

Heterotic integrable deformation of the principal chiral model

David Osten^{*}

*Institute for Theoretical Physics (IFT), University of Wrocław,
Plac Maxa Borna 9, 50-204 Wrocław, Poland*

 (Received 11 February 2024; accepted 16 April 2024; published 24 May 2024)

A novel classically integrable model is proposed. It is a deformation of the two-dimensional principal chiral model, embedded into a heterotic σ model by a particular heterotic gauge field. This is inspired by the bosonic part of the heterotic σ model and its recent Hamiltonian formulation in terms of $O(d, d+n)$ -generalized geometry in Hatsuda *et al.* [Gauged double field theory, current algebras and heterotic sigma models, *J. High Energy Phys.* **05** (2023) 220]. Classical integrability is shown by the construction of a Lax pair and a classical \mathcal{R} matrix. The latter is almost of the canonical form with a twist function and solves the classical Yang-Baxter equation.

DOI: 10.1103/PhysRevD.109.106021

I. INTRODUCTION

One of the main impacts of the study of integrable model in high energy physics is the connection of quantities of integrable Green-Schwarz superstring σ models in supercoset anti-de Sitter (AdS) backgrounds and those of dual integrable conformal field theories [1,2] in context of the AdS/CFT duality. In the last years, some effort has been spent to generalize this to other, less symmetric settings. On the world sheet side of the duality, new integrable models were found that are deformations of such (semi)symmetric (super) cosets, namely Yang-Baxter deformations [3–5], λ -[6] and Wess-Zumino (WZ)-term [7] deformations and their interplay [8], or generalizations to nonsymmetric cosets [9–11]. So far, this survey was focusing on conventional two-dimensional nonlinear σ models, i.e., the embedding of a two-dimensional world sheet in some d -dimensional target space. See Refs. [10,12] for two recent reviews.

This article aims to slightly step out of this fairly well explored context and suggest that also the heterotic string [13–16] might have interesting integrable models. In context of this paper, the bosonic part of a heterotic string σ model is a d -dimensional nonlinear σ model, characterized by a metric and a B field, together with n chiral currents, characterized by a gauge field A and a given n -dimensional gauge group.

A. A heterotic \mathcal{E} model

Recently, progress in understanding the space of two-dimensional classically integrable σ models was made

^{*}david.osten@uwr.edu.pl

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

using two quite different constructions: in terms of as affine Gaudin models [17,18], coming from a four-dimensional Chern-Simons theory [19–24], or as \mathcal{E} models [25–31]. The latter does not directly clarify the origin of the integrable structure but its data relates directly to the σ -model target space geometry in the generalized flux formulation. Concretely, an \mathcal{E} model is defined in terms of $O(d, d)$ -covariant world sheet currents J_A with Poisson brackets—the so-called current algebra—

$$\{J_A(\sigma), J_B(\sigma')\} = \eta_{AB} \delta'(\sigma - \sigma') - F^C{}_{AB}(\sigma) J_C(\sigma) \delta(\sigma - \sigma'),$$

where $A = 1, \dots, 2d$, d is the dimension of the target space, F_{ABC} are the so-called generalized fluxes, which are constant for an \mathcal{E} model, and η_{AB} is the $O(d, d)$ -invariant metric, together with a Hamiltonian $H = \frac{1}{2} \int d\sigma \delta^{AB} J_A J_B$.

Similar to that, $O(d, d)$ generalized geometry appeared first in the Hamiltonian formulation and the analysis of the current algebra of the world sheet string [32–35] before it was considered as a tool for supergravity [36–42].

For the heterotic string the duality group is $O(d, d+n)$ [43,44]. Heterotic supergravity was phrased in terms of $O(d, d+n)$ generalized geometry [42,45–47]. World sheet formulations of the heterotic string related to $O(d, d+n)$ generalized geometry have been suggested in [48] and recently [49]. This is the latest addition to the program of duality covariant formulations of world-volume theories either via actions [50–55] or via Hamiltonian formulations [56–63]. Following [49], a heterotic \mathcal{E} model is defined in the Introduction.

B. A new integrable deformation of the principal chiral model

The new integrable model proposed here is a deformation of the principal chiral model, embedded into a heterotic

string σ model. The deformation is characterized by a heterotic gauge field:

$$A_a^\alpha = \epsilon A_a^\alpha, \quad \text{with} \quad \mathcal{A}_a^\alpha \mathcal{A}_b^\beta \kappa_{\alpha\beta} = \kappa_{ab}, \quad (1.1)$$

which is constant in flat indices a, b, c . Here ϵ is the deformation parameter and $\kappa_{ab}, \kappa_{\alpha\beta}$ are the Killing forms on \mathfrak{g} (the Lie algebra from the principal model) and a symmetric nondegenerate bilinear form on $\mathfrak{u}(1)^n$ (the Abelian gauge algebra), respectively. The Lax pair and the classical \mathcal{R} matrix, to show Hamiltonian integrability in Maillet's formalism [64,65], are computed in the second section. It also shown that this deformation can be combined with the introduction of a WZ term with arbitrary coefficient while maintaining classical integrability.

II. THE CLASSICAL BOSONIC HETEROTIC STRING IN THE HAMILTONIAN FORMALISM

In a slight stretch of the concept, a (classical) bosonic heterotic string here will be a d -dimensional nonlinear σ model together with n (Abelian) chiral currents. The d coordinate fields x , their canonical duals p , and n chiral currents k can be arranged into an $O(d, d+n)$ vector:

$$\mathcal{P}_M = (p_m(\sigma), k^\mu(\sigma), \partial x^m(\sigma)). \quad (2.1)$$

Latin indices $k, l, m, \dots = 1, \dots, d$ will be reserved for the d -dimensional coordinates, and greek letters $\kappa, \lambda, \mu, \dots = 1, \dots, n$ for the n -dimensional coordinates. The latter can be raised and lowered using $\kappa_{\mu\nu}$, the Killing metric on $\mathfrak{u}(1)^n$. The canonical Poisson brackets of these fields can (up to topological terms) also be written in an $O(d, d+n)$ -covariant way:

$$\{\mathcal{P}_M(\sigma), \mathcal{P}_N(\sigma')\} = \eta_{MN} \delta'(\sigma - \sigma') \quad (2.2)$$

with the $O(d, d+n)$ -invariant metric

$$\eta_{MN} = \begin{pmatrix} 0 & 0 & \delta_m^n \\ 0 & \kappa_{\mu\nu} & 0 \\ \delta_n^m & 0 & 0 \end{pmatrix}, \quad (2.3)$$

which is also used to raise and lower the $\mathcal{K}, \mathcal{L}, \mathcal{N}, \dots$ indices. The action on functions of these currents is $\{\mathcal{P}_M, f(x)\} = -\partial_M f(x) = (-\partial_m f(x), 0, 0)$. The derivative ∂_M is subject to the section condition $\eta^{MN} \partial_M * \partial_N * = 0 = \eta^{MN} \partial_M \partial_N *$, which is an $O(d, d+n)$ -covariant way to express that the functions only depend on x .

