Higher-order NLO initial state QED radiative corrections to e^+e^- annihilation revisited

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Radiative corrections due to initial state radiation in electron-positron annihilation are calculated within the QED structure function approach. Results are shown in the next-to-leading logarithmic approximation up to $\mathcal{O}(\alpha^4L^3)$ order, where $L = \ln(s/m_e^2)$ is the large logarithm. Several mistakes in previous calculations are corrected. The results are relevant for future high-precision experiments at e^+e^- colliders.

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I. INTRODUCTION

The physical program of future electron-positron colliders such as FCCee [1], CEPC [2], and ILC [3], foresee extremely high experimental precision in measurements of e^+e^- annihilation and scattering processes. In particular, it is planned to collect up to 10^{12} events with production of Z bosons in the so-called TeraZ operation mode [1] at the Z peak. The foreseen precision of the experimental measurements requires for increasing accuracy of theoretical predictions [4].

Computation of the complete $\mathcal{O}(\alpha^2)$ electroweak and even QED radiative corrections to realistic observables is still a difficult problem. The QED structure function [5] approach [6] allows taking systematically into account the terms enhanced by the so-called large logarithm

$$L = \ln\frac{\mu_F^2}{\mu_R^2}, \tag{1}$$

where μ_F is the factorization scale and μ_R is the renormalization scale. The natural choice of μ_R in QED is the electron mass. For e^+e^- annihilation into a Z boson, μ_F can be chosen equal to the Z-boson mass. We use the standard modified minimal subtraction scheme ($\overline{\text{MS}}$) for treatment of factorization. Other schemes like Frixione-Kunst-Signer [7] and deep inelastic scattering [8] can be applied in the same way.

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The method of structure functions in QED was developed on the base of the QCD parton distribution function approach. The Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations were reduced to QED by Kuraev and Fadin [6]. There are numerous applications and further developments of the method within the leading logarithmic approximation; see, e.g., Refs. [9–15]. Application of the method in the next-to-leading-order (NLO) approximation was for the first time demonstrated in [16] for derivation of QED radiative corrections due to the initial state radiation (ISR) in electron-positron annihilation. Then it was applied for calculations of $\mathcal{O}(\alpha^2L)$ corrections to a few other processes including muon decay [17], deep inelastic scattering [18], and Bhabha scattering [19]. Recently, the calculations of next-to-leading ISR corrections to e^+e^- annihilation were extended to higher orders up to $\mathcal{O}(\alpha^6L^5)$ [20]. The details on derivation of the electron NLO parton distribution functions (PDFs) can be found in [21,22].

Because of the importance of higher-order ISR corrections to e^+e^- annihilation, we decided to perform an independent calculation of them. In particular, in [23] we have already noticed a discrepancy in the $\mathcal{O}(\alpha^3L^3)$ singlet contribution to the electron PDF with respect to [11]. Below we will show the corresponding effect in the ISR corrections. We also perform here a detailed comparison with the results presented in [20].

II. CALCULATIONS

A. Master formula

The cross section of electron-positron annihilation into a virtual photon or Z boson $e^+e^- \to \gamma^*(Z^*)$ can be represented in the form of convolution of two electron PDFs and partonic cross sections [16],

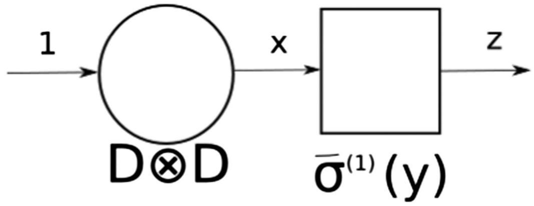

FIG. 1. The scheme of energy fractions in the process.

$$\sigma^{\rm NLO}_{\bar{e}e}(s') = \sum_{i,j=e,\bar{e},\gamma} \int_{\bar{z}_1}^{1}\int_{\bar{z}_2}^{1} dz_1 dz_2 D^{\rm str}_{ie}\left(z_1, \frac{\mu_R^2}{\mu_F^2}\right) \times D^{\rm str}_{j\bar{e}}\left(z_2, \frac{\mu_R^2}{\mu_F^2}\right)\left(\sigma^{(0)}_{ij}(sz_1z_2) + \bar{\sigma}^{(1)}_{ij}(sz_1z_2) + \mathcal{O}(\alpha^2L^0)\right)\delta(s'-sz) + \mathcal{O}\left(\frac{\mu_R^2}{\mu_F^2}\right), \quad (2)$$

where $\bar{e} \equiv e^+$ is the positron and $e \equiv e^-$ is the electron, $\sigma^{(0,1)}_{ij}$ are the Born (0) and one-loop (1) cross sections of annihilation to $\gamma^*(Z^*)$ at the parton level, s is the initial center-of-mass energy squared, and s' is the invariant mass of the produced virtual photon (or Z boson), $s' = sz$. For $D \otimes D \otimes \sigma^{(0)}$ (see Fig. 1), $z = z_1 z_2 \equiv x$ because of the absence of the radiation in the Born-level partonic cross section. In the case of the one-loop contribution, we have to introduce a variable describing possible energy losses due to radiation in the one-loop partonic cross section. Let us assume that $y = z/x$ is the ratio of the squared invariant mass of the produced virtual photon to the squared invariant mass of colliding partons i and j. So, the condition $s' = sz$ takes the form $s' = sz_1z_2y = sxy = sz$.

The process is schematically shown in Fig. 1.

The master formula for the cross section in terms of convolutions from the evolution equation reads

$$\frac{d\sigma_{\bar{e}e}}{ds'} = \sigma^{(0)}\big[D_{ee}\otimes D_{\bar{e}\bar{e}}\otimes\sigma_{e\bar{e}} + D_{ee}\otimes D_{\gamma\bar{e}}\otimes\sigma_{e\gamma} + D_{ee}\otimes D_{e\bar{e}}\otimes\sigma_{ee} + D_{\gamma e}\otimes D_{\bar{e}\bar{e}}\otimes\sigma_{\gamma\bar{e}} + D_{\gamma e}\otimes D_{\gamma\bar{e}}\otimes\sigma_{\gamma\gamma} + D_{\gamma e}\otimes D_{e\bar{e}}\otimes\sigma_{\gamma e} + D_{\bar{e}e}\otimes D_{\bar{e}\bar{e}}\otimes\sigma_{\bar{e}\bar{e}} + D_{\bar{e}e}\otimes D_{\gamma\bar{e}}\otimes\sigma_{\bar{e}\gamma} + D_{\bar{e}e}\otimes D_{e\bar{e}}\otimes\sigma_{\bar{e}e}\big], \quad (3)$$

where D_{ia} are the parton distribution functions of parton i in the initial particle a and σ_{ij} are the partonic cross sections, which in QCD are called Wilson coefficients [16]. The symbol ⊗ means convolution operation; see, e.g., [22].

