Editors' Suggestion

# In-in correlators and scattering amplitudes on a causal set

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Causal set theory is an approach to quantum gravity in which spacetime is fundamentally discrete at the Planck scale and takes the form of an irregular Lorentzian lattice, or "causal set," from which continuum spacetime emerges in a large-scale (low-energy) approximation. In this work, we present new developments in the framework of interacting quantum field theory on causal sets. We derive a diagrammatic expansion for in-in correlators in local scalar field theories with finite polynomial interactions. We outline how these same correlators can be computed using the double-path integral, which acts as a generating functional for the in-in correlators. We modify the in-in generating functional to obtain a generating functional for in-out correlators. We define a notion of scattering amplitudes on causal sets with noninteracting past and future regions and verify that they are given by $S$-matrix elements (matrix elements of the time-evolution operator). We describe how these formal developments can be implemented to compute early Universe observables under the assumption that spacetime is fundamentally discrete.

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## I. INTRODUCTION

The challenge of quantum gravity is bridging the mathematical and conceptual disparities between quantum mechanics and general relativity [1–4]. These disparities can also be credited for the proliferation of potential resolutions, since the resolution that we will reach will depend on the building blocks we used to construct it (a canonical or path integral formalism, a continuum or a discretuum etc.). But ultimately, this division between opposing principles should be reconciled as they all emerge from a unified theory of quantum gravity.

Causal set theory (CST) is an approach to quantum gravity in which spacetime is fundamentally discrete at the Planck scale [1,5–9]. Taking its cue from theorems in Lorentzian geometry that state that in past- and future-distinguishing spacetimes (of which globally hyperbolic spacetimes are a subclass) the causal structure determines the metric uniquely up to the conformal factor [10,11], CST elevates the causal structure of spacetime to be its principal feature [5,12–14]. The missing ingredient—the conformal factor or volume measure—is accounted for by the discreteness that replaces spacetime points with spacetime "atoms," which need only be counted in order to compute spacetime volumes. Concretely, in CST, spacetime takes the form of a Lorentzian lattice or *causal set*, a collection of elements arranged in a partial order $\prec$, which encodes the causal relations between them, where $x \prec y$ reads as "$x$ precedes $y$" or "$x$ lies in the causal past of $y$." The continuum geometries of general relativity must emerge from this discrete structure via some large-scale approximation and coarse graining [5,15]. Much progress has been made in recent years in extracting continuum quantities, such as dimension and curvature, directly from the causal set (see, for example, [16–21]).

One of the important achievements of this line of work has been the formulation of quantum field theory on a causal set [22–28]. Acting as a bridge between the discrete and the continuum, this work was able to reproduce the Green functions of continuum geometries from underlying causal sets through combinatorial means [22,29]. It also established a distinguished vacuum state on a causal set [24]. This last achievement is revealing of the fact that a causal set is more akin to a curved geometry than to a flat one, sharing the challenges posed by curvature—there may be no distinguished vacuum state or no natural way to define asymptotic states or an $S$ matrix. Indeed, the construction of a distinguished vacuum on a causal set prompted the Sorkin-Johnston proposal for a distinguished vacuum state in continuum spacetimes [30–34], which coincides with the ground state of the Hamiltonian in the case of static spacetimes. It also offered new insight into the computation of entanglement entropy [35–37].

In this work, we build on [26–28] in extending the formalism of quantum field theory on causal sets to include

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interactions. We develop a systematic diagrammatic method for computing in-in correlators and define a notion of scattering on a causal set.

### A. In-in formalism

The in-in formalism is conceptually and mathematically well-suited to CST: It is well-known to be manifestly causal [38–40]; it requires only the notions of a state and of local operators, both of which are readily available on a causal set; and it can be phrased in terms of the Keldysh-Schwinger double path integral, a close relative of the decoherence functional that describes spacetime dynamics in CST [41,42]. These aspects come into play in our work and make the adaptation from the continuum to the discrete possible. In particular, we see direct analogs between our manifestly causal diagrammatic framework for computing in-in correlators on a causal set with the work of [38,39] in the continuum. We also find one major difference: The fundamental discreteness of the causal set acts as a natural cutoff, eliminating the UV divergences that characterize the continuum (although IR divergences are present when the interacting region is infinite). This is because each diagram in our expansion is a subcausal set of the interaction region. Since a finite interaction region contains only finitely many subcausal sets, the associated expansion contains only finitely many diagrams and terminates at a finite order in the interaction coupling.