The coupling to the bosonic part of heterotic supergravity, namely a d -dimensional metric G and two-form gauge field B and the heterotic gauge field A can be encoded in a Hamiltonian that is quadratic in the currents \mathcal{P}_M and contains the couplings in the so-called generalized metric \mathcal{H}_{MN} :

$$H = \frac{1}{2} \int d\sigma \mathcal{H}^{MN}(G, B, A) \mathcal{P}_M(\sigma) \mathcal{P}_N(\sigma). \quad (2.4)$$

The spatial diffeomorphism constraint also takes a simple form, $0 = \mathcal{P}_M \mathcal{P}_N \eta^{MN}$, but will not play a role in the

further discussion. The explicit generalized metric $\mathcal{H}(G, B, A)$ and the connection to a Lagrangian picture, and also the generalization to non-Abelian currents has been discussed in detail in [49].

For the purpose of this paper the generalized flux formulation is better suited. To this end, we diagonalize the generalized metric $\mathcal{H}_{AB} = E_A^M E_B^N \mathcal{H}_{MN}$ with a generalized vielbein E_A^M and some suitable constant \mathcal{H}_{AB} . The coupling to the background is encoded in the Poisson brackets of the currents $J_A = E_A^M \mathcal{P}_M$:

$$\{J_A(\sigma), J_B(\sigma')\} = \eta_{AB} \delta'(\sigma - \sigma') - F^C{}_{AB}(\sigma) J_C(\sigma) \delta(\sigma - \sigma') \quad (2.5)$$

via the so-called generalized fluxes $F_{ABC} = 3\partial_{[A} E_B^N E_{C]N}$. In the particular case that these F_{ABC} can be chosen to be constant, we would call this construction heterotic \mathcal{E} model, in analogy to the construction for ordinary $2d$ σ models and $O(d, d)$ -generalized geometry. For a generic heterotic supergravity background these fluxes are not constant.

In the standard (geometric) parametrization of the background in terms of metric G , two-form gauge field B , and heterotic gauge field A the generalized vielbein is [45]

$$E_A^M = \begin{pmatrix} \delta_a^b & A_a^\alpha & B_{ab} - \frac{1}{2} A_{\alpha a} A_b^\alpha \\ 0 & \delta_\alpha^\beta & -A_{ab} \\ 0 & 0 & \delta_b^a \end{pmatrix} \begin{pmatrix} e_b^m & 0 & 0 \\ 0 & \delta_\beta^\mu & 0 \\ 0 & 0 & e_b^m \end{pmatrix} \quad (2.6)$$

with $e_a^m e_b^n G_{mn} = \kappa_{ab}$, which will be used to raise and lower a, b, \dots indices. Then, the nonvanishing components of F_{ABC} are

$$\begin{aligned} H_{abc} &= F_{abc} \\ &= \left[\partial_a B_{bc} + f^d{}_{ab} \left(B_{cd} - \frac{1}{2} A_c^\alpha A_{da} \right) + (\partial_{[a} A_{b]}) A_{ac} \right] \\ &\quad + \text{cyclic permutations of } abc, \end{aligned}$$

$$\begin{aligned} f^c{}_{ab} &= F^c{}_{ab} = 2e_n{}^c \partial_{[a} e_{b]}^n, \\ F^\gamma{}_{ab} &= 2\partial_{[a} A_{b]}^\gamma - f^c{}_{ab} A_c^\gamma. \end{aligned} \quad (2.7)$$

In hindsight of what is to come, we write $J_A = (J_a, J^\alpha, J^a) = (j_{0a}, k^\alpha, j_1^a)$. The subscripts 0,1 refer to the world sheet coordinates τ, σ . In this decomposition the current algebra (2.5) is

$$\begin{aligned} \{j_{0a}(\sigma), j_{0b}(\sigma')\} &= (-f^c{}_{ab} j_{0c} - F^\gamma{}_{ab} k_\gamma(\sigma) - H_{abc} j_1^c) \\ &\quad \times \delta(\sigma - \sigma'), \\ \{j_{0a}(\sigma), j_1^b(\sigma')\} &= \delta_a^b \delta'(\sigma - \sigma') - f^b{}_{ca} j_1^c(\sigma) \delta(\sigma - \sigma'), \\ \{j_{0a}(\sigma), k^\beta(\sigma')\} &= -F^\beta{}_{ca} j_1^c \delta(\sigma - \sigma'), \\ \{k^\alpha(\sigma), k^\beta(\sigma')\} &= \kappa^{\alpha\beta} \delta'(\sigma - \sigma'), \\ \{j_1^a(\sigma), j_1^b(\sigma')\} &= 0 = \{j_1^a(\sigma), k^\beta(\sigma')\}. \end{aligned} \quad (2.8)$$

The resulting equations of motion $\partial_t \mathcal{O} = \{\mathcal{O}, H\}$ with the Hamiltonian (2.4) in the generalized metric frame,

$$H = \frac{1}{2} \int d\sigma (\kappa^{ab} j_{0a}(\sigma) j_{0b}(\sigma) + \kappa_{ab} j_1^a(\sigma) j_1^b(\sigma) + \kappa_{\alpha\beta} k^\alpha(\sigma) k^\beta(\sigma)), \quad (2.9)$$

are

$$\begin{aligned} 0 &= \partial_+ j_{-a} + \partial_- j_{+a} - 2F_{ca}^b k_\beta j_-^c - H_{abc} j_+^b j_-^c, \\ 0 &= \partial_+ j_-^a - \partial_- j_+^a + f^a_{bc} j_+^b j_-^c, \\ 0 &= 2\partial_- k^\alpha - F^{\alpha}_{bc} j_+^b j_-^c, \end{aligned} \quad (2.10)$$

in conventions with $V_\pm = V_0 \pm V_1$. The generalization of these equations of motion to a generic F_{ABC} , corresponding to choices of frame than (2.6), is straightforward as for $O(d, d)$, e.g., in [28].

