambda}}_{zz} = P_{\Lambda z}P_{\bar{\Lambda}z}$ if only quark-quark and antiquark-antiquark spin correlations are considered.

B. With other degrees of freedom

It is clear that in this case the six-quark (antiquark) spin density matrix $\hat{\rho}^{(1\ldots6)}(\alpha_n)$ takes the same form as $\hat{\rho}^{(1\ldots6)}$ in (74) except that all $\hat{\rho}^{(n)}$ $(n = 1, \ldots, 6)$ depends on α_n and that all correlation coefficients $c^{(1\cdots n)}_{i_1\cdots i_n}$ with $n \le 6$ depend on α_n. The spin density matrix for $H_1\bar{H}_2$ now becomes $\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m'_{H_1}m'_{\bar{H}_2}}(\alpha_{H_1}, \alpha_{\bar{H}_2})$ that depends on α_{H_1} and $\alpha_{\bar{H}_2}$. Then we obtain a similar formula for $\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m'_{H_1}m'_{\bar{H}_2}}(\alpha_{H_1}, \alpha_{\bar{H}_2})$ to Eq. (77). By assuming a factorization condition similar to Eq. (76) for H_1 and $\bar{H}_2$ and that for the spin and α parts of wave functions, we obtain

$$\begin{aligned}&\rho^{H_1\bar{H}_2}_{m_{H_1}m_{\bar{H}_2};m'_{H_1}m'_{\bar{H}_2}}(\alpha_{H_1}, \alpha_{\bar{H}_2}) \\ &\quad= N_{H_1\bar{H}_2}(\alpha_{H_1}, \alpha_{\bar{H}_2})\sum_{m_n,m'_n}\langle m_{H_1}m_{\bar{H}_2}|m_n\rangle \\ &\quad\quad\times \langle m_n|\hat{\bar{\rho}}^{(1\ldots6)}(\alpha_{H_1}, \alpha_{\bar{H}_2})|m'_n\rangle\langle m'_n|m'_{H_1}m'_{\bar{H}_2}\rangle,\end{aligned} \tag{81}$$

where the effective density matrix is given by

$$\hat{\bar{\rho}}^{(1\ldots6)}(\alpha_{H_1}, \alpha_{\bar{H}_2}) = \sum_{\alpha_n}\sum_{\alpha_m}\hat{\rho}^{(1\ldots6)}(\alpha_n, \alpha_m)|\langle\alpha_n|\alpha_{H_1}\rangle|^2|\langle\alpha_m|\alpha_{\bar{H}_2}\rangle|^2. \tag{82}$$

We see the difference between Eq. (77) for the case with only the spin degree of freedom, and Eq. (81) is the replacement $\hat{\rho}^{(1...6)} \to \hat{\bar{\rho}}^{(1...6)}(\alpha_{H_1}, \alpha_{\bar{H}_2})$.

We emphasize the average in Eq. (82) can be carried out inside H_1 and $\bar{H}_2$ for quarks and antiquarks successively. This is different from the case for vector meson discussed in Sec. III B, where the average inside V is carried out for the quark and antiquark, simultaneously. More precisely, we have

$$\bar{P}_{q_l}(\alpha_{H_1}) = \sum_{\alpha_n} P_{q_l}(\alpha_{q_l}) |\langle \alpha_n | \alpha_{H_1} \rangle|^2 = \langle P_{q_l}(\alpha_{q_l}) \rangle_{H_1}, \quad (83)$$

$$\bar{P}_{\bar{q}_l}(\alpha_{\bar{H}_2}) = \sum_{\alpha_m} P_{\bar{q}_l}(\alpha_{\bar{q}_l}) |\langle \alpha_m | \alpha_{\bar{H}_2} \rangle|^2 = \langle P_{\bar{q}_l}(\alpha_{\bar{q}_l}) \rangle_{\bar{H}_2}. \quad (84)$$