In recent years, the in-in formalism has become increasingly important as the modern sky surveys whose data it describes continue to make our cosmological observations increasingly precise (e.g., [43,44]), enabling us for the first time to glimpse into the early Universe and probe regimes in which quantum gravity effects become important—the regimes in which CST predicts we will detect the Planckian discreteness of spacetime. To make the most of these observational advances, we must develop robust and predictive theoretical frameworks through which the observations can be interpreted. This is one of the main motivations behind this work, and our intention is for the construction that we present here to be applied in computing cosmological predictions for sky surveys (e.g., primordial non-Gaussianities [45,46]).

### B. Discrete/continuum correspondence

Our formalism enables the computation of in-in correlators on any causal set, but not every causal set is a physically sensible background. We are interested in those causal sets that can give rise to cosmological geometries through the so-called discrete/continuum correspondence [5,47,48]. We say that a causal set $(\mathcal{C}, \prec)$ is well approximated by a continuum geometry $(\mathcal{M}, g)$ of dimension $d$ if there exists a *faithful embedding* of $\mathcal{C}$ in $\mathcal{M}$, that is a map $f\colon \mathcal{C} \to \mathcal{M}$ such that

(i) $q \prec p \Leftrightarrow f(q) \in J^-(f(p))$, where $J^-(x)$ denotes the causal past of $x \in \mathcal{M}$,

(ii) The points $f(\mathcal{C})$ are distributed in $\mathcal{M}$ according to the Poisson distribution at some fixed density $\rho$, and

(iii) The discreteness length $l = \rho^{-\frac{1}{d}}$ is small compared to any curvature length scale in $\mathcal{M}$.

Condition (*i*) requires that $f$ preserves the causal order, and condition (*ii*) requires that the number of elements mapped by $f$ into an interval of volume $V$ in $\mathcal{M}$ is approximately equal to $\rho V$. There are other ways in which one could imagine evenly distributing the points in $\mathcal{M}$, for example, in a regular square grid, but in flat space, such a grid picks a preferred frame: Once the grid is embedded in the continuum, a Lorentz boost will render it no longer a regular grid. The Poisson process has the advantage of Lorentz invariance: A Lorentz transformation merely changes a Poisson distribution of points into another Poisson distribution with the same density [47,49,50].

"Sprinkling" is a method for numerical implementation of the discrete/continuum correspondence: Given a geometry $(\mathcal{M}, g)$, one generates causal sets that faithfully embed into it by sampling points from $\mathcal{M}$ according to the Poisson distribution (i.e., by "sprinkling" into $\mathcal{M}$) [47,51]. This process is particularly suited to a cosmological setting where the observable spacetime volume is finite.

Given some combinatorial quantity $\mathbf{Q}$ (such as a causal set in-in correlator), its average $\mathbb{E}_\rho(\mathbf{Q})$ over a sample of causal sets produced via sprinkling into $\mathcal{M}$ at fixed $\rho$ is a quantity that we can associate with the continuum $(\mathcal{M}, g)$. The result will depend on $\rho$, and the phenomenological effects of spacetime discreteness will be encoded in the high (but finite) $\rho$ corrections. In some cases, one can compute the ensemble average analytically and show that its $\rho \to \infty$ limit is exactly equal to an invariant of the continuum (e.g., this is the case for curvature invariants [52]), suggesting that our framework may prove a useful computational tool independently of whether or not spacetime is fundamentally discrete. In particular, our framework may offer a novel regularization of the UV divergences of the continuum.

### C. Scattering amplitudes

Scattering between asymptotic states is what we measure in particle accelerator experiments, and the associated $S$ matrix has been an invaluable tool for studying quantum field theories on flat space [53]. Its definition on de Sitter has also received some attention, where the lack of a conserved positive energylike quantity poses an obstruction to the definition of asymptotic particle states [54–57]. Therefore, it is natural to ask how one might define the $S$ matrix on a causal set. There are many continuum concepts and structures that are lacking on a causal set, and in particular, rather than continuum hypersurfaces labeled by $t$, we simply have a discrete label (taking its values in the positive integers) for each causal set site. These labels play the role of coordinates and their permutation (the analog of a coordinate transformation) should leave the $S$ matrix invariant. This complicates the question of how analogs of asymptotic states should be defined. Should these states be associated with a specific site, and if so, which particular site would be appropriate? We suggest an answer to this question in the setup where the interaction region is confined to an area between noninteracting "past" and "future" regions.