III. THE A DEFORMATION OF THE PRINCIPAL CHIRAL MODEL

There is a plethora of integrable deformations of the principal chiral model: λ -, η -homogeneous Yang-Baxter and WZ-term deformation and combinations of those. Here, a new deformation of a d -dimensional principal chiral model by the heterotic gauge field A is proposed.

A. Integrability of the undeformed model

As starting point we take the principal chiral model on a group G with Lie algebra \mathfrak{g} , embedded into the bosonic heterotic σ model. In the language of the previous section, this will be characterized by the components of the Maurer-Cartan form on G as vielbein e and $B = A = 0$ in the generalized vielbein (2.6). This corresponds to the generalized fluxes (2.7)

$$f^c_{ab}, \quad H_{abc} = 0, \quad F^\gamma_{ab} = 0, \quad (3.1)$$

where f^c_{ab} are structure constants to the Lie algebra \mathfrak{g} . This corresponds to the following equations of motion

$$\begin{aligned} \partial_+ j_{-a} + \partial_- j_{+a} &= 0, & \partial_+ j_-^a - \partial_- j_+^a + f^a_{bc} j_+^b j_-^c &= 0, \\ \partial_- k^\alpha &= 0. \end{aligned} \quad (3.2)$$

This model is classically integrable, as the currents k and the d -dimensional principal chiral model with the currents j decouple. Classical integrability is shown by providing a Lax representation and a classical \mathcal{R} matrix, proving Hamiltonian integrability, i.e., involution of the charges produced from the Lax representation.

The Lax representation of this model is

$$\begin{aligned} L_+(\lambda) &= \frac{1}{1-\lambda} (j_+^a t_a + 2k^\alpha t_\alpha), \\ L_-(\lambda) &= \frac{1}{1+\lambda} j_-^a t_a \quad \in \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{u}(1)^n, \end{aligned} \quad (3.3)$$

where λ is the spectral parameter. (t_a, t_α) are generators of $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{u}(1)^n$, which is defined by

$$[t_a, t_b] = f^c_{ab} t_c, \quad [t_\alpha, t_\beta] = 0 = [t_\alpha, t_a].$$

A method to show involution of the associated tower of conserved charges is Maillet's formalism [64,65]. Central for this is the Lax matrix

$$\begin{aligned} \mathcal{L}(\lambda) &= L_+(\lambda) - L_-(\lambda) \\ &= \frac{\lambda j_0^a + j_1^a}{1-\lambda^2} t_a + \frac{1}{1-\lambda} k^\alpha t_\alpha. \end{aligned} \quad (3.4)$$

The aim is to construct a so-called classical \mathcal{R} matrix, i.e., an object $\mathcal{R}(\lambda, \mu) \in \tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$ such that

$$\begin{aligned} \{\mathcal{L}_1(\lambda, \sigma), \mathcal{L}_2(\mu, \sigma')\} \\ = ([\mathcal{R}_{12}(\lambda, \mu), \mathcal{L}_1(\lambda, \sigma)] - [\mathcal{R}_{21}(\mu, \lambda), \mathcal{L}_2(\mu, \sigma')]) \delta(\sigma - \sigma') \\ - (\mathcal{R}_{12}(\lambda, \mu) + \mathcal{R}_{21}(\mu, \lambda)) \delta'(\sigma - \sigma'). \end{aligned} \quad (3.5)$$

A bold subscript indicates the place in the tensor product, e.g., $X_1 = X \otimes \mathbb{1}$. A sufficient condition for the involution of the conserved charges constructed from the Lax pair is the classical Yang-Baxter equation:

$$\begin{aligned} [\mathcal{R}_{12}(\lambda_1, \lambda_2), \mathcal{R}_{13}(\lambda_1, \lambda_3)] + [\mathcal{R}_{12}(\lambda_1, \lambda_2), \mathcal{R}_{23}(\lambda_2, \lambda_3)] \\ + [\mathcal{R}_{32}(\lambda_3, \lambda_2), \mathcal{R}_{31}(\lambda_3, \lambda_1)] = 0. \end{aligned} \quad (3.6)$$

The classical \mathcal{R} matrix for the undeformed model (3.2) is

$$\mathcal{R}(\lambda, \mu) = \frac{t_a \otimes t^a}{\lambda - \mu} \phi_{PCM}^{-1}(\mu) + \frac{t_\alpha \otimes t^\alpha}{\lambda - \mu} \phi_{\text{chiral}}^{-1}(\mu), \quad (3.7)$$

with the so-called twist functions

$$\phi_{PCM}(\lambda) = \frac{1-\lambda^2}{\lambda^2}, \quad \phi_{\text{chiral}}(\lambda) = 1-\lambda. \quad (3.8)$$

This is close to the twist function form $\mathcal{R}(\lambda, \mu) = \mathcal{R}^{(0)}(\lambda, \mu) \phi^{-1}(\lambda)$ with a solution of the classical Yang-Baxter equation $\mathcal{R}^{(0)}$, like $\mathcal{R}(\lambda, \mu) = \frac{t_a \otimes t^a}{\lambda - \mu}$ and a twist function $\phi(\mu)$. This form is a sufficient condition in order for \mathcal{R} to be a solution of the classical Yang-Baxter equation (3.6) and a typical assumption in the classification of integrable σ models [66]. From this point of view it is noteworthy that this simple model has no \mathcal{R} matrix of this form. Nevertheless, it is easy to see that (3.7) solves the classical Yang-Baxter equation, as $t_\alpha \otimes t^\alpha$ is a central element of $\tilde{\mathfrak{g}} \otimes \tilde{\mathfrak{g}}$, and all its contributions drop out of the classical Yang-Baxter equation (3.6).