In these formulas, all possible contributions to the NLO order are taken into account. In Table I, these contributions and their leading powers of α and the large logarithm are shown. In the table, the symbol of convolution (⊗) is omitted for convenience.

TABLE I. Orders of different contributions.

i \ j	$\bar{e}$	γ	e
e	$D_{ee}D_{\bar{e}\bar{e}}\sigma_{e\bar{e}}$ LO (1)	$D_{ee}D_{\gamma\bar{e}}\sigma_{e\gamma}$ NLO (α^2L)	$D_{ee}D_{e\bar{e}}\sigma_{ee}$ NNLO (α^4L^2)
γ	$D_{\gamma e}D_{\bar{e}\bar{e}}\sigma_{\gamma\bar{e}}$ NLO (α^2L)	$D_{\gamma e}D_{\gamma\bar{e}}\sigma_{\gamma\gamma}$ NNLO (α^4L^2)	$D_{\gamma e}D_{e\bar{e}}\sigma_{\gamma e}$ NLO (α^4L^3)
$\bar{e}$	$D_{\bar{e}e}D_{\bar{e}\bar{e}}\sigma_{\bar{e}\bar{e}}$ NNLO (α^4L^2)	$D_{\bar{e}e}D_{\gamma\bar{e}}\sigma_{\bar{e}\gamma}$ NLO (α^4L^3)	$D_{\bar{e}e}D_{e\bar{e}}\sigma_{\bar{e}e}$ LO (α^4L^4)

In works [16,20] only the following four contributions $D_{ee}\otimes D_{\bar{e}\bar{e}}\otimes\sigma_{e\bar{e}}$, $D_{ee}\otimes D_{\gamma\bar{e}}\otimes\sigma_{e\gamma}$, $D_{\gamma e}\otimes D_{\bar{e}\bar{e}}\otimes\sigma_{\gamma\bar{e}}$, and $D_{\gamma e}\otimes D_{\gamma\bar{e}}\otimes\sigma_{\gamma\gamma}$ were taken into account, i.e., the transitions from electrons to positrons (and vice versa) were omitted. In paper [16], this limitation was well justified since the authors were interested only in $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(\alpha^2)$ corrections to which the electron-into-positron transitions do not contribute. Meanwhile, for higher-order corrections calculated in [20], the transitions become relevant even in the leading logarithmic approximation.

B. Evolution equations

Let us consider QED evolution equations for PDFs in the spacelike region. The equations are induced by the renormalization group and have the following form, e.g., see Ref. [24]:

$$D_{ba}\left(x, \frac{\mu_R^2}{\mu_F^2}\right) = \delta(1-x)\delta_{ba} + \sum_{i=e,\bar{e},\gamma}\int_{\mu_0^2}^{\mu^2}\frac{dt\alpha(t)}{2\pi t} \times \int_x^1 \frac{dy}{y} D_{ia}\left(y, t, \frac{\mu_R^2}{\mu_F^2}\right) P_{bi}\left(\frac{x}{y}\right), \quad (4)$$

where index a corresponds to the initial particle, e.g., an electron; and indices b and i mark QED partons that can be photons (γ) or massless electrons (e) and positrons ($\bar{e}$). Note that transition into all three types of partons have to be taken into account.

Every splitting function, PDF, or radiator can be divided into Θ and Δ parts as

$$F(z) = \lim_{\Delta\to 0}(F_\Theta(z)\Theta(1-\Delta-z) + \delta(1-z)F_\Delta).$$

Both appear in the process of Δ regularization of the functions divergent at $z \to 1$. The Θ part depends on the energy fraction z and corresponds to hard photon or pair emission. The Δ part provides the contribution of virtual and soft radiation with the emitted energy fraction not more than Δ. Details can be found in [22].

The splitting functions $P_{ji}(x)$ can be expanded in the fine structure constant α

$$P_{ji}(x) = P_{ji}^{(0)}(x) + \frac{\alpha(t)}{2\pi} P_{ji}^{(1)}(x) + \mathcal{O}(\alpha^2). \tag{5}$$

We took the expressions for the NLO QCD splitting functions $P_{ji}^{(1)}$ from Refs. [25,26] for the spacelike case and from Refs. [26,27] for the timelike ones. We reduced the functions to the (Abelian) QED case by taking the appropriate values of constants: $C_F = 1$, $C_G = 0$, $T_R \cdot N_F = 1$ for [26] and $C_F = 1, N_C = 0$, and $T_f = 1$ for [25]. Note that the expressions for $P_{ij}^{(1)}$ must be consistent with the running of α to avoid double counting, see Sec. II D below. Details of the analytical iterative solution of evolution equations can be found in [22].

The initial conditions for the QED DGLAP evolution equations can be found in, e.g., Refs. [17,28,29]. For the timelike case, one can use the Abelian part of the initial conditions for QCD fragmentation functions [27]. Details on derivation of the $d_{ee}^{(1)}$ function were given in [30].

C. Factorization at NLO

The cross section in the NLO approximation takes into account the QED radiative corrections enhanced by the large logarithms and reads

$$d\sigma_{ab\to cd}^{\mathrm{NLO}} = d\sigma_{ab\to cd}^{(0)} \left\{ 1 + \sum_{k=1}^{\infty} \left(\frac{\alpha}{2\pi}\right)^k \sum_{l=k-1}^{k} c_{kl} L^l + \mathcal{O}(\alpha^k L^{k-2}) \right\}, \tag{6}$$

where c_{kl} are the coefficients to be computed. The terms of the type $\alpha^k L^k$ provide the leading-order (LO) logarithmic approximation, and the ones of the type $\alpha^k L^{k-1}$ yield the NLO contribution.

