Irreversible vierbein postulate: Emergence of spacetime from quantum phase transition

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We formulate a model for quantum gravity based on the local Lorentz symmetry and general-coordinate invariance. A key idea is the irreversible vierbein postulate that a tree-level action for the model at a certain energy scale does not contain an inverse vierbein. Under this postulate, only the spinor becomes a dynamical field, and no gravitational background field is introduced in the tree-level action. In this paper, after explaining the transformation rules of the local Lorentz and general-coordinate transformations in detail, a tree-level action is defined. We show that fermionic fluctuations can induce a nonvanishing gravitational background field.

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I. INTRODUCTION

Quantum gravity—how to consistently quantize interacting spacetime fluctuations—is one of the most profound mysteries that has defied human challenges over the past century. On the other hand, it has been established experimentally that there indeed exist spacetime fluctuations that propagate over cosmic distances with the speed of light, namely, the gravitational waves, consistently described by Einstein's general relativity; see, e.g., Refs. [1,2] for classic examples. Given the tremendous success of the Standard Model (SM) of particle physics based on quantum field theory (see, e.g., Ref. [3] for a review), it is natural to expect that the gravitational field governing the observed spacetime fluctuation must be quantized too. Whether it is really quantized or not will be experimentally explored within the forthcoming decades as a form of quantized free spacetime fluctuations, gravitons, on a curved classical background during inflation in observations of the B-mode polarization of the cosmic microwave background [4–6] and further in direct observation of the cosmic gravitational-wave background [7]; see also Ref. [8].

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Recent advances in cooling, control, and measurement of mechanical systems in the quantum regime, particularly using matter-wave and optomechanical systems, have set the stage for potential first observations of quantum gravitational effects, as predicted by various low-energy quantum-gravity models, though with certain challenges [9]. Concurrently, recent table-top experiments in quantum-gravity phenomenology reassess classical descriptions by focusing on gravitational effects from delocalized quantum sources, aiming to uncover interactions beyond the Newtonian potential and deepen our understanding of gravity's quantum nature [10].

It is well known, however, that the quantization of the metric based on Einstein's general relativity is perturbatively nonrenormalizable, requiring an infinite number of counterterms and thus spoiling its predictability at the quantum level due to the infinite number of free parameters; see, e.g., Refs. [11–13], and also Ref. [14] for a review. Furthermore, the truncation of the gravitational action up to the dimension-two Einstein-Hilbert term with the Ricci curvature scalar $\mathcal{R}$ yields a conformal mode that has a wrong-sign kinetic term, which makes Euclidean quantum gravity ill-defined for both directions of Wick rotations such that either the wrong-sign mode or the other fields become exponentially growing along the imaginary time direction; see, e.g., Refs. [15,16] and Appendix A in Ref. [17] for a simple review.

It is known that a higher-derivative gravity involving the $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}^2$ terms, in addition to the Einstein-Hilbert term, is perturbatively renormalizable. However, this leads to a loss of unitarity in the theory [18], though this issue of nonunitarity is under an attempt to be circumvented by recent works in Refs. [19,20] and references therein; see also Refs. [21,22] for further discussions.

If we allow the theory to discard the Lorentz symmetry, it could be perturbatively renormalizable [23], though the speed of light depends on particle species, and hence we need additional fine-tunings; see also Refs. [24–27]. In any case, it appears that the realization of a renormalizable theory of gravity in perturbation theory is difficult if we retain all the essential symmetries and properties, in particular, both the Lorentz symmetry and unitarity.

The above perturbative nonrenormalizability argument is based on quantum theory around a free theory. In the Wilsonian viewpoint, the perturbative gravity is constructed around the Gaussian (trivial) fixed point. On the other hand, the notion of renormalizability in quantum field theory is generalized to the nonperturbative realm. This scenario is known as asymptotically safe gravity [28–30]; see also Refs. [14,31–43] for reviews. There is accumulating evidence that there exists a nontrivial (interacting) ultraviolet (UV) fixed point in gravitational systems by means of the functional renormalization group method. Quantum gravity is perturbatively nonrenormalizable but might be nonperturbatively renormalizable. This situation is similar to the $O(N)$ nonlinear sigma model. (This will be discussed in Sec. II.)

The statements mentioned above are based on an assumption that the metric field (spin-2 symmetric tensor field) is the fundamental degree of freedom. The view that Einstein's general relativity is a local Lorentz (LL) gauge theory [44] is found almost at the same time as the (non-Abelian) gauge theory itself [45] and has been developed in Refs. [46,47]. To write down the LL symmetry, it is essential to rewrite the metric degrees of freedom by the vierbein (tetrad) ones. The vierbein is also indispensable to writing down a spinor field on a curved space, namely, the matter field in our Universe.[1] In this sense, the vierbein degrees of freedom are more fundamental than the metric ones. In this paper, we postulate that the dynamical degrees of freedom that describe spacetime fluctuation are the vierbein and the LL-gauge field.[2] The simplest gravitational model with the vierbein and the LL-gauge fields is the Einstein-Cartan gravity; see, e.g., Ref. [51] for a review on classical Einstein-Cartan gravity and Refs. [52,53] for its quantization.

In this paper, we consider a model for gravity and matter based on the LL-gauge symmetry as well as the invariance under the general-coordinate (GC) transformation [sometimes interchangeably called diffeomorphism (diff)] at a certain energy scale $\Lambda_{\rm G}$ [54]. In particular, we postulate that its tree-level action admits the degenerate limit of the vierbein [55,56]. This forbids inverse vierbeins in the action, and therefore we call it the "irreversible vierbein postulate."

Under the irreversible vierbein postulate, only spinor fields can have kinetic terms, while the other fields become dynamical due to the quantum effects of spinor fields below $\Lambda_{\rm G}$. The main purpose of this work is to demonstrate possible generation of a spacetime background, i.e., the emergence of a nonvanishing background vierbein field, due to quantum fluctuations of the spinor field. This idea might also be viewed along the direction of pregeometry; see, e.g., Refs. [54,57–68].

This paper is organized as follows: We start with a brief overview of degrees of freedom and symmetries in gravitational theories in Sec. II. In Sec. III, we introduce our notation and explain transformation laws under the LL and GC transformations in detail. In particular, together with Appendixes A and B, we intend to highlight differences between earlier works and ours. Then, we implement the degenerate limit on the action in Sec. IV, where we refer to Appendixes C and D for detailed calculations. After briefly explaining transformation laws for the background fields in Sec. V, we demonstrate a generation of a nonvanishing flat background field of vierbein due to quantum effects of fermionic degrees of freedom in Sec. VI. In Sec. VII, we summarize this work and discuss future prospects.

II. DEGREES OF FREEDOM OF GRAVITATIONAL FIELDS

In this section, we first review the ordinary minimal Einstein gravity in the metric formalism and then in the vielbein formalism.

A. Minimal Einstein gravity in metric formalism

It is known that the Einstein-Hilbert action as the metric formalism

$$S_{\rm EH} = \int {\rm d}^D x \sqrt{-g}\left[-\Lambda_{\rm cc} + \frac{M_{\rm P}^2}{2}\mathcal{R}(g)\right] \tag{1}$$

well describes the classical gravitational interactions in $D = 4$. Here, $M_{\rm P}^2 = 1/(8\pi G_N)$ is the Planck mass squared or inverse Newtonian coupling constant and $\Lambda_{\rm cc}$ is the cosmological constant. The Ricci scalar curvature $\mathcal{R}(g)$ is given by the metric field $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. The metric

[1]One may consider replacing the vierbein degrees of freedom by promoting the gamma matrices $\gamma_\mu(x) = e^{\mathbf{a}}{}_\mu(x)\gamma_{\mathbf{a}}$ as dynamical variables [48,49]. The fluctuation of $\gamma_\mu(x)$ can be decomposed into that of metric and $SL(4,\mathbb{C})$ transformation. If this $SL(4,\mathbb{C})$ transformation is not anomalous, the corresponding degrees of freedom become redundant, unless there is a higher-dimensional operator that includes derivatives of $\gamma_\mu(x)$ in the action. We do not delve into this issue in this paper, and choose to take the vierbein as the fundamental degrees of freedom.

[2]It is worth noting that supergravity also uses the vierbein and the LL-gauge field as the fundamental (bosonic) degrees of freedom; contrary to the simplest model presented here, supergravity induces torsion in general due to the presence of a (fermionic) gravitino; see, e.g., Ref. [50] for a review.

field is a symmetric tensor, so it classically has $D(D+1)/2$ degrees of freedom in D-dimensional spacetimes.

It is known that the quantum theory based on the action (1) is nonrenormalizable in terms of the perturbative expansion of G_N [11]. The simplest explanation for the perturbative nonrenormalizability is the negative mass dimensionality of G_N. This may however be somewhat naive. Indeed, the Einstein-Hilbert action in three-dimensional spacetime is renormalizable even though the Newton coupling has the negative mass dimensionality [69]. This is because the Einstein-Hilbert action in three-dimensional spacetime becomes a topological theory and thus can be formulated as a Chern-Simons theory due to the peculiarity of the three-dimensional spacetime. In other words, this is because there are no propagating degrees of freedom of a graviton. The simple dimensional counting of the coupling constant cannot fully capture the property of renormalizability.

Another viewpoint why the perturbation theory for the Einstein-Hilbert action becomes nonrenormalizable is the existence of the inverse metric which is defined by

$$g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu_\mu. \tag{2}$$

This reversibility condition (2) for the metric field induces an infinite number of interactions: When one considers the metric fluctuation field $h_{\mu\nu}$ around a background field $\bar{g}_{\mu\nu}$, namely, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, the inverse metric is expanded so as to satisfy Eq. (2) and is given by

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\alpha h^{\alpha\nu} + \cdots. \tag{3}$$

This series continues infinitely around a certain background field. That is, once the inverse metric is defined by Eq. (2), the Ricci scalar curvature in the action (1) generally contains an infinite number of vertices of metric fluctuations, whereas all vertices have a common coupling constant G_N. In general, one cannot remove all the UV divergences arising from quantum loops including an arbitrary number of vertices by only a single coupling constant.

B. Nonlinear and linear sigma models

The above situation is quite similar to the $O(N)$ nonlinear sigma model which is a low-energy effective model of pions, π^i ($i = 1, \ldots, N-1$). Its action is given by

$$S_{\rm NLS} = \frac{f_\pi^2}{2}\int d^D x\left[-\partial_\mu\pi^i\partial^\mu\pi^i - (\pi^i\partial^2\pi^i)^2 + \cdots\right], \tag{4}$$

where f_π is the pion decay constant. This theory can be obtained from the spontaneous symmetry breaking in the $O(N)$ linear sigma model whose action reads

$$S_{\rm LS} = \int d^D x\left[-\frac{1}{2}(\partial_\mu\phi^i)^2 - \frac{m^2}{2}(\phi^i\phi^i) - \frac{\lambda}{4}(\phi^i\phi^i)^2\right]. \tag{5}$$

Here, $\phi^i = (\pi^j, \sigma)$ with $i = 1, \ldots, N$ and $j = 1, \ldots, N-1$. For $m^2 < 0$, a nontrivial vacuum

$$\langle\phi^i\phi^i\rangle = \frac{2|m^2|}{\lambda} \tag{6}$$

is realized and $O(N)$ symmetry is broken into $O(N-1)$. As a consequence, the σ mode becomes massive and decouples from the low-energy dynamics, while π^i are massless and remain as effective degrees of freedom in the low-energy regime. In this case, one has the constraint on fields (6) with which integrating out the σ mode in Eq. (5) [with the constraint $\sigma = \sqrt{f_\pi^2 - (\pi^i)^2}$] results in the action of the nonlinear sigma model (4). The decay constant just corresponds to the vacuum expectation value $f_\pi = \sqrt{2|m^2|/\lambda}$. To summarize, the nonlinear sigma model (4) is obtained from the expansion of the linear sigma model (5) around the vacuum (6).

An important fact is that for $D > 2$, the nonlinear sigma model is perturbatively nonrenormalizable, while the linear sigma model is perturbatively renormalizable. The parameter f_π, which arises from the consequence of the $O(N)$ symmetry breaking in the linear sigma model, is a free parameter in the nonlinear sigma model. In particular, the massless pions are realized only at the vacuum at $\langle\sigma\rangle = f_\pi$ in the linear sigma model as a consequence of the Nambu-Goldstone theorem. In this viewpoint, one has an inconsistency in the nonlinear sigma model between the massless pion condition and a free choice of f_π, and there is a range of validity for pion field fluctuations: $\pi^i \lesssim f_\pi$. This makes the system nonrenormalizable.[3]

C. Minimal Einstein gravity in vielbein formalism

In the metric theory describing gravity, Eq. (2) may be regarded as the constraint analogous to Eq. (6). Following the argument above, the inconsistency at high energies in the metric formalism may be between an expansion of the metric field around a background field, e.g., a flat background metric $\langle g_{\mu\nu}\rangle = \bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and the existence of massless metric fields. We expect that there exists an appropriate high-energy theory of the metric theory, and the generation of a vacuum $\langle g_{\mu\nu}\rangle = \bar{g}_{\mu\nu} \neq 0$ may imply the appearance of the massless metric field. The reversibility condition (2) for the metric field enforces a finite domain where quantum fields can fluctuate analogously to $\pi^i \lesssim f_\pi$

[3]Note here that the nonlinear sigma model in $D = 3$ is an asymptotically safe theory; i.e., there exists a nontrivial UV fixed point at which a nonperturbatively renormalizable theory is constructed [70–72]. Quantum gravity as metric theories could be an asymptotically safe theory as well [28–30].

in the nonlinear sigma model. Beyond such a domain, we expect that new degrees of freedom appear and participate in the dynamics.

We notice at this point that the gravitational theory with the action (1) is similar to the nonlinear sigma model (5). Hence, we intend to construct a gravitational theory with new additional degrees of freedom analogous to the meson σ in the linear sigma model.

Having this viewpoint in mind, we are motivated to consider a theory for gravity with more degrees of freedom. Let us here deal with a formulation for the gravitational theory with vielbein $e^{\mathbf{a}}{}_{\mu}(x)$ and LL-gauge field $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$ known as the Einstein-Cartan gravity based on $SO(1,d)$ LL symmetry. Its minimal form of action is given by

$$S_{\rm EC} = \int \mathrm{d}^D x |e(x)| \left[-\Lambda_{\rm cc} + \frac{M_{\rm P}^2}{2} e_{\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}}{}^{\nu}(x) \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x) \right], \tag{7}$$

where $|e(x)| = \det_{\mathbf{a},\mu} e^{\mathbf{a}}{}_{\mu}(x)$ is the determinant of the vielbein, and $\overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x)$ is the field strength of LL-gauge field $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$. Note here that LL indices $\mathbf{a}, \mathbf{b}, \ldots$ are antisymmetric in $\omega_{\mathbf{ab}\mu}(x)$ and $\overset{\omega}{\mathcal{F}}_{\mathbf{ab}\mu\nu}(x)$, namely, $\omega_{\mathbf{ab}\mu}(x) = -\omega_{\mathbf{ba}\mu}(x)$ and $\overset{\omega}{\mathcal{F}}_{\mathbf{ab}\mu\nu}(x) = -\overset{\omega}{\mathcal{F}}_{\mathbf{ba}\mu\nu}(x)$, due to the $SO(1,d)$ algebra; see Sec. III D below for details. It is worth stressing that the action (7) does not contain the LL-gauge kinetic term $\overset{\omega}{\mathcal{F}}_{\mathbf{ab}\mu\nu}(x)\overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}\mu\nu}(x)$, whereas the existence of the term $e_{\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}}{}^{\nu}(x) \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x)$ is peculiar in the Einstein-Cartan theory, as compared to an ordinary gauge theory that does not have such a term.

In this formulation, it seems that there are apparently $D(3D-1)/2$ independent classical degrees of freedom because $e^{\mathbf{a}}{}_{\mu}(x)$ and $\omega^{\mathbf{a}}_{\mathbf{b}\mu}(x)$ have D^2 and $D(D-1)/2$ degrees of freedom, respectively. In the action (7), however, there are no apparent kinetic terms for the vielbein or the LL-gauge field, so these fields are auxiliary fields at this stage; i.e., they are not dynamical degrees of freedom yet. Imposing the equation of motion on $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$, i.e., $\delta S_{\rm EC}/\delta\omega_{\mu} = 0$, one obtains its solution to $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$ as a function of the vielbein, namely, the Levi-Civita spin connection: $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) = \overset{e}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$; see Eq. (73) below for its explicit definition. By substituting this into $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$ in the field strength, the LL-gauge field disappears from the action, and the action is written in terms of only the vielbein with the kinetic term. At this point, the second term on the right-hand side of Eq. (7) just turns into the Einstein-Hilbert term written in terms of the vielbein, and the number of dynamical degrees of freedom is $D(D+1)/2$ which is the same as that of the symmetric metric field, being the composite of vielbein fields $g_{\mu\nu}(x) = \eta_{\mathbf{ab}} e^{\mathbf{a}}{}_{\mu}(x) e^{\mathbf{b}}{}_{\nu}(x)$.

From the fact above, it turns out that a certain condition between $e^{\mathbf{a}}{}_{\mu}(x)$ and $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$, such as the equation of motion for $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$, reduces the original (tree-level) auxiliary degrees of freedom to the "dynamical" ones that have the kinetic term. A question now is whether such a condition can be generalized or not.

In this paper, we advocate the irreversible vierbein postulate [54]: At a certain energy scale $\Lambda_{\rm G}$, the action for gravity must admit the degenerate limit of the vielbein [55,56,73] in which an arbitrary set of eigenvalues of the vielbein goes to zero, and hence the inverse vielbein cannot be defined. In a sense, the action at $\Lambda_{\rm G}$ corresponds to the linear sigma model (5). The irreversible vierbein postulate shares the same assumption that, in the language of the linear sigma model, we do not take into account inverse powers of $O(N)$ invariant $\phi^i\phi^i$ such as $(\phi^i\phi^i)^{-1}$ and $(\phi^i\phi^i)^{-2}$. Even though they do not spoil renormalizability in terms of power counting, they do prevent defining the symmetric phase $\langle\phi^i\phi^i\rangle = 0$. In this sense, the irreversible vierbein postulate ensures a well-defined symmetric phase $\langle g_{\mu\nu}\rangle = 0$.

Indeed, solving the equation of motion for the auxiliary field $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$ requires the inverse vielbein. Thus, at $\Lambda_{\rm G}$, we cannot impose the equation of motion. If one introduced the inverse vielbein *a priori*, it could kinematically reduce the degrees of freedom. By contrast, in our postulate, we claim that the reduction of degrees of freedom takes place dynamically below the scale $\Lambda_{\rm G}$. Hence, the inverse vielbein is defined thanks to quantum dynamics.

We consider a gravitational theory which is based on $SO(1,d)\times\mathrm{GC}$ in the degenerate limit which entails $\langle g_{\mu\nu}\rangle = 0$ at the tree level. Its dynamics realizes $\langle g_{\mu\nu}\rangle \neq 0$ and the massless metric field as a consequence of spontaneous symmetry breaking: $SO(1,d)\times\mathrm{GC}\to\mathrm{GC}$. In the following sections, we explain the transformation laws under the LL and GC transformations and introduce corresponding gauge fields in detail.

III. LOCAL LORENTZ AND GENERAL-COORDINATE TRANSFORMATIONS

In this section, we clarify how the fields transform under the LL and GC symmetries in details. In Sec. III A, we spell out the field content. In Secs. III B and III C, we present their transformation laws under the LL and GC symmetries, respectively. In Sec. III D, we show the field strength for the LL-gauge field and argue that we do not need an extra GC-gauge field or its field strength. Through this section, we work in $d+1$ spacetime dimensions. Later, we will specify $d=3$ when constructing a concrete action.

This section is intended to be mainly a review; see, e.g., Refs. [51,74] for further details. Nevertheless, to our best knowledge, the following are the first to be clearly stated in our paper comparing with the literature:

(i) The reduction condition of $GL(4)$ to GC in Eq. (45)
(ii) The fact that the antisymmetric part becomes unnecessary for GC as shown in Eq. (69)
(iii) The $GL(4)$ field strength being a differential operator as shown in Eq. (82)
(iv) The distinction between our GC and what we call the LD transformations

A. Field content

We introduce the fields and symmetries to clarify our notations and to construct an action. Our starting assumption is that at a certain scale Λ_G, the action enjoys the LL and GC symmetries. In particular, the gravitational sector consists of the vielbein (vierbein in four-dimensional spacetime) and the LL-gauge field.

The gravity sector consists of the vielbein field $e^{\mathbf{a}}{}_\mu(x)$ and the LL-gauge field $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$, where $\mu, \nu, \ldots$ ($\mathbf{a}, \mathbf{b}, \ldots$) run for the spacetime (tangent-space) indices $0, \ldots, d$ ($\mathbf{0}, \ldots, \boldsymbol{d}$). Here and hereafter, we make the dependence on a specific coordinate system x^μ explicit on each chart, unless otherwise stated, since it is anyway necessary for any realistic calculation of a dynamical quantity; this will make a distinction between a variable and constant more apparent. From the vielbein, we construct the metric field

$$g_{\mu\nu}(x) = \eta_{\mathbf{ab}} e^{\mathbf{a}}{}_\mu(x) e^{\mathbf{b}}{}_\nu(x), \tag{8}$$

where the tangent-space metric and its inverse are

$$\left[\eta_{\mathbf{ab}}\right]_{\mathbf{a},\mathbf{b}=\mathbf{0},\ldots,\boldsymbol{d}} = \left[\eta^{\mathbf{ab}}\right]_{\mathbf{a},\mathbf{b}=\mathbf{0},\ldots,\boldsymbol{d}} = \mathrm{diag}(-1, 1, \ldots, 1), \tag{9}$$

in which "diag" denotes the corresponding diagonal matrix. We note that the metric field $g_{\mu\nu}(x)$ can be constructed without using the inverse vielbein field $e_{\mathbf{a}}{}^\mu(x)$, whereas construction of an inverse metric field $g^{\mu\nu}(x)$ does require the inverse vielbein.

The matter sector of an effective field theory consists of scalar, spinor, and 1-form fields $\phi(x)$, $\psi(x)$, and $\mathcal{A}_\mu(x)$ with spin-0, $-1/2$, and -1, respectively.[4] Precisely speaking, $\mathcal{A}_\mu(x)$ are the components of the 1-form field $\mathcal{A}(x) \coloneqq \mathcal{A}_\mu(x)\mathrm{d}x^\mu$, but we sloppily call these components a 1-form field too. Below, $\Psi(x)$ will denote either $\phi(x)$ or $\psi(x)$ fields collectively. Also, $\Phi(x)$ will denote any one of the fields, including both the gravity and matter sectors.