We obtain two-particle spin correlations as

$$\begin{aligned} \bar{c}_{ij}^{(q_1 q_2)}(\alpha_{H_1}, \alpha_{\bar{H}_2}) &= \sum_{\alpha_n, \alpha_m} [c_{ij}^{(q_1 q_2)}(\alpha_{q_1}, \alpha_{q_2}) + P_{q_1 i}(\alpha_{q_1}) P_{q_2 j}(\alpha_{q_2})] |\langle \alpha_n | \alpha_{H_1} \rangle|^2 |\langle \alpha_m | \alpha_{\bar{H}_2} \rangle|^2 - \bar{P}_{q_1 i}(\alpha_{H_1}) \bar{P}_{q_2 j}(\alpha_{H_1}), \\ &= \sum_{\alpha_n} [c_{ij}^{(q_1 q_2)}(\alpha_{q_1}, \alpha_{q_2}) + P_{q_1 i}(\alpha_{q_1}) P_{q_2 j}(\alpha_{q_2})] |\langle \alpha_n | \alpha_{H_1} \rangle|^2 - \bar{P}_{q_1 i}(\alpha_{H_1}) \bar{P}_{q_2 j}(\alpha_{H_1}), \end{aligned} \quad (85)$$

$$\begin{aligned} \bar{c}_{ij}^{(\bar{q}_1 \bar{q}_2)}(\alpha_{H_1}, \alpha_{\bar{H}_2}) &= \sum_{\alpha_n, \alpha_m} [c_{ij}^{(\bar{q}_1 \bar{q}_2)}(\alpha_{\bar{q}_1}, \alpha_{\bar{q}_2}) + P_{\bar{q}_1 i}(\alpha_{\bar{q}_1}) P_{\bar{q}_2 j}(\alpha_{\bar{q}_2})] |\langle \alpha_n | \alpha_{H_1} \rangle|^2 |\langle \alpha_m | \alpha_{\bar{H}_2} \rangle|^2 - \bar{P}_{\bar{q}_1 i}(\alpha_{\bar{H}_2}) \bar{P}_{\bar{q}_2 j}(\alpha_{\bar{H}_2}), \\ &= \sum_{\alpha_m} [c_{ij}^{(\bar{q}_1 \bar{q}_2)}(\alpha_{\bar{q}_1}, \alpha_{\bar{q}_2}) + P_{\bar{q}_1 i}(\alpha_{\bar{q}_1}) P_{\bar{q}_2 j}(\alpha_{\bar{q}_2})] |\langle \alpha_m | \alpha_{\bar{H}_2} \rangle|^2 - \bar{P}_{\bar{q}_1 i}(\alpha_{\bar{H}_2}) \bar{P}_{\bar{q}_2 j}(\alpha_{\bar{H}_2}), \end{aligned} \quad (86)$$

$$\begin{aligned} \bar{c}_{ij}^{(q_1 \bar{q}_2)}(\alpha_{H_1}, \alpha_{\bar{H}_2}) &= \sum_{\alpha_n, \alpha_m} [c_{ij}^{(q_1 \bar{q}_2)}(\alpha_{q_1}, \alpha_{\bar{q}_2}) + P_{q_1 i}(\alpha_{q_1}) P_{\bar{q}_2 j}(\alpha_{\bar{q}_2})] |\langle \alpha_n | \alpha_{H_1} \rangle|^2 |\langle \alpha_m | \alpha_{\bar{H}_2} \rangle|^2 - \bar{P}_{q_1 i}(\alpha_{H_1}) \bar{P}_{\bar{q}_2 j}(\alpha_{\bar{H}_2}) \\ &= \sum_{\alpha_n, \alpha_m} c_{ij}^{(q_1 \bar{q}_2)}(\alpha_{q_1}, \alpha_{\bar{q}_2}) |\langle \alpha_n | \alpha_{H_1} \rangle|^2 |\langle \alpha_m | \alpha_{\bar{H}_2} \rangle|^2. \end{aligned} \quad (87)$$