B. Introducing the deformation

The deformation is characterized by a constant (in flat indices) heterotic gauge field

$$A_a^\alpha = \epsilon \mathcal{A}_a^\alpha, \quad \text{with} \quad \mathcal{A}_a^\alpha \mathcal{A}_b^\beta \kappa_{\alpha\beta} = \kappa_{ab}, \quad (3.9)$$

so $\mathcal{A} \in \text{O}(d) \subset \text{Mat}(n, d)$. For this one needs that $n > d$ which would be the case for the bosonic part of the heterotic string where $n = 16$.¹ We use κ_{ab} (as the Killing metric on \mathfrak{g}) and $\kappa_{\alpha\beta}$ to raise and lower indices on \mathcal{A} , e.g., to define the object \mathcal{A}_a^α . Care needs to be taken that when contracting \mathcal{A} s in the n -dimensional indices one will produce $\mathcal{A}_a^\alpha \mathcal{A}_{ba}^\beta = \kappa_{ab}$ by definition (3.9), but contractions of the d -dimensional flat indices will result in a projector on the part of the n -dimensional currents, that couple to the j_+ (see below in the equations of motions)

$$\mathcal{P}^{\alpha\beta} = \mathcal{A}_a^\alpha \mathcal{A}^{a\beta}. \quad (3.10)$$

In particular, one has that $\mathcal{P}^\alpha{}_\gamma \mathcal{P}^\gamma{}_\beta = \mathcal{P}^{\alpha\beta}$ and $\mathcal{P}^\alpha{}_\beta \mathcal{A}_a^\beta = \mathcal{A}_a^\alpha$. So contractions like $\mathcal{A}_{aa}^\alpha k^\alpha$ will project out the decoupled components of k as well. One could, of course, decompose the $\mathfrak{u}(1)^n$ currents into coupled and decoupled currents immediately, by choosing the generators of $\mathfrak{u}(1)^n$ accordingly. Also, in this sense, there is only one inequivalent \mathcal{A} deformation. All $\mathcal{A} \in \text{O}(d)$ are related by a basis change of $\mathfrak{u}(1)^n$. Keeping in mind possible generalizations, the n -dimensional covariance by introducing \mathcal{A} is kept in the following.

As a result, any choice of gauge field satisfying (3.9) should result in the same deformation of the fluxes

$$\begin{aligned} H_{abc} &= \left(\tilde{h} - \frac{3}{2} \epsilon^2 \right) f_{abc} = h f_{abc}, \\ f^c{}_{ab} &= f^c{}_{ab}, \\ F^{\gamma}{}_{ab} &= -\epsilon \mathcal{A}_c^\gamma f^c{}_{ab}, \end{aligned} \quad (3.11)$$

where an WZ term with some free coefficient \tilde{h} was allowed as well, i.e., a B field such that $H \sim f$. Effectively, this results in a two-parameter deformation where the WZ part is, of course, well understood but it interesting to note that one can combine it with the novel \mathcal{A} deformation. The equations of motion are

$$\begin{aligned} 0 &= \partial_+ j_-^a + \partial_- j_+^a + 2\epsilon \mathcal{A}_b^\alpha f_{bc}^\alpha k_+^b j_-^c - h f^a{}_{bc} j_+^b j_-^c, \\ 0 &= \partial_+ j_-^a - \partial_- j_+^a + f^a{}_{bc} j_+^b j_-^c, \\ 0 &= 2\partial_- k^\alpha + \epsilon \mathcal{A}_a^\alpha f_{bc}^\alpha j_+^b j_-^c. \end{aligned} \quad (3.12)$$

It is convenient to decompose the last equation into an d -dimensional part of k that couples to the j , and the $(n-d)$ -dimensional part that is still decoupled:

$$0 = \mathcal{A}_a^\alpha (2\partial_- k^\alpha + \epsilon f^a{}_{bc} j_+^b j_-^c), \quad 0 = (1-P)_{\alpha\beta} \partial_- k^\beta. \quad (3.13)$$

¹This is due the fact that $\text{U}(1)^n$ is the Cartan subgroup of $E_8 \times E_8$ or $\text{SO}(32)$.

These decoupled components and the associated generators will be denoted with a bar in the following:

$$(1-P)^\alpha{}_\beta k^\beta = \bar{k}^\alpha, \quad (1-P)_{\alpha\beta} t^\beta = \bar{t}_\alpha.$$

C. The Lax pair

In order to show the existence of a Lax pair $L_\pm \in \tilde{\mathfrak{g}}$, we make the ansatz is

$$\begin{aligned} L_-(\lambda) &= a(\lambda) j_-^a t_a, \\ L_+ &= (b(\lambda) j_+^a + c(\lambda) \epsilon \mathcal{A}_a^\alpha k^\alpha) t_a + \frac{2}{1-\lambda} \bar{k}^\alpha \bar{t}_\alpha. \end{aligned} \quad (3.14)$$

where the coefficient in front of the decoupled currents \bar{k} is taken from the undeformed case (3.3) and λ is the spectral parameter. The coefficients $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ are determined such that

$$0 = \partial_+ L_- - \partial_- L_+ + [L_+, L_-] \in \tilde{\mathfrak{g}} \quad (3.15)$$

is valid by virtue of the equations of motion (3.12). This results in the under determined system of equations:

$$0 = \frac{a+b}{2} - ab - \frac{1}{2}(\epsilon^2 c + h(a-b)), \quad 0 = -(a-b) + ac.$$

A convenient parametrization of the solution to this is

$$a = \frac{1}{1+\lambda}, \quad b = \frac{1 - [\epsilon^2(1+\lambda) + h]}{f_{\epsilon,h}(\lambda)}, \quad c = -\frac{2\lambda}{f_{\epsilon,h}(\lambda)}, \quad (3.16)$$

with $f_{\epsilon,h}(\lambda) = 1 - \lambda - (1+\lambda)[\epsilon^2(1+\lambda) + h]$. In that parametrization one can recognize a typical parametrization of the Lax pair of the principal chiral model in the undeformed limit; i.e., for the \mathfrak{g} -valued part of $L(\lambda)$

$$L_\pm^a(\lambda) \longrightarrow \frac{j_\pm^a}{1 \mp \lambda}, \quad \text{for } \epsilon, h \rightarrow 0. \quad (3.17)$$

But the full Lax pair $L_\pm \in \tilde{\mathfrak{g}}$ in the limit $\epsilon \rightarrow 0$ does not give the full undeformed Lax pair (3.3)—the part of the k^α that is coupled in the deformed models disappears from the Lax pair and the equations of motion reconstructed from it.