Here, we take the 1-form field $\mathcal{A}_\mu(x)$ rather than the corresponding vector field $\mathcal{A}^\mu(x) \coloneqq g^{\mu\nu}(x)\mathcal{A}_\nu(x)$ as a fundamental degree of freedom because the former rather than the latter primarily appears in a gauge covariant derivative[5]

$$\overset{\mathcal{A}}{\mathcal{D}}_\mu \coloneqq \partial_\mu + \mathcal{A}_\mu(x). \tag{10}$$

More explicitly, on a field $\Psi(x)$ in the fundamental representation of a gauge group,

$$\Psi(x) \to U(x)\Psi(x), \tag{11}$$

the covariant derivative $\overset{\mathcal{A}}{\mathcal{D}}_\mu = \partial_\mu + \mathcal{A}_\mu(x)$ transforms covariantly,

$$\overset{\mathcal{A}}{\mathcal{D}}_\mu \Psi(x) \to U(x) \overset{\mathcal{A}}{\mathcal{D}}_\mu \Psi(x), \tag{12}$$

due to the gauge transformation of the gauge field $\mathcal{A}_\mu(x)$:

$$\mathcal{A}_\mu(x) \to U(x)\mathcal{A}_\mu(x)U^{-1}(x) - \partial_\mu U(x) U^{-1}(x), \tag{13}$$

or for an infinitesimal transformation $U(x) = I + \vartheta(x)$,

$$\mathcal{A}_\mu(x) \to \mathcal{A}_\mu(x) + \left[\vartheta(x), \mathcal{A}_\mu(x)\right] - \partial_\mu \vartheta(x), \tag{14}$$

where I is the identity matrix and the commutator is $\left[A, B\right] \coloneqq AB - BA$.

In the following sections, for totally symmetric and antisymmetric tensors denoted here by $\mathcal{S}$ and $\mathcal{A}$, respectively, we use the notation

$$\mathcal{S}_{\mu_1 \cdots \mu_n} = \mathcal{S}_{(\mu_1 \cdots \mu_n)}, \qquad \mathcal{A}_{\mu_1 \cdots \mu_n} = \mathcal{A}_{[\mu_1 \cdots \mu_n]}. \tag{15}$$

In particular, second-rank tensors are given by[6]

$$\mathcal{S}_{\mu\nu} = \mathcal{S}_{(\mu\nu)} = \frac{\mathcal{S}_{\mu\nu} + \mathcal{S}_{\nu\mu}}{2}, \quad \mathcal{A}_{\mu\nu} = \mathcal{A}_{[\mu\nu]} = \frac{\mathcal{A}_{\mu\nu} - \mathcal{A}_{\nu\mu}}{2}. \tag{16}$$

B. Local Lorentz transformation

Here, we review the LL transformations on various fields to spell out our notation. The LL transformation is a local rotation of the tangent-space basis; therefore, in particular, it does not act on a spacetime-scalar field.

[4] The existence of a nearly massless spin-$3/2$ field, gravitino, implies nearly unbroken local supersymmetry, supergravity, which does not seem to be realized in our Universe at low energies. It may still be interesting to include it since our scale Λ_G is supposed to be much higher than the electroweak one; see Appendix C 3.

[5] For a compact gauge group G with the corresponding Lie algebra $\mathfrak{g}$, one usually writes $\mathcal{A}_\mu(x) = i g_G A_\mu(x) = i g_G A_\mu^a(x) T^a$ where g_G is the gauge coupling and T^a ($a = 1, \ldots, \dim \mathfrak{g}$) are the Hermitian generators of the gauge symmetry. In particular, the kinetic terms of $\mathcal{A}_\mu(x)$ and $A_\mu^a(x)$ have opposite signs.

[6] Note that in this notation, $\{\mathcal{S}_\mu, \mathcal{S}_\nu\} = 2\mathcal{S}_{(\mu}\mathcal{S}_{\nu)}$ and $[\mathcal{A}_\mu, \mathcal{A}_\nu] = 2\mathcal{A}_{[\mu}\mathcal{A}_{\nu]}$.

1. LL transformation on the gravity sector

Under an LL transformation that satisfies the defining relation of the $SO(1,d)$ symmetry

$$\Lambda^{\mathbf{c}}{}_{\mathbf{a}}(x)\eta_{\mathbf{cd}}\Lambda^{\mathbf{d}}{}_{\mathbf{b}}(x) = \eta_{\mathbf{ab}}, \tag{17}$$

the gravitational fields transform as

$$e^{\mathbf{a}}{}_{\mu}(x) \xrightarrow{\mathrm{LL}} \Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x)e^{\mathbf{b}}{}_{\mu}(x), \tag{18}$$

$$\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \xrightarrow{\mathrm{LL}} \Lambda^{\mathbf{a}}{}_{\mathbf{c}}(x)\omega^{\mathbf{c}}{}_{\mathbf{d}\mu}(x)(\Lambda^{-1})^{\mathbf{d}}{}_{\mathbf{b}}(x) - \partial_\mu\Lambda^{\mathbf{a}}{}_{\mathbf{c}}(x)(\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}}(x), \tag{19}$$

where $(\Lambda^{-1})^{\mathbf{a}}{}_{\mathbf{b}} = \Lambda_{\mathbf{b}}{}^{\mathbf{a}}$, with their indices being lowered and raised by the tangent-space metric and its inverse (9). Here and hereafter, a derivative such as $\partial_\mu \coloneqq \frac{\partial}{\partial x^\mu}$ acts only on its neighbor:

$$\partial_\mu AB \coloneqq (\partial_\mu A)B, \qquad \partial_\mu(AB)C \coloneqq (\partial_\mu(AB))C, \tag{20}$$

etc.

We will also use a matrix notation such as

$$[\omega_\mu(x)]^{\mathbf{a}}{}_{\mathbf{b}} \coloneqq \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x), \tag{21}$$

leading to

$$\Lambda^{\mathrm{t}}(x)\eta\Lambda(x) = \eta, \tag{22}$$

and

$$e_\mu(x) \xrightarrow{\mathrm{LL}} \Lambda(x)e_\mu(x), \tag{23}$$

$$\omega_\mu(x) \xrightarrow{\mathrm{LL}} \Lambda(x)\omega_\mu(x)\Lambda^{-1}(x) - \partial_\mu\Lambda(x)\Lambda^{-1}(x), \tag{24}$$

where the superscript "t" denotes the transpose.

For an infinitesimal transformation $\Lambda(x) = I + \theta(x)$ in the matrix notation, or more explicitly, $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x) = \delta^{\mathbf{a}}_{\mathbf{b}} + \theta^{\mathbf{a}}{}_{\mathbf{b}}(x)$, we have

$$\omega_\mu(x) \xrightarrow{\mathrm{LL}} \omega_\mu(x) + [\theta(x), \omega_\mu(x)] - \partial_\mu\theta(x), \tag{25}$$

or more explicitly,

$$\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \xrightarrow{\mathrm{LL}} \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) + \theta^{\mathbf{a}}{}_{\mathbf{c}}(x)\omega^{\mathbf{c}}{}_{\mathbf{b}\mu}(x) - \omega^{\mathbf{a}}{}_{\mathbf{c}\mu}(x)\theta^{\mathbf{c}}{}_{\mathbf{b}}(x) - \partial_\mu\theta^{\mathbf{a}}{}_{\mathbf{b}}(x). \tag{26}$$

Note that the defining relation for $SO(1,d)$ in Eq. (17), or (22), implies the antisymmetry $\theta_{\mathbf{ba}}(x) = -\theta_{\mathbf{ab}}(x)$.

To summarize, the vielbein transforms as a fundamental representation of the LL symmetry, while being a spacetime 1-form. Recall that the Higgs field transforms as a fundamental representation of gauge symmetry while being a spacetime scalar. On the other hand, the LL-gauge field transforms just as an $SO(1,d)$ gauge field under the LL symmetry.

It is the transformation (19), or (24), that makes the LL-covariant derivative on an LL-vector (spacetime-scalar) field $V^{\mathbf{a}}(x)$,

$$\overset{\omega}{\mathcal{D}}_\mu V^{\mathbf{a}}(x) \coloneqq \partial_\mu V^{\mathbf{a}}(x) + \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)V^{\mathbf{b}}(x), \tag{27}$$

to be covariant:

$$\overset{\omega}{\mathcal{D}}_\mu V^{\mathbf{a}}(x) \xrightarrow{\mathrm{LL}} \Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x)\overset{\omega}{\mathcal{D}}_\mu V^{\mathbf{b}}(x). \tag{28}$$

In the matrix notation, the above equations read

$$\overset{\omega}{\mathcal{D}}_\mu V(x) \coloneqq [\partial_\mu + \omega_\mu(x)]V(x) \tag{29}$$

and

$$\overset{\omega}{\mathcal{D}}_\mu V(x) \xrightarrow{\mathrm{LL}} \Lambda(x)\overset{\omega}{\mathcal{D}}_\mu V(x). \tag{30}$$

2. LL transformation on the matter sector

Now we turn to the matter fields. The bosonic matter fields transform as a scalar (namely, do not transform) under the LL symmetry:

$$\phi(x) \xrightarrow{\mathrm{LL}} \phi(x), \tag{31}$$

$$\mathcal{A}_\mu(x) \xrightarrow{\mathrm{LL}} \mathcal{A}_\mu(x). \tag{32}$$

We here comment on the relation to the irreversible vierbein postulate which we will impose on the action. The 1-form field $\mathcal{A}_\mu(x)$ can be regarded as a composite field $\mathcal{A}_\mu(x) = \mathcal{A}_{\mathbf{a}}(x)e^{\mathbf{a}}{}_\mu(x)$ made of the vielbein $e^{\mathbf{a}}{}_\mu(x)$ and an LL-vector[7] spacetime scalar $\mathcal{A}_{\mathbf{a}}(x)$ that transforms as

$$\mathcal{A}_{\mathbf{a}}(x) \xrightarrow{\mathrm{LL}} \mathcal{A}_{\mathbf{b}}(x)\Lambda^{\mathbf{b}}{}_{\mathbf{a}}(x). \tag{33}$$

Even when we regard $\mathcal{A}_{\mathbf{a}}(x)$ as a fundamental degree of freedom, we can always construct $\mathcal{A}_\mu(x)$ without contradicting the irreversible vierbein postulate. In contrast, if starting from the 1-form field $\mathcal{A}_\mu(x)$, we need the inverse vielbein field $e_{\mathbf{a}}{}^\mu(x)$ to construct the LL-vector spacetime-scalar field $\mathcal{A}_{\mathbf{a}}(x) = e_{\mathbf{a}}{}^\mu(x)\mathcal{A}_\mu(x)$. That is, we cannot reconstruct $\mathcal{A}_{\mathbf{a}}(x)$ from $\mathcal{A}_\mu(x)$ under the irreversible vierbein postulate at the scale Λ_{G}.

The fermionic matter field, spinor, transforms nontrivially under the LL symmetry. In the matrix notation, we may parametrize an LL transformation as

[7]For the LL symmetry, we call both the covariant vector $V_{\mathbf{a}}$ and the contravariant vector $V^{\mathbf{a}}$ the LL-covariant vectors.

$$\Lambda(x) = e^{\theta(x)}, \tag{34}$$

that is,

$$\Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x) = \delta^{\mathbf{a}}_{\mathbf{b}} + \theta^{\mathbf{a}}{}_{\mathbf{b}}(x) + \frac{1}{2!}\theta^{\mathbf{a}}{}_{\mathbf{c}}(x)\theta^{\mathbf{c}}{}_{\mathbf{b}}(x) + \cdots. \tag{35}$$

Now the spinor field transforms as

$$\psi(x) \xrightarrow{\text{LL}} S(\Lambda(x))\psi(x), \tag{36}$$

where we define

$$S(e^{\theta(x)}) \coloneqq e^{\frac{1}{2}\theta_{\mathbf{ab}}(x)\sigma^{\mathbf{ab}}}, \tag{37}$$

in which the LL generators on the spinor representation are

$$\sigma^{\mathbf{ab}} \coloneqq \frac{[\gamma^{\mathbf{a}}, \gamma^{\mathbf{b}}]}{4}, \tag{38}$$

with $\gamma^{\mathbf{a}}$ being the gamma matrices that obey the Clifford algebra:

$$\{\gamma^{\mathbf{a}}, \gamma^{\mathbf{b}}\} = 2\eta^{\mathbf{ab}} I. \tag{39}$$

Here the anticommutator is defined by $\{A, B\} \coloneqq AB + BA$.

Among the matter fields, the spinor field is the only nontrivial representation under the LL symmetry. As stressed in the Introduction, the LL symmetry is necessary to define a spinor field at all on a curved spacetime. Note also that in our treatment, the LL symmetry is no different from the ordinary gauge symmetry other than it is under a noncompact group $SO(1, d)$.

C. General coordinate transformation

Next, we discuss the GC transformation $x^\mu \to x'^\mu(x)$, where, throughout this paper, the prime symbol $'$ exclusively denotes a quantity after the GC transformation and not a derivative. Accordingly, the bases for 1-form and spacetime vector transform as

$$\mathrm{d}x^\mu \xrightarrow{\text{GC}} \mathrm{d}x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\mathrm{d}x^\nu, \tag{40}$$

$$\partial_\mu \xrightarrow{\text{GC}} \partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\partial_\nu. \tag{41}$$

The GC transformation is generally identified with diff, while it is often said that the transformation under diff is given by the Lie-derivative (LD) transformation. One might regard that these three transformations were equivalent. Strictly speaking, however, GC/diff and the LD transformation should be distinguished. Indeed, the GC transformation introduced in this section is not given by the LD. A detailed comparison between the GC transformation and diff is given in Appendix A. In the main body of this paper, we use the terminology "GC" rather than diff.

In a matrix notation

$$M^\mu{}_\nu(x) \coloneqq \frac{\partial x'^\mu}{\partial x^\nu}, \qquad (M^{-1})^\nu{}_\lambda(x) = \frac{\partial x^\nu}{\partial x'^\lambda}, \tag{42}$$

and writing similarly to Eq. (21) such as $[M(x)]^\mu{}_\nu \coloneqq M^\mu{}_\nu(x)$, the above GC transformations on the bases read

$$\mathrm{d}x^\mu \xrightarrow{\text{GC}} \mathrm{d}x'^\mu = M^\mu{}_\nu(x)\mathrm{d}x^\nu, \tag{43}$$

$$\partial_\mu \xrightarrow{\text{GC}} \partial'_\mu = [M^{-1}(x)]^\nu{}_\mu\partial_\nu. \tag{44}$$

It is important that the "matrix" M satisfies the extra $\frac{d(d+1)^2}{2}$ conditions

$$\partial_{[\lambda}M^\mu{}_{\nu]}(x) = 0 \tag{45}$$

for the GC transformation (42). Conversely, it is also true that any function $M^\mu{}_\nu(x)$ that satisfies the condition (45) can always be written (locally) in terms of $(d+1)$ functions $x'^\mu(x)$ $(\mu = 0, \ldots, d)$ as in Eq. (42). The transformation by $M^\mu{}_\nu$ without the condition (45) corresponds to the general linear (GL) transformation, i.e., $GL(d+1)$.

From $\partial_\lambda(M^{-1}M) = 0$, we obtain $\partial_\lambda M^{-1}M = -M^{-1}\partial_\lambda M$, or specifying indices, it is given by $\partial_\lambda(M^{-1})^\mu{}_\rho M^\rho{}_\nu = -(M^{-1})^\mu{}_\rho\partial_\lambda M^\rho{}_\nu$. By antisymmetrizing λ and ν, we get

$$M^\rho{}_{[\nu}\partial_{\lambda]}(M^{-1})^\mu{}_\rho = 0. \tag{46}$$

Similarly, the derivative of the inverse function gives

$$0 = \frac{\partial^2 x^\lambda}{\partial x'^{[\mu}\partial x'^{\nu]}} = \frac{\partial x^\alpha}{\partial x'^{[\mu|}}\frac{\partial}{\partial x^\alpha}\frac{\partial x^\lambda}{\partial x'^{|\nu]}} = \partial_\alpha(M^{-1})^\lambda{}_{[\nu}(M^{-1})^\alpha{}_{\mu]}, \tag{47}$$

where vertical lines (between the antisymmetrization symbols in indices) denote that the indices between the vertical lines are not antisymmetrized. For instance, in Eq. (47), the index α between the vertical lines is not antisymmetrized. In actual computations such as will be done in Eq. (72), it is more convenient to use the coordinate notation on the right-hand sides in Eq. (42) rather than to use these relations (42), (46), (47), etc., in the matrix notation on the left-hand sides in Eq. (42). The matrix notation is of use for more conceptual understanding.

1. GC transformation on fields

Under the GC transformation, the gravitational fields transform as

$$e^{\mathbf{a}}{}_{\mu}(x) \xrightarrow{\text{GC}} e'^{\mathbf{a}}{}_{\mu}(x') = e^{\mathbf{a}}{}_{\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}}, \tag{48}$$

$$\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \xrightarrow{\text{GC}} \omega'^{\mathbf{a}}{}_{\mathbf{b}\mu}(x') = \omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}}, \tag{49}$$

and the matter fields as

$$\phi(x) \xrightarrow{\text{GC}} \phi'(x') = \phi(x), \tag{50}$$

$$\psi(x) \xrightarrow{\text{GC}} \psi'(x') = \psi(x), \tag{51}$$

$$\mathcal{A}_{\mu}(x) \xrightarrow{\text{GC}} \mathcal{A}'_{\mu}(x') = \mathcal{A}_{\nu}(x)\frac{\partial x^{\nu}}{\partial x'^{\mu}}. \tag{52}$$

The transformed scalar field $\phi'(x')$ is defined to satisfy $\phi'(x'(x)) = \phi(x)$ such that the pullback of the function $\phi'(x')$ by the function $x'(x)$ becomes $\phi(x)$. Equivalently, the pullback of the function $\phi(x)$ by the inverse function $x(x')$ is $\phi'(x')$, namely, $\phi(x(x')) = \phi'(x')$. Here, the spinor field also transforms the same as the scalar field (namely, as the pullback of the function) under the GC transformation; see Appendix B for another point of view and our opinion on it. Note that the spinor field transforms as scalar under the GC transformation because it does not have any spacetime index, while the "LD transformation" gives different transformation laws from Eq. (51). They are discussed in Appendix B.

In the matrix notation, with Eq. (42), the GC transformation is, on the gravitational fields,

$$e^{\mathbf{a}}{}_{\mu}(x) \xrightarrow{\text{GC}} e'^{\mathbf{a}}{}_{\mu}(x') = e^{\mathbf{a}}{}_{\nu}(x)\left[M^{-1}(x)\right]^{\nu}{}_{\mu}, \tag{53}$$

$$\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \xrightarrow{\text{GC}} \omega'^{\mathbf{a}}{}_{\mathbf{b}\mu}(x') = \omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x)\left[M^{-1}(x)\right]^{\nu}{}_{\mu}, \tag{54}$$

and, on the matter fields,

$$\phi(x) \xrightarrow{\text{GC}} \phi'(x') = \phi(x), \tag{55}$$

$$\psi(x) \xrightarrow{\text{GC}} \psi'(x') = \psi(x), \tag{56}$$

$$\mathcal{A}_{\mu}(x) \xrightarrow{\text{GC}} \mathcal{A}'_{\mu}(x') = \mathcal{A}_{\nu}(x)\left[M^{-1}(x)\right]^{\nu}{}_{\mu}. \tag{57}$$

In the matrix notation, a spacetime vector V^{μ} transforms like a fundamental representation under the GC transformation: $V^{\mu} \to M^{\mu}{}_{\nu}V^{\nu}$.

2. GC-gauge field

Now we want to define a GC-covariant derivative. To this end, let us first suppose that there exists a GC-gauge field that transforms as

$$\Upsilon_{\mu}(x) \xrightarrow{\text{GC}} \Upsilon'_{\mu}(x') = \left[M(x)\Upsilon_{\nu}(x)M^{-1}(x) - \partial_{\nu}M(x)M^{-1}(x)\right]\left[M^{-1}(x)\right]^{\nu}{}_{\mu}, \tag{58}$$

where we employ the matrix notation $\left[\Upsilon_{\mu}(x)\right]^{\alpha}{}_{\beta} \coloneqq \Upsilon^{\alpha}{}_{\beta\mu}(x)$ similarly to Eq. (21). More explicitly, the transformation (58) means

$$\Upsilon^{\alpha}{}_{\beta\mu}(x) \xrightarrow{\text{GC}} \Upsilon'^{\alpha}{}_{\beta\mu}(x') = \Big(M^{\alpha}{}_{\gamma}(x)\Upsilon^{\gamma}{}_{\delta\nu}(x)(M^{-1})^{\delta}{}_{\beta}(x) - \partial_{\nu}M^{\alpha}{}_{\gamma}(x)(M^{-1})^{\gamma}{}_{\beta}(x)\Big)(M^{-1})^{\nu}{}_{\mu}(x). \tag{59}$$

Here, we stress that the difference from the gauge transformations of the ordinary and LL-gauge fields in Eqs. (13) and (19) [or (24)], respectively, is the last M^{-1} factor that rotates the spacetime index too.