We see that $\bar{c}_{ij}^{(q_1 q_2)}$ is independent of $\alpha_{\bar{H}_2}$ and $\bar{c}_{ij}^{(\bar{q}_1 \bar{q}_2)}$ is independent of α_{H_1}, while $\bar{c}_{ij}^{(q_1 \bar{q}_2)}$ just reduces to

$$\bar{c}_{ij}^{(q_1 \bar{q}_2)}(\alpha_{H_1}, \alpha_{\bar{H}_2}) = \langle c_{ij}^{(q_1 \bar{q}_2)}(\alpha_{q_1}, \alpha_{\bar{q}_2}) \rangle_{H_1 \bar{H}_2}, \quad (88)$$

because

$$\langle P_{q_1 i}(\alpha_{q_1}) P_{\bar{q}_2 j}(\alpha_{\bar{q}_2}) \rangle_{H_1 \bar{H}_2} = \langle P_{q_1 i}(\alpha_{q_1}) \rangle_{H_1} \langle P_{\bar{q}_2 j}(\alpha_{\bar{q}_2}) \rangle_{\bar{H}_2}. \quad (89)$$

Here, we neglect the overlap of H_1 and $\bar{H}_2$ in α space. In this case, we have no contributions from the induced spin correlation between the quark and antiquark, i.e., $\bar{c}_{ij}^{(q_1 \bar{q}_2;0)}(\alpha_{H_1}, \alpha_{\bar{H}_2}) = 0$. Also, because α_{q_i} is inside H_1 while $\alpha_{\bar{q}_j}$ is inside $\bar{H}_2$, we do not have contributions from local spin correlations between quarks and antiquarks. This can be seen more clearly if we assume all genuine two-particle correlations vanish. In this case, the average of each term in Eqs. (82) can be separated into a product of two factors, one is inside H_1 for quarks and the other is inside $\bar{H}_2$ for antiquarks. This shows explicitly that we have contributions from local quark-quark and antiquark-antiquark correlations but no contribution from local quark-antiquark correlations. Hence, in spin correlations between the hyperon and anti-hyperon, there is no contribution from local correlations between the quark and antiquark.

Now we compute the spin correlation of $\Lambda\bar{\Lambda}$ with above formula. With only two-particle spin correlations, the result is just those given by Eqs. (78)–(80) with the replacement of all quantities by the corresponding effective ones, i.e.,

$$\begin{aligned} c_{zz}^{\Lambda\bar{\Lambda}}(\alpha_\Lambda, \alpha_{\bar{\Lambda}}) \approx{}& P_{\Lambda z}(\alpha_\Lambda) P_{\bar{\Lambda} z}(\alpha_{\bar{\Lambda}}) + \bar{c}_{zz}^{(s\bar{s})} \\ &- \frac{\bar{P}_{sz}}{\bar{C}_\Lambda} [\bar{c}_{iz}^{(d\bar{s})} \bar{P}_{ui} + \bar{c}_{iz}^{(u\bar{s})} \bar{P}_{di}] \\ &- \frac{\bar{P}_{\bar{s}z}}{\bar{C}_{\bar{\Lambda}}} [\bar{c}_{zi}^{(s\bar{d})} \bar{P}_{\bar{u}i} + \bar{c}_{zi}^{(s\bar{u})} \bar{P}_{\bar{d}i}]. \end{aligned} \quad (90)$$

We emphasize in particular that all quantities for quarks and/or antiquarks on the rhs of Eq. (90) are effective ones and are functions of α_Λ and/or $\alpha_{\bar{\Lambda}}$. More precisely, $\bar{P}_{qi}$ and $\bar{P}_{\bar{q}i}$ are functions of α_Λ and $\alpha_{\bar{\Lambda}}$, respectively, while $\bar{c}_{ij}^{(q_1 \bar{q}_2)}$ is a function of $(\alpha_\Lambda, \alpha_{\bar{\Lambda}})$. These results are valid in the case that we neglect the overlap of the wave function of Λ with that of $\bar{\Lambda}$.