So the expression for the one-loop correction to electron-positron annihilation (with reduced energy due to the initial state radiation) reads

$$\delta_{\bar{e}e}^{(1)}(sx) \equiv \frac{\sigma_{\bar{e}e}^{(1)}(sx)}{\sigma_{\bar{e}e}^{(0)}(sx)} = \frac{\alpha}{\pi} \left\{ \left[\frac{1+y^2}{1-y}\right]_+ \left(\ln\frac{sx}{m_e^2} - 1 \right) + \delta(1-y)\left(2\zeta_2 - \frac{1}{2}\right) \right\}, \qquad y = \frac{z}{x}, \tag{7}$$

and, analogously,

$$\delta_{e\gamma}^{(0)}(sx) \equiv \frac{\sigma_{e\gamma}^{(0)}(sx)}{\sigma_{\bar{e}e}^{(0)}(sx)}. \tag{8}$$

Formula (7) for $x = 1$ represents one-loop ISR correction to the process of electron-positron annihilation for the center-of-mass energy $\sqrt{s}$. By looking at this expression, we can see that the (*á la* Brodsky-Lepage-Mackenzie) factorization scale choice $\mu_F = \sqrt{s}$ is well motivated, since it absorbs the bulk of the one-loop correction. So in our calculations, we adapt the latter factorization scale. In work [16] and later in [20] the factorization scale $\mu_F = \sqrt{zs}$ was chosen, which is the invariant mass of the final state. The latter choice looks not optimal, especially for small z.

This choice of $\bar{\delta}_{\bar{e}e}^{(1)}(x)$ satisfies the matching equality

$$\delta_{\bar{e}e}^{(1)}(sx) = \bar{\delta}_{\bar{e}e}^{(1)}(sx) + 2\frac{\alpha}{2\pi}\left[P_{ee}^{(0)}(y)L + d_{ee}^{(1)}(y)\right] + \mathcal{O}\left(\frac{m_e^2}{s}\right), \tag{9}$$

where on the left-hand side we have the known one-loop ISR correction [16].

After the subtraction of mass singularities within the standard modified minimal subtraction scheme ($\overline{\mathrm{MS}}$), we get

$$\bar{\delta}_{\bar{e}e}^{(1)}(sx) = \frac{\alpha}{\pi}\left\{ \left[\frac{1+y^2}{1-y}\right]_+ (\ln z - \ln y) + 2(1+y^2)\left[\frac{\ln(1-y)}{1-y}\right]_+ + \delta(1-y)\left(2\zeta_2 - \frac{1}{2}\right) \right\}. \tag{10}$$

Note that the "bar" over δ here means that the latter is calculated for massless partons. Note also that variable z above is the energy fraction of the produced virtual photon or Z boson, and it is not a variable of integration.

In the works [16,20] the factorization scale is implicitly chosen as $\mu_F^2 = sz$ [31]. So, the large logarithm in the electron PDFs is $\ln(s/m_e^2) + \ln z$. But in the expression for the one-loop partonic cross section used in Refs. [16,20], variable $y = z/x$ was occasionally replaced by x. So they had

$$\left[\bar{\delta}_{\bar{e}e}^{(1)}(x)\right]^* = \frac{\alpha}{\pi}\left\{ \left[\frac{1+x^2}{1-x}\right]_+ \ln x + 2(1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+ + \delta(1-x)\left(2\zeta_2 - \frac{1}{2}\right) \right\}. \tag{11}$$

The result calculated with this deformation of the factorization scale choice occasionally coincides with the known result of direct two-loop calculation in the leading and subleading logarithmic contributions [16,29], but in higher orders the two schemes give significantly different results. In particular, our $\mathcal{O}(\alpha^3 L^2)$ result for function $c_{32}(z)$ [Eq. (A4) in the Appendix] considerably differs from the one given in [20].

The coefficient c_{21}, see Eq. (6), calculated via solving the evolution equation in terms of convolutions contains the one-loop correction

$$c_{21}(z) = \sigma^{(0)}_{e\bar{e}}\left[2\delta^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} + \frac{2}{3}\bar{\delta}^{(1)}_{\bar{e}e} + 2\bar{\delta}^{(1)}_{\bar{e}e} \otimes P^{(0)}_{ee} + 2P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e} + 2P^{(1)}_{ee} - \frac{20}{9}P^{(0)}_{ee} + 4P^{(0)}_{ee} \otimes d^{(1)}_{ee}\right]. \tag{12}$$

It agrees with the corresponding result in [16]. And its Θ part as function of z reads

$$\begin{aligned} c^{\Theta}_{21}(z) = {} & \ln z\left(-\frac{37}{6} - \frac{2}{3z} + \frac{29}{6(1-z)} + 2\frac{\ln(1-z)}{1-z} + \ln(1-z) - \frac{11}{3}z + z\ln(1-z) + \frac{4}{3}z^2\right) + \ln^2 z\left(\frac{1}{4} - \frac{2}{1-z} + \frac{1}{4}z\right) + \frac{2}{9} \\ & - \frac{8}{9z} + \frac{4}{3z}\ln(1-z) - \frac{73}{9(1-z)} + 2\mathrm{Li}_2(1-z) - \frac{20}{3}\frac{\ln(1-z)}{1-z} + \frac{13}{3}\ln(1-z) + \frac{4\zeta_2}{1-z} - 2\zeta_2 + \frac{71}{9}z + 2z\mathrm{Li}_2(1-z) \\ & + \frac{7}{3}z\ln(1-z) - 2z\zeta_2 + \frac{8}{9}z^2 - \frac{4}{3}z^2\ln(1-z), \end{aligned} \tag{13}$$

and the Δ part is

$$\begin{aligned} c^{\Delta}_{21} = {} & -\frac{203}{12} + 22\zeta_2 + 12\zeta_3 - \frac{40}{3}\ln^2\Delta \\ & + \left(16\zeta_2 - \frac{292}{9}\right)\ln\Delta. \end{aligned} \tag{14}$$

D. Running coupling

We use the expression for the running coupling in the $\overline{\mathrm{MS}}$ scheme

$$\alpha(\mu^2) = \frac{\alpha(\mu_0)}{1 + \bar{\Pi}(\mu, \mu_0, \alpha(0))}, \tag{15}$$

that can be found, e.g., in [32,33] with

$$\begin{aligned} \bar{\Pi}(\mu, \mu_0, \alpha(0)) = {} & \frac{\alpha(0)}{\pi}\left(\frac{5}{9} - \frac{L}{3}\right) + \left(\frac{\alpha(0)}{\pi}\right)^2\left(\frac{55}{48} - \zeta_3 - \frac{L}{4}\right) \\ & + \left(\frac{\alpha(0)}{\pi}\right)^3\left(\frac{-L^2}{24}\right) + \ldots, \end{aligned} \tag{16}$$

where $L = \ln(\mu_R^2/\mu_F^2)$ is again the large logarithm. After expansion, we get

$$\begin{aligned} \alpha(\mu_R^2) = \alpha(0)\biggl\{ & 1 + \frac{\alpha(0)}{2\pi}\left(-\frac{10}{9} + \frac{2}{3}L\right) + \left(\frac{\alpha(0)}{2\pi}\right)^2 \\ & \times\left(-\frac{1085}{324} + 4\zeta_3 - \frac{13}{27}L + \frac{4}{9}L^2\right) + \mathcal{O}(\alpha^3(0))\biggr\}, \end{aligned} \tag{17}$$

where $\zeta_n \equiv \zeta(n)$ is the Riemann ζ function. Here we put $\mu_R = m_e$ and assume $\alpha(\mu_R^2) \approx \alpha(0) \equiv \alpha$.