D. Hamiltonian integrability

One should check whether the conserved charges from the Lax pair are also in Poisson involution for arbitrary choices of ϵ and h . The relevant quantity for this is the Lax matrix

$$\begin{aligned} \mathcal{L} &\equiv L_1 \\ &= \frac{1}{(1+\lambda)f_{\epsilon,h}(\lambda)} [\lambda j_0^a + (f_{\epsilon,h}(\lambda) + \lambda) j_1^a - \epsilon \lambda (1+\lambda) \mathcal{A}_a^\alpha k_+^\alpha] t_a \\ &\quad + \frac{1}{1-\lambda} \bar{k}^\alpha \bar{t}_\alpha. \end{aligned} \quad (3.18)$$

A corresponding classical \mathcal{R} matrix is

$$\mathcal{R}_{12}(\lambda, \mu) = \frac{t_a \otimes t^a}{\lambda - \mu} \phi_{\epsilon, h}^{-1}(\mu) + \frac{\bar{t}^a \otimes \bar{t}_a}{\lambda - \mu} \frac{1}{1 - \mu}, \quad \text{with}$$

$$\phi_{\epsilon, h}(\mu) = \frac{(1 + \mu)f_{\epsilon, h}(\mu)}{\mu^2}. \quad (3.19)$$

Again, \mathcal{R} is almost of the form with twist function and solves the classical Yang-Baxter equation (3.6) by the same argument as in the undeformed case. This \mathcal{R} matrix could be guessed relatively easily from inspiration from

$$\frac{t^a \otimes t^b}{(1 + \lambda)f_{\epsilon, h}(\lambda)(1 + \mu)f_{\epsilon, h}(\mu)} (f^c{}_{ab} \delta(\sigma - \sigma') ((-\lambda\mu)j_{0c}(\sigma) + (\lambda\mu)\epsilon \mathcal{A}^c{}_{\gamma} k_{\gamma}(\sigma) - [(\lambda + \mu) - \epsilon^2(\lambda + \mu + 2\lambda\mu) - h(\lambda + \mu + \lambda\mu)]j_{1c}(\sigma)) + \kappa_{\alpha\beta} \delta'(\sigma - \sigma') ((\lambda + \mu) - \epsilon^2(\lambda + \mu + 3\mu\lambda - \lambda^2\mu^2) - h(\lambda + \mu + 2\mu\lambda))). \quad (3.20)$$

The part valued in $\mathfrak{u}(1)^n \otimes \mathfrak{u}(1)^n$ remains unchanged from the discussion in the undeformed case.

IV. DISCUSSION

In this article a new integrable deformation of the principal chiral model inspired by the heterotic string was proposed. It builds on the recent Hamiltonian formulation of the bosonic part of the heterotic string σ model in terms of $O(d, d + n)$ -generalized geometry in [49]. Building on that the analog to the so-called \mathcal{E} model in $O(d, d)$ -generalized geometry was proposed in the Introduction. The deformation was introduced in the second section and is characterized by a certain choice of heterotic gauge field $A^\alpha = \epsilon \mathcal{A}^\alpha$. The concrete choice of $\mathcal{A} \in O(d)$ can be absorbed into a change of basis for the n -dimensional current k . Hence, this deformation of A really only results in a one-parameter deformation given by the fluxes (3.11). It is also shown that this deformation can be extended to a two-parameter deformation together with a WZ-term-like deformation, while preserving integrability. Classical integrability was shown by the explicit construction of a Lax pair (3.18) and a classical \mathcal{R} matrix (3.19) that solves the classical Yang-Baxter equation.

To the knowledge of the author this is uncharted territory. From here several direction should be taken to further explore this space of novel integrable σ models:

- (1) Generalization to the full (supersymmetric) heterotic string
- (2) Different starting points for the deformation:
 - (i) symmetric space coset σ models, Yang-Baxter or λ deformation, instead of the principal chiral model for a Lie group G
 - (ii) non-Abelian chiral currents k^α , i.e., some non-trivial gauge algebra \mathfrak{h} instead of $\mathfrak{u}(1)^n$

the form of the \mathcal{R} matrix of the principal chiral model. It is easy to see that the $(\mathfrak{g} \otimes \mathfrak{g})$ -valued part is reproduced for $\epsilon, h \rightarrow 0$, as then $f_{\epsilon, h}(\lambda) \rightarrow (1 - \lambda)$. But, as also for the Lax pair, the full \mathcal{R} matrix (3.19) does not reproduce the full undeformed \mathcal{R} matrix (3.7) in the nondeformation limit.

One checks that (3.19) and (3.18) fit into Maillet's formalism, by showing that for the $(\mathfrak{g} \otimes \mathfrak{g})$ -valued part both the left-hand and right-hand sides of Maillet bracket (3.5) correspond to

- (3) Dual version of the A deformation.² The natural starting point would be the dual vielbein to (2.6) with dual fields α and β to A and B :

$$E_{\mathcal{A}}^{\mathcal{M}} = \begin{pmatrix} \delta_a^b & 0 & 0 \\ \alpha_a^b & \delta_a^b & 0 \\ \beta^{ab} - \frac{1}{2} \alpha_a^\alpha \alpha^{b\alpha} & -\alpha^{a\beta} & \delta_b^a \end{pmatrix} \times \begin{pmatrix} e_b^m & 0 & 0 \\ 0 & \delta_{\beta^\mu} & 0 \\ 0 & 0 & e^b_m \end{pmatrix}. \quad (4.1)$$

- (4) The deformed models as reduction of known integrable σ models.

From the point of view of the systematics of integrable models, the question is this: How new is this model really? The present Lax representation is valued in $\mathfrak{g} \oplus \mathfrak{u}(1)^n$. A natural candidate for a more conventional integrable σ model that would reproduce the same equations of motion (2.10) would be something like a principal chiral model or a Wess-Zumino-Witten (WZW) model with target space $G \times T^n$, together with some constraint that makes k_- (which did not appear in the construction here) vanish. This is clearly possible for the undeformed model, but does not seem possible straightforward for the deformed model. This deserves further attention.