We now discuss a simple case where both quark polarizations and quark spin correlations are small so that we can neglect the last two terms in Eq. (90) compared with the first two. In this case, when further averaged over α_Λ and $\alpha_{\bar{\Lambda}}$ in a given kinematic region, we obtain

$$\begin{aligned}\langle c_{zz}^{\Lambda\bar{\Lambda}}\rangle &\approx \langle P_{\Lambda z}(\alpha_\Lambda)P_{\bar{\Lambda}z}(\alpha_{\bar{\Lambda}})\rangle + \langle \bar{c}_{zz}^{(s\bar{s})}\rangle \\ &\approx \langle P_{\Lambda z}\rangle\langle P_{\bar{\Lambda}z}\rangle + \langle c_{zz}^{s\bar{s};0}\rangle + \langle \bar{c}_{zz}^{(s\bar{s})}\rangle,\end{aligned} \tag{91}$$

where $\langle \bar{c}_{zz}^{(s\bar{s};0)}\rangle = \langle \bar{P}_{sz}(\alpha_\Lambda)\bar{P}_{\bar{s}z}(\alpha_{\bar{\Lambda}})\rangle - \langle \bar{P}_{sz}(\alpha_\Lambda)\rangle\langle \bar{P}_{\bar{s}z}(\alpha_{\bar{\Lambda}})\rangle$. Compared to the correlation inside a hadron, we call this long range correlation. We see that $\langle \bar{c}_{zz}^{(s\bar{s};0)}\rangle$ is the contribution from the induced spin correlation, while $\langle \bar{c}_{zz}^{(s\bar{s})}\rangle$ is from the genuine quark spin correlation of $s\bar{s}$.

Since we do not consider the overlap of the wave functions of the hyperon and antihyperon, the results obtained above can be extended directly to hyperon-hyperon spin correlations. In particular, those given by Eqs. (90)–(91) can be extended to $\Lambda\Lambda$ spin correlations if we neglect the overlap of the wave function of the two Λ's. In this case, we need only to replace $\bar{P}_q(\alpha_\Lambda)$ and $\bar{P}_{\bar{q}}(\alpha_{\bar{\Lambda}})$ by $\bar{P}_q(\alpha_{\Lambda_1})$ and $\bar{P}_q(\alpha_{\Lambda_2})$, respectively, in order to obtain $c_{zz}^{\Lambda\Lambda}(\alpha_{\Lambda_1},\alpha_{\Lambda_2})$,

$$\begin{aligned}c_{zz}^{\Lambda\Lambda}(\alpha_{\Lambda_1},\alpha_{\Lambda_2}) &\approx P_{\Lambda z}(\alpha_{\Lambda_1})P_{\Lambda z}(\alpha_{\Lambda_2}) + \bar{c}_{zz}^{(ss)}(\alpha_{\Lambda_1},\alpha_{\Lambda_2}) \\ &\quad - \frac{\bar{P}_{sz}(\alpha_{\Lambda_1})}{\bar{C}_\Lambda(\alpha_{\Lambda_1})}[\bar{c}_{iz}^{(ds)}(\alpha_{\Lambda_1},\alpha_{\Lambda_2})\bar{P}_{ui}(\alpha_{\Lambda_1}) + \bar{c}_{iz}^{(us)}(\alpha_{\Lambda_1},\alpha_{\Lambda_2})\bar{P}_{di}(\alpha_{\Lambda_1})] \\ &\quad - \frac{\bar{P}_{sz}(\alpha_{\Lambda_2})}{\bar{C}_\Lambda(\alpha_{\Lambda_2})}[\bar{c}_{zi}^{(sd)}(\alpha_{\Lambda_1},\alpha_{\Lambda_2})\bar{P}_{ui}(\alpha_{\Lambda_2}) + \bar{c}_{zi}^{(su)}(\alpha_{\Lambda_1},\alpha_{\Lambda_2})\bar{P}_{di}(\alpha_{\Lambda_2})].\end{aligned} \tag{92}$$