Note that, in the traditional way of $\overline{\mathrm{MS}}$ scheme application in QCD calculations, the expansion for the running coupling constant takes into account only the terms proportional to large logs (via β_0, β_1, and so on). The effects due to constant (nonlogarithmic) terms, like $-10/9$ of the $\mathcal{O}(\alpha)$ order in the above formula, are kept in higher-order splitting functions, e.g., in $P^{(1)}_{ij}$. Here we apply a QED-like scheme in which the nonlogarithmic terms are preserved in the running α and thus we modify the NLO splitting functions in the following way: $[P^{(1)}_{ij}(x)]_{\mathrm{QED}} = [P^{(1)}_{ij}(x)]_{\mathrm{QCD}} + \frac{10}{9}P^{(0)}_{ij}(x)$, see details in [22]. One can verify that this scheme choice does not affect the final results.

III. RESULTS IN TERMS OF CONVOLUTIONS

Parton distribution functions of the types $D_{e\bar{e}}$ and $D_{\bar{e}e}$ start to give their contribution to cross sections from the order α^2L^2.

The complete results for c_{33}, c_{44}, c_{32}, and c_{43} in terms of convolutions read

$$\begin{aligned} c_{33}(z) = {} & \frac{2}{3}P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} + \frac{1}{3}P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\gamma} + \frac{8}{27}P^{(0)}_{ee} \\ & + \frac{5}{3}P^{(0)}_{ee} \otimes P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} + \frac{4}{3}P^{(0)\otimes 2}_{ee} + \frac{4}{3}P^{(0)\otimes 3}_{ee}, \end{aligned} \tag{18}$$

$$c_{44}(z) = \sigma^{(0)}_{e\bar{e}}\Big(\frac{11}{27}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{1}{12}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\bar{e}}\otimes P^{(0)}_{\bar{e}\gamma} + \frac{1}{3}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{1}{12}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)\otimes 2}_{\gamma\gamma}$$
$$+ \frac{1}{4}P^{(0)\otimes 2}_{e\gamma}\otimes P^{(0)\otimes 2}_{\gamma\bar{e}} + \frac{1}{3}P^{(0)\otimes 2}_{e\gamma}\otimes P^{(0)\otimes 2}_{\gamma e} + \frac{4}{27}P^{(0)}_{ee} + \frac{5}{3}P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{1}{2}P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma}$$
$$+ \frac{17}{12}P^{(0)\otimes 2}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{22}{27}P^{(0)\otimes 2}_{ee} + \frac{4}{3}P^{(0)\otimes 3}_{ee} + \frac{2}{3}P^{(0)\otimes 4}_{ee}\Big), \tag{19}$$

$$c_{32}(z) = \sigma^{(0)}_{e\bar{e}}\Big\{\delta^{(0)}_{e\gamma}\otimes\Big(P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + 3P^{(0)}_{ee}\otimes P^{(0)}_{\gamma e} + 2P^{(0)}_{\gamma e}\Big) + \bar{\delta}^{(1)}_{\bar{e}e}\otimes\Big(P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + 2P^{(0)}_{ee} + 2P^{(0)}_{ee}\otimes P^{(0)}_{ee}\Big)$$
$$+ \frac{4}{9}\bar{\delta}^{(1)}_{\bar{e}e} + P^{(0)}_{\gamma e}\otimes P^{(1)}_{e\gamma} + P^{(0)}_{e\gamma}\otimes P^{(1)}_{\gamma e} + \frac{2}{3}P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e} - \frac{20}{9}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{4}{3}P^{(1)}_{ee}$$
$$+ 2d^{(1)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} - \frac{13}{27}P^{(0)}_{ee} + 3P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e} + 4P^{(0)}_{ee}\otimes P^{(1)}_{ee} + \frac{4}{3}P^{(0)}_{ee}\otimes d^{(1)}_{ee} - \frac{40}{9}P^{(0)}_{ee}\otimes P^{(0)}_{ee}$$
$$+ 4P^{(0)}_{ee}\otimes P^{(0)}_{ee}\otimes d^{(1)}_{ee}\Big\}, \tag{20}$$