A related question would be whether these models can be obtained from 4D Chern-Simons theory [19–24]. The not quite “twist function” form of

²In generalization of the fact that one can have integrable deformations based on the Kalb-Ramond field B (WZ term) or their duals β (η and homogeneous Yang-Baxter deformations), there could be also a dual version of A deformation.

the \mathcal{R} matrices (3.7) and (3.19) might suggest that this is not the case.

Besides these questions, two issues should be addressed and will be briefly discussed below: the questions about the embedding of these models into string theory and their incorrect nondeformation limit. Both questions certainly require more study, in particular if the present approach can be extended to the above mentioned more nontrivial examples.

A. Embedding into heterotic supergravity

The usual question for the integrable models and their deformations in context of string theory is whether they are also supergravity backgrounds. A detailed discussion is left for future research when a similar analysis to the one in this article has been done with the inclusion of fermions.

This question consists of two parts: (1) When is the undeformed model a solution of heterotic supergravity? (2) Is the A deformation a solution to heterotic supergravity, given that the undeformed model is?

The principal chiral model: In comparison to the type II σ models, the heterotic principal chiral model cannot be completed a string σ model by simply choosing a supergroup with vanishing Killing form [67].

B. The WZW-type undeformed model

A valid starting point for the deformation in heterotic supergravity would be the WZW-type model, i.e., target space geometry corresponding to the fluxes (3.11) with $h = 1$, $\epsilon = 0$. This is clearly a solution of heterotic supergravity, as it only contains NS-NS (Neveu-Schwarz) fields and is a conformal model. WZW models have been applied in several circumstances in heterotic string theory, for example for the heterotic string on AdS_3 [68] or compactifications on group manifolds with torsion [69]. Some of these discussions actually require (chiral) gauged WZW models [70–72]. For this, the present approach has to be extended from group to coset manifolds as discussed above.

C. The deformed model

One can check that an integrable deformation of such a WZW model will correspond to a solution of heterotic supergravity, given that the two deformation parameters h and ϵ in (3.11) are related in a certain way.

For example, one of the equations of motion of heterotic supergravity [15]—in the frame formalism and for constant dilaton—is

$$0 = R_{ab}^{(+)} - 2\alpha'(\text{tr}(F_{ac}F_b{}^c) - R_{ac}^{(+)}R^{(+)}{}_b{}^c). \quad (4.2)$$

Here, $R^{(+)}$ is, as usual, the Ricci tensor of the torsionful connection $\Gamma + \frac{1}{2}H$, where Γ is the Christoffel connection. The choice of fluxes (3.11) means that

$$\text{tr}(F_{ac}F_b{}^c) = \epsilon^2\kappa_{ab}, \quad R_{ab}^{(+)} = -\chi^2\kappa_{ab}$$

with the Killing form κ on \mathfrak{g} and $\chi^2 = \frac{1}{4}(h-1)^2(2+(h-1)^2)$ for simplicity of notation.

So far, we ignored dimensions. Only for the following analysis, these are briefly reintroduced. In particular, $[\chi^2] = [\epsilon] = [\alpha']^{-1}$. The heterotic supergravity equation (4.2) at $\mathcal{O}(\alpha')$ is satisfied, given that deformation parameters are related:

$$\chi^2 = \frac{1}{4\alpha'} - \sqrt{\frac{1}{16\alpha'^2} + \epsilon^2} \approx 2\alpha'\epsilon^2 + (\alpha')^{-1}\mathcal{O}((\alpha'\epsilon)^4). \quad (4.3)$$

The other heterotic supergravity equations hold in a similar way. Also, by virtue of the Jacobi identity of \mathfrak{g} , the anomaly cancellation condition

$$dH = 0 = \text{tr}(F \wedge F) \quad (4.4)$$

is automatically fulfilled for fluxes of the deformed model (3.11).

D. Nondeformation limit

As discussed briefly in the main text, the objects of the Lax construction, i.e., the Lax pair and the classical \mathcal{R} matrix, do not have the correct nondeformation limit. The \mathfrak{g} -valued part of the Lax connection (3.18) correctly reproduces the one of the principal chiral model, but the full $(\mathfrak{g} \oplus \mathfrak{u}(n^n))$ -valued Lax connection does not reproduce the full undeformed one (3.3). A similar problem also arises for the poles and zeros of the twist function. Whereas $\phi_{\epsilon,h} \rightarrow \phi_{PCM}$ and both have double poles at 0 and infinity, $\phi_{\epsilon,h} = (1+\lambda)f_{\epsilon,h}(\lambda)\lambda^{-2}$ has three zeros:

$$\begin{aligned} \lambda_1 &= -1, \\ \lambda_{2,3} &= -\frac{1}{2\epsilon^2} \left(2\epsilon^2 + (1+h) \pm \sqrt{8\epsilon^2 + (h+1)^2} \right). \end{aligned} \quad (4.5)$$

For an analysis of the $h, \epsilon \rightarrow 0$ limit, a concrete ϵ dependence of $h = \tilde{h} - \frac{3}{2}\epsilon^2$, respectively, also of \tilde{h} needs to be specified. As an example, let us assume that $h = h_0$ has no ϵ dependence (so $\tilde{h} = h_0 + \frac{3}{2}\epsilon^2$). Then one of the $\lambda_{2,3}$ has the expected $\epsilon \rightarrow 0$ limit:

$$\begin{aligned} \lambda_2 &= -\frac{1+h_0}{\epsilon^2} - \frac{3}{1+h_0} + \mathcal{O}(\epsilon^2) \xrightarrow{\epsilon, h_0 \rightarrow 0} -\infty, \\ \lambda_3 &= \frac{1}{1+h_0} + \mathcal{O}(\epsilon^2) \xrightarrow{\epsilon, h_0 \rightarrow 0} = 1. \end{aligned} \quad (4.6)$$

The latter gives the expected zero in comparison to the twist function of the principal chiral model, but the role and origin of the additional zero at infinity is not clear at this stage. Let us summarize these issues:

deformed Lax pair (2.18) \rightleftharpoons undeformed Lax pair (2.3)
 deformed \mathcal{R} matrix (2.19) \rightleftharpoons undeformed \mathcal{R} matrix (2.7)
 for $\epsilon, h \rightarrow 0$ (4.7)

3 zeros of $\phi_{\epsilon, h} \rightleftharpoons 2$ zeros of ϕ_{PCM} .