In the simple case considered above for $\langle c_{zz}^{\Lambda\bar{\Lambda}}\rangle$, we obtain similar result for $c_{zz}^{\Lambda\Lambda}$ as

$$\langle c_{zz}^{\Lambda\Lambda}\rangle \approx \langle P_{\Lambda z}\rangle^2 + \langle \bar{c}_{zz}^{(ss;0)}\rangle + \langle \bar{c}_{zz}^{(ss)}\rangle. \tag{93}$$

We see that in this case the spin correlation between two Λ's measures the spin correlation between two s quarks.

To compare with the results obtained in Sec. III, we see clearly that the spin alignment of the ϕ meson probes the spin correlations between s and $\bar{s}$ inside the vector meson. In contrast, $\Lambda\bar{\Lambda}$ and $\Lambda\Lambda$ spin correlations probes the spin correlations between ss or $s\bar{s}$ in the whole QGP system [13,54,55]. The former is in general short ranged while the latter includes long range contributions. The strength of such correlations is determined by the dynamics of the system and is an important direction for future study.

VI. NUMERICAL ESTIMATES

The global quark spin polarizations and correlations are determined by the QCD dynamics in heavy-ion collisions and can be calculated using QCD-based theoretical models. Having the relationships between measurable quantities at the hadron level and global spin properties at the quark level, we can also extract them from data available and make predictions for other measurable quantities. The available data are however still far from enough to make high precision predictions. In this section, we just present a rough estimate based on the data available [1–6,41].

We use Eqs. (70), (45), (91) and take approximately,

$$\langle P_\Lambda\rangle \sim \langle P_s\rangle, \tag{94}$$

$$\langle \rho_{00}^\phi\rangle \sim \frac{1 - \bar{c}_{zz;\phi}^{(s\bar{s})} - \langle P_s\rangle^2}{3 + \bar{c}_{zz;\phi}^{(s\bar{s})} + \langle P_s\rangle^2}, \tag{95}$$

$$\langle c_{zz}^{\Lambda\bar{\Lambda}}\rangle \sim \bar{c}_{zz;\Lambda\bar{\Lambda}}^{(s\bar{s})} + \langle P_s\rangle^2, \tag{96}$$

where all quark spin correlations are effective ones and are in general sums of genuine and induced contributions. We use these equations to extract $\langle P_s\rangle$ and $\bar{c}_{zz;\phi}^{(s\bar{s})}$ from data for $\langle P_\Lambda\rangle$ and $\langle \rho_{00}^\phi\rangle$ [1–6,41], and make estimates of $\langle c_{zz}^{\Lambda\bar{\Lambda}}\rangle$.

We take the following forms of $\langle P_s\rangle$ and $\bar{c}_{zz;\phi}^{(s\bar{s})}$ as functions of $\sqrt{s_{NN}}$:

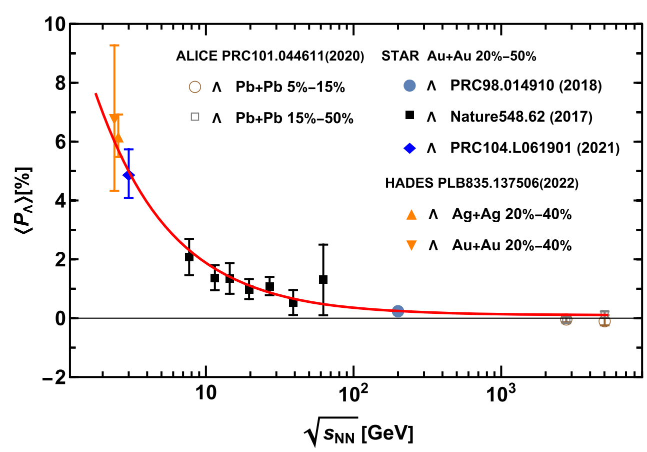

FIG. 1. Fit to the global Λ polarization as a function of energy $\sqrt{s_{NN}}$. The data are taken from Refs. [1–6].