$$c_{43}(z) = \sigma^{(0)}_{e\bar{e}}\Big\{\delta^{(0)}_{e\gamma}\otimes\Big(\frac{44}{27}P^{(0)}_{\gamma e} + \frac{1}{3}P^{(0)}_{\gamma e}\otimes P_{\gamma\bar{e}}\otimes P_{\bar{e}\gamma} + \frac{4}{3}P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{1}{3}P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma}\otimes P^{(0)}_{\gamma\gamma} + 4P^{(0)}_{ee}\otimes P^{(0)}_{\gamma e}$$
$$+ \frac{4}{3}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma e} + \frac{4}{3}P^{(0)}_{ee}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{7}{3}P^{(0)}_{ee}\otimes P^{(0)}_{ee}\otimes P^{(0)}_{\gamma e}\Big) + \frac{8}{27}\bar{\delta}^{(1)}_{e\bar{e}} + \bar{\delta}^{(1)}_{e\bar{e}}\Big(\frac{4}{3}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}$$
$$+ \frac{1}{3}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{5}{3}P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{44}{27}P^{(0)}_{ee} + \frac{8}{3}P^{(0)}_{ee}\otimes P^{(0)}_{ee} + \frac{5}{6}P^{(0)}_{\gamma e}\otimes P^{(1)}_{e\bar{e}}\otimes P_{\bar{e}\gamma}$$
$$+ \frac{4}{3}P^{(0)\otimes 3}_{ee}\Big) + \frac{10}{9}\sigma^{(0)}_{e\bar{e}}\otimes P^{(0)}_{\gamma e}P^{(1)}_{e\gamma} + \frac{1}{3}P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma}\otimes P^{(1)}_{e\gamma} + \frac{5}{6}P^{(0)}_{e\gamma}\otimes P^{(1)}_{\bar{e}\bar{e}}\otimes P^{(0)}_{\gamma\bar{e}} + \frac{8}{9}P^{(0)}_{e\gamma}\otimes P^{(1)}_{\gamma e} + \frac{8}{27}P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e}$$
$$+ \frac{1}{3}P^{(0)}_{e\gamma}\otimes P^{(1)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{5}{6}P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e}\otimes P^{(0)}_{\gamma\bar{e}}\otimes P^{(0)}_{\bar{e}\gamma} + \frac{2}{3}P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{1}{3}P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma}\otimes P^{(0)}_{\gamma\gamma}$$
$$- \frac{11}{9}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{1}{3}P^{(0)}_{e\gamma}P^{(0)}_{\gamma e}\otimes P^{(1)}_{\gamma\gamma} + \frac{8}{9}P^{(1)}_{ee} - \frac{10}{9}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{1}{2}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes d^{(1)}_{\gamma e}P_{\bar{e}\gamma}$$
$$+ \frac{4}{3}P^{(0)}_{e\gamma}\otimes P^{(0)\otimes 2}_{e\gamma}\otimes d^{(1)}_{\gamma e} + \frac{5}{3}P^{(1)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{4}{3}d^{(1)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + \frac{10}{81}P^{(0)}_{ee} + \frac{5}{3}P^{(0)}_{ee}\otimes P^{(0)}_{\gamma e}\otimes P^{(1)}_{e\gamma}$$
$$+ \frac{2}{3}d^{(1)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\gamma} + \frac{5}{3}P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(1)}_{\gamma e} + \frac{4}{3}d^{(1)}_{\gamma e}\otimes P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma\gamma} + 2P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e}$$
$$- \frac{50}{9}P^{(0)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} + 4P^{(0)}_{ee}\otimes P^{(1)}_{ee} + \frac{16}{27}P^{(0)}_{ee}\otimes d^{(1)}_{ee} + \frac{10}{3}P^{(0)}_{ee}\otimes d^{(1)}_{ee}\otimes P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e} - \frac{22}{9}P^{(0)\otimes 2}_{ee} + 4P^{(0)\otimes 2}_{ee}$$
$$\otimes P^{(1)}_{ee} + \frac{7}{3}P^{(0)\otimes 2}_{ee}\otimes P^{(0)}_{e\gamma}\otimes d^{(1)}_{\gamma e} + \frac{8}{3}P^{(0)\otimes 2}_{ee}\otimes d^{(1)}_{ee} + \frac{8}{3}\sigma^{(0)}_{e\bar{e}}\otimes d^{(1)}_{ee}\otimes P^{(0)\otimes 3}_{ee}\Big\}, \tag{21}$$

where $P^{(0)\otimes n}_{ij}$ is the successive convolution of n $P^{(0)}_{ij}$ functions. There are no positron-induced contributions in c_{32} because they appear only starting from the $\mathcal{O}(\alpha^4)$ order. The formula for $c32$ in terms of convolutions coincides with the one given in [20]. But we do not agree in the final result for this contribution (as a function of z) given in the Appendix below because of the difference in treatment of NLO factorization.

In the $\mathcal{O}(\alpha^4 L^4)$ contribution, we have the difference with respect to the result from the work [20],

$$\Delta c_{44} = \frac{1}{3}\sigma^{(0)}_{e\bar{e}}P^{(0)}_{e\gamma}\otimes P^{(0)}_{\gamma e}\otimes P^{(0)}_{\gamma\bar{e}}\otimes P^{(0)}_{\bar{e}\gamma} \tag{22}$$

because of taking into account electron-into-positron transitions. We have two sources of these transitions: including such transitions in evolution equations and including the term proportional to $D_{e\bar{e}}\otimes D_{\bar{e}e}$ in the master formula (3). If we exclude both parts, our result for c_{44} completely coincides with the result from the work [20]. From the evolution equation, when we include the transitions into

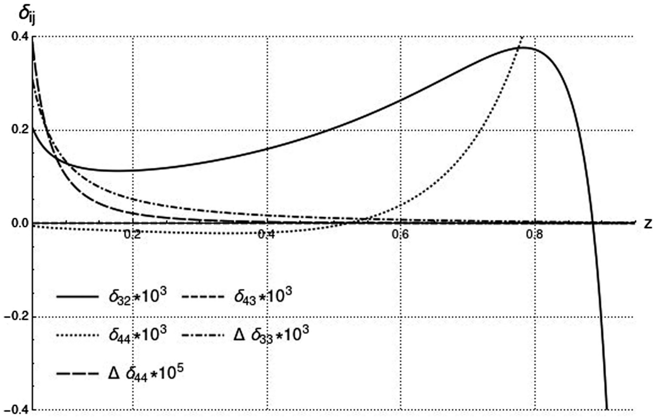

FIG. 2. Higher-order contribution to the ISR radiator function.

positrons in the equations for D_{ee} and $D_{\gamma e}$, we get $\frac{1}{12}\sigma^{(0)}_{e\bar{e}} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\bar{e}} \otimes P^{(0)}_{\bar{e}\gamma}$, and the last term in Eq. (3) yields $\frac{1}{4}\sigma^{(0)}_{e\bar{e}} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e} \otimes P^{(0)}_{\gamma\bar{e}} \otimes P^{(0)}_{\bar{e}\gamma}$.

Numerical illustrations of our results in the orders $\mathcal{O}(\alpha^3 L^2)$, $\mathcal{O}(\alpha^4 L^4)$, and $\mathcal{O}(\alpha^4 L^3)$ are shown in Fig. 2. We plot the values

$$\delta_{ij} = \left(\frac{\alpha}{2\pi}\right)^i L^j \frac{c_{ij}}{\sigma^{(0)}_{e\bar{e}}} \tag{23}$$

as functions of z (we put $L = 24$, i.e., $\mu_F \approx M_Z$). Note that these quantities are contributions to the ISR radiator functions, which have to be later integrated with the Born-level cross section over z in an interval defined by the experiment. We also show the difference $\Delta\delta_{33}$ in the $\mathcal{O}(\alpha^3 L^3)$ order, which comes from the correction in the singlet part of D_{ee} with respect to the result given in [11], and the difference $\Delta\delta_{44}$ between our fourth-order leading logarithmic contribution δ_{44} and the one from [20], which is due to the electron-into-positron transitions. One can see that all shown contributions are relevant for future experiments with the precision tag of the order 10^{-5}. The radiator function contributions typically diverge for $z \to 1$, but taking into account Δ parts cancels out this divergence in the total correction. The differences $\Delta\delta_{33}$ and $\Delta\delta_{44}$ are enhanced at small z since both are related to singlet transitions.