This result might seem puzzling, as for the known integrable deformations the nondeformation limit gives the expected undeformed Lax pairs, see, e.g., [10]. An explanation for this might be the following: It is well known in perturbation theory that deformations (or perturbations) of partial differential equations do not necessarily correspond to deformations of the solution space. One

might interpret this result in this context, as the Lax pair corresponds to a solution of the system. Nevertheless, this question deserves further attention.

ACKNOWLEDGMENTS

The author thanks Falk Hassler, Eric Lescano, and the anonymous referee for PRD for helpful comments on the draft. This research is part of the Project No. 2022/45/P/ST2/03995 cofunded by the National Science Centre and the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 945339.

-
- [1] N. Beisert *et al.*, Review of AdS/CFT integrability: An overview, *Lett. Math. Phys.* **99**, 3 (2012).
- [2] D. Bombardelli, A. Cagnazzo, R. Frassek, F. Levkovich-Maslyuk, F. Loebbert, S. Negro, I.M. Szecsenyi, A. Sfondrini, S.J. van Tongeren, and A. Torrielli, An integrability primer for the gauge-gravity correspondence: An introduction, *J. Phys. A* **49**, 320301 (2016).
- [3] C. Klimčík, Yang-Baxter sigma models and dS/AdS T duality, *J. High Energy Phys.* **12** (2002) 051.
- [4] C. Klimčík, On integrability of the Yang-Baxter sigma-model, *J. Math. Phys. (N.Y.)* **50**, 043508 (2009).
- [5] F. Delduc, M. Magro, and B. Vicedo, An integrable deformation of the $AdS_5 \times S^5$ superstring action, *Phys. Rev. Lett.* **112**, 051601 (2014).
- [6] K. Sfetsos, Integrable interpolations: From exact CFTs to non-Abelian T-duals, *Nucl. Phys.* **B880**, 225 (2014).
- [7] C. Klimčík, Yang-Baxter σ -model with WZNW term as \mathcal{E} -model, *Phys. Lett. B* **772**, 725 (2017).
- [8] F. Delduc, M. Magro, and B. Vicedo, Integrable double deformation of the principal chiral model, *Nucl. Phys.* **B891**, 312 (2015).
- [9] C. A. S. Young, Non-local charges, $Z(m)$ gradings and coset space actions, *Phys. Lett. B* **632**, 559 (2006).
- [10] B. Hoare, Integrable deformations of sigma models, *J. Phys. A* **55**, 093001 (2022).
- [11] D. Osten, Lax pairs for new Z_N -symmetric coset σ -models and their Yang-Baxter deformations, *Nucl. Phys.* **B981**, 115856 (2022).
- [12] S. Driezen, Modave lectures on classical integrability in 2d field theories, *Proc. Sci. Modave2021* (2022) 002 [arXiv:2112.14628].
- [13] M.B. Green and J.H. Schwarz, Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory, *Phys. Lett. B* **149**, 117 (1984).
- [14] D.J. Gross, J.A. Harvey, E.J. Martinec, and R. Rohm, Heterotic string theory. I. The free heterotic string, *Nucl. Phys.* **B256**, 253 (1985).
- [15] A. Sen, Equations of motion for the heterotic string theory from the conformal invariance of the sigma model, *Phys. Rev. Lett.* **55**, 1846 (1985).
- [16] A. Sen, The heterotic string in arbitrary background field, *Phys. Rev. D* **32**, 2102 (1985).
- [17] B. Vicedo, On integrable field theories as dihedral affine Gaudin models, *Int. Math. Res. Not.* **2020**, 4513 (2020).
- [18] S. Lacroix, Integrable models with twist function and affine Gaudin models, Ph.D. thesis, Lyon, Ecole Normale Supérieure, 2018, arXiv:1809.06811.
- [19] K. Costello, E. Witten, and M. Yamazaki, Gauge theory and integrability, I, *ICCM Not.* **06**, 46 (2018).
- [20] K. Costello, E. Witten, and M. Yamazaki, Gauge theory and integrability, II, *ICCM Not.* **06**, 120 (2018).
- [21] R. Bittleston and D. Skinner, Twistors, the ASD Yang-Mills equations and 4d Chern-Simons theory, *J. High Energy Phys.* **02** (2023) 227.
- [22] S. Lacroix and B. Vicedo, Integrable \mathcal{E} -Models, 4d Chern-Simons theory and affine Gaudin models. I. Lagrangian aspects, *SIGMA* **17**, 058 (2021).
- [23] N. Levine, Equivalence of 1-loop RG flows in 4d Chern-Simons and integrable 2d sigma-models, arXiv:2309.16753.
- [24] L. T. Cole, R. A. Cullinan, B. Hoare, J. Liniado, and D. C. Thompson, Integrable deformations from twistor space, arXiv:2311.17551.
- [25] C. Klimčík and P. Ševera, Dual non-Abelian duality and the Drinfeld double, *Phys. Lett. B* **351**, 455 (1995).
- [26] K. Sfetsos, Duality-invariant class of two-dimensional field theories, *Nucl. Phys.* **B561**, 316 (1999).
- [27] S. Demulder, F. Hassler, and D. C. Thompson, Doubled aspects of generalised dualities and integrable deformations, *J. High Energy Phys.* **02** (2019) 189.
- [28] D. Osten, Current algebras, generalised fluxes and non-geometry, *J. Phys. A* **53**, 265402 (2020).
- [29] F. Hassler, RG flow of integrable \mathcal{E} -models, *Phys. Lett. B* **818**, 136367 (2021).
- [30] R. Borsato and S. Driezen, Supergravity solution-generating techniques and canonical transformations of σ -models from $O(D, D)$, *J. High Energy Phys.* **05** (2021) 180.
- [31] R. Borsato, S. Driezen, and F. Hassler, An algebraic classification of solution generating techniques, *Phys. Lett. B* **823**, 136771 (2021).