$$\langle P_s \rangle = a s_{NN}^{-b} + c, \tag{97}$$

$$\bar{c}_{zz;\phi}^{(s\bar{s})} = e s_{NN}^{-f} + d. \tag{98}$$

By taking $a = 0.123$, $b = 0.42$, $c = 0.002$, $d = 0.032$, $e = -0.25$, and $f = 0.18$, we obtain the fits to $\langle P_\Lambda \rangle$ and

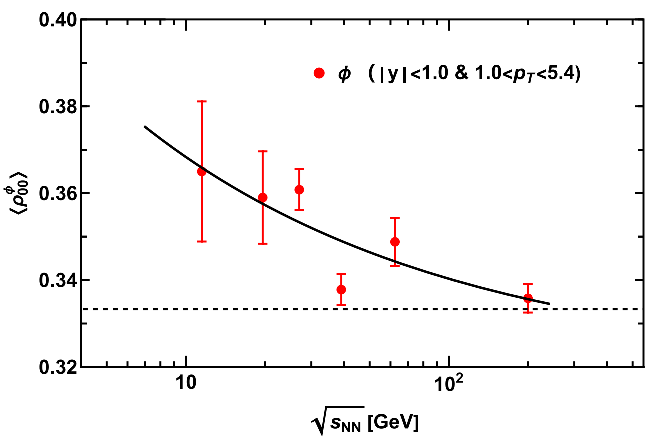

FIG. 2. Fit to ρ_{00}^{ϕ} as a function of energy $\sqrt{s_{NN}}$. The data are taken from Ref. [41].

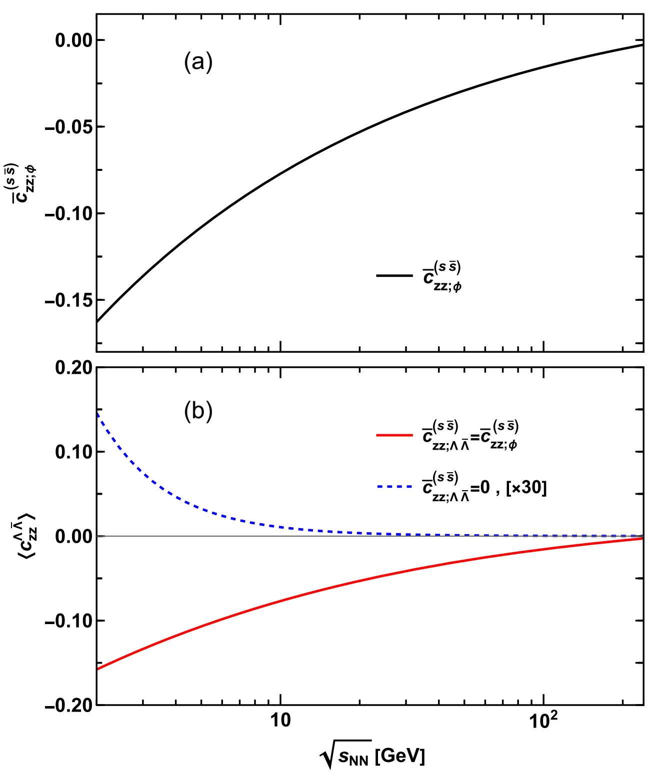

FIG. 3. (a) The effective global spin correlation $\bar{c}_{zz}^{(s\bar{s})}$ between s and $\bar{s}$ as a function of energy $\sqrt{s_{NN}}$ obtained by fitting the data [1–6,41] using Eqs. (94) and (95). (b) Estimates of $\langle c_{zz}^{\Lambda\bar{\Lambda}} \rangle$ as functions of $\sqrt{s_{NN}}$ in two extreme cases described in the text.