IV. CONCLUSIONS

In this way, we revisited the application of the QED structure function method for calculations of higher-order NLO ISR radiative corrections to e^+e^- annihilation. A bug in the singlet part of the $\mathcal{O}(\alpha^3 L^3)$ of the electron PDF is corrected.

Several other issues that arose in earlier calculations of these corrections are clarified and improved. The relevance of positron in electron (and vice versa) PDF is demonstrated explicitly. Taking into account splitting of electron into positron (and vice versa) appears to be also significant in solutions of the QED evolution equations. Moreover, treatment of the factorization scale in NLO was refined. The issues listed above lead to a considerable difference of our results (both the leading and next-to-leading ones) from the ones given in [20].

The obtained results will be implemented into the ZFITTER computer code [34]. We would like to underline that the applied method leads to results being integrated over angular variables of the ISR radiation, and hence they can not take into account all possible experimental cuts. Nevertheless, first of all, one can use our results as benchmarks to verify the precision of Monte Carlo codes. Moreover, after implementation of the completely differential two-loop radiative corrections in a Monte Carlo code, one can add certain higher-order corrections in the collinear approximation without spoiling the theoretical precision too much. Extension of the presented here results to higher orders, like c_{55} and c_{54}, is straightforward. The corresponding results will be presented elsewhere.

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APPENDIX: RESULTS AS FUNCTIONS OF z

Here, we show explicit results for the higher-order coefficients c_{kl} from Eq. (6) as functions of the energy fraction z.

$$\begin{aligned} c^{\Theta}_{33} = {} & -\frac{461}{18} - \frac{7z}{18} + \frac{710}{27(1-z)} + \frac{16}{27z} + \zeta_2\left(16z - \frac{32}{1-z} + 16\right) + \left(\frac{44z}{3} + \frac{44}{3}\right)\mathrm{Li}_2(1-z) + \frac{8z^2}{27(1-z)} - \frac{16z^2}{27} \\ & + \left(\frac{40z^2}{9} + \frac{128z}{9} - \frac{88}{3(1-z)} + \frac{242}{9}\right)\ln(z) + \left[-\frac{40z^2}{9} - \frac{50z}{3} + \frac{176}{3(1-z)} + \frac{40}{9z} + \left(\frac{92z}{3} - \frac{32}{1-z} + \frac{92}{3}\right)\right. \\ & \left. \times \ln(z) - 42\right]\ln(1-z) + \left(-16z + \frac{32}{1-z} - 16\right)\ln^2(1-z) + \left(-\frac{19z}{3} + \frac{16}{3(1-z)} - \frac{19}{3}\right)\ln^2(z), \end{aligned} \tag{A1}$$

$$
\begin{aligned}
c_{44}^{\Theta}(z) = &-\frac{18779}{432}+\frac{4z^2}{27(1-z)}+\frac{328z^2}{81}+\frac{1115z}{432}+\frac{1108}{27(1-z)}-\frac{328}{81z}+\zeta_2\left(\frac{68z^2}{9}+\frac{113z}{3}-\frac{128}{1-z}-\frac{68}{9z}+\frac{271}{3}\right)\\
&+\zeta_3\left(-\frac{64z}{3}+\frac{128}{3(1-z)}-\frac{64}{3}\right)+\ln z\left[\frac{1145}{24}+\frac{64z^2}{27}-\frac{241z}{72}-\frac{1708}{27(1-z)}-\frac{32}{27z}+\zeta_2\left(-\frac{32z}{3}+\frac{64}{3(1-z)}\right.\right.\\
&\left.-\frac{32}{3}\right)+\left(\frac{136z^2}{9}+\frac{194z}{3}-\frac{128}{1-z}+\frac{352}{3}\right)\ln(1-z)+\left.\left(\frac{178z}{3}-\frac{64}{1-z}+\frac{178}{3}\right)\ln^2(1-z)\right]+\ln(1-z)\left[-\frac{32z^2}{9}-\frac{659z}{54}\right.\\
&\left.+\frac{32}{9z}+\frac{3416}{27(1-z)}-\frac{6173}{54}+\zeta_2\left(64z-\frac{128}{1-z}+64\right)\right]+\left[27+\frac{68z^2}{9}+27z+\frac{68}{9z}+\left(\frac{164z}{3}+\frac{164}{3}\right)\right.\\
&\left.+\left(-38z+\frac{64}{3(1-z)}-38\right)\ln z\right]\mathrm{Li}_2(1-z)+\left(-\frac{164z}{3}-\frac{164}{3}\right)\mathrm{Li}_3(1-z)+\left(-\frac{146z}{3}+\frac{128}{3(1-z)}-\frac{146}{3}\right)\mathrm{Li}_3(z)\\
&+\left[-\frac{68z^2}{9}-\frac{113z}{3}+\frac{128}{1-z}+\frac{68}{9z}-\frac{271}{3}\right]\ln^2(1-z)+\left[-\frac{34z^2}{9}-\frac{275z}{24}+\frac{64}{3(1-z)}+\left(-\frac{146z}{3}+\frac{128}{3(1-z)}\right.\right.\\
&\left.\left.-\frac{146}{3}\right)\ln(1-z)-\frac{187}{8}\right]\ln^2(z)+\left(6z+\frac{128}{3(1-z)}+6\right)\ln^3(1-z)+\left(\frac{61z}{36}-\frac{16}{9(1-z)}+\frac{61}{36}\right)\ln^3(z),
\end{aligned}
\tag{A2}
$$

$$
\Delta c_{44}^{\Theta}(z)=\frac{1}{3}\left[\ln z\left(-21-\frac{16}{9z}-21z-\frac{16}{9}z^2\right)+\ln^2 z(-2+2z)+\ln^3 z\left(-\frac{2}{3}-\frac{2}{3}z\right)-26-\frac{176}{27z}+26z+\frac{176}{27}z^2\right].
\tag{A3}
$$