Then, one can construct a GC-covariant derivative on a spacetime-vector field $V^{\mu}(x)$ and a 1-form field $W_{\mu}(x)$: In the matrix notation, we write

$$\left[\overset{\Upsilon}{\nabla}_{\mu}V(x)\right]^{\alpha} \coloneqq \left[(\partial_{\mu} + \Upsilon_{\mu}(x))V(x)\right]^{\alpha}, \tag{60}$$

$$\left[\overset{\Upsilon}{\nabla}_{\mu}W(x)\right]_{\alpha} \coloneqq \left[W(x)(\overleftarrow{\partial_{\mu}} - \Upsilon_{\mu}(x))\right]_{\alpha}, \tag{61}$$

where the left derivative reads $A\overleftarrow{\partial_{\mu}} \coloneqq \partial_{\mu}A$, with the neighboring notation $AB\overleftarrow{\partial_{\mu}} \coloneqq A(\partial_{\mu}B)$, $A(BC)\overleftarrow{\partial_{\mu}} \coloneqq A(\partial_{\mu}(BC))$, etc., similar to Eq. (20). More explicitly, they are expressed as

$$\overset{\Upsilon}{\nabla}_{\mu}V^{\alpha}(x) \coloneqq \partial_{\mu}V^{\alpha}(x) + \Upsilon^{\alpha}{}_{\beta\mu}(x)V^{\beta}(x), \tag{62}$$

$$\overset{\Upsilon}{\nabla}_{\mu}W_{\alpha}(x) \coloneqq \partial_{\mu}W_{\alpha}(x) - W_{\beta}(x)\Upsilon^{\beta}{}_{\alpha\mu}(x). \tag{63}$$

It is straightforward to check their covariance under the GC transformation: In the matrix notation,

$$\begin{aligned}\overset{\Upsilon}{\nabla}_{\mu}V(x) &\xrightarrow{\text{GC}} \overset{\Upsilon}{\nabla}{}'_{\mu}V'(x')\\ &= (\partial'_{\mu} + \Upsilon'_{\mu}(x'))V'(x')\\ &= \left[(\partial_{\nu} + M\Upsilon_{\nu}M^{-1} - \partial_{\nu}MM^{-1})(MV)\right](M^{-1})^{\nu}{}_{\mu}\\ &= \left[M\overset{\Upsilon}{\nabla}_{\nu}V\right](M^{-1})^{\nu}{}_{\mu},\end{aligned} \tag{64}$$

$$\begin{aligned}\overset{\Upsilon}{\nabla}_{\mu}W(x) &\xrightarrow{\text{GC}} \overset{\Upsilon}{\nabla}{}'_{\mu}W'(x')\\ &= W'(x')(\overleftarrow{\partial'_{\mu}} - \Upsilon'_{\mu}(x'))\\ &= \left[(WM^{-1})(\overleftarrow{\partial_{\nu}} - M\Upsilon_{\nu}M^{-1} + \partial_{\nu}MM^{-1})\right](M^{-1})^{\nu}{}_{\mu}\\ &= \left[\overset{\Upsilon}{\nabla}_{\nu}WM^{-1}\right](M^{-1})^{\nu}{}_{\mu},\end{aligned} \tag{65}$$

where we have suppressed the dependence on x on the right-hand side and have used the identity $\partial_\mu(MM^{-1}) = \partial_\mu MM^{-1} + M\partial_\mu M^{-1} = 0$. Recall that we are employing the neighboring notation for derivatives as given in Eq. (20). More explicitly, the above transformations read

$$\overset{\Upsilon}{\nabla}_\mu V^\alpha(x) \xrightarrow{\text{GC}} M^\alpha{}_\beta(x)\overset{\Upsilon}{\nabla}_\nu V^\beta(x)(M^{-1})^\nu{}_\mu(x), \tag{66}$$

$$\overset{\Upsilon}{\nabla}_\mu W_\alpha(x) \xrightarrow{\text{GC}} \overset{\Upsilon}{\nabla}_\nu W_\beta(x)(M^{-1})^\beta{}_\alpha(x)(M^{-1})^\nu{}_\mu(x). \tag{67}$$

We may separate the GC-gauge field Υ_μ into symmetric and antisymmetric parts:

$$\Upsilon^\alpha{}_{\beta\mu}(x) = \Upsilon^\alpha{}_{(\beta\mu)}(x) + \Upsilon^\alpha{}_{[\beta\mu]}(x), \tag{68}$$

where the parentheses and square brackets for the indices are defined in Eq. (16). Note that we have mixed the indices β and μ that correspond to an internal gauge index and a spacetime index, respectively, for the case of the ordinary/LL-gauge field. The symmetric and antisymmetric parts in the first and second terms of Eq. (68) have $\frac{(d+1)^2(d+2)}{2}$ and $\frac{d(d+1)^2}{2}$ degrees of freedom, respectively. The number of degrees of freedom of the antisymmetric part $\Upsilon^\alpha{}_{[\beta\mu]}$ is the same as that of the GC conditions (45). This fact suggests that it is redundant for the GC symmetry.

Let us see that this is indeed the case. Under the GC transformation (59), the antisymmetric part of the GC-gauge field transforms homogeneously:

$$\begin{aligned}\Upsilon^\alpha{}_{[\beta\mu]}(x) \xrightarrow{\text{GC}} \Upsilon'^\alpha{}_{[\beta\mu]}(x') &= M^\alpha{}_\gamma \Upsilon^\gamma{}_{\delta\nu}(M^{-1})^\delta{}_{[\beta}(M^{-1})^\nu{}_{\mu]} - \partial_\nu M^\alpha{}_\gamma(M^{-1})^\gamma{}_{[\beta}(M^{-1})^\nu{}_{\mu]} \\ &= M^\alpha{}_\gamma \Upsilon^\gamma{}_{[\delta\nu]}(M^{-1})^{[\delta}{}_\beta(M^{-1})^{\nu]}{}_\mu - \partial_{[\nu}M^\alpha{}_{\gamma]}(M^{-1})^{[\gamma}{}_\beta(M^{-1})^{\nu]}{}_\mu \\ &= M^\alpha{}_\gamma \Upsilon^\gamma{}_{[\delta\nu]}(M^{-1})^\delta{}_\beta(M^{-1})^\nu{}_\mu,\end{aligned} \tag{69}$$

where we have omitted the dependence on x on the right-hand side for simplicity and have used the GC condition (45) in the last step. That is, the GC covariance of the GC-covariant derivative is maintained even if we do not include the antisymmetric part $\Upsilon^\alpha{}_{[\beta\mu]}(x)$. (Though it means that we do not need the antisymmetric part at all in order to realize the GC covariance of the GC-covariant derivative, this argument itself does not prohibit having the antisymmetric part.)

3. Levi-Civita (spin) connection

Conventionally, the Levi-Civita connection $\overset{g}{\Gamma}$ has been used as the GC-gauge field[8]

$$\overset{g}{\Gamma}{}^\alpha{}_{\beta\mu}(x) := \frac{g^{\alpha\gamma}(x)}{2}(-\partial_\gamma g_{\beta\mu}(x) + \partial_\beta g_{\mu\gamma}(x) + \partial_\mu g_{\gamma\beta}(x)), \tag{70}$$

which is the solution to the metricity condition on Υ:

$$\overset{\Upsilon}{\nabla}_\alpha g_{\beta\mu}(x) = 0. \tag{71}$$

By construction, it has only the symmetric part $\overset{g}{\Gamma}{}^\alpha{}_{\beta\mu}(x) = \overset{g}{\Gamma}{}^\alpha{}_{(\beta\mu)}(x)$; recall the discussion in the paragraphs containing Eqs. (68) and (69). The transformation of the Levi-Civita connection can be found, as in any textbook of general relativity, e.g., Ref. [75], to be the same as Eq. (58), or (59):

$$\begin{aligned}\overset{g}{\Gamma}{}^\alpha{}_{\beta\mu}(x) \xrightarrow{\text{GC}} &\left[M(x)\overset{g}{\Gamma}_\nu(x)M^{-1}(x)\right]{}^\alpha{}_\beta\left[M^{-1}(x)\right]{}^\nu{}_\mu + \frac{\partial x'^\alpha}{\partial x^\gamma}\frac{\partial^2 x^\gamma}{\partial x'^\mu \partial x'^\beta} \\ &= \left[M(x)\overset{g}{\Gamma}_\nu(x)M^{-1}(x)\right]{}^\alpha{}_\beta\left[M^{-1}(x)\right]{}^\nu{}_\mu - \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial^2 x'^\alpha}{\partial x^\nu \partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\beta} \\ &= \left[M(x)\overset{g}{\Gamma}_\nu(x)M^{-1}(x)\right]{}^\alpha{}_\beta\left[M^{-1}(x)\right]{}^\nu{}_\mu - (M^{-1})^\nu{}_\mu(x)\partial_\nu M^\alpha{}_\gamma(x)(M^{-1})^\gamma{}_\beta(x) \\ &= \left[M(x)\overset{g}{\Gamma}_\nu(x)M^{-1}(x) - \partial_\nu M(x)M^{-1}(x)\right]{}^\alpha{}_\beta(M^{-1})^\nu{}_\mu(x).\end{aligned} \tag{72}$$

[8]This is the case in supergravity too [50]: "One has four choices: ω or $\omega(e)$ for Lorentz connection, and Γ or $\Gamma(g)$ for the other connection. The choice appropriate for local supersymmetry is ω and $\Gamma(g)$. Any other choice would do as well, but one would need extra complicated terms in the action."

We note that the Levi-Civita connection requires an inverse metric $g^{\mu\nu}$ and hence an inverse vielbein $e_{\mathbf{a}}{}^{\mu}$. Therefore, it cannot be used under the irreversible vierbein postulate imposed on our action at the scale Λ_{G}. That is, the GC-gauge field is absent at Λ_{G} since we do not further introduce it as extra degrees of freedom; see also footnote 8. We will come back to this point below.

Once the Levi-Civita connection $\overset{g}{\Gamma}_{\mu}$ is introduced (in our scenario, it is induced by quantum fluctuations below the scale Λ_{G}), then another LL-gauge field can also be induced, namely, the Levi-Civita spin connection $\overset{e}{\Omega}_{\mu}$:

$$\overset{e}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \coloneqq e^{\mathbf{a}}{}_{\lambda}(x)\overset{g}{\nabla}_{\mu}e_{\mathbf{b}}{}^{\lambda}(x) \coloneqq e^{\mathbf{a}}{}_{\lambda}(x)(\partial_{\mu}e_{\mathbf{b}}{}^{\lambda}(x) + \overset{g}{\Gamma}{}^{\lambda}{}_{\sigma\mu}(x)e_{\mathbf{b}}{}^{\sigma}(x)). \quad (73)$$

It is straightforward to check that the Levi-Civita spin connection $\overset{e}{\Omega}_{\mu}(x)$ transforms in the same way as the LL-gauge field $\omega_{\mu}(x)$ under the LL and GC transformations.

D. Field strengths, Riemann tensor, and $GL(d+1)$

Now we come back to considering the general gravitational gauge fields ω and Υ.

The field strengths for the ordinary and LL-gauge fields are given in the matrix notation, respectively, as[9]

$$\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x) \coloneqq \partial_{\mu}\mathcal{A}_{\nu}(x) - \partial_{\nu}\mathcal{A}_{\mu}(x) + [\mathcal{A}_{\mu}(x), \mathcal{A}_{\nu}(x)], \quad (74)$$

$$\overset{\omega}{\mathcal{F}}_{\mu\nu}(x) \coloneqq \partial_{\mu}\omega_{\nu}(x) - \partial_{\nu}\omega_{\mu}(x) + [\omega_{\mu}(x), \omega_{\nu}(x)]. \quad (75)$$

More explicitly, the LL field strength reads

$$\overset{\omega}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x) = \partial_{\mu}\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) - \partial_{\nu}\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) + \omega^{\mathbf{a}}{}_{\mathbf{c}\mu}(x)\omega^{\mathbf{c}}{}_{\mathbf{b}\nu}(x) - \omega^{\mathbf{a}}{}_{\mathbf{c}\nu}(x)\omega^{\mathbf{c}}{}_{\mathbf{b}\mu}(x). \quad (76)$$

We can rewrite the field strength as a commutator of the covariant derivatives (10) and (27), or (29), on the fundamental representation:

$$\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x) = \left[\overset{\mathcal{A}}{\mathcal{D}}_{\mu}, \overset{\mathcal{A}}{\mathcal{D}}_{\nu}\right], \quad (77)$$

$$\overset{\omega}{\mathcal{F}}_{\mu\nu}(x) = \left[\overset{\omega}{\mathcal{D}}_{\mu}, \overset{\omega}{\mathcal{D}}_{\nu}\right]. \quad (78)$$

It is important that the field strengths reduce to functions (74) and (75) when deriving their covariance

$$\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x) \to U(x)\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x)U^{-1}(x), \quad (79)$$

$$\overset{\omega}{\mathcal{F}}_{\mu\nu}(x) \xrightarrow{\mathrm{LL}} \Lambda(x)\overset{\omega}{\mathcal{F}}_{\mu\nu}(x)\Lambda^{-1}(x) \quad (80)$$

from the covariance of the covariant derivatives (12) and (30), respectively, as follows:

$$\begin{aligned}
\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x)\Psi(x) &= \left[\overset{\mathcal{A}}{\mathcal{D}}_{\mu}, \overset{\mathcal{A}}{\mathcal{D}}_{\nu}\right]\Psi(x) \to U(x)\left[\overset{\mathcal{A}}{\mathcal{D}}_{\mu}, \overset{\mathcal{A}}{\mathcal{D}}_{\nu}\right]\Psi(x) = U(x)\left[\overset{\mathcal{A}}{\mathcal{D}}_{\mu}, \overset{\mathcal{A}}{\mathcal{D}}_{\nu}\right](U^{-1}(x)U(x)\Psi(x)) \\
&= \left(U(x)\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x)U^{-1}(x)\right)U(x)\Psi(x), \\
\overset{\omega}{\mathcal{F}}_{\mu\nu}(x)V(x) &= \left[\overset{\omega}{\mathcal{D}}_{\mu}, \overset{\omega}{\mathcal{D}}_{\nu}\right]V(x) \xrightarrow{\mathrm{LL}} \Lambda(x)\left[\overset{\omega}{\mathcal{D}}_{\mu}, \overset{\omega}{\mathcal{D}}_{\nu}\right]V(x) = \Lambda(x)\left[\overset{\omega}{\mathcal{D}}_{\mu}, \overset{\omega}{\mathcal{D}}_{\nu}\right](\Lambda^{-1}(x)\Lambda(x)V(x)) \\
&= \left(\Lambda(x)\overset{\omega}{\mathcal{F}}_{\mu\nu}(x)\Lambda^{-1}(x)\right)\Lambda(x)\Psi(x). \quad (81)
\end{aligned}$$

Now let us take the commutator of the GC-covariant derivative (60), or (62), on a spacetime-vector field $V(x) = V^{\mu}(x)\partial_{\mu}$ that transforms as a fundamental representation under GC transformation: In the matrix notation,

$$\begin{aligned}
\overset{\Upsilon}{\mathcal{F}}_{\mu\nu}(x)V(x) &\coloneqq \left[\overset{\Upsilon}{\nabla}_{\mu}, \overset{\Upsilon}{\nabla}_{\nu}\right]V(x) \\
&= \Big(\partial_{\mu}\Upsilon_{\nu}(x) - \partial_{\nu}\Upsilon_{\mu}(x) + \left[\Upsilon_{\mu}(x), \Upsilon_{\nu}(x)\right] \\
&\quad + 2I\Upsilon^{\rho}{}_{[\nu\mu]}(x)\overset{\Upsilon}{\nabla}_{\rho}\Big)V(x), \quad (82)
\end{aligned}$$

or more explicitly,

[9]For a compact gauge group such as that introduced in footnote 5, it is more common to use $F_{\mu\nu}(x) = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x) + ig_G[A_{\mu}(x), A_{\nu}(x)]$ that follows from $\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x) = ig_G F^{a}_{\mu\nu}(x)T^{a}$.

$$\overset{\Upsilon}{\mathcal{F}}{}^{\alpha}{}_{\beta\mu\nu}(x)V^{\beta}(x) = (\partial_\mu\Upsilon^{\alpha}{}_{\beta\nu}(x) - \partial_\nu\Upsilon^{\alpha}{}_{\beta\mu}(x) + \Upsilon^{\alpha}{}_{\gamma\mu}(x)\Upsilon^{\gamma}{}_{\beta\nu}(x) - \Upsilon^{\alpha}{}_{\gamma\nu}(x)\Upsilon^{\gamma}{}_{\beta\mu}(x) + 2\delta^{\alpha}_{\beta}\Upsilon^{\rho}{}_{[\nu\mu]}(x)\overset{\Upsilon}{\nabla}_{\rho})V^{\beta}(x). \quad (83)$$

The last term is peculiar to the GC-field strength: The antisymmetric part $\Upsilon^{\rho}{}_{[\nu\mu]}(x)$ is not only in vain in covariantizing the GC-covariant derivative, but also, it is an obstacle to making the GC-field strength a function rather than a differential operator. This fact disfavors an introduction of the extra GC-gauge field Υ_μ with the antisymmetric part. See also the discussion around Eq. (69) for the redundancy of the antisymmetric part.

Let us comment on the relation to the Levi-Civita (spin) connection. It is noteworthy that if the GC-gauge field is identified with the Levi-Civita connection (70),

$$\Upsilon^{\alpha}{}_{\beta\mu} \equiv \overset{g}{\Gamma}{}^{\alpha}{}_{\beta\mu}, \quad (84)$$

the GC-field strength becomes the Riemann tensor itself:

$$\overset{g}{\mathcal{F}}{}^{\alpha}{}_{\beta\mu\nu}(x) = \mathcal{R}^{\alpha}{}_{\beta\mu\nu}(x), \quad (85)$$

where $\overset{g}{\mathcal{F}}_{\mu\nu}(x) \coloneqq \overset{\overset{g}{\Gamma}}{\mathcal{F}}_{\mu\nu}(x)$. This can be shown as follows:

$$\overset{g}{\mathcal{F}}_{\mu\nu}(x)V(x) \coloneqq \left[\overset{g}{\nabla}_\mu, \overset{g}{\nabla}_\nu\right]V(x) = 2\left(\partial_{[\mu} + \overset{g}{\Gamma}_{[\mu}(x)\right)\left(\partial_{\nu]} + \overset{g}{\Gamma}_{\nu]}(x)\right)V(x) = \left(\partial_\mu\overset{g}{\Gamma}_\nu(x) - \partial_\nu\overset{g}{\Gamma}_\mu(x) + \overset{g}{\Gamma}_\mu(x)\overset{g}{\Gamma}_\nu(x) - \overset{g}{\Gamma}_\nu(x)\overset{g}{\Gamma}_\mu(x)\right)V(x), \quad (86)$$

or more explicitly,

$$\left[\overset{g}{\mathcal{F}}_{\mu\nu}(x)\right]{}^{\alpha}{}_{\beta} = \left[\partial_\mu\overset{g}{\Gamma}_\nu(x) - \partial_\nu\overset{g}{\Gamma}_\mu(x) + \overset{g}{\Gamma}_\mu(x)\overset{g}{\Gamma}_\nu(x) - \overset{g}{\Gamma}_\nu(x)\overset{g}{\Gamma}_\mu(x)\right]{}^{\alpha}{}_{\beta},$$

$$\overset{g}{\mathcal{F}}{}^{\alpha}{}_{\beta\mu\nu}(x) = \partial_\mu\overset{g}{\Gamma}{}^{\alpha}{}_{\beta\nu}(x) - \partial_\nu\overset{g}{\Gamma}{}^{\alpha}{}_{\beta\mu}(x) + \overset{g}{\Gamma}{}^{\alpha}{}_{\gamma\mu}(x)\overset{g}{\Gamma}{}^{\gamma}{}_{\beta\nu}(x) - \overset{g}{\Gamma}{}^{\alpha}{}_{\gamma\nu}(x)\overset{g}{\Gamma}{}^{\gamma}{}_{\beta\mu}(x); \quad (87)$$

the right-hand side is nothing but the Riemann tensor. Under the assumption (84), the antisymmetric part of the GC-gauge field $\Upsilon^{\alpha}{}_{\beta\mu}$ does not take part and play any role.

In the same manner, we can define the Riemann tensor from the LL-field strength with the Levi-Civita spin connection, i.e., $\Upsilon^{\alpha}{}_{\beta\mu} \equiv \overset{e}{\Omega}{}^{\alpha}{}_{\beta\mu}$, together with a vielbein and its inverse:

$$e_{\mathbf{a}}{}^{\alpha}(x)e^{\mathbf{b}}{}_{\beta}(x)\overset{e}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x) = \mathcal{R}^{\alpha}{}_{\beta\mu\nu}(x),$$
$$\overset{e}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x) = e^{\mathbf{a}}{}_{\alpha}(x)e_{\mathbf{b}}{}^{\beta}(x)\mathcal{R}^{\alpha}{}_{\beta\mu\nu}(x), \quad (88)$$

where $\overset{e}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x) \coloneqq \overset{\overset{e}{\Omega}}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x)$, namely, in the matrix notation,

$$\overset{e}{\mathcal{F}}_{\mu\nu}(x) \coloneqq \left[\overset{e}{\mathcal{D}}_\mu, \overset{e}{\mathcal{D}}_\nu\right] = 2\left(\partial_{[\mu} + \overset{e}{\Omega}_{[\mu}(x)\right)\left(\partial_{\nu]} + \overset{e}{\Omega}_{\nu]}(x)\right) = \partial_\mu\overset{e}{\Omega}_\nu(x) - \partial_\nu\overset{e}{\Omega}_\mu(x) + \overset{e}{\Omega}_\mu(x)\overset{e}{\Omega}_\nu(x) - \overset{e}{\Omega}_\nu(x)\overset{e}{\Omega}_\mu(x), \quad (89)$$

in which $\overset{e}{\mathcal{D}}_\mu \coloneqq \partial_\mu + \overset{e}{\Omega}_\mu$ [see Eq. (73)], or more explicitly,

$$\left[\overset{e}{\mathcal{F}}_{\mu\nu}(x)\right]{}^{\mathbf{a}}{}_{\mathbf{b}} = \left[\partial_\mu\overset{e}{\Omega}_\nu(x) - \partial_\nu\overset{e}{\Omega}_\mu(x) + \overset{e}{\Omega}_\mu(x)\overset{e}{\Omega}_\nu(x) - \overset{e}{\Omega}_\nu(x)\overset{e}{\Omega}_\mu(x)\right]{}^{\mathbf{a}}{}_{\mathbf{b}},$$

$$\overset{e}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x) = \partial_\mu\overset{e}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) - \partial_\nu\overset{e}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) + \overset{e}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{c}\mu}(x)\overset{e}{\Omega}{}^{\mathbf{c}}{}_{\mathbf{b}\nu}(x) - \overset{e}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{c}\nu}(x)\overset{e}{\Omega}{}^{\mathbf{c}}{}_{\mathbf{b}\mu}(x). \quad (90)$$

Under the irreversible-vielbein postulate, we expect that physically the following scenario takes place: In the action at Λ_G, there is no specific background of the vielbein. The quantum dynamics induces a nontrivial background vielbein $\bar{e}^{\mathbf{a}}{}_{\mu}(x)$ that has its inverse $\bar{e}_{\mathbf{a}}{}^{\mu}(x)$ everywhere, namely, a nondegenerate $\bar{e}^{\mathbf{a}}{}_{\mu}(x)$, on which the expectation value of the LL-gauge field should become the Levi-Civita spin connection:

$$\langle\omega_\mu(x)\rangle_{\bar{e}(x)} \overset{!}{=} \bar{\Omega}_\mu(x), \quad (91)$$

where

$$\bar{\Omega}_\mu(x) \coloneqq \overset{e}{\Omega}_\mu(x)\Big|_{e(x)\to\bar{e}(x)}. \quad (92)$$

This way, our formulation would reproduce the conventional covariant derivative acting on the spinor field in the metric formulation. It suffices therefore that only the LL-field strength exists to construct the Riemann tensor at lower energies below Λ_G if the physical expectation (91) is met. We do not need to prepare the GC-field strength as a source for the Riemann tensor from the beginning at Λ_G.

We comment on the application order of transformations, namely, the GC transformation after a gauge transformation, or the other way around. The GC transformation acts on all spacetime indices, so that gauge transformations are affected by it. In other words, one may think that elements of their transformations do not commute. Fortunately, our definition of the GC transformations (48)–(52) commute with any gauge transformation, while diff defined by the LD transformation does not. This means that the former is given by a direct product "GC × gauge," while the latter is by the semidirect product "GC ⋉ gauge." These facts are discussed in Appendix A 3.