- [32] A. A. Tseytlin, Duality symmetric closed string theory and interacting chiral scalars, *Nucl. Phys.* **B350**, 395 (1991).
- [33] A. A. Tseytlin, Duality symmetric formulation of string world sheet dynamics, *Phys. Lett. B* **242**, 163 (1990).
- [34] W. Siegel, Superspace duality in low-energy superstrings, *Phys. Rev. D* **48**, 2826 (1993).
- [35] W. Siegel, Manifest duality in low-energy superstrings, in *Proceedings of the International Conference on Strings 93 Berkeley, California, 1993* (1993), pp. 353–363, arXiv: hep-th/9308133.
- [36] C. Hull and B. Zwiebach, Double field theory, *J. High Energy Phys.* **09** (2009) 099.
- [37] B. Zwiebach, Double field theory, T-duality, and courant brackets, *Lect. Notes Phys.* **851**, 265 (2012).
- [38] G. Aldazabal, D. Marqués, and C. Nuñez, Double field theory: A pedagogical review, *Classical Quantum Gravity* **30**, 163001 (2013).
- [39] O. Hohm, D. Lüst, and B. Zwiebach, The spacetime of double field theory: Review, remarks, and outlook, *Fortschr. Phys.* **61**, 926 (2013).
- [40] D. Geißbühler, D. Marqués, C. Nuñez, and V. Penas, Exploring double field theory, *J. High Energy Phys.* **06** (2013) 101.
- [41] E. Plauschinn, Non-geometric backgrounds in string theory, *Phys. Rep.* **798**, 1 (2019).
- [42] E. Lescano, α' -corrections and their double formulation, *J. Phys. A* **55**, 053002 (2022).
- [43] J. Maharana and J. H. Schwarz, Noncompact symmetries in string theory, *Nucl. Phys.* **B390**, 3 (1993).
- [44] J. H. Schwarz and A. Sen, Duality symmetries of 4-D heterotic strings, *Phys. Lett. B* **312**, 105 (1993).
- [45] O. Hohm and S. K. Kwak, Double field theory formulation of heterotic strings, *J. High Energy Phys.* **06** (2011) 096.
- [46] M. Graña and D. Marqués, Gauged double field theory, *J. High Energy Phys.* **04** (2012) 020.
- [47] O. A. Bedoya, D. Marques, and C. Nunez, Heterotic α' -corrections in double field theory, *J. High Energy Phys.* **12** (2014) 074.
- [48] W. Siegel, Two vierbein formalism for string inspired axionic gravity, *Phys. Rev. D* **47**, 5453 (1993).
- [49] M. Hatsuda, H. Mori, S. Sasaki, and M. Yata, Gauged double field theory, current algebras and heterotic sigma models, *J. High Energy Phys.* **05** (2023) 220.
- [50] Y. Sakatani and S. Uehara, Branes in extended spacetime: Brane worldvolume theory based on duality symmetry, *Phys. Rev. Lett.* **117**, 191601 (2016).
- [51] C. D. A. Blair and E. T. Musaev, Five-brane actions in double field theory, *J. High Energy Phys.* **03** (2018) 111.
- [52] Y. Sakatani and S. Uehara, Exceptional M-brane sigma models and η -symbols, *Prog. Theor. Exp. Phys.* **2018**, 033B05 (2018).
- [53] A. S. Arvanitakis and C. D. Blair, The exceptional sigma model, *J. High Energy Phys.* **04** (2018) 064.
- [54] C. D. A. Blair, Open exceptional strings and D-branes, *J. High Energy Phys.* **07** (2019) 083.
- [55] Y. Sakatani and S. Uehara, Born sigma model for branes in exceptional geometry, *Prog. Theor. Exp. Phys.* **2020**, 073B05 (2020).
- [56] M. J. Duff and J. X. Lu, Duality rotations in membrane theory, *Nucl. Phys.* **B347**, 394 (1990).
- [57] D. S. Berman and M. J. Perry, Generalized geometry and M theory, *J. High Energy Phys.* **06** (2011) 074.
- [58] M. Hatsuda and K. Kamimura, SL(5) duality from canonical M2-brane, *J. High Energy Phys.* **11** (2012) 001.
- [59] M. J. Duff, J. X. Lu, R. Percacci, C. N. Pope, H. Samtleben, and E. Sezgin, Membrane duality revisited, *Nucl. Phys.* **B901**, 1 (2015).
- [60] Y. Sakatani and S. Uehara, Non-Abelian U -duality for membranes, *Prog. Theor. Exp. Phys.* **2020**, 073B01 (2020).
- [61] Y. Sakatani, Extended Drinfel'd algebras and non-Abelian duality, *Prog. Theor. Exp. Phys.* **2021**, 063B02 (2021).
- [62] D. Osten, Currents, charges and algebras in exceptional generalised geometry, *J. High Energy Phys.* **06** (2021) 070.
- [63] D. Osten, On exceptional QP-manifolds, *J. High Energy Phys.* **01** (2024) 028.
- [64] J. M. Maillet, Hamiltonian structures for integrable classical theories from graded Kac-moody algebras, *Phys. Lett.* **167B**, 401 (1986).
- [65] J.-M. Maillet, New integrable canonical structures in two-dimensional models, *Nucl. Phys.* **B269**, 54 (1986).
- [66] S. Lacroix, Lectures on classical Affine Gaudin models, arXiv:2312.13849.
- [67] M. Bershadsky, S. Zhukov, and A. Vaintrob, PSL($n|n$) sigma model as a conformal field theory, *Nucl. Phys.* **B559**, 205 (1999).
- [68] S. Hohenegger, C. A. Keller, and I. Kirsch, Heterotic AdS₃/CFT₂ duality with (0,4) spacetime supersymmetry, *Nucl. Phys.* **B804**, 193 (2008).
- [69] A. Chatzistavarakidis and G. Zoupanos, Dimensional reduction of the heterotic string over nearly-Kaehler manifolds, *J. High Energy Phys.* **09** (2009) 077.
- [70] T. Gannon, Partition functions for heterotic WZW conformal field theories, *Nucl. Phys. B* **402**, 729 (1993).
- [71] S.-w. Chung and S. H. H. Tye, Chiral gauged WZW theories and coset models in conformal field theory, *Phys. Rev. D* **47**, 4546 (1993).
- [72] K. Sfetsos and A. A. Tseytlin, Chiral gauged WZNW models and heterotic string backgrounds, *Nucl. Phys.* **B415**, 116 (1994).