$$
\begin{aligned}
c_{32}^{\Theta}(z) = &\frac{1453}{9}-\frac{935z^2}{54(1-z)}-\frac{413z^2}{27}-\frac{910z}{9}-\frac{4967}{54(1-z)}+\frac{584}{27z}+\left(\frac{10z^2}{9(1-z)}+\frac{32z^2}{9}-\frac{20z}{9}+\frac{146}{9(1-z)}-\frac{32}{9z}\right.\\
&\left.-\frac{110}{9}\right)\zeta_2+\left(40+40z-\frac{40}{1-z}+\frac{24z^2}{1-z}\right)\zeta_3+\ln z\left[\frac{67}{9}-\frac{1597z}{18}+\frac{214}{3(1-z)}-\frac{20}{9z}-\frac{100z^2}{9}+\left(8z-\frac{64}{3(1-z)}+8\right)\zeta_2\right.\\
&+\left(-28z+\frac{32}{1-z}-28\right)\mathrm{Li}_2(1-z)+\left(\frac{104z^2}{3}-\frac{154z}{3}+\frac{224}{3(1-z)}-\frac{8}{3z}-\frac{328}{3}\right)\ln(1-z)+\left(10z+\frac{32}{1-z}+10\right)\\
&\left.\times\ln^2(1-z)\right]+\ln(1-z)\left[\frac{88z^2}{9(1-z)}+\frac{136z^2}{9}+\frac{1678z}{9}-\frac{1768}{9(1-z)}-\frac{100}{9z}+\frac{142}{9}+\zeta_2\left(-\frac{16z}{3}+\frac{32}{3(1-z)}-\frac{16}{3}\right.\right.\\
&\left.\left.+(52z+52)\mathrm{Li}_2(1-z)\right)\right]+\left(\frac{64z^2}{3}-40z+\frac{32}{3z}-46\right)\mathrm{Li}_2(1-z)+(-52z-52)\mathrm{Li}_3(1-z)\\
&+\left(-40z+\frac{64}{1-z}-40\right)\mathrm{Li}_3(z)+\left(-\frac{20z^2}{3}+\frac{16z}{3}-\frac{92}{3(1-z)}+\left(-4z-\frac{16}{1-z}-4\right)\ln(1-z)+\frac{185}{6}\right)\ln^2(z)\\
&+\left(-\frac{52z^2}{3}+19z-\frac{64}{1-z}+\frac{52}{3z}+45\right)\ln^2(1-z)+\left(-\frac{23z}{3}+\frac{32}{3(1-z)}-\frac{23}{3}\right)\ln^3(z),
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
c_{43}^{\Theta} = &\frac{183425}{648}-\frac{35281}{162(1-z)}+\frac{1366}{27z}-\frac{3259z^2}{162(1-z)}-\frac{730z^2}{27}-\frac{76361z}{648}+\left(\frac{1292z^2}{27(1-z)}-\frac{512z^2}{27}-\frac{3245z}{9}\right.\\
&\left.+\frac{12524}{27(1-z)}-\frac{19}{9}-\frac{208}{27z}\right)\zeta_2+\left(-\frac{56z^2}{3(1-z)}-\frac{80z^2}{9}-\frac{632z}{9}-\frac{392}{3(1-z)}+\frac{244}{9}+\frac{1168}{9z}\right)\zeta_3\\
&+\left(-\frac{2092z}{3}+\frac{320}{1-z}-\frac{2042}{3}\right)\zeta_4+\left(\frac{77z}{12}-\frac{16}{3(1-z)}+\frac{211}{36}\right)\ln^4(z)+\left[\frac{152z^2}{27}-\frac{329z}{27}+\frac{376}{9(1-z)}-\frac{1693}{54}\right.\\
&\left.+\left(-176z+\frac{512}{3(1-z)}-176\right)\ln(1-z)\right]\ln^3(z)+\left[\frac{364z^2}{27}-\frac{2780}{27(1-z)}+\frac{1297}{9}+\frac{88}{27z}+\frac{947z}{9}\right.\\
&+\left(\frac{152z}{3}-\frac{192}{1-z}+\frac{152}{3}\right)\ln^2(1-z)+\zeta_2\left(14z+\frac{64}{1-z}+\frac{62}{3}\right)+\left(-80z^2+\frac{620z}{9}-\frac{368}{3(1-z)}+\frac{1556}{9}-\frac{160}{9z}\right)\ln(1-z)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(-\frac{40z^2}{9}+\frac{10z}{3}+\frac{10}{3}-\frac{40}{9z}\right)\ln(1+z)\Bigg]\ln^2(z)+\Bigg[-\frac{7795}{108}+\frac{344}{27z}-\frac{1850z^2}{81}+\frac{13813z}{54}+\frac{6947}{27(1-z)}+\left(12z+\frac{64}{1-z}+12\right)\\
&\times\ln^3(1-z)+\left(88z^2-\frac{994z}{9}+\frac{688}{3(1-z)}-\frac{2356}{9}+\frac{16}{3z}\right)\ln^2(1-z)+\Bigg(-\frac{1000z^2}{27}-\frac{17582z}{27}+\frac{12100}{27(1-z)}-\frac{6950}{27}\\
&-\frac{296}{27z}+\zeta_2\left(\frac{1208z}{3}-\frac{448}{1-z}+\frac{1208}{3}\right)\Bigg)\ln(1-z)+\left(-\frac{224z^2}{9}+\frac{974z}{9}-\frac{1472}{3(1-z)}+\frac{3410}{9}+\frac{272}{9z}\right)\zeta_2\\
&+\left(-\frac{40z^2}{3}+\frac{320z}{9}+\frac{320}{9}-\frac{40}{3z}\right)\ln(1+z)+\left(60z+\frac{160}{1-z}+\frac{260}{3}\right)\zeta_3\Bigg]\ln(z)+\Bigg[-\frac{88z^2}{3}+\frac{62z}{3}\\
&-\frac{256}{3(1-z)}+\frac{194}{3}+\frac{88}{3z}\Bigg]\ln^3(1-z)+\left(\frac{80z^2}{27}-\frac{20z}{9}-\frac{20}{9}+\frac{80}{27z}\right)\ln^3(1+z)+\Bigg[\frac{584z^2}{27}+\frac{7675z}{18}-\frac{1376}{3(1-z)}+\frac{341}{18}\\
&-\frac{224}{27z}+\zeta_2\left(-224z+\frac{448}{1-z}-224\right)\Bigg]\ln^2(1-z)+\left(-168z+\frac{64}{1-z}-168\right)\mathrm{Li}_2^2(1-z)+\left(\frac{40}{3}-\frac{40z}{3}\right)\mathrm{H}(-3,0,z)\\
&+\left(-\frac{1348z}{3}+\frac{1088}{3(1-z)}-\frac{1348}{3}\right)\mathrm{H}(3,0,z)+\left(\frac{80z}{3}-\frac{80}{3}\right)\mathrm{H}(-2,-1,0,z)+\left(\frac{40}{3}-\frac{40z}{3}\right)\mathrm{H}(-2,0,0,z)\\
&+\left(-320z+\tfrac{640}{1-z}-320\right)\mathrm{H}(1,2,0,z)+\left(-336z+\tfrac{1088}{3(1-z)}-336\right)\mathrm{H}(2,0,0,z)+\left(-592z+\tfrac{640}{1-z}-592\right)\mathrm{H}(2,1,0,z)\\
&+\left(-96z+\frac{192}{1-z}-96\right)\mathrm{H}(1,0,0,0,z)+\left(-256z+\frac{512}{1-z}-256\right)\mathrm{H}(1,1,0,0,z)+\left(-320z+\frac{640}{1-z}-320\right)\mathrm{H}(1,1,1,0,z)\\
&+\left(-\frac{80z^2}{9}+\frac{20z}{3}+\frac{20}{3}-\frac{80}{9z}\right)\zeta_2\ln(1+z)+\Bigg[-\frac{208z^2}{9}-\frac{2089z}{9}+(132z+132)\ln^2(1-z)\\
&+\left(-\frac{400z}{3}+\frac{256}{3(1-z)}-\frac{400}{3}\right)\ln^2 z+\left(\frac{752z^2}{9}-\frac{1412z}{9}-\frac{1160}{9}+\frac{928}{9z}\right)\ln(1-z)+\zeta_2\left(\frac{536z}{3}+\frac{536}{3}\right)+\Bigg(-\frac{608z^2}{9}\\
&+\frac{344z}{9}+\left(-\frac{464z}{3}+\frac{128}{1-z}-\frac{464}{3}\right)\ln(1-z)+\frac{256}{3(1-z)}-\frac{40}{9}-\frac{272}{9z}\Bigg)\ln(z)-\frac{725}{3}-\frac{104}{9z}\Bigg]\mathrm{Li}_2(1-z)+\Bigg[-\frac{40z^2}{3}\\
&+\frac{320z}{9}+\zeta_2\left(\frac{40}{3}-\frac{40z}{3}\right)+\left(\frac{80z^2}{9}+\frac{40z}{3}-\frac{40}{3}-\frac{80}{9z}\right)\ln(z)+\frac{320}{9}-\frac{40}{3z}\Bigg]\mathrm{Li}_2(-z)+\Bigg[-\frac{752z^2}{9}+\frac{1412z}{9}-(264z+264)\\
&\times\ln(1-z)+\left(-\frac{544z}{3}-\frac{544}{3}\right)\ln(z)+\frac{1160}{9}-\frac{928}{9z}\Bigg]\mathrm{Li}_3(1-z)+\left(-\frac{80z^2}{3}-20z+\frac{100}{3}+\frac{80}{9z}\right)\mathrm{Li}_3(-z)\\
&+\Bigg[-\frac{608z^2}{9}+\frac{74z}{9}+\left(256z-\frac{512}{1-z}+256\right)\ln(1-z)+\left(-176z+\frac{512}{3(1-z)}-176\right)\ln(z)+\frac{512}{3(1-z)}-\frac{334}{9}\\
&-\frac{320}{9z}\Bigg]\mathrm{Li}_3(z)+\left(-\frac{160z^2}{9}+\frac{40z}{3}+\frac{40}{3}-\frac{160}{9z}\right)\mathrm{Li}_3\left(\frac{1}{1+z}\right)+\left(584z-\frac{640}{1-z}+584\right)\mathrm{Li}_4(1-z)+\Bigg(-284z+\frac{192}{1-z}\\
&-284\Bigg)\mathrm{Li}_4(z)+\left(532z-\frac{768}{1-z}+532\right)\mathrm{S}_{2,2}(z)+\Bigg[\frac{572z^2}{27(1-z)}+\frac{1226z^2}{81}-\frac{5680}{9(1-z)}-\frac{3205z}{27}+\frac{2345}{3}-\frac{2090}{81z}+\Bigg(\frac{752z^2}{9}\\
&-\frac{64z^2}{3(1-z)}+-\frac{340z}{3}+\frac{1376}{3(1-z)}-\frac{1100}{3}-\frac{752}{9z}\Bigg)\zeta_2+\left(-288z+\frac{576}{1-z}-288\right)\zeta_3\Bigg]\ln(1-z).
\end{aligned}
\tag{A5}
$$