Finally, let us comment on the $GL(d+1)$ theory. If we do not impose the GC conditions (45)–(47), etc., and we regard $M^\mu{}_\nu$ as a general $(d+1)\times(d+1)$ matrix, then the theory becomes a $GL(d+1)$ gauge theory; see, e.g., Refs. [76,77]. This theory might be of interest in itself, but we do not go in this direction and do not introduce the extra GC-gauge field Υ_μ at $\Lambda_{\rm G}$ because of the above-mentioned points: (i) the nonnecessity of its antisymmetric part for the covariance of the GC-covariant derivative, (ii) the GC-field strength becoming a differential operator rather than a function due to the antisymmetric part, and (iii) the nonnecessity as a source for constructing the Riemann tensor.

E. Summary on covariant derivatives

For general LL and GC-gauge fields ω and Υ, respectively, we summarize our notation for the covariant derivatives:

$$\text{LL only}\quad \overset{\omega}{\mathcal{D}}_\mu e^{\mathbf{a}}{}_\nu(x) = \partial_\mu e^{\mathbf{a}}{}_\nu(x) + \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) e^{\mathbf{b}}{}_\nu(x), \tag{93}$$

$$\text{GC only}\quad \overset{\Upsilon}{\nabla}_\mu e^{\mathbf{a}}{}_\nu(x) = \partial_\mu e^{\mathbf{a}}{}_\nu(x) - e^{\mathbf{a}}{}_\lambda(x) \Upsilon^\lambda{}_{\nu\mu}(x), \tag{94}$$

$$\begin{aligned}\text{LL and GC}\quad \overset{\omega,\Upsilon}{\mathscr{D}}_\mu e^{\mathbf{a}}{}_\nu(x) &:= \partial_\mu e^{\mathbf{a}}{}_\nu(x) + \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) e^{\mathbf{b}}{}_\nu(x) - e^{\mathbf{a}}{}_\lambda(x)\Upsilon^\lambda{}_{\nu\mu}(x) \\ &= \overset{\omega}{\mathcal{D}}_\mu e^{\mathbf{a}}{}_\nu(x) - e^{\mathbf{a}}{}_\lambda(x)\Upsilon^\lambda{}_{\nu\mu}(x) \\ &= \overset{\Upsilon}{\nabla}_\mu e^{\mathbf{a}}{}_\nu(x) + \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) e^{\mathbf{b}}{}_\nu(x).\end{aligned} \tag{95}$$

IV. ACTION UNDER IRREVERSIBLE VIERBEIN POSTULATE

In this section, we construct a "tree" action given at a certain scale $\Lambda_{\rm G}$ based on the LL and GC symmetries, i.e., GC × $SO(1,3)$. As discussed in the Introduction, a central assumption at $\Lambda_{\rm G}$ is the irreversible vierbein postulate that forbids the action at $\Lambda_{\rm G}$ to contain an inverse of the vielbein.

In Sec. IV A, we start with the introduction of the Levi-Civita tensor which is independent of the vielbein and inverse vielbein. Then, in Sec. IV B, the definition of the irreversible vierbein postulate is explained, and the action respecting this postulate is shown.

This section fully explains, for the first time, the idea briefly sketched in the preceding Letter [54] to this paper.

A. Levi-Civita tensor

To write down the action, we spell out our notation on the totally antisymmetric tensor, etc. We first introduce the Levi-Civita symbol:

$$\epsilon[\mu_0 \ldots \mu_d] = \begin{cases} 1 & \text{when } (\mu_0, \ldots, \mu_d) \text{ is even permutation of } (0, \ldots, d), \\ -1 & \text{when } (\mu_0, \ldots, \mu_d) \text{ is odd permutation of } (0, \ldots, d), \\ 0 & \text{otherwise,} \end{cases} \tag{96}$$

and similarly,

$$\epsilon[a_0 \ldots a_d] = \begin{cases} 1 & \text{when } (\boldsymbol{a_0}, \ldots, \boldsymbol{a_d}) \text{ is even permutation of } (\mathbf{0}, \ldots, \boldsymbol{d}), \\ -1 & \text{when } (\boldsymbol{a_0}, \ldots, \boldsymbol{a_d}) \text{ is odd permutation of } (\mathbf{0}, \ldots, \boldsymbol{d}), \\ 0 & \text{otherwise.} \end{cases} \tag{97}$$

We write the determinant of the vielbein and metric as

$$\begin{aligned} |e(x)| := \det_{\mathbf{a},\mu} e^{\mathbf{a}}{}_\mu(x) &= \epsilon[a_0 \ldots a_d]\, e^{\boldsymbol{a_0}}{}_0(x) \cdots e^{\boldsymbol{a_d}}{}_d(x) \\ &= \epsilon[\mu_0 \ldots \mu_d]\, e^{\mathbf{0}}{}_{\mu_0}(x) \cdots e^{\boldsymbol{d}}{}_{\mu_d}(x) \\ &= \frac{1}{(d+1)!}\epsilon[a_0 \ldots a_d]\,\epsilon[\mu_0 \ldots \mu_d]\, e^{\boldsymbol{a_0}}{}_{\mu_0} \cdots e^{\boldsymbol{a_d}}{}_{\mu_d}, \end{aligned} \tag{98}$$

$$|g(x)| \coloneqq \det_{\mu,\nu}(x) = \epsilon[\mu_0 \ldots \mu_d] g_{\mu_0 0}(x) \cdots g_{\mu_d d}(x)$$
$$= \epsilon[\nu_0 \ldots \nu_d] g_{0\nu_0}(x) \cdots g_{d\nu_d}(x)$$
$$= \frac{1}{(d+1)!} \epsilon[\mu_0 \ldots \mu_d] \epsilon[\nu_0 \ldots \nu_d] g_{\mu_0\nu_0}(x) \cdots g_{\mu_d\nu_d}(x), \tag{99}$$

where the summation over repeated indices is understood for the Levi-Civita symbol as well. It follows that

$$|e(x)| = \sqrt{-|g(x)|}, \tag{100}$$

where $|g(x)|$ is always negative due to the Lorentzian signature.

From the Levi-Civita symbol, we define the Levi-Civita tensor for the LL transformation,

$$\epsilon_{\boldsymbol{a}_0 \ldots \boldsymbol{a}_d} \coloneqq \epsilon[a_o \ldots a_d], \tag{101}$$

$$\epsilon^{\boldsymbol{a}_0 \ldots \boldsymbol{a}_d} \coloneqq \eta^{\boldsymbol{a}_0 \boldsymbol{b}_0} \cdots \eta^{\boldsymbol{a}_d \boldsymbol{b}_d} \epsilon_{\boldsymbol{b}_0 \ldots \boldsymbol{b}_d} = -\epsilon[a_0 \ldots a_d], \tag{102}$$

and for the GC transformation,

$$\varepsilon_{\mu_0 \ldots \mu_d}(x) \coloneqq |e(x)| \epsilon[\mu_0 \ldots \mu_d], \tag{103}$$

$$\varepsilon^{\mu_0 \ldots \mu_d}(x) \coloneqq g^{\mu_0\nu_0}(x) \cdots g^{\mu_d\nu_d}(x) \varepsilon_{\nu_0 \ldots \nu_d}(x) = -\frac{\epsilon[\mu_0 \ldots \mu_d]}{|e(x)|}. \tag{104}$$

Note that the Lorentzian signature leads to

$$\frac{1}{(d+1)!} \epsilon_{\boldsymbol{a}_0 \ldots \boldsymbol{a}_d} \epsilon^{\boldsymbol{a}_0 \ldots \boldsymbol{a}_d} = -1, \tag{105}$$

$$\frac{1}{(d+1)!} \varepsilon_{\mu_0 \ldots \mu_d}(x) \varepsilon^{\mu_0 \ldots \mu_d}(x) = -1, \tag{106}$$

which follows from the $p = d$ case of more general identities: for $0 \leq p \leq d$,

$$\frac{1}{(p+1)!} \epsilon[\mu_0 \ldots \mu_p \mu_{p+1} \ldots \mu_d] \epsilon[\mu_0 \ldots \mu_p \mu'_{p+1} \ldots \mu'_d] = (d+1-p)! \delta^{\mu'_{p+1}}_{[\mu_{p+1}} \cdots \delta^{\mu'_d}_{\mu_d]}, \tag{107}$$

etc.[10] Using the Levi-Civita tensor, we can write down the volume element in terms of the local coordinate system of each chart:

[10]See footnote 6 for the normalization.

$$\star 1 = \frac{1}{(d+1)!} \varepsilon_{\mu_0 \ldots \mu_d}(x) \mathrm{d}x^{\mu_0} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}$$
$$= |e(x)| \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^d = |e(x)| \mathrm{d}^{d+1}x, \tag{108}$$

where $\star$ is the Hodge dual, which is defined for a p-form $\alpha(x) = \frac{1}{p!} \alpha_{\mu_0 \ldots \mu_{p-1}}(x) \mathrm{d}x^{\mu_0} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p-1}}$ by[11]

$$\star\alpha(x) \coloneqq \frac{1}{(d+1-p)!} (\star\alpha)_{\mu_p \ldots \mu_d}(x) \mathrm{d}x^{\mu_p} \wedge \cdots \wedge \mathrm{d}x^{\mu_d}, \tag{109}$$

with

$$(\star\alpha)_{\mu_p \ldots \mu_d}(x) \coloneqq \frac{1}{p!} \alpha_{\nu_0 \ldots \nu_{p-1}}(x) \varepsilon^{\nu_0 \ldots \nu_{p-1}}{}_{\mu_p \ldots \mu_d}(x). \tag{110}$$

Note that from the vielbein, the LL-gauge field, and its field strength, we can construct GC-scalar 1- and 2-form fields such that

$$e^{\mathbf{a}}(x) \coloneqq e^{\mathbf{a}}{}_\mu(x) \mathrm{d}x^\mu, \tag{111}$$

$$\omega^{\mathbf{a}}{}_{\mathbf{b}}(x) \coloneqq \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \mathrm{d}x^\mu, \tag{112}$$

$$\overset{\omega}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}}(x) \coloneqq \frac{1}{2} \overset{\omega}{\mathcal{F}}{}^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}(x) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu. \tag{113}$$

B. Irreversible vierbein and action

Let us now construct an action invariant under GC $\times$ $SO(1,3)$ symmetry. Hereafter, we work in the $d = 3$ spatial dimensions, assuming that it is already settled down to be so at Λ_{G}. We call the vielbein for the $d + 1 = 4$ spacetime dimensions the vierbein, accordingly.

The formulation of the irreversible vierbein postulate starts by imposing regularity under the limit of any zero eigenvalues $\lambda_a \to 0$ of the vierbein among four eigenvalues in four-dimensional spacetime. Obviously, in such a case, the inverse vierbein cannot be defined. In other words, the inverse vierbein contains divergences. Then, the irreversible

[11]Or else, one may first define

$$\star(\mathrm{d}x^{\mu_0} \wedge \cdots \mathrm{d}x^{\mu_{p-1}}) \coloneqq \frac{1}{(d+1-p)!} \varepsilon^{\mu_0 \ldots \mu_{p-1}}{}_{\nu_p \ldots \nu_d}(x) \mathrm{d}x^{\nu_p} \wedge \cdots \wedge \mathrm{d}x^{\nu_d}$$

so that

$$\star\alpha(x) \coloneqq \frac{1}{p!} \alpha_{\mu_0 \ldots \mu_{p-1}}(x) \star(\mathrm{d}x^{\mu_0} \wedge \cdots \mathrm{d}x^{\mu_{p-1}}).$$

vierbein postulate at Λ_G states that the action at Λ_G does not diverge even for the (not necessarily simultaneous) zero eigenvalue limit of vierbein. We call this kind of limit the degenerate limit [55,56,73].

There are some cases where, even when a term is apparently written down using the inverse vierbein, the irreversible vierbein postulate does not forbid such a term in the degenerate limit. An important observation for this is that some inverse vierbeins can be absorbed into the determinant $|e(x)|$ from the volume element. More specifically, from the identities (107) and

$$|e(x)|\epsilon[\mu\nu\rho\sigma] = \epsilon[\mathbf{abcd}]\, e^{\mathbf{a}}{}_{\mu}(x) e^{\mathbf{b}}{}_{\nu}(x) e^{\mathbf{c}}{}_{\rho}(x) e^{\mathbf{d}}{}_{\sigma}(x), \tag{114}$$

we obtain

$$|e(x)| e_{\mathbf{a}}{}^{\mu}(x) = \frac{1}{3!}\epsilon[\mathbf{abcd}]\epsilon[\mu\nu\rho\sigma] e^{\mathbf{b}}{}_{\nu}(x) e^{\mathbf{c}}{}_{\rho}(x) e^{\mathbf{d}}{}_{\sigma}(x), \tag{115}$$

$$|e(x)| e_{[\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}]}{}^{\nu}(x) = \frac{1}{2}\epsilon[\mathbf{abcd}]\epsilon[\mu\nu\rho\sigma] e^{\mathbf{c}}{}_{\rho}(x) e^{\mathbf{d}}{}_{\sigma}(x), \tag{116}$$

$$|e(x)| e_{[\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}}{}^{\nu}(x) e_{\mathbf{c}]}{}^{\rho}(x) = \epsilon[\mathbf{abcd}]\epsilon[\mu\nu\rho\sigma] e^{\mathbf{d}}{}_{\sigma}(x), \tag{117}$$

$$|e(x)| e_{[\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}}{}^{\nu}(x) e_{\mathbf{c}}{}^{\rho}(x) e_{\mathbf{d}]}{}^{\sigma}(x) = \epsilon[\mathbf{abcd}]\epsilon[\mu\nu\rho\sigma]. \tag{118}$$

Though the left-hand sides of these equations appear to have the inverse vierbeins, the right-hand sides do not. Only these combinations of the inverse vierbeins can be used to write down the action at Λ_G without contradicting the irreversible vierbein postulate.

Let us now write down the action explicitly. First, it turns out that the kinetic terms for the GC-scalar and vector fields contain the inverse vierbeins, all of which cannot be simultaneously absorbed into the volume element via Eqs. (115)–(118). They contain the inverse metric $g^{\mu\nu}$ that is symmetric for its indices (see Appendix C for more a explicit discussion)[12]:

$$S_{\mathrm{boson}} = \int \mathrm{d}^4x |e(x)| \Bigg[-\frac{1}{2} g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + \frac{1}{2g_G^2} g^{\mu\rho}(x) g^{\nu\sigma}(x) \mathrm{tr}\Big(\overset{\mathcal{A}}{\mathcal{F}}_{\mu\nu}(x)\overset{\mathcal{A}}{\mathcal{F}}_{\rho\sigma}(x)\Big)\Bigg]. \tag{119}$$

Therefore, these terms are forbidden, and thus the GC-scalar and vector fields are not dynamical at Λ_G. The kinetic terms for vierbein $e^{\mathbf{a}}{}_{\mu}$ and LL-gauge fields $\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}$ cannot be introduced due to the same reason.

On the other hand, the spinor kinetic term is consistent with the irreversible vierbein postulate because it contains only a single inverse vierbein, and thus we can use Eq. (115)[13]:

$$\begin{aligned} S_{\mathrm{spinor}} &= \int \mathrm{d}^4x |e(x)| \Bigg[-\bar\psi(x) e_{\mathbf{a}}{}^{\mu}(x)\gamma^{\mathbf{a}} \times \left(\partial_\mu + \frac{1}{2}\omega_{\mathbf{bc}\mu}(x)\sigma^{\mathbf{bc}}\right)\psi(x)\Bigg] \\ &= \int \mathrm{d}^4x \Bigg[-\frac{1}{3!}\epsilon[\mathbf{abcd}]\epsilon[\mu\nu\rho\sigma] e^{\mathbf{b}}{}_{\nu}(x) e^{\mathbf{c}}{}_{\rho}(x) e^{\mathbf{d}}{}_{\sigma}(x) \times \bar\psi(x)\gamma^{\mathbf{a}}\left(\partial_\mu + \frac{1}{2}\omega_{\mathbf{a'b'}\mu}(x)\sigma^{\mathbf{a'b'}}\right)\psi(x)\Bigg]. \end{aligned} \tag{120}$$

Unlike the ordinary gauge theory, the LL symmetry has a fundamental representation that is a spacetime vector, the vierbein. It allows one to write down an LL-invariant term constructed from a single field strength:

[12]See footnote 5 for the sign of the gauge kinetic term.

[13]One may regard the action (120) as $\propto \int \epsilon[\mathbf{abcd}] e^{\mathbf{a}}(x) \wedge e^{\mathbf{b}}(x) \wedge e^{\mathbf{c}}(x) \wedge \Delta^{\mathbf{d}}(x)$, where $\Delta^{\mathbf{a}}{}_{\mu}(x) \coloneqq \bar\psi(x)\gamma^{\mathbf{a}}\overset{\omega}{\mathcal{D}}_\mu\psi(x)$. This is nothing but a replacement from the action (122), being $\propto \int \epsilon[\mathbf{abcd}] e^{\mathbf{a}}(x) \wedge e^{\mathbf{b}}(x) \wedge e^{\mathbf{c}}(x) \wedge e^{\mathbf{d}}(x)$, of a single vierbein 1-form: $e^{\mathbf{d}}(x) \to \Delta^{\mathbf{d}}(x)$. In principle, one may replace any vierbein $e^{\mathbf{a}}{}_{\mu}(x)$ by $\Delta^{\mathbf{a}}{}_{\mu}(x)$ without contradicting the irreversible vierbein postulate. When one replaces all of the four vierbeins to the fermion bilinear in the cosmological constant term (122), one obtains the action for the spinor gravity $\propto \int \epsilon[\mathbf{abcd}]\Delta^{\mathbf{a}}(x) \wedge \Delta^{\mathbf{b}}(x) \wedge \Delta^{\mathbf{c}}(x) \wedge \Delta^{\mathbf{d}}(x)$ [68,78]. In this paper, we restrict ourselves to the lowest-derivative terms up to single $\Delta^{\mathbf{a}}{}_{\mu}(x)$. Further generalizations will be presented in a separate publication.

$$S_{\rm LL} = \int \mathrm{d}^4x |e(x)| \left[\frac{M_{\rm P}^2}{2} e_{[\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}]}{}^{\nu}(x) \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x)\right]$$
$$= \int \mathrm{d}^4x \left[\frac{M_{\rm P}^2}{4} \epsilon[\mathbf{abcd}] \epsilon[\mu\nu\rho\sigma] e^{\mathbf{c}}{}_{\rho}(x) e^{\mathbf{d}}{}_{\sigma}(x) \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x)\right]. \quad (121)$$

This term has become compatible with the irreversible vierbein postulate thanks to Eq. (116). Note that as mentioned in the paragraph containing Eq. (92), it is not necessary to introduce the term with the field strength corresponding to the GC transformation $\overset{\Upsilon}{\mathcal{F}}$.

Finally, one can also write down the cosmological constant term:

$$S_{\rm cc} = \int \mathrm{d}^4x |e(x)| \Lambda_{\rm cc}. \quad (122)$$

Barring the topological terms (see discussion below) as well as the higher-derivative terms (see footnote 13), the terms (120)–(122) are the only combinations that are consistent with the irreversible vierbein postulate:

$$S_{\Lambda_{\rm G}} = \int \mathrm{d}^4x |e(x)| \left[-Z_\psi \bar{\psi}(x) e_{\mathbf{a}}{}^{\mu}(x) \gamma^{\mathbf{a}} \times \left(\partial_\mu + \frac{1}{2}\omega_{\mathbf{bc}\mu}(x)\sigma^{\mathbf{bc}}\right)\psi(x) + X_\omega \frac{M_{\rm P}^2}{2} e_{[\mathbf{a}}{}^{\mu}(x) e_{\mathbf{b}]}{}^{\nu}(x) \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x) - V\right], \quad (123)$$

where Z_ψ, X_ω, and V are arbitrary functions of spacetime scalars at x constructed by matter fields, say, $\phi(x)$, $\bar{\psi}(x)\psi(x)$, etc. For example, V includes the mass term for spinor $M_\psi \bar{\psi}(x)\psi(x)$ and the cosmological constant $\Lambda_{\rm cc}$ as well as the ordinary scalar potential. Note that the kinetic term for the Rarita-Schwinger field, which is a spin-$3/2$ field, is also compatible with the degenerate limit (see Appendix C 3). In this work, we do not take this into account.

Various combinations of fields can be contracted with the LL metric $\eta_{\mathbf{ab}}$ and the totally antisymmetric LL tensor $\epsilon[\mathbf{abcd}]$ to yield topological terms of the action, which are summarized in Appendix D. In general, one may multiply, on these "topological" terms, arbitrary functions of GC scalars such as Z_ψ, X_ω, and V in Eq. (123) so that they become dynamical (nontopological). Since all such interactions are higher dimensional, we neglect them in this paper. The inclusion of these terms might be of interest in itself, which we leave for future study.

Finally, we stress again the reason why we impose the irreversible vierbein postulate at the tree level. A typical criticism may be as follows: The existence of inverse vierbein in the tree action is harmless since terms with inverse vierbeins behave as $e^{-\mathcal{O}(e^{-1})} \to 0$ for $e \to 0$ within the path integral. On the other hand, the Standard Model of particle physics assumes that the action does not have inverse power of the fields such as $1/H^\dagger H$, where H is the Higgs field, even though such terms can be perfectly consistent with all the gauge and spacetime symmetries. The absence of inverse power is particularly noteworthy in the effective field theory picture because such negative-power terms are more relevant than the normal ones toward the IR direction. We can interpret this as the requirement of the existence of the weak-field limit $H \to 0$ for the action such that the symmetric phase $\langle H \rangle = 0$ is well defined.[14]

The irreversible vierbein postulate introduces a well-defined symmetric phase in our quantum-gravity framework. Conventional quantum field theories, like the Standard Model, also assume the existence of the symmetric phase. Our framework facilitates the exploration of quantum spacetime dynamics where the spacetime metric $g_{\mu\nu}$ approaches zero. By excluding inverse vierbeins, which are undefined in this degenerate limit, the postulate offers a novel approach to studying quantum gravitational phenomena under extreme conditions. This method extends beyond mathematical convenience, providing a framework that enhances our understanding of spacetime in scenarios such as near singularities or strong gravitational fields, and may offer valuable insights into phenomena like the early Universe and black hole interiors.

V. LOCAL LORENTZ AND GENERAL-COORDINATE TRANSFORMATIONS FOR BACKGROUND FIELDS

One of thte main purposes in this paper is to demonstrate the generation of nontrivial background fields due to quantum effects in four-dimensional spacetimes. This will be done in Sec. VI. In this section, assuming that background vielbein and LL-gauge fields are induced in arbitrary spacetime dimensions, we discuss their transformation laws and covariance. Those are important for understanding of the low-energy effective theory from the action (123). After these summary reviews, in Sec. V C, we fully explain the idea briefly sketched in the preceding Letter [54].