$\langle \rho_{00}^{\phi} \rangle$ as in Figs. 1 and 2, respectively. The obtained $\bar{c}_{zz;\phi}^{(s\bar{s})}$ as a function of $\sqrt{s_{NN}}$ is shown in Fig. 3(a).

We take two extreme examples, i.e., $\bar{c}_{zz;\Lambda\bar{\Lambda}}^{(s\bar{s})} = \bar{c}_{zz;\phi}^{(s\bar{s})}$ or $\bar{c}_{zz;\Lambda\bar{\Lambda}}^{(s\bar{s})} = 0$ and draw the results for $\langle c_{zz}^{\Lambda\bar{\Lambda}} \rangle$ as functions of $\sqrt{s_{NN}}$ in Fig. 3(b). We see that the results in these two extreme cases are quite different from each other and they can be tested by future experiments.

We stress that from Eqs. (45), the vector meson spin alignment for ϕ mesons is determined mainly by the local spin correlation between s and $\bar{s}$ while those for $\langle c_{zz}^{\Lambda\bar{\Lambda}} \rangle$ depends mainly on the long range spin correlation between them. They are in general quite different from each other. The results in Fig. 3(b) are just for two extreme cases. Similarly, if we consider, e.g., other vector mesons, the results are determined by the local spin correlations between the quark and antiquark with corresponding flavors. In this sense, measurements, e.g., of K^{*0} by the STAR Collaboration [41] at RHIC seem to suggest that the local spin correlations between d and $\bar{s}$ are much smaller than those between s and $\bar{s}$.

VII. SUMMARY AND OUTLOOK

The STAR measurements of the global spin alignment of vector mesons ρ_{00}^{ϕ} [41] indicate that there are strong global quark-antiquark spin correlations in relativistic heavy-ion collisions. It opens a new window to study properties of QGP and reaction mechanisms of relativistic heavy ion collisions. We propose a systematic way of describing quark and/or antiquark spin correlations in the QGP. We show that effective quark spin correlations contain contributions from genuine spin correlations from dynamics and induced spin correlations due to average over other degrees of freedom. We derive the relationships between these spin correlations at the quark level and those for hyperons and vector mesons that are measurable in experiments. We show in particular that the vector meson's spin density matrix elements, either diagonal or off diagonal, are sensitive to local spin correlations between the quark and antiquark, while hyperon-anti-hyperon spin correlations are sensitive to long rang quark spin correlations. We present a rough estimate of spin correlations based on available data [1–6,41] to guide future measurements.

We point out that genuine spin correlations have never been considered in most theoretical studies of spin phenomena in heavy-ion collisions to our knowledge [46–64]. The global vector meson's spin alignment in previous studies comes only from induced quark correlations. We note that genuine spin correlations exist in general for quarks and/or antiquarks produced in elementary high energy processes such as $e^+e^- \to q\bar{q}$ [65,66] and have been discussed in connection with dihadron spin correlations in e^+e^-, lp or pp collisions [67–70]. The STAR data [41] suggest a strong global quark-antiquark spin correlation, so the study of global genuine quark spin correlations in heavy-ion collisions at the dynamical level can be an important direction in the future.

When other degrees of freedom characterized by α are considered, we assume that the spin and α part of the wave function are factorizable. This is general true in the nonrelativistic case. However, in the relativistic case, spin and other degrees of freedom such as momentum are usually coupled in an intrinsic way so that such a factorization is impossible. In such cases, the calculation is more complicated but can be done, which we reserve for a future study.

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