$$
c_{33}^{\Delta}=\left(\frac{718}{27}-32\zeta_2\right)\ln\Delta+\frac{88}{3}\ln^2\Delta+\frac{32}{3}\ln^3\Delta+\frac{143}{18}+\frac{64}{3}\zeta_3-\frac{88}{3}\zeta_2,
\tag{A6}
$$

$$
c_{44}^{\Delta}=\left(\frac{1112}{27}+\frac{256}{3}\zeta_3-128\zeta_2\right)\ln\Delta+\left(\frac{1708}{27}-64\zeta_2\right)\ln^2\Delta+\frac{128}{3}\ln^3\Delta+\frac{32}{3}\ln^4\Delta+\frac{715}{72}+16\zeta_4+\frac{256}{3}\zeta_3-\frac{1708}{27}\zeta_2,
\tag{A7}
$$

$$c_{32}^{\Delta} = \left(-\frac{2951}{27} + 48\zeta_3 + 104\zeta_2\right)\ln\Delta + \left(32\zeta_2 - \frac{280}{3}\right)\ln^2\Delta - \frac{64}{3}\ln^3\Delta - \frac{1387}{36} - 80\zeta_4 + \frac{4}{3}\zeta_3 + \frac{928}{9}\zeta_2, \tag{A8}$$

$$c_{43}^{\Delta} = -\frac{13969}{216} + \frac{1504\zeta_2\zeta_3}{3} + \frac{1465\zeta_2}{27} + \frac{314\zeta_3}{9} - \frac{488\zeta_4}{3} - 1024\zeta_5 - \frac{64}{3}\ln^4\Delta + \left(\frac{896\zeta_2}{3} - \frac{1376}{9}\right)\ln^3\Delta + \left(\frac{1616\zeta_2}{3} - 672\zeta_3 - \frac{8234}{27}\right)\ln^2\Delta + \left(\frac{16408\zeta_2}{27} - \frac{1856\zeta_3}{3} + 320\zeta_4 - \frac{19270}{81}\right)\ln\Delta. \tag{A9}$$

Here $H(a_1, \ldots, a_k; z)$ are harmonic polylogarithms [35,36].

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