A. Invertible background vielbein and Levi-Civita (spin) connection

To begin with, we introduce a certain gravitational background $\bar{\Phi} = (\bar{e}, \bar{\omega})$, while we do not assume a classical background for the matter fields for simplicity: $\bar{\Psi} = (\bar{\phi}, \bar{\psi}, \bar{A}_\mu) = 0$.

[14] The weak-field limit $H \to 0$ here is different from that in the gravitational literature (see, e.g., Ref. [79]) in the sense that the latter means the limit of zero fluctuation around a background, that is, $\delta g_{\mu\nu} \to 0$ for $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$. For the former, the whole Higgs $H = \bar{H} + \delta H$ goes to zero rather than $H \to \bar{H}$. The existence of limit $H \to \bar{H} \neq 0$ cannot forbid the negative power such as $1/H^\dagger H$.

Here, an important assumption is that the background vielbein field $\bar{e}^{\mathbf{a}}{}_{\mu}(x)$ is invertible that allows the inverse background vielbein $\bar{e}_{\mathbf{a}}{}^{\mu}(x)$ and the inverse background metric

$$\bar{g}^{\mu\nu}(x) \coloneqq \eta^{\mathbf{ab}}\bar{e}_{\mathbf{a}}{}^{\mu}(x)\bar{e}_{\mathbf{b}}{}^{\nu}(x), \tag{124}$$

where the background vielbein is defined to satisfy

$$\bar{e}_{\mathbf{a}}{}^{\mu}(x)\bar{e}^{\mathbf{a}}{}_{\nu}(x) = \delta^{\mu}_{\nu}, \qquad \bar{e}_{\mathbf{a}}{}^{\mu}(x)\bar{e}^{\mathbf{b}}{}_{\mu}(x) = \delta^{\mathbf{b}}_{\mathbf{a}},$$
$$\bar{g}^{\mu\nu}(x)\bar{g}_{\nu\rho}(x) = \delta^{\mu}_{\rho}. \tag{125}$$

In general, the full vielbein and metric fields

$$e^{\mathbf{a}}{}_{\mu}(x) = \bar{e}^{\mathbf{a}}{}_{\mu}(x) + \mathfrak{e}^{\mathbf{a}}{}_{\mu}(x), \tag{126}$$

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + \mathfrak{g}_{\mu\nu}(x) \tag{127}$$

will also become invertible, where $\mathfrak{e}$ and $\mathfrak{g}$ represent their quantum fluctuations, respectively. However, we hereafter raise and lower the spacetime indices by the background vielbein and/or metric. In particular, we can now give

$$A_{\mathbf{a}}(x) \coloneqq \bar{e}_{\mathbf{a}}{}^{\mu}(x)A_{\mu}(x); \tag{128}$$

recall the argument below Eq. (33).

The invertible background vielbein also allows us to write down the background Levi-Civita (spin) connection:

$$\bar{\Gamma}^{\mu}{}_{\rho\sigma} \coloneqq \frac{\bar{g}^{\mu\nu}}{2}(-\partial_{\nu}\bar{g}_{\rho\sigma} + \partial_{\rho}\bar{g}_{\sigma\nu} + \partial_{\sigma}\bar{g}_{\nu\rho}), \tag{129}$$

$$\bar{\Omega}^{\mathbf{a}}{}_{\mathbf{b}\mu} \coloneqq \bar{e}^{\mathbf{a}}{}_{\lambda}\overline{\nabla}_{\mu}\bar{e}_{\mathbf{b}}{}^{\lambda} = \bar{e}^{\mathbf{a}}{}_{\lambda}\partial_{\mu}\bar{e}_{\mathbf{b}}{}^{\lambda} + \bar{\Gamma}^{\mathbf{a}}{}_{\mathbf{b}\mu}, \tag{130}$$

where

$$\overline{\nabla}_{\mu}\bar{e}_{\mathbf{b}}{}^{\lambda} \coloneqq \partial_{\mu}\bar{e}_{\mathbf{b}}{}^{\lambda} + \bar{\Gamma}^{\lambda}{}_{\sigma\mu}\bar{e}_{\mathbf{b}}{}^{\sigma}, \tag{131}$$

$$\bar{\Gamma}^{\mathbf{a}}{}_{\mathbf{b}\mu} \coloneqq \bar{e}^{\mathbf{a}}{}_{\lambda}\bar{\Gamma}^{\lambda}{}_{\sigma\mu}\bar{e}_{\mathbf{b}}{}^{\sigma}, \tag{132}$$

etc. Note that $\bar{\Gamma}$ and $\bar{\Omega}$ are solely made of the vielbein and its inverse. In the previous language,

$$\bar{\Gamma}^{\mu}{}_{\rho\sigma}(x) = \overset{\bar{g}}{\Gamma}{}^{\mu}{}_{\rho\sigma}(x), \qquad \bar{\Omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) = \overset{\bar{e}}{\Omega}{}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x). \tag{133}$$

The background spin connection $\bar{\Omega}$ transforms the same as the background LL-gauge field $\bar{\omega}$ under the background LL and GC transformations.

B. Background covariance

For the given background field of the gravitational fields $\bar{\Phi}$, we define the following LL-background-covariant (only) derivatives denoted by $\bar{\mathcal{D}}_{\mu}$ for matter fields

$$\bar{\mathcal{D}}_{\mu}\phi(x) \coloneqq \partial_{\mu}\phi(x), \tag{134}$$

$$\bar{\mathcal{D}}_{\mu}\psi(x) \coloneqq \partial_{\mu}\psi(x) + \frac{\bar{\omega}_{\mathbf{ab}\mu}(x)}{2}\Sigma^{\mathbf{ab}}\psi(x), \tag{135}$$

$$\bar{\mathcal{D}}_{\mu}A^{\mathbf{a}}(x) \coloneqq \partial_{\mu}A^{\mathbf{a}}(x) + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)A^{\mathbf{b}}(x), \tag{136}$$

and for gravitational fields

$$\bar{\mathcal{D}}_{\mu}e^{\mathbf{a}}{}_{\nu}(x) \coloneqq \partial_{\mu}e^{\mathbf{a}}{}_{\nu}(x) + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)e^{\mathbf{b}}{}_{\nu}(x), \tag{137}$$

$$\bar{\mathcal{D}}_{\mu}\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) \coloneqq \partial_{\mu}\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{c}\mu}(x)\omega^{\mathbf{c}}{}_{\mathbf{b}\nu}(x) - \omega^{\mathbf{a}}{}_{\mathbf{c}\nu}(x)\bar{\omega}^{\mathbf{c}}{}_{\mathbf{b}\mu}(x). \tag{138}$$

We also define the LL-and-GC-background-covariant derivative $\bar{\mathscr{D}}_{\mu}$. It acts the same as $\bar{\mathcal{D}}_{\mu}$ on the matter fields without the spacetime indices Ψ,

$$\bar{\mathscr{D}}_{\mu}\Psi(x) \coloneqq \bar{\mathcal{D}}_{\mu}\Psi(x), \tag{139}$$

whereas on the gravitational fields,[15]

$$\begin{aligned}\bar{\mathscr{D}}_{\mu}e^{\mathbf{a}}{}_{\nu}(x) &= \bar{\mathcal{D}}_{\mu}e^{\mathbf{a}}{}_{\nu}(x) - e^{\mathbf{a}}{}_{\lambda}(x)\bar{\Upsilon}^{\lambda}{}_{\nu\mu}(x)\\ &= \partial_{\mu}e^{\mathbf{a}}{}_{\nu}(x) + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)e^{\mathbf{b}}{}_{\nu}(x) - e^{\mathbf{a}}{}_{\lambda}(x)\bar{\Upsilon}^{\lambda}{}_{\nu\mu}(x),\\ \bar{\mathscr{D}}_{\mu}\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) &= \bar{\mathcal{D}}_{\mu}\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) - \omega^{\mathbf{a}}{}_{\mathbf{b}\lambda}(x)\bar{\Upsilon}^{\lambda}{}_{\nu\mu}(x)\\ &= \partial_{\mu}\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{c}\mu}(x)\omega^{\mathbf{c}}{}_{\mathbf{b}\nu}(x)\\ &\quad - \omega^{\mathbf{a}}{}_{\mathbf{c}\nu}(x)\bar{\omega}^{\mathbf{c}}{}_{\mathbf{b}\mu}(x) - \omega^{\mathbf{a}}{}_{\mathbf{b}\lambda}(x)\bar{\Upsilon}^{\lambda}{}_{\nu\mu}(x).\end{aligned}$$

So far, $\bar{\omega}$ is pretty much unconstrained.[16] In this paper, we assume that the background vielbein should obey the metricity

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[15]For a given background $\bar{e}$ and $\bar{\omega}$, we may always construct a background GC-gauge field

$$\bar{\gamma}^{\lambda}{}_{\nu\mu}(x) \coloneqq \bar{\omega}^{\lambda}{}_{\nu\mu}(x) - \bar{e}^{\mathbf{c}}{}_{\nu}(x)\partial_{\mu}\bar{e}_{\mathbf{c}}{}^{\lambda}(x)$$

that is defined to satisfy the metricity:

$$\overset{\bar{\omega},\bar{\gamma}}{\mathscr{D}}_{\mu}\bar{e}^{\mathbf{a}}{}_{\nu}(x) = 0.$$

We can explicitly check that $\bar{\gamma}^{\lambda}{}_{\nu\mu}$ is invariant under the background LL transformation.

[16]We might end up with the following expression:

$$\bar{\mathscr{D}}_{\mu}\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\nu} = (\partial_{\mu}\bar{\omega}^{\mathbf{a}}{}_{\mathbf{bc}})\bar{e}^{\mathbf{c}}{}_{\nu} + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{c}\mu}\bar{\omega}^{\mathbf{c}}{}_{\mathbf{b}\nu} - \bar{\omega}^{\mathbf{a}}{}_{\mathbf{c}\nu}\bar{\omega}^{\mathbf{c}}{}_{\mathbf{b}\mu};$$

see, e.g., Ref. [49]. Using $\bar{\mathscr{D}}\,\bar{e} = 0$, we may rewrite

$$\bar{\mathscr{D}}_{\mu}\bar{\omega}^{\mathbf{a}}{}_{\mathbf{bc}} = \partial_{\mu}\bar{\omega}^{\mathbf{a}}{}_{\mathbf{bc}} + \bar{\omega}^{\mathbf{a}}{}_{\mathbf{d}\mu}\bar{\omega}^{\mathbf{d}}{}_{\mathbf{bc}} - \bar{\omega}^{\mathbf{a}}{}_{\mathbf{dc}}\bar{\omega}^{\mathbf{d}}{}_{\mathbf{b}\mu}.$$

In the language of differential forms,

$$\bar{\mathscr{D}} \coloneqq \mathrm{d}x^{\mu}\bar{\mathscr{D}}_{\mu}, \qquad \bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}}(x) \coloneqq \bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\nu}(x)\mathrm{d}x^{\nu},$$
$$\mathrm{d} \coloneqq \mathrm{d}x^{\mu}\partial_{\mu}, \qquad \bar{e}^{\mathbf{c}}(x) \coloneqq \bar{e}^{\mathbf{c}}{}_{\nu}(x)\mathrm{d}x^{\nu}.$$

This can be written as

$$\bar{\mathscr{D}}\,\bar{\omega} = \mathrm{d}\bar{\omega} \wedge \bar{e} + 2\bar{\omega} \wedge \bar{\omega}.$$

$$\bar{\mathscr{D}}_\mu \bar{e}^{\mathbf{a}}{}_\nu(x) = 0, \tag{140}$$

the background LL-gauge field $\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)$ be Eq. (91), and the background GC connection be the Levi-Civita one

$$\bar{\Gamma}^\lambda{}_{\nu\mu}(x) = \overset{g}{\Gamma}{}^\lambda{}_{\nu\mu}(x)\Big|_{g=\bar{g}}; \tag{141}$$

recall Eq. (70).

C. Global background Lorentz invariance after spontaneous symmetry breaking

For a general background $\bar{e}^{\mathbf{a}}{}_\mu$, there remains a local $SO(1,3) \times \mathrm{GC}$ background symmetry:

$$\bar{\phi}(x) \overset{SO(1,3)\times\mathrm{GC}}{\longrightarrow} \bar{\phi}'(x'), \tag{142}$$

$$\bar{\psi}(x) \overset{SO(1,3)\times\mathrm{GC}}{\longrightarrow} \bar{\psi}'(x') = S(\Lambda(x'))\bar{\psi}(x'), \tag{143}$$

$$\bar{\mathcal{A}}_\mu(x) \overset{SO(1,3)\times\mathrm{GC}}{\longrightarrow} \bar{\mathcal{A}}'_\mu(x') = \bar{\mathcal{A}}_\nu(x)\frac{\partial x^\nu}{\partial x'^\mu}, \tag{144}$$

and

$$\bar{e}^{\mathbf{a}}{}_\mu(x) \overset{SO(1,3)\times\mathrm{GC}}{\longrightarrow} \bar{e}'^{\mathbf{a}}{}_\mu(x') = \Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x)\bar{e}^{\mathbf{b}}{}_\nu(x)(M^{-1})^\nu{}_\mu(x), \tag{145}$$

$$\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) \overset{SO(1,3)\times\mathrm{GC}}{\longrightarrow} \bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}(x') = \left[\Lambda(x)\bar{\omega}_\nu(x)\Lambda^{-1}(x) - (\partial_\nu\Lambda(x))\Lambda^{-1}(x)\right]^{\mathbf{a}}{}_{\mathbf{b}}(M^{-1})^\nu{}_\mu(x). \tag{146}$$

The Lorentz transformation under the global $SO(1,3)$ is nothing but an accidental symmetry arising only when we take the flat background $\bar{e}^{\mathbf{a}}{}_\mu = \delta^{\mathbf{a}}_\mu$ for which $\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu} = 0$. That is, the gobal $SO(1,3)$ is the "diagonal subgroup": From Eq. (145), one has[17]

$$\delta^{\mathbf{a}}_\mu \to \Lambda^{\mathbf{a}}{}_{\mathbf{b}}\delta^{\mathbf{b}}_\nu(\Lambda^{-1})^\nu{}_\mu = \delta^{\mathbf{a}}_\mu, \tag{147}$$

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu \quad \text{(this is mere a reparametrization)}, \tag{148}$$

so to say

$$SO(1,3) \times \mathrm{GC} \to SO(1,3)_{\mathrm{diag}}. \tag{149}$$

Hence, under this transformation in the diagonal subgroup, the kinetic term of spinors transforms as usual, even though we assign them only a trivial representation under GC.

[17] See the discussion around Eq. (B5) in Appendix B for the reduction of $(M^{-1})^\nu{}_\mu(x) \to (\Lambda^{-1})^\nu{}_\mu$.

VI. DYNAMICAL GENERATION OF FLAT SPACETIME FROM A SPINOR LOOP

In this section, we derive the effective potential for a vierbein background field, assuming it to be a flat spacetime background, and then demonstrate that indeed a nonvanishing flat vierbein background is induced by quantum effects of the fermion field.

As emphasized in the preceding Letter [54], both the vierbein and LL-gauge fields are auxiliary at Λ_{G}, and both of them are shown to acquire the kinetic terms below Λ_{G}. This situation is in accordance with the emergence of the hidden local symmetry applied in QCD; see, e.g., Ref. [80] for a review. Now we show the effective potential for the vierbein as a (linearly realized) Higgs field [68,73,81–83].

A. Effective action for conformal mode

We study now the dynamical symmetry breaking of the LL symmetry in four-dimensional spacetime. A central object for the observation of such a symmetry breaking is the effective potential for the vierbein field. To obtain it, we assume a background field for vierbein $\bar{e}^{\mathbf{a}}{}_\mu$ and investigate the effective potential. The degenerate limit would enforce such a minimum to be located at $\bar{e}^{\mathbf{a}}{}_\mu = 0$. What we want to see in this section is whether quantum effects generate a nontrivial expectation value of $\bar{e}^{\mathbf{a}}{}_\mu$ or not. In this section, we consider the quantum effects of the spinor fields at the one-loop level on the effective potential for the vierbein, while we deal with the vierbein and the LL-gauge fields as classical fields. The one-loop approximation might be justified by a large number of spinor degrees of freedom in the SM that is 90 without including the right-handed neutrinos, whereas we do not exclude the possibility of large effects from other sectors; we will come back to this point later.

We start with separating the vierbein field into the background and fluctuation as in Eq. (126). For simplicity, we concentrate on a constant background field in this paper. At the tree level of our action (123), the potential of the (constant) vierbein field is simply given by

$$V_{\mathrm{tree}}(\bar{e}) = \Lambda_{\mathrm{cc}}|\bar{e}|. \tag{150}$$

Here, we suppose that the kinetic term of the vierbein, which will be generated dynamically below Λ_{G}, will take a "correct" (negative) sign so that the action is given by $S_{\mathrm{eff}} = \int \mathrm{d}^4x\left[-(\partial_\mu\bar{e}^{\mathbf{a}}{}_\nu)^2 - V_{\mathrm{eff}}(\bar{e})\right]$. This assumption will be discussed at the end of this section. For a negative cosmological constant $\Lambda_{\mathrm{cc}} < 0$, we have an unbounded potential, and $|\bar{e}| = 0$ is an unstable extremum, while for $\Lambda_{\mathrm{cc}} > 0$, the potential is bounded and has a minimum at $\bar{e} = 0$. The former induces the spacetime background already at the tree level. How about the latter?

We next consider quantum corrections to the effective potential. At the level of the action (123) at Λ_{G}, only the spinor fields are dynamical and give the leading effects on the effective potential. Here, we compute the effective potential for an assumed background vierbein field value $\bar{e}^{\mathbf{a}}{}_{\mu}(x)$, taking into account the spinor one-loop correction. In order to evaluate the spinor action concretely, let us here assume a flat spacetime background, i.e., we parametrize

$$\bar{e}^{\mathbf{a}}{}_{\mu} = C\delta^{\mathbf{a}}_{\mu}, \tag{151}$$

where C is a dimensionless constant. This parametrization is a special case in that all eigenvalues λ_a of the vierbein take the same value, namely, $\bar{e}^{\mathbf{a}}{}_{\mu} = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = \mathrm{diag}(C, C, C, C)$. For such a flat background, the equation of motion may entail $\bar{\omega}_{\mathbf{ab}\mu} = 0$.

Hence, the point $C = 0$ corresponds to the degenerate limit for the eigenvalues, and thus is identified with a "symmetric phase" of GC $\times\, SO(1,3)$ in analogy to the ordinary Higgs mechanism [81]. Needless to say, the point $C = 0$ has no background spacetime at all and is intractable. Our strategy to handle this "symmetric" point to compare with the broken phase[18] $C \neq 0$ is first computing the effective potential for $C > 0$ and then examining the limit $C \searrow 0$.

Let us derive the effective potential for C arising from the spinor loop. The assumption (151) for the background field gives $|\bar{e}| = C^4$ and $\bar{e}_{\mathbf{a}}{}^{\mu} = C^{-1}\delta^{\mu}_{\mathbf{a}}$. In order to obtain the one-loop effects from a spinor under the background (151), it suffices to take its quadratic terms:

$$S_{\mathrm{kin}} = \int \mathrm{d}^4x \left[-Z_\psi C^3 \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - C^4 M_\psi \bar{\psi}(x)\psi(x)\right], \tag{152}$$

where we have included the spinor mass term coming from V and have written $\gamma^\mu \coloneqq \delta^{\mu}_{\mathbf{a}}\gamma^{\mathbf{a}}$.[19] Now we redefine the spinor field as $\psi(x) \to Z_\psi^{-1/2} C^{-3/2}\psi(x)$:

$$S_{\mathrm{kin}} = \int \mathrm{d}^4x \left[-\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - C m_\psi \bar{\psi}(x)\psi(x)\right], \tag{153}$$

where $m_\psi \coloneqq M_\psi / Z_\psi$.

Integrating out the spinor field yields the effective action

$$\Gamma_{\mathrm{eff}}(C) = -\int \mathrm{d}^4x C^4 \Lambda_{\mathrm{cc}} - i\mathrm{Tr}\log\left[-\gamma^\mu\partial_\mu - m_\psi C\right], \tag{154}$$

where Tr acts on all internal degrees of freedom involved in the spinor field, e.g., eigenvalues of the covariant derivative, spinor space, and so on. The second term in Eq. (154) may give a nontrivial form of the effective potential.

One can perform the Fourier transformation to obtain

$$\Gamma_{\mathrm{eff}}(C) = -\int \mathrm{d}^4x C^4 \left[\Lambda_{\mathrm{cc}} + i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\log\left[-i\gamma^\mu p_\mu - m_\psi C\right]\right], \tag{155}$$

where we have taken the spacetime volume from the momentum-space delta function $\delta^4(0)$ in the trace as $(2\pi)^4\delta^4(0) = C^4\int \mathrm{d}^4x$, which can be naturally understood by first performing the heat-kernel expansion and then taking the limit (151). Thus, the effective potential for C under the flat background (151) is given by

$$V_{\mathrm{eff}}(C) = -\frac{\Gamma_{\mathrm{eff}}(C)}{\int \mathrm{d}^4x} = \Lambda_{\mathrm{cc}}C^4 - C^4 \times \int \frac{\mathrm{d}^4p_{\mathrm{E}}}{(2\pi)^4}\log\left[-i\gamma^\mu p_{\mathrm{E}\mu} - m_\psi C\right], \tag{156}$$

where we have performed the Wick rotation such that $p^0 = ip_{0\mathrm{E}}$ and $p^i = p_{i\mathrm{E}}$. If we neglect the second term corresponding to the quantum correction from the spinor field, the effective potential would be simply $V_{\mathrm{eff}}(C) = \Lambda_{\mathrm{cc}}C^4$ and hence, for $\Lambda_{\mathrm{cc}} > 0$, the effective potential would have vacuum $C = 0$, implying $\bar{e}^{\mathbf{a}}{}_{\mu} = \langle e^{\mathbf{a}}{}_{\mu}\rangle = 0$.

Let us now perform the loop momentum integral in Eq. (156). To this end, we need regularizations. Here, we attempt to employ the momentum-cutoff and dimensional regularization. The use of the momentum cutoff such that $0 < p_{\mathrm{E}} < \Lambda_{\mathrm{G}}$ and $C^2 m_f^2 \ll \Lambda_{\mathrm{G}}^2$ gives

$$\begin{aligned} V_{\mathrm{eff}}(C) &= \Lambda_{\mathrm{cc}}C^4 + \frac{C^4}{2(4\pi)^2}\int_0^{\Lambda_{\mathrm{G}}^2} \mathrm{d}(p_{\mathrm{E}}^2)(p_{\mathrm{E}}^2) \times \left[-\frac{1}{2}\log(p_{\mathrm{E}}^2 + (m_f C)^2)\right] \\ &= \Lambda_{\mathrm{cc}}C^4 + \frac{m_f^4}{2(4\pi)^2}C^4(\log\Lambda_{\mathrm{G}}^2 - \log(m_f^2C^2)) \\ &\quad + \frac{C^4}{2(4\pi)^2}\left[-\frac{\Lambda_{\mathrm{G}}^4}{8} + \frac{\Lambda_{\mathrm{G}}^4}{4}\log(\Lambda_{\mathrm{G}}^2)\right] + \frac{\Lambda_{\mathrm{G}}^2}{8(4\pi)^4}m_f^2C^2, \end{aligned} \tag{157}$$

while by dimensional regularization, we obtain

$$\begin{aligned} V_{\mathrm{eff}}(C) &= \Lambda_{\mathrm{cc}}C^4 - C^4\int \frac{\mathrm{d}^{4-\epsilon}p_{\mathrm{E}}}{(2\pi)^{4-\epsilon}}\left[-\frac{1}{2}\log(p_{\mathrm{E}}^2 + (m_f C)^2)\right] \\ &= \Lambda_{\mathrm{cc}}C^4 + \frac{m_f^4}{2(4\pi)^2}C^4\left(\frac{2}{\bar{\epsilon}} - \log(m_f^2C^2)\right), \end{aligned} \tag{158}$$

where $2/\bar{\epsilon} = 2/\epsilon - \gamma_E - \log 4\pi$ with $\epsilon = 4 - d$.

[18] This phase is sometimes called the Einstein phase in the literature [55].

[19] Once we identify ψ as one of the SM fermions, each M_ψ should be regarded as the one from the electroweak symmetry breaking, with $M_\psi \ll \Lambda_{\mathrm{G}}$. Here we treat it as a massive Dirac spinor for simplicity since the generalization is straightforward.

The momentum regularization case (157) is more complicated than the dimensional regularization case (158). To understand Eq. (157), let us consider the chiral limit ($m_f \to 0$) for which

$$V_{\rm eff}(C) = \Lambda_{\rm cc} C^4 + \frac{C^4}{2(4\pi)^2}\left[-\frac{\Lambda_{\rm G}^4}{8} + \frac{\Lambda_{\rm G}^4}{4}\log(\Lambda_{\rm G}^2)\right]. \quad (159)$$

The quartically divergent terms $\sim\Lambda_{\rm G}^4$ can be subtracted by the additive renormalization for the cosmological constant $\Lambda_{\rm cc}$. More specifically, we prepare counterterms for $\Lambda_{\rm cc}$ such that $\delta\Lambda_{\rm cc} + \Lambda_{\rm cc,R}\delta_{\Lambda_{\rm cc}}$ where $\delta\Lambda_{\rm cc}$ additively subtracts terms proportional to $\Lambda_{\rm G}^4$, while the counterterm $\delta_{\Lambda_{\rm cc}}$ multiplicatively subtracts divergent terms. Therefore, by employing an appropriate renormalization condition, we would obtain

$$\delta\Lambda_{\rm cc} = -\frac{1}{2(4\pi)^2}\left[-\frac{\Lambda_{\rm G}^4}{8} + \frac{\Lambda_{\rm G}^4}{4}\log(\Lambda_{\rm G}^2)\right]. \quad (160)$$

This counterterm does not contribute to the running of the cosmological constant. Indeed, such a prescription is analogous to the mass-independent scheme in scalar theories: For the scalar mass term, we give $\delta m^2 + m_R^2\delta_{m^2}$ and subtract quadratic divergences $\sim\Lambda^2$ by δm^2, while logarithmic divergences are removed by $m_R^2\delta_{m^2}$ and the running effects of scalar mass originate from $m_R^2\delta_{m^2}$, but not from quadratic divergences. The cancellation between the $\Lambda_{\rm G}^4$ terms and $\delta\Lambda_{\rm cc}$ is nothing but the cosmological constant problem [84]. We do not intend to solve this problem in this work.

Next, we consider the last term on the right-hand side of Eq. (157). It seems that this term cannot be subtracted because no counterterm proportional to C^2 exists in the original action (123). However, this is not the case. The momentum-cutoff regularization explicitly breaks the symmetries $\mathrm{GC}\times SO(1,3)$. In such a case, one has to estimate symmetry-breaking effects from the corresponding Ward-Takahashi identity and add counterterms to the action. Therefore, the last term on the right-hand side of Eq. (157) should be also removed from the effective action. To summarize, the effective potential under the momentum-cutoff regularization reads

$$V_{\rm eff}(C) = \Lambda_{\rm cc} C^4 + \frac{m_f^4}{2(4\pi)^2}C^4\left(\log\Lambda_{\rm G}^2 - \log(m_f^2C^2)\right). \quad (161)$$

This is compatible with the result from dimensional regularization (158) together with the identification $\log\Lambda_{\rm G} \leftrightarrow 2/\bar{\epsilon}$.

After removing the power divergences, which do not affect the running of theory parameters, the effective potential for C becomes

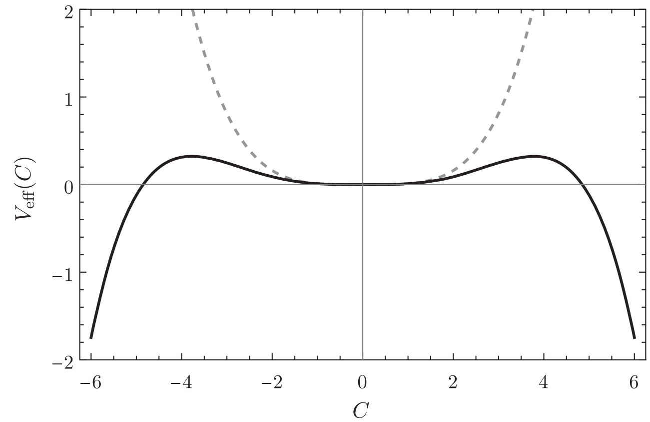

FIG. 1. Plots of the potential as a function of C. The dashed line shows the tree-level potential (150), $V_{\rm tree}(C) = \Lambda_{\rm cc}C^4$, while the effective potential (162) is depicted by the solid line. We set $\Lambda_{\rm G} = m_f = 1$ and $\Lambda_{\rm cc} = 0.01$ and assume a correct-sign kinetic term for C.

$$V_{\rm eff}(C) = \Lambda_{\rm cc} C^4 - \frac{m_f^4}{2(4\pi)^2}C^4\log\left(\frac{m_f^2C^2}{\Lambda_{\rm G}^2}\right), \quad (162)$$

in the sense of bare perturbation theory around the scale $\Lambda_{\rm G}$. In Fig. 1, we plot the effective potential (162) as a function of C. Here, we set $\Lambda_{\rm G} = m_f = 1$ and $\Lambda_{\rm cc} = 0.01$ for displaying the effective potential. It is expected that the effective potential could yield a nonzero expectation value of C, i.e., $\langle e^{\mathbf{a}}{}_{\mu}\rangle \neq 0$ due to the quantum tunneling effects.

However, at this level of the approximation, the effective potential becomes a "runaway" form. Thus, a non-trivial stable vacuum cannot be determined. So far, there are mainly two interpretations within the current model:

(i) Inclusion of higher-order effects such as loop effects of the vierbein and the LL-gauge field stabilize the effective potential.
(ii) The runaway behavior of C implies the cosmological evolution of the scale factor [85].

First we comment on the possibility (i). In the tree-level action (123), there are no kinetic terms for the vierbein and the LL-gauge field, so these fields do not contribute to the effective potential at the leading order. In possibility (i), after the kinetic terms are induced by the fermionic quantum effects, the effective potential receives loop effects from the vierbein and the LL-gauge field and could be stabilized. As a demonstration, we plot in Fig. 2 the potential with a possible higher-order correction $\propto |e|^2$,

$$\tilde{V}_{\rm eff}(C) = \Lambda_{\rm cc} C^4 - \frac{m_f^4}{2(4\pi)^2}C^4\log\left(\frac{m_f^2C^2}{\Lambda_{\rm G}^2}\right) + \frac{\lambda_8}{(4\pi)^4}C^8, \quad (163)$$

with a sample value $\lambda_8 = 0.02$. For other parameters, we use the same value as Fig. 1, i.e., $\Lambda_{\rm G} = m_f = 1$ and

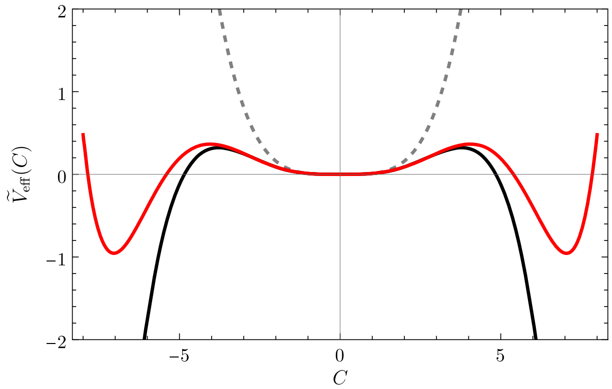

FIG. 2. Plots of the sample potential (red line) (163) as a function of C with $\lambda_8 = 0.02$ and the same values of other parameters as in Fig. 1. The black solid and gray dashed lines are the same as Fig. 1.

$\Lambda_{\rm cc} = 0.01$. In this case, there is a stable global vacuum at $\langle C\rangle = 7.04$. However, if λ_8 is large, the origin becomes the global minimum, and we do not get the emergence of spacetime.

Second, we comment on possibility (ii). The field C can be actually regarded as a conformal factor because the parametrization (151) with $C = e^\sigma$ gives the Weyl rescaling

$$\bar{g}_{\mu\nu} = \eta_{\mathbf{ab}}\bar{e}^{\mathbf{a}}{}_\mu \bar{e}^{\mathbf{b}}{}_\nu = e^{2\sigma}\eta_{\mu\nu}. \tag{164}$$

Here, σ is called the dilaton, the conformal field, or the scalaron, depending on the situation, and is associated with the scale symmetry. Indeed, in the potential (162), powers of C reflect the canonical scaling of dimensionful parameters such as $\Lambda_{\rm cc}$ and m_f. Another possible interpretation is therefore that C is regarded as a renormalization scale. The change of C may give the running of renormalized couplings. The evolution of C may be reasonable for realizing the expanding universe in cosmology.[20] In this sense, the runaway potential is not excluded from possible scenarios. Moreover, the runaway potential may be also related to the wave function of the Universe [86]. In any case, an important fact is that the solution $\langle C\rangle = 0$ is an unstable vacuum, and then a nonvanishing vacuum is realized.

Finally, we comment on the kinetic term of the conformal mode C in the vierbein. In the argument above, we have assumed a "correct-sign" kinetic term of the conformal mode and then have obtained a runaway effective potential. Conversely, if the conformal mode has a wrong-sign kinetic term, the effective potential is unstabilized by the cosmological constant term and is bounded by the fermion loop effect. Consequently, we obtain a stable vacuum at

$$\langle C\rangle = \frac{\Lambda_{\rm G}}{m_f}\exp\left(\frac{16\pi^2\Lambda_{\rm cc}}{m_f^4}\right). \tag{165}$$

The sign of the kinetic term for the conformal mode highly depends on interactions between gravitational fields and matter fields. Indeed, depending on the interaction between the scalar curvature $\mathcal{R} = \mathcal{R}^{\alpha\beta}{}_{\alpha\beta}$ [see Eq. (85) for the definition of the Riemann tensor] and scalars, the sign of the kinetic term for the conformal mode varies; see, e.g., Appendix A in Ref. [17]. In our model, it depends on whether or not the Planck mass in Eq. (121) is regarded as a function of scalars. Therefore, we do not specify the sign of the kinetic term for the conformal mode. Nonetheless, we stress that in any case, background vierbein has a nontrivial expectation value thanks to the fermion loop effect.

VII. SUMMARY AND DISCUSSION

In this paper, we have proposed a model for quantum gravity based on the LL gauge and GC symmetries. In Sec. II, we have explained our viewpoint on constructing a quantum-gravity model. We have summarized transformation laws under these symmetries and the corresponding covariance carefully in Sec. III. Our main claim is to impose the irreversible vierbein postulate on the tree-level action at the scale $\Lambda_{\rm G}$ such that the action does not contain an inverse vierbein in invariant operators under the LL gauge and GC symmetries. This postulate also prohibits, at $\Lambda_{\rm G}$, the background field of the gravitational fields, especially the vierbein. It has been shown in Sec. IV that with matter fields, only three types of terms are admitted among operators up to dimension four in the tree-level action (123): the cosmological constant, the linear term in the field strength of the LL-gauge field, and the kinetic term for the spinor, possibly with their couplings being GC- and LL-invariant functions of matter fields. This means that only the spinor can behave as the dynamical quantum field at the lowest level. Transformation laws for background fields have been summarized in Sec. V. In Sec. VI, we have argued the generation of a nonvanishing background vierbein field. There, supposing that a flat background vierbein field is induced, $\bar{e}^{\mathbf{a}}{}_\mu = C\delta^{\mathbf{a}}_\mu$, we have discussed the effective action for C. Depending on the signs of the kinetic term of C and of the cosmological constant, the symmetric vacuum $\langle C\rangle = 0$, which is consistent with the irreversible vierbein postulate, can be a stable minimum of the potential at the tree level, and then the fermionic fluctuations can make it unstable so that a nonvanishing value of $\langle C\rangle$ is realized. This implies the generation of the spacetime background.

[20] Obtaining a runaway potential for the vierbein has been discussed in Ref. [85] in a different mechanism: A background metric is assumed to be a cylinder of topology $\mathbb{R}\times\mathcal{M}^3$ with an arbitrary three manifold $\mathcal{M}^3$, i.e., $\mathrm{d}s^2 = C^2(t)(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2)$ with time-dependent factor $C(t)$ and the effective potential for $C^2(t)$ is derived. In this scenario, a "wrong-sign" kinetic term of $C(t)$ is crucial to obtain an unbounded potential for $C(t)$.

With the nontrivial background vierbein generated by the quantum dynamics, we can discuss an effective theory. In the low-energy regime, gravitational interactions are well described by metric theories in which only 2 degrees of freedom within 10 degrees of freedom of symmetric tensor are physical, while our model (123) has $4^2 + 6 \times 4 = 40$ degrees of freedom at the tree level since the vierbein and the LL-gauge fields have 16 and $6 \times 4 = 24$ degrees of freedom, respectively. This discrepancy can be understood by the Higgs mechanism: The vierbein field plays the role of the Higgs field in generating the background spacetime as its vacuum expectation value. The LL-gauge field eats 6 degrees of freedom in the vierbein field and becomes massive [54]. The remaining 10 degrees of freedom of the vierbein field are the same as those of the symmetric tensor field. At the quantum level, only 2 degrees of freedom remain due to the subtraction of 8 degrees of freedom by the gauge fixing and the ghost field. This picture provides a similar analogy to the nonlinear sigma model from the linear sigma model as discussed in Sec. II. Hence, in high energy, there are heavy modes for describing the gravitational interactions which cannot be observed in low-energy experiments; see Refs. [73,81,83,87–91] for a similar mechanism in the $GL(4)$ case.

In this paper, we have not explored renormalizability of the theory because the existence of its further UV theories is unclear. String theory might be a high-energy theory for our model. Another possible candidate within the realm of quantum field theory is spinor gravity [78,92–94], in which vierbein and LL-gauge fields are composites of spinor fields. This can be studied by the renormalization group with the compositeness condition [95].

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APPENDIX A: LIE DERIVATIVE, GENERAL COORDINATE, AND GAUGE TRANSFORMATIONS

In this appendix, we summarize the transformation laws under the GC transformation and the Lie derivative. In particular, the difference between them is clarified in Appendixes A 1 and A 2.

1. Infinitesimal general-coordinate transformation

For an infinitesimal GC transformation

$$x'^{\mu}(x) = x^{\mu} + \xi^{\mu}(x), \tag{A1}$$

we have

$$M^{\mu}{}_{\nu}(x) = \delta^{\mu}_{\nu} + \Theta^{\mu}{}_{\nu}(x), \tag{A2}$$

$$(M^{-1})^{\mu}{}_{\nu}(x) = \delta^{\mu}_{\nu} - \Theta^{\mu}{}_{\nu}(x), \tag{A3}$$

where

$$\Theta^{\mu}{}_{\nu}(x) \coloneqq \partial_{\nu}\xi^{\mu}(x). \tag{A4}$$

This implies

$$\partial_{[\rho}\Theta^{\mu}{}_{\nu]}(x) = 0. \tag{A5}$$

Conversely, a function $\Theta^{\mu}{}_{\nu}(x)$ that satisfies the condition (A5) can always be written (locally) as Eq. (A4). The condition (A5) is the infinitesimal version of the GC condition (45).

For the infinitesimal transformation (A1), the variation of the bases becomes

$$\delta_{\mathrm{GC}}\mathrm{d}x^{\mu} = \Theta^{\mu}{}_{\nu}(x)\mathrm{d}x^{\nu} = \mathrm{d}x^{\nu}\partial_{\nu}\xi^{\mu}(x), \tag{A6}$$

$$\delta_{\mathrm{GC}}\partial_{\mu} = -\Theta^{\nu}{}_{\mu}(x)\partial_{\nu} = -\partial_{\mu}\xi^{\nu}(x)\partial_{\nu} \tag{A7}$$

of the gravitational fields

$$\delta_{\mathrm{GC}}e^{\mathbf{a}}{}_{\mu}(x) = -e^{\mathbf{a}}{}_{\nu}(x)\Theta^{\nu}{}_{\mu}(x) = -\partial_{\mu}\xi^{\nu}(x)e^{\mathbf{a}}{}_{\nu}(x), \tag{A8}$$

$$\delta_{\mathrm{GC}}\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) = -\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x)\Theta^{\nu}{}_{\mu}(x) = -\partial_{\mu}\xi^{\nu}(x)\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x) \tag{A9}$$

and of the matter fields

$$\delta_{\mathrm{GC}}\phi(x) = 0, \tag{A10}$$

$$\delta_{\mathrm{GC}}\psi(x) = 0, \tag{A11}$$

$$\delta_{\mathrm{GC}}\mathcal{A}_{\mu}(x) = -\mathcal{A}_{\nu}(x)\Theta^{\nu}{}_{\mu}(x) = -\partial_{\mu}\xi^{\nu}(x)\mathcal{A}_{\nu}(x). \tag{A12}$$

The 1-forms $e^{\mathbf{a}}(x) = e^{\mathbf{a}}{}_{\mu}(x)\mathrm{d}x^{\mu}$, $\omega^{\mathbf{a}}{}_{\mathbf{b}}(x) = \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)\mathrm{d}x^{\mu}$, and $\mathcal{A}(x) = \mathcal{A}_{\mu}(x)\mathrm{d}x^{\mu}$ are trivially invariant under the GC transformation:

$$\delta_{\mathrm{GC}}e^{\mathbf{a}}(x) = -\partial_{\mu}\xi^{\nu}(x)e^{\mathbf{a}}{}_{\nu}(x)\mathrm{d}x^{\mu} + e^{\mathbf{a}}{}_{\mu}(x)\mathrm{d}x^{\nu}\partial_{\nu}\xi^{\mu}(x) = 0, \tag{A13}$$

$$\delta_{\rm GC}\omega^{\mathbf{a}}{}_{\mathbf{b}}(x) = -\partial_\mu \xi^\nu(x)\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x)\mathrm{d}x^\mu + \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)\mathrm{d}x^\nu \partial_\nu \xi^\mu(x) = 0, \quad \text{(A14)}$$

$$\delta_{\rm GC}\mathcal{A}(x) = -\partial_\mu \xi^\nu(x)\mathcal{A}_\nu(x)\mathrm{d}x^\mu + \mathcal{A}_\mu(x)\mathrm{d}x^\nu \partial_\nu \xi^\mu(x) = 0. \quad \text{(A15)}$$

For the infinitesimal GC-gauge transformation $M^\alpha{}_\beta(x) = \delta^\alpha_\beta + \Theta^\alpha{}_\beta(x)$, the GC-gauge field (as well as the Levi-Civita spin connection, both if they exist) transforms as

$$\delta_{\rm GC}\Upsilon^\alpha{}_{\beta\mu}(x) = \Theta^\alpha{}_\gamma(x)\Upsilon^\gamma{}_{\beta\mu}(x) - \Upsilon^\alpha{}_{\delta\mu}(x)\Theta^\delta{}_\beta(x) - \Upsilon^\alpha{}_{\beta\nu}(x)\Theta^\nu{}_\mu(x) - \partial_\mu \Theta^\alpha{}_\beta(x). \quad \text{(A16)}$$

The next-to-last term comes from the rotation of the spacetime index and is peculiar to the GC-gauge field, compared to the ordinary and LL-gauge transformations in Eqs. (14) and (25) [or (26)], respectively. The last term is the inhomogeneous transformation that characterizes the gauge-field transformation.

It is important that the antisymmetric part has a vanishing inhomogeneous term under the GC transformation:

$$\delta_{\rm GC}\Upsilon^\alpha{}_{[\beta\mu]}(x) = \Theta^\alpha{}_\gamma(x)\Upsilon^\gamma{}_{[\beta\mu]}(x) - \Upsilon^\alpha{}_{\delta[\mu}(x)\Theta^\delta{}_{\beta]}(x) - \Theta^\nu{}_{[\mu}(x)\Upsilon^\alpha{}_{\beta]\nu}(x) \quad \text{(A17)}$$

due to the condition (A5) for the GC transformation. That is, the antisymmetric part is in vain for covariantizing the GC-covariant derivative. This is the infinitesimal version of the discussion in the last paragraph in Sec. III C 2.

2. Lie-derivative transformation

Choosing the coordinate system (chart) x for each open subset U of a manifold $\mathcal{M}$ can be regarded as a diffeomorphism from U to $\mathbb{R}^{d+1}$. The GC transformation between two different coordinate systems is a map between two different diffeomorphisms $U \to \mathbb{R}^{d+1}$ from the same U; see the upper panel in Fig. 3.

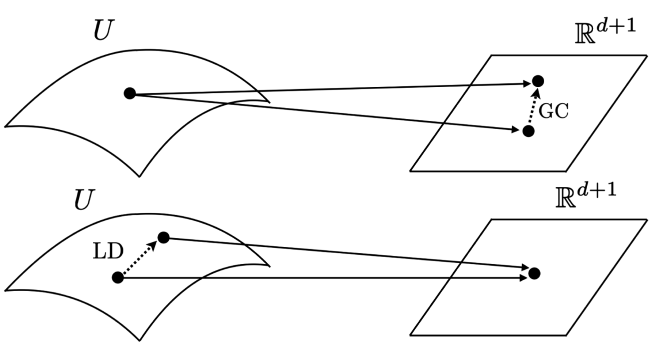

FIG. 3. Schematic figure for the GC (upper) and LD (lower) transformations.

Instead, one may consider two different maps from U to the same coordinate values in $\mathbb{R}^{d+1}$. The transformation between these two maps can also be regarded as a diffeomorphism. We can define a derivative of such a diffeomorphism, namely, the Lie derivative, which we will call the LD transformation below; see the lower panel in Fig. 3. In the literature [52,53,96,97], the self-diffeomorphism group on $\mathcal{M}$ is called the diffeomorphism, or diff in short, and this LD transformation can be regarded as its infinitesimal.[21]

A Lie derivative along an infinitesimal vector field $\xi^\mu(x)$ is defined as the difference between fields on two distinct spacetime points that happen to have the same coordinate value x' before and after the infinitesimal GC transformation (A1):

$$\begin{aligned}\mathcal{L}_\xi \Phi(x) &\coloneqq \Phi(x') - \Phi'(x') = \Phi(x+\xi(x)) - \Phi'(x') \\ &= (\Phi(x) + \xi(x)\Phi(x)) - \Phi'(x') \\ &= \xi(x)\Phi(x) + (\Phi(x) - \Phi'(x')) \\ &= \xi(x)\Phi(x) - \delta_{\rm GC}\Phi(x),\end{aligned} \quad \text{(A18)}$$

where $\xi(x)$ in an argument (of Φ in the first line, in this case) denotes the $(d+1)$ variables $(\xi^0(x), \ldots, \xi^d(x))$, whereas those in other places denote the differential operator $\xi(x) \coloneqq \xi^\mu(x)\partial_\mu$. In other words, the Lie derivative (A18) is the difference between the original field at the GC-transformed point and the GC-transformed field. In terms of the Lie derivative, the GC transformation can be written as

$$\delta_{\rm GC}\Phi(x) = \xi(x)\Phi(x) - \mathcal{L}_\xi \Phi(x). \quad \text{(A19)}$$

By construction, the Lie derivative does not change the basis, unlike the GC transformation (A6) and (A7).

We spell out the concrete forms using Eqs. (A8)–(A12):

$$\mathcal{L}_\xi e^{\mathbf{a}}{}_\mu(x) = \xi(x)e^{\mathbf{a}}{}_\mu(x) + \partial_\mu \xi^\nu(x) e^{\mathbf{a}}{}_\nu(x), \quad \text{(A20)}$$

$$\mathcal{L}_\xi \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) = \xi(x)\omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x) + \partial_\mu \xi^\nu(x)\omega^{\mathbf{a}}{}_{\mathbf{b}\nu}(x), \quad \text{(A21)}$$

and

$$\mathcal{L}_\xi \phi(x) = \xi(x)\phi(x), \quad \text{(A22)}$$

$$\mathcal{L}_\xi \psi(x) = \xi(x)\psi(x), \quad \text{(A23)}$$

$$\mathcal{L}_\xi \mathcal{A}_\mu(x) = \xi(x)\mathcal{A}_\mu(x) + \partial_\mu \xi^\nu(x)\mathcal{A}_\nu(x). \quad \text{(A24)}$$

In the language of differential geometry,

[21] Such a transformation can be interpreted as either "passive" or "active," and it is claimed that they are distinct concepts; see, e.g., Fig. 10 and Sec. 4 in Ref. [98]. The GC and LD transformations in our language may correspond to the passive and active interpretations, respectively.

$$\mathcal{L}_\xi e^{\mathbf{a}}(x) = \xi(x) e^{\mathbf{a}}(x) + \langle e^{\mathbf{a}}(x), \mathrm{d}\xi(x)\rangle, \tag{A25}$$

$$\mathcal{L}_\xi \omega^{\mathbf{a}}{}_{\mathbf{b}}(x) = \xi(x) \omega^{\mathbf{a}}{}_{\mathbf{b}}(x) + \langle \omega^{\mathbf{a}}{}_{\mathbf{b}}(x), \mathrm{d}\xi(x)\rangle, \tag{A26}$$

$$\mathcal{L}_\xi \mathcal{A}(x) = \xi(x) \mathcal{A}(x) + \langle \mathcal{A}(x), \mathrm{d}\xi(x)\rangle, \tag{A27}$$

where the exterior derivative on the vector field $\xi(x) = \xi^\mu(x)\partial_\mu$ is given by $\mathrm{d}\xi(x) \coloneqq (\partial_\mu \xi^\nu(x) \mathrm{d}x^\mu)\partial_\nu$ and the inner product defined through that of the bases $\langle \mathrm{d}x^\lambda, \partial_\nu\rangle = \delta^\lambda_\nu$, reads

$$\begin{aligned}\langle \mathcal{A}(x), \mathrm{d}\xi(x)\rangle &= \langle \mathcal{A}_\lambda(x)\mathrm{d}x^\lambda, (\partial_\mu\xi^\nu(x)\mathrm{d}x^\mu)\partial_\nu\rangle \\ &= \mathcal{A}_\lambda(x)(\partial_\mu\xi^\nu(x)\mathrm{d}x^\mu)\langle \mathrm{d}x^\lambda, \partial_\nu\rangle \\ &= \partial_\mu\xi^\nu(x)\mathcal{A}_\nu(x)\mathrm{d}x^\mu,\end{aligned} \tag{A28}$$

etc.

One might find it uneasy to transform the coordinate bases as in Eqs. (A6) and (A7). Then it might be tempting to define another transformation, which we call the LD transformation:

$$\delta_{\mathrm{LD}}\Phi(x) \coloneqq \delta_{\mathrm{GC}}\Phi(x) - \xi(x)\Phi(x) = -\mathcal{L}_\xi\Phi(x). \tag{A29}$$

An advantage of the GC transformation over the LD one is that the former commutes with gauge symmetries, whereas the latter does not [52,53,96,97] because the latter is the difference between two distinct spacetime points, which are gauge transformed differently.

3. (Semi)direct product of general-coordinate and gauge transformations

Since the GC transformation acts on spacetime indices, it acts on those of the gauge field. Therefore, when we apply both GC and gauge transformations for a system, one may worry about the order of transformations, that is, a GC transformation after a gauge transformation or the other way around. To clarify this, we show explicit computations.

We first perform a gauge transformation and then a GC transformation:

$$\begin{aligned}\Psi(x) &\xrightarrow{\text{gauge}} \check{\Psi}(x) = U(x)\Psi(x) \\ &\xrightarrow{\text{GC}} \check{\Psi}'(x') = U'(x')\Psi'(x') = U(x)\Psi(x),\end{aligned} \tag{A30}$$

$$\begin{aligned}\mathcal{A}_\mu(x) &\xrightarrow{\text{gauge}} \check{\mathcal{A}}_\mu(x) = U(x)\mathcal{A}_\mu(x)U^{-1}(x) - \partial_\mu U(x)U^{-1}(x) \\ &\xrightarrow{\text{GC}} \check{\mathcal{A}}'_\mu(x') = U'(x')\mathcal{A}'_\mu(x')U'^{-1}(x') - \partial'_\mu U'(x')U'^{-1}(x') \\ &\qquad = \left[U(x)\mathcal{A}_\nu(x)U^{-1}(x) - \partial_\nu U(x)U^{-1}(x)\right]\left[M^{-1}(x)\right]^\nu{}_\mu,\end{aligned} \tag{A31}$$

where $U'(x')$ is the pullback defined by $U'(x'(x)) = U(x)$. In the opposite order, we obtain

$$\begin{aligned}\Psi(x) &\xrightarrow{\text{GC}} \Psi'(x') = \Psi(x) \\ &\xrightarrow{\text{gauge}} \check{\Psi}'(x') = U(x)\Psi(x),\end{aligned} \tag{A32}$$

$$\begin{aligned}\mathcal{A}_\mu(x) &\xrightarrow{\text{GC}} \mathcal{A}'_\mu(x') = \mathcal{A}_\nu(x)\left[M^{-1}(x)\right]^\nu{}_\mu \\ &\xrightarrow{\text{gauge}} \check{\mathcal{A}}'_\mu(x') = \left[U(x)\mathcal{A}_\nu(x)U^{-1}(x) - \partial_\mu U(x)U^{-1}(x)\right]\left[M^{-1}(x)\right]^\nu{}_\mu.\end{aligned} \tag{A33}$$

Obviously, the GC and gauge groups commute each other: The generators of the gauge and GC transformations $\delta_\theta^{\text{gauge}}$ and δ_ξ^{LD} satisfy

$$\left[\delta_\theta^{\text{gauge}}, \delta_\xi^{\text{LD}}\right]\Psi(x) = 0, \tag{A34}$$

$$\left[\delta_\theta^{\text{gauge}}, \delta_\xi^{\text{LD}}\right]\mathcal{A}_\mu(x) = 0, \tag{A35}$$

where $\theta(x)$ and $\xi_\mu(x)$ are their transformation parameters, respectively. Thus, they form a direct product:

$$\text{GC} \times \text{gauge}. \tag{A36}$$

For instance, our action is invariant under $\text{GC} \times SO(1,d)$. Indeed, the commutativity between the GC and gauge groups is because of the definition of the GC transformations (48)–(52).

In the literature (see, e.g., Refs. [52,53,96,97]), instead of GC, one has imposed the symmetry (A29) that acts only on path-integrated quantum fields. We write it $\mathsf{diff}^{\mathrm{LD}}$. In $\mathsf{diff}^{\mathrm{LD}}$, the first term in Eq. (A19) is absent, which results in the nonvanishing commutator of $\mathsf{diff}^{\mathrm{LD}}$ and elements of a gauge transformation $\mathfrak{g}$. In particular, their commutator becomes the generator of the gauge transformation $\mathfrak{g}$ with gauge parameters $-\xi^\nu(x)\partial_\nu\theta(x)$. In this case, the group becomes a semidirect product:

$$\mathsf{diff}^{\mathrm{LD}} \ltimes \text{gauge}. \tag{A37}$$

This can be seen explicitly. To this end, let us deal with infinitesimal transformations. First, $\mathsf{diff}^{\mathrm{LD}}$ and subsequent gauge transformations yield

$$\begin{aligned}\Psi(x) &\xrightarrow{\mathsf{diff}^{\mathrm{LD}}} \Psi(x) - \xi^\mu(x)\partial_\mu\Psi(x)\\ &\xrightarrow{\text{gauge}} \Psi(x) + \theta(x)\Psi(x) - \xi^\mu(x)\partial_\mu\theta(x)\Psi(x) - \xi^\mu(x)\partial_\mu\Psi(x) - \xi^\mu(x)\theta(x)\partial_\mu\Psi(x),\end{aligned} \tag{A38}$$

$$\mathcal{A}_\mu(x) \xrightarrow{\mathsf{diff}^{\mathrm{LD}}} \mathcal{A}_\mu(x) - \xi^\nu(x)\partial_\nu\mathcal{A}_\mu(x) - \mathcal{A}_\nu(x)\partial_\mu\xi^\nu(x) \tag{A39}$$

$$\begin{aligned}&\xrightarrow{\text{gauge}} \mathcal{A}_\mu(x) + \theta(x)\mathcal{A}_\mu(x) - \mathcal{A}_\mu(x)\theta(x) - \partial_\mu\theta(x)\\ &\quad - \xi^\nu(x)\partial_\nu(\mathcal{A}_\mu(x) + \theta(x)\mathcal{A}_\mu(x) - \mathcal{A}_\mu(x)\theta(x) - \partial_\mu\theta(x))\\ &\quad - (\mathcal{A}_\nu(x) + \theta(x)\mathcal{A}_\nu(x) - \mathcal{A}_\nu(x)\theta(x) - \partial_\nu\theta(x))\partial_\mu\xi^\nu(x)\\ &= \mathcal{A}_\mu + \theta\mathcal{A}_\mu - \mathcal{A}_\mu\theta - \partial_\mu\theta - \xi\mathcal{A}_\mu - \xi\theta\mathcal{A}_\mu - \theta\xi\mathcal{A}_\mu + \xi\mathcal{A}_\mu\theta + \mathcal{A}_\mu\xi\theta + \xi\partial_\mu\theta\\ &\quad - (\mathcal{A}_\nu + \theta\mathcal{A}_\nu - \mathcal{A}_\nu\theta - \partial_\nu\theta)\partial_\mu\xi^\nu.\end{aligned} \tag{A40}$$

Note that $\mathcal{A}_\mu = \mathcal{A}_\mu^a T^a$ and $\theta = \theta^a T^a$ do not commute here. On the other hand, a gauge transformation and a subsequent $\mathsf{diff}^{\mathrm{LD}}$ yield

$$\begin{aligned}\Psi(x) &\xrightarrow{\text{gauge}} \Psi(x) + \theta(x)\Psi(x)\\ &\xrightarrow{\mathsf{diff}^{\mathrm{LD}}} \Psi(x) - \xi^\mu(x)\partial_\mu\Psi(x) + \theta(x)\Psi(x) - \theta(x)\xi^\mu(x)\partial_\mu\Psi(x),\end{aligned} \tag{A41}$$

$$\begin{aligned}\mathcal{A}_\mu(x) &\xrightarrow{\text{gauge}} \mathcal{A}_\mu(x) + \theta(x)\mathcal{A}_\mu(x) - \mathcal{A}_\mu(x)\theta(x) - \partial_\mu\theta(x)\\ &\xrightarrow{\mathsf{diff}^{\mathrm{LD}}} \mathcal{A}_\mu(x) - \xi^\nu(x)\partial_\nu\mathcal{A}_\mu(x) - \mathcal{A}_\nu(x)\partial_\mu\xi^\nu(x) + \theta(x)(\mathcal{A}_\mu(x) - \xi^\nu(x)\partial_\nu\mathcal{A}_\mu(x) - \mathcal{A}_\nu(x)\partial_\mu\xi^\nu(x))\\ &\quad - (\mathcal{A}_\mu(x) - \xi^\nu(x)\partial_\nu\mathcal{A}_\mu(x) - \mathcal{A}_\nu(x)\partial_\mu\xi^\nu(x))\theta(x) - \partial_\mu\theta(x)\\ &= \mathcal{A}_\mu - \xi\mathcal{A}_\mu - \mathcal{A}_\nu\partial_\mu\xi^\nu + \theta(\mathcal{A}_\mu - \xi\mathcal{A}_\mu - \mathcal{A}_\nu\partial_\mu\xi^\nu) - (\mathcal{A}_\mu - \xi\mathcal{A}_\mu - \mathcal{A}_\nu\partial_\mu\xi^\nu)\theta - \partial_\mu\theta.\end{aligned} \tag{A42}$$

Subtracting these two, we obtain

$$\left[\delta_\theta^{\text{gauge}}, \delta_\xi^{\mathrm{LD}}\right]\Psi(x) = -\xi^\mu(x)\partial_\mu\theta(x)\Psi(x), \tag{A43}$$

$$\left[\delta_\theta^{\text{gauge}}, \delta_\xi^{\mathrm{LD}}\right]\mathcal{A}_\mu(x) = -\xi^\nu(x)\partial_\nu\theta(x)\mathcal{A}_\mu(x) + \mathcal{A}_\mu(x)\xi^\nu(x)\partial_\nu\theta(x) + \xi^\nu(x)\partial_\nu\partial_\mu\theta(x) + \partial_\nu\theta(x)\partial_\mu\xi^\nu(x). \tag{A44}$$

The commutator becomes the extra gauge transformation with the gauge parameter $\mathcal{L}_\xi\theta(x) = -\xi\theta = -\xi^\nu(x)\partial_\nu\theta(x)$. For this noncommutativity, it is important that $\mathsf{diff}^{\mathrm{LD}}$ does not transform $\theta(x)$ by assumption. To avoid noncommutativity of LD and diff, one may further introduce a modified $\mathsf{diff}^{\mathrm{LD}}$ denoted by $\tilde{\delta}_\xi^{\mathrm{LD}}$ to make it commute with $\mathfrak{g}$ [52,53,96,97].

We comment on the global Poincaré transformation $ISO(1,d)$ in the Minkowski space $\mathbf{M}^{1,d}$. The global Poincaré transformation $ISO(1,d)$ contains the translation in $\mathbf{M}^{1,d}$ as a normal subgroup in the sense that $x \to \Lambda x \to \Lambda x + a \to \Lambda^{-1}(\Lambda x + a) = x + \Lambda^{-1}a$. Since $SO(1,d) \simeq ISO(1,d)/T(1,d)$, we write[22]

$$ISO(1,d) = T(1,d) \rtimes SO(1,d). \tag{A47}$$

The local version of $T(1,d) \rtimes SO(1,d)$ is given by

$$\mathsf{diff} \rtimes SO(1,d), \tag{A48}$$

which is opposite Eq. (A37). In a gravitational theory based on the global Poincaré transformation, we infer to realize the symmetry breaking

$$ISO(1,d) = T(1,d) \rtimes SO(1,d) \to \mathsf{diff}. \tag{A49}$$

APPENDIX B: COMMENT ON THE LIE DERIVATIVE ON THE SPINOR

In this appendix, several definitions of the Lie derivative acting on the spinor are argued.

Once the background-covariant derivative $\overline{\mathscr{D}}$ is defined for the matter field Ψ, we may consider a parallel transport with respect to $\bar{\mathscr{D}}$:

$$\Psi'(x+\xi) = \Psi(x) - \xi^\mu \bar{\mathscr{D}}_\mu \Psi(x). \tag{B1}$$

Here, we stress that this parallel transport differs from the GC transformation in Eqs. (50)–(52) in the sense that the transport (B1) compares fields on physically distinct points that happen to have the same coordinate values before and after the GC transformation.

The parallel transport (B1) induces another Lie derivative of Ψ:

$$L_\xi \Psi(x) = \xi^\mu(x) \bar{\mathscr{D}}_\mu \Psi(x). \tag{B2}$$

On spinors, this is nothing but the Lie derivative introduced by Weyl [99],

$$L_\xi \psi(x) = \xi^\mu(x)\left(\partial_\mu \psi(x) + \frac{\bar{\omega}_{\mathbf{ab}\mu}(x)}{2}\Sigma^{\mathbf{ab}}\psi(x)\right), \tag{B3}$$

if $\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}$ is identified to the Levi-Civita spin connection (130).[23]

One may further extend the above definition to the following form [100] (see also Ref. [101][24])

$$\mathbb{L}_\xi \psi(x) \coloneqq \xi^\mu(x) \bar{\mathscr{D}}_\mu \psi(x) + \frac{\bar{\mathscr{D}}_{[\mu}\xi_{\nu]}(x)}{4} \bar{e}^\mu{}_{\mathbf{a}}(x) \bar{e}^\nu{}_{\mathbf{b}}(x) \Sigma^{\mathbf{ab}} \psi(x). \tag{B4}$$

This extension is motivated by the fact that on the flat Minkowski background $\bar{e}^{\mathbf{a}}{}_\mu = \delta^{\mathbf{a}}_\mu$ and $\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu} = 0$, there remains the global $SO(1,d)$ invariance under

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu = (\delta^\mu_\nu + \theta^\mu{}_\nu + \cdots)x^\nu = x^\mu + \xi^\mu, \tag{B5}$$

namely, $\xi^\mu(x) = \theta^\mu{}_\nu x^\nu + \cdots$. This is the same as Eq. (A2) with Eq. (A4). Hence, Eq. (B4) becomes equivalent to the Lie derivative obtained when the background GC transformation reduces to the global $SO(1,d)$ on the flat background, namely, when $M^\mu{}_\nu(x) \to \Lambda^\mu{}_\nu$:

$$\psi(x) \to \psi'(x') = \left(1 + \frac{\theta_{\mu\nu}}{2}\Sigma^{\mu\nu} + \cdots\right)\psi(x). \tag{B6}$$

The transformation (B6) corresponds just to the global $SO(1,d)$ transformation for the spinor. The definition (B4) may however be a detour notation in our case. As discussed in Sec. V C, in our formulation, the global Lorentz $SO(1,d)$ transformation is accidentally realized as a diagonal subgroup of $SO(1,d) \times \mathrm{GC}$, so that the detour notation is not necessary.

APPENDIX C: DEGENERATE LIMIT OF THE VIERBEIN

In this appendix, we explain the detailed definition of the degenerate limit and show its application for explicit several examples.

1. General definition of degenerate limit

As discussed in the Introduction, we assume that the action at Λ_{G} admits the weak-field limit $e^{\mathbf{a}}{}_\mu(x) \to 0$ just as the SM action does for the limit $H(x) \to 0$, etc. In particular, we postulate that the action admits the degenerate limit for any combination of the $(d+1)^2 = 16$ components of the vierbein [54]. This requirement puts a more severe constraint on the action than just requiring a simultaneous limit for all the components, as we will

[22]In the case of $SU(2) \to U(1)$ breaking, i.e., $T \simeq SU(2)/U(1)$, we write

$$SU(2) = T \ltimes U(1). \tag{A45}$$

For the SM $SU(2) \times U(1)_Y \to U(1)_Q$, we have $T \simeq (SU(2) \times U(1)_Y)/U(1)_Q$ and then write

$$SU(2) \times U(1)_Y = T \ltimes U(1)_Q. \tag{A46}$$

[23]As said above, whether or not $\bar{\omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}$ coincides with $\bar{\Omega}^{\mathbf{a}}{}_{\mathbf{b}\mu}$ is to be determined dynamically in our formalism.

[24]In Ref. [101], the original Lie derivative by Weyl is said to be $\mathcal{L}_U \psi = U^\mu \partial_\mu \psi$ without the background LL connection, corresponding to the transformation (51).

see. The degenerate configuration of the vierbein appears in the topology change of the background spacetime and is expected to play an important role in quantum gravity [55,56].

In general, a vierbein $e^{\mathbf{a}}{}_{\mu}$ has four eigenvalues in four-dimensional spacetime. In the degenerate limit, at least one eigenvalue goes to zero, resulting in the determinant of the vierbein to be zero: $|e| \to 0$. Note that the degenerate limit does not necessarily mean the null limit $e^{\mathbf{a}}{}_{\mu} \to 0$ for all 16 components. Let us see this fact using a specific example. By a rescaling $e^{\mathbf{a}}{}_{\mu} \mapsto Ce^{\mathbf{a}}{}_{\mu}$, we obtain $|e| \mapsto C^4|e|$ and $g^{\mu\nu} \mapsto C^{-2}g^{\mu\nu}$, and hence the term

$$|e|g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \mapsto C^2|e|g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{C1}$$

does not diverge in the null limit $C \to 0$. However, this term is in general divergent in a degenerate limit as we will see below.

Now, we define the degenerate limit. The power for the degenerate limit can be counted by $|e|$ rather than by the overall normalization factor C: The inverse of the vielbein and metric, $e_{\mathbf{a}}{}^{\mu}$ and $g^{\mu\nu}$, contains one and two factors of $|e|^{-1}$, respectively,

$$e_{\mathbf{a}}{}^{\mu} = \frac{C_{\mathbf{a}}{}^{\mu}}{|e|}, \qquad g^{\mu\nu} = \frac{\eta^{\mathbf{ab}}C_{\mathbf{a}}{}^{\mu}C_{\mathbf{b}}{}^{\nu}}{|e|^2}, \tag{C2}$$

where the transpose of the cofactor matrix of $e^{\mathbf{a}}{}_{\mu}$ is denoted by $C_{\mathbf{a}}{}^{\mu}$, which remains finite in the degenerate limit. Hereafter, we write the power of $|e|$ as

$$\lfloor e_{\mathbf{a}}{}^{\mu}\rfloor = -1, \qquad \lfloor g^{\mu\nu}\rfloor = -2, \quad \text{etc.} \tag{C3}$$

Each upper greek index of the vielbein or metric serves an extra -1 power of $|e|$, and its power -1 cancels the power $+1$ from a lower index of the metric or vielbein.

This can be explicitly seen as follows. The $D^2 = (d+1)^2$ degrees of freedom of the vielbein can be parametrized as

$$\left[e^{\mathbf{a}}{}_{\mu}\right]_{\mathbf{a},\mu=0,\ldots,d} = \Lambda\mathrm{diag}(\lambda_0,\ldots,\lambda_d)M^{\mathrm{t}}, \tag{C4}$$

where $(\lambda_0,\ldots,\lambda_d) \in \mathbb{R}^D$ with $\lambda_0 < 0$ and $\lambda_i > 0$ $(i = 1,\ldots,d)$ and each of $\Lambda, M \in SO_+(1,d)$ has $\frac{D(D-1)}{2}$ degrees of freedom.[25] In the degenerate limit, an eigenvalue λ_a goes to zero: $\lambda_a \to 0$. The determinant reads

$$|e| = \lambda_0\cdots\lambda_d, \tag{C5}$$

while the metric and its inverse are

$$\begin{aligned}\left[g_{\mu\nu}\right]_{\mu,\nu=0,\ldots,d} &= (\Lambda\,\mathrm{diag}(\lambda_0,\ldots,\lambda_d)M^{\mathrm{t}})^{\mathrm{t}}\eta \\ &\quad\times(\Lambda\,\mathrm{diag}(\lambda_0,\ldots,\lambda_d)M^{\mathrm{t}}) \\ &= M\,\mathrm{diag}(-\lambda_0^2,\lambda_1^2,\ldots,\lambda_d^2)M^{\mathrm{t}},\end{aligned} \tag{C6}$$

$$\left[g^{\mu\nu}\right]_{\mu,\nu=0,\ldots,d} = M\,\mathrm{diag}(-\lambda_0^{-2},\lambda_1^{-2},\ldots,\lambda_d^{-2})M^{\mathrm{t}}. \tag{C7}$$

We see that a contraction cancels a power: For example,

$$\left[e^{\mathbf{a}}{}_{\mu}g^{\mu\nu}\right]_{\mathbf{a},\nu=0,\ldots,d} = \Lambda\,\mathrm{diag}(-\lambda_0^{-1},\lambda_1^{-1},\ldots,\lambda_d^{-1})M^{\mathrm{t}} \tag{C8}$$

gives the power $\lfloor e^{\mathbf{a}}{}_{\mu}g^{\mu\nu}\rfloor = -1$. In one more example, for $\nabla_\mu e^{\mathbf{a}}{}_{\nu} = \partial_\mu e^{\mathbf{a}}{}_{\nu} - e^{\mathbf{a}}{}_{\lambda}\Gamma^{\lambda}{}_{\nu\mu}$ with the Levi-Civita connection (70), we have

$$e^{\mathbf{a}}{}_{\lambda}\Gamma^{\lambda}{}_{\nu\mu} = e^{\mathbf{a}}{}_{\lambda}\frac{g^{\lambda\lambda'}}{2}(-\partial_{\lambda'}g_{\nu\mu} + \partial_\nu g_{\mu\lambda'} + \partial_\mu g_{\lambda'\nu}), \tag{C9}$$

which gives $\lfloor e^{\mathbf{a}}{}_{\lambda}\Gamma^{\lambda}{}_{\nu\mu}\rfloor = -1$ because the first term does not contain a vierbein whose λ' leg is to be contracted.

Other examples are in order: The Levi-Civita connection $\Gamma^{\lambda}{}_{\nu\mu}$ contains two extra inverse powers of $|e|$ coming from $g^{\mu\nu}$, $\lfloor\Gamma^{\mu}{}_{\rho\sigma}\rfloor = -2$, while the Levi-Civita spin connection has $\lfloor\Omega^{\mathbf{a}}{}_{\mathbf{b}\mu}\rfloor = -2$ because of $\lfloor e^{\mathbf{a}}{}_{\lambda}g^{\lambda\rho}\rfloor = -1$ and hence, $\lfloor e^{\mathbf{a}}{}_{\lambda}\Gamma^{\lambda}{}_{\sigma\mu}\rfloor = -1$ (contraction of $e_{\mathbf{b}}{}^{\sigma}$ with $\Gamma^{\lambda}{}_{\sigma\mu}$ does give the additional power -1 because the latter contains the index from ∂_σ, which does not come from the vielbein or from metric). Note that the contraction of $\Gamma^{\lambda}{}_{\nu\mu}$ with $e_{\mathbf{a}}{}^{\nu}$ does raise the power by 1 because the former contains the second term in the parentheses in Eq. (70), $\partial_\nu g_{\mu\lambda}$, whose lower ρ index comes from the derivative.

2. Concrete examples

Here, we more explicitly show the degenerate limit on various terms and list the terms prohibited by having a negative power of eigenvalues of $e^{\mathbf{a}}{}_{\mu}$.

(i) The kinetic term of a scalar

$$S_\phi = \int \mathrm{d}^4x|e|\left[-\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\right] \tag{C10}$$

is prohibited because from Eqs. (C5) and (C7)

$$\begin{aligned}|e|g^{\mu\nu} &\propto (\lambda_0\cdots\lambda_3)\mathrm{diag}(-\lambda_0^{-2},\lambda_1^{-2},\lambda_2^{-2},\lambda_3^{-2}) \\ &= \mathrm{diag}\left(-\frac{\lambda_1\lambda_2\lambda_3}{\lambda_0},\frac{\lambda_0\lambda_2\lambda_3}{\lambda_1},\frac{\lambda_0\lambda_1\lambda_3}{\lambda_2},\frac{\lambda_0\lambda_1\lambda_2}{\lambda_3}\right).\end{aligned} \tag{C11}$$

When some eigenvalues become zero $\lambda_a \to 0$, the matrix contains a divergent component $\propto \lambda_a^{-1}$. We note that Eq. (116) cannot be used in Eq. (C11) since

[25]Percacci has generalized M to be $M \in GL(D)$ having D^2 degrees of freedom.

$|e|g^{\mu\nu} = |e|e_{(\mathbf{a}}{}^{\mu}e_{\mathbf{b})}{}^{\nu}\eta^{\mathbf{ab}}$ which is different from $|e|e_{[\mathbf{a}}{}^{\mu}e_{\mathbf{b}]}{}^{\nu}$ to be vanishing for the contraction with $\eta^{\mathbf{ab}}$.

(ii) The symmetric kinetic term for the vierbein

$$S_e = \int \mathrm{d}^4x|e|\left[-\frac{Z_e}{2}g^{\mu\mu'}g^{\nu\nu'}\eta_{\mathbf{ab}}(\nabla_\mu e^{\mathbf{a}}{}_\nu)(\nabla_{\mu'}e^{\mathbf{b}}{}_{\nu'})\right] \tag{C12}$$

(with a factor Z_e being mass dimension 2) is prohibited because

$$\begin{aligned}&|e|g^{\nu\nu'}\eta_{\mathbf{ab}}e^{\mathbf{a}}{}_\nu e^{\mathbf{b}}{}_{\nu'}\\ &\propto (\lambda_0\cdots\lambda_3)\mathrm{diag}(-\lambda_0^{-2},\lambda_1^{-2},\lambda_2^{-2},\lambda_3^{-2})\\ &\quad\times\mathrm{diag}(-\lambda_0,\lambda_1,\lambda_2,\lambda_3)\mathrm{diag}(-\lambda_0,\lambda_1,\lambda_2,\lambda_3)\\ &=\mathrm{diag}\left(-\frac{\lambda_1\lambda_2\lambda_3}{\lambda_0},\frac{\lambda_0\lambda_2\lambda_3}{\lambda_1},\frac{\lambda_0\lambda_1\lambda_3}{\lambda_2},\frac{\lambda_0\lambda_1\lambda_2}{\lambda_3}\right).\end{aligned} \tag{C13}$$

The ath diagonal element of this matrix has λ_a^{-1} which diverges for $\lambda_a \to 0$.

(iii) The antisymmetrized kinetic term

$$S_{e,\text{anti-sym}} = \int \mathrm{d}^Dx|e|\left[-\frac{1}{2}g^{\mu\mu'}g^{\nu\nu'}(\partial_{[\mu}e_{\nu]})(\partial_{[\mu'}e_{\nu']})\right] \tag{C14}$$

is prohibited. Note that this term is GC invariant because $\nabla_{[\mu}e^{\mathbf{a}}{}_{\nu]} = \partial_{[\mu}e^{\mathbf{a}}{}_{\nu]}$ due to the torsion-free identity of the Levi-Civita connection $\Gamma^\mu{}_{[\rho\sigma]} = 0$. Even though the power $\lfloor\nabla_\mu e^{\mathbf{a}}{}_\nu\rfloor = -1$ is raised to $\lfloor\partial_{[\mu}e^{\mathbf{a}}{}_{\nu]}\rfloor = 0$, we still have the total power -1.

(iv) The gauge kinetic term

$$S_A = \int \mathrm{d}^4x|e|\left[+\frac{1}{2}g^{\mu\mu'}g^{\nu\nu'}F^{\mathbf{a}}{}_{\mathbf{b}\mu\nu}F^{\mathbf{b}}{}_{\mathbf{a}\mu'\nu'}\right] \tag{C15}$$

is proportional to λ_a^{-3} because

$$\begin{aligned}|e|g^{\mu\mu'}g^{\nu\nu'} &\propto (\lambda_0\cdots\lambda_3)\mathrm{diag}(-\lambda_0^{-2},\lambda_1^{-2},\lambda_2^{-2},\lambda_3^{-2})\\ &\quad\times\mathrm{diag}(-\lambda_0^{-2},\lambda_1^{-2},\lambda_2^{-2},\lambda_3^{-2})\\ &=\mathrm{diag}\left(-\frac{\lambda_1\lambda_2\lambda_3}{\lambda_0^3},\frac{\lambda_0\lambda_2\lambda_3}{\lambda_1^3},\frac{\lambda_0\lambda_1\lambda_3}{\lambda_2^3},\frac{\lambda_0\lambda_1\lambda_2}{\lambda_3^3}\right)\end{aligned} \tag{C16}$$

and thus diverges for $\lambda_a \to 0$. Therefore, the gauge kinetic term is not compatible with the degenerate limit.

(v) The Einstein-Hilbert action solely made of e,

$$S_{\mathrm{EH}} = \int \mathrm{d}^Dx|e|R. \tag{C17}$$

The Riemann and Ricci tensors that are solely made of e are $R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\lambda\rho}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\mu{}_{\lambda\sigma}\Gamma^\lambda{}_{\nu\rho}$ and $R_{\nu\sigma} = \partial_\mu\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\mu} + \Gamma^\mu{}_{\lambda\mu}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\mu{}_{\lambda\sigma}\Gamma^\lambda{}_{\nu\mu}$. Both give the power -4. The Ricci scalar $R = g^{\nu\sigma}(\partial_\mu\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\mu} + \Gamma^\mu{}_{\lambda\mu}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\mu{}_{\lambda\sigma}\Gamma^\lambda{}_{\nu\mu})$, which is solely made of e, gives the same power -4.[26] Therefore, it has the power -3.

These terms are forbidden in the action consistent with the degenerate limit.

In contrast, the following operators can admit the degenerate limit:

(i) The cosmological constant term

$$\int \mathrm{d}^Dx|e| \tag{C18}$$

is obviously not divergent for $\lambda_a \to 0$ thanks to Eq. (C5).

(ii) The linear term in $F^{\mathbf{ab}}{}_{\mu\nu}$, namely,

$$S = \int \mathrm{d}^Dx|e|e_{[\mathbf{a}}{}^{\mu}e_{\mathbf{b}]}{}^{\nu}F^{\mathbf{ab}}{}_{\mu\nu} \tag{C19}$$

does not diverge because the use of Eq. (116) yields

$$\begin{aligned}|e|e_{[\mathbf{a}}{}^{\mu}e_{\mathbf{b}]}{}^{\nu} &= \frac{1}{2}\epsilon[\mathbf{abcd}]e^{\mathbf{c}}{}_\rho e^{\mathbf{d}}{}_\sigma\epsilon[\mu\nu\rho\sigma]\\ &\propto (\mathrm{diag}(-\lambda_0,\lambda_1,\lambda_2,\lambda_3))^2\\ &= \mathrm{diag}(\lambda_0^2,\lambda_1^2,\lambda_2^2,\lambda_3^2).\end{aligned} \tag{C20}$$

(iii) The kinetic term of the spinor field

$$S = \int \mathrm{d}^Dx|e|\bar{\psi}e_{\mathbf{a}}{}^{\mu}\gamma^{\mathbf{a}}\mathcal{D}_{\mathbf{a}}\psi \tag{C21}$$

has

$$\begin{aligned}|e|e_{\mathbf{a}}{}^{\mu} &= \frac{1}{3!}\epsilon[\mathbf{abcd}]e^{\mathbf{b}}{}_\nu e^{\mathbf{c}}{}_\rho e^{\mathbf{d}}{}_\sigma\epsilon[\mu\nu\rho\sigma]\\ &\propto (\mathrm{diag}(-\lambda_0,\lambda_1,\lambda_2,\lambda_3))^3\\ &= \mathrm{diag}(-\lambda_0^3,\lambda_1^3,\lambda_2^3,\lambda_3^3).\end{aligned} \tag{C22}$$

This term does not contain any inverse of eigenvalues of the vierbein field, so that it is free from

[26] In $\Gamma^\mu{}_{\lambda\sigma}\Gamma^\lambda{}_{\nu\mu}$, the power -4 term is one in which both the lower indices μ and λ come from derivatives. In that case, the lower indices ν and σ are from metrics, and hence erase the power from $g^{\nu\sigma}$.

divergences for $\lambda_a \to 0$. Note that the mass term of the spinor field is also accepted.

Hence, imposing the degenerate limit on the action accepts only these three terms. In particular, only spinor fields have their kinetic terms, i.e., become dynamical.

3. Rarita-Schwinger field

Here we examine if the spin-3/2 Rarita-Schwinger field is compatible with the irreversible vierbein postulate.

When we regard the lower-indexed ψ_μ as the fundamental field, its free action is

$$S = \int d^4x|e(x)|\left[-i\varepsilon^{\mu\nu\rho\sigma}(x)\bar{\psi}_\mu(x)\gamma_5\gamma_\nu(x)\partial_\rho\psi_\sigma(x)\right]$$
$$= \int d^4x\left[i\epsilon\left[\mu\nu\rho\sigma\right]\bar{\psi}_\mu(x)\gamma_5\gamma_\nu(x)\partial_\rho\psi_\sigma(x)\right]. \quad \text{(C23)}$$

When we regard the upper-indexed one ψ^μ fundamental, its free action is

$$S = \int d^4x|e(x)|\left[-i\varepsilon_{\mu\nu\rho\sigma}(x)\bar{\psi}^\mu(x)\gamma_5\gamma^\nu(x)\partial^\rho\psi^\sigma(x)\right]$$
$$= \int d^4x|e(x)|^2\left[i\epsilon\left[\mu\nu\rho\sigma\right]\bar{\psi}^\mu(x)\gamma_5 e_{\mathbf{a}}{}^\nu(x)\gamma^{\mathbf{a}}(x)g^{\rho\rho'}(x)\partial_{\rho'}\psi^\sigma(x)\right]. \quad \text{(C24)}$$

The former action is consistent with the irreversible vierbein postulate, whereas the latter is not. The compatibility of the Rarita-Schwinger field with the irreversible vierbein postulate is contingent upon whether we consider the field with upper or lower indices as the fundamental entity.

APPENDIX D: TOPOLOGICAL TERMS

There are four topological terms: (i) the Immirzi term, (ii) the Nieh-Yan invariant, (iii) the Pontryagin index, and (iv) the Euler number; see, e.g., Ref. [102] for others.[27] More specifically, they are given in the language of differential forms by

$$S_{\text{Immirzi}} = \frac{1}{2}\int \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}} \wedge e_{\mathbf{a}} \wedge e_{\mathbf{b}} = \int d^4x\epsilon\left[\mu\nu\rho\sigma\right]\frac{1}{4}\overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x)e_{\mathbf{a}\rho}(x)e_{\mathbf{b}\sigma}(x), \quad \text{(D1)}$$

$$S_{\text{Nieh-Yan}} = \frac{1}{2}\int d(e^{\mathbf{a}} \wedge T_{\mathbf{a}}) = \frac{1}{2}\int (T^{\mathbf{a}} \wedge T_{\mathbf{a}} - \overset{\omega}{\mathcal{F}}_{\mathbf{ab}} \wedge e^{\mathbf{a}} \wedge e^{\mathbf{b}})$$
$$= \int d^4x\epsilon\left[\mu\nu\rho\sigma\right]\left(\frac{1}{8}T^{\mathbf{a}}{}_{\mu\nu}(x)T_{\mathbf{a}\rho\sigma}(x) - \frac{1}{4}\overset{\omega}{\mathcal{F}}_{\mathbf{ab}\mu\nu}(x)e^{\mathbf{a}}{}_\rho(x)e^{\mathbf{b}}{}_\sigma(x)\right), \quad \text{(D2)}$$

$$S_{\text{Pontryagin}} = \frac{1}{2}\int \overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}} \wedge \overset{\omega}{\mathcal{F}}_{\mathbf{ab}} = \int d^4x\epsilon\left[\mu\nu\rho\sigma\right]\frac{1}{8}F^{\mathbf{ab}}{}_{\mu\nu}(x)F_{\mathbf{ab}\rho\sigma}(x), \quad \text{(D3)}$$

$$S_{\text{Euler}} = \frac{1}{8}\int \epsilon_{\mathbf{abcd}}\overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}} \wedge \overset{\omega}{\mathcal{F}}{}^{\mathbf{cd}} = \frac{1}{32}\int d^4x\epsilon\left[\mu\nu\rho\sigma\right]\epsilon_{\mathbf{abcd}}\overset{\omega}{\mathcal{F}}{}^{\mathbf{ab}}{}_{\mu\nu}(x)\overset{\omega}{\mathcal{F}}{}^{\mathbf{cd}}{}_{\rho\sigma}(x), \quad \text{(D4)}$$

where we have omitted the coupling constants. We define

$$T^{\mathbf{a}} \coloneqq de^{\mathbf{a}} + \omega^{\mathbf{a}}{}_{\mathbf{b}} \wedge e^{\mathbf{b}} = \frac{1}{2}(\partial_\mu e^{\mathbf{a}}{}_\nu(x) + \omega^{\mathbf{a}}{}_{\mathbf{b}\mu}(x)e^{\mathbf{b}}{}_\nu(x))dx^\mu \wedge dx^\nu, \quad \text{(D5)}$$

namely,

$$T^{\mathbf{a}}{}_{\mu\nu}(x) = \partial_{[\mu}e^{\mathbf{a}}{}_{\nu]}(x) + \omega^{\mathbf{a}}{}_{\mathbf{b}[\mu}(x)e^{\mathbf{b}}{}_{\nu]}(x), \quad \text{(D6)}$$

and we have used $dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = \epsilon\left[\mu\nu\rho\sigma\right]d^4x$. More specifically, the Nieh-Yan terms are computed as

[27]Recently, another topological term $\int d(e^{\mathbf{a}} \wedge \star T_{\mathbf{a}})$ is proposed [103]. This term would be interesting to study on its own, though it is incompatible with the irreversible vierbein postulate because it contains the Hodge dual of the 2-form and hence two inverse metrics.

$$\begin{aligned} \mathrm{d}(e^{\mathbf{a}} \wedge T_{\mathbf{a}}) &= \mathrm{d}e^{\mathbf{a}} \wedge T_{\mathbf{a}} - e^{\mathbf{a}} \wedge \mathrm{d}T_{\mathbf{a}} = (T^{\mathbf{a}} - \omega^{\mathbf{a}}{}_{\mathbf{b}} \wedge e^{\mathbf{b}}) \wedge T_{\mathbf{a}} - e^{\mathbf{a}} \wedge (\mathrm{d}\omega_{\mathbf{ab}} \wedge e^{\mathbf{b}} - \omega_{\mathbf{ab}} \wedge \mathrm{d}e^{\mathbf{b}}) \\ &= T^{\mathbf{a}} \wedge T_{\mathbf{a}} - \omega^{\mathbf{a}}{}_{\mathbf{b}} \wedge e^{\mathbf{b}} \wedge T_{\mathbf{a}} - e^{\mathbf{a}} \wedge ((\overset{\omega}{\mathcal{F}}_{\mathbf{ab}} - \omega_{[\mathbf{a}|\mathbf{c}|} \wedge \omega^{\mathbf{c}}{}_{\mathbf{b}]}) \wedge e^{\mathbf{b}} - \omega_{\mathbf{ab}} \wedge (T^{\mathbf{b}} - \omega^{\mathbf{b}}{}_{\mathbf{c}} \wedge e^{\mathbf{c}})) \\ &= T^{\mathbf{a}} \wedge T_{\mathbf{a}} - \omega^{\mathbf{a}}{}_{\mathbf{b}} \wedge e^{\mathbf{b}} \wedge T_{\mathbf{a}} - e^{\mathbf{a}} \wedge e^{\mathbf{b}} \wedge \overset{\omega}{\mathcal{F}}_{\mathbf{ab}} + e^{\mathbf{a}} \wedge e^{\mathbf{b}} \wedge \omega_{[\mathbf{a}|\mathbf{c}|} \wedge \omega^{\mathbf{c}}{}_{\mathbf{b}]} + e^{\mathbf{a}} \wedge \omega_{\mathbf{ab}} \wedge T^{\mathbf{b}} \\ &\quad - e^{\mathbf{a}} \wedge \omega_{\mathbf{ab}} \wedge \omega^{\mathbf{b}}{}_{\mathbf{c}} \wedge e^{\mathbf{c}} \\ &= T^{\mathbf{a}} \wedge T_{\mathbf{a}} - e^{\mathbf{a}} \wedge e^{\mathbf{b}} \wedge \overset{\omega}{\mathcal{F}}_{\mathbf{ab}}. \end{aligned} \tag{D7}$$

These terms are compatible with the degenerate limit. However, barring the field-dependent couplings as discussed in the second-to-last paragraph in Sec. IV, they do not affect the quantum dynamics of the vierbein and LL-gauge fields since these topological terms give the propagators of neither the vierbein nor the LL-gauge fields.

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