### D. Main results

Our results are restricted to interacting scalar field theories on a causal set background, under the assumption that all interaction terms are local polynomials of a finite order. With this caveat, the main results of this work are as follows:

(i) *A diagrammatic expansion for the Heisenberg field.* Our derivation of the expansion relies on the commutator representation of time evolution (attributed to Weinberg [40]), which in the continuum, is equivalent to solving the full equation of motion for the Heisenberg field. We show that when the interaction region is finite, the expansion terminates at a finite order. Hence, in this scenario, the Heisenberg field is a finite-order polynomial in the interaction picture fields and vice versa.

(ii) *A diagrammatic expansion for in-in correlators on a causal set.* Our expansion makes use of Wick's theorem and is therefore limited to the case when the in-state is a Gaussian state. Aside from this caveat, our expansion is generic and can be used to compute "nonequal time" in-in correlators of composite operators (as there are no issues with coincident operators in the discrete). Our diagrams have two kinds of legs corresponding to the Feynman and the retarded propagator. Each interaction vertex must be connected to an external vertex via a path of retarded propagators; thus, causality is manifest. When the interaction region is finite, the expansion terminates at a finite order.

(iii) *A generating functional for in-out correlators.* We review the generating functional for in-in correlators and modify it to obtain a generating functional for in-out correlators. These correlators have an infinite expansion even on a finite causal set.

(iv) *The S matrix on a causal set.* We define the analog of asymptotic states in the case where the interaction region is bounded between noninteracting past and future regions. We use this to define the $S$ matrix and give its diagrammatic expansion. The expansion contains the same diagrams as in the continuum (and hence does not terminate).

### E. Outline

In Sec. II, we summarize some useful causal set terminology and review the formalism for a free quantum field theory on a causal set. In Sec. III, we define the Heisenberg field in interacting theories in terms of the causal set time evolution operator and give its diagrammatic expansion. In Sec. IV, we give our diagrammatic rules for in-in correlators on a causal set (leaving the detailed derivation to the Appendix). In Sec. V, we outline how the in-in correlators of the previous section can be computed using a double-path integral whose measure is given by a decoherence functional. We modify this generating functional to give a generating functional for in-out correlators. In Sec. VI, we define the $S$ matrix on a causal set. We conclude with discussion of future directions in Sec. VII.

## II. BACKGROUND

In this section, we introduce the causal set terminology, which we will need and review the operator formalism for a free scalar quantum field theory on a causal set. For a review of causal set theory and related terminology, see [5] and references therein.

### A. Causal sets

A *partially ordered set* or *poset* is a set (called the *underlying set*) together with an irreflexive, antisymmetric and transitive relation, denoted by $\prec$, on it. Given a poset and a pair of elements $x$ and $y$ in it, which satisfy $x \prec y$, the associated *interval*, $\mathrm{int}(x, y)$, is

$$\mathrm{int}(x, y) = \{z | x \prec z \prec y\}.$$

A poset is *locally finite* if every interval of the poset is finite. A *causal set* or *causet* is a locally finite poset. By the standard abuse of notation, we will use $\mathcal{C}$ to denote both the causal set and its underlying set. Given a causet $\mathcal{C}$ and $x, y \in \mathcal{C}$ such that $x \prec y$, we say that $y$ *covers* $x$ and write $x \prec\cdot y$ if the interval $\mathrm{int}(x, y)$ is empty (i.e., if there is no element $z$, which lies between $x$ and $y$ in the partial order). If $x$ and $y$ are unordered by $\prec$, we write $x \natural y$.

A Hasse diagram of a causet $\mathcal{C}$ is a graph whose vertices represent the elements of $\mathcal{C}$ and whose edges represent the covering relations in $\mathcal{C}$, where there is an upward-going edge from $x$ to $y$ if and only if $x \prec\cdot y$ (the other relations are implied by transitivity). A Hasse diagram is the transitive reduction of a directed acyclic graph.

Using a Hasse diagram, we can think of a causal set as a Lorentzian lattice. The lattice sites play the role of spacetime points, while the causal structure is given by the direction of the edges: The past (future) of a lattice site $x$ are all the sites $y$ such that there exists an upward-going (downward-going) path from $y$ to $x$, and if a pair of sites $x$ and $y$ are such that there is no directed path from one to the other, then $x$ and $y$ are spacelike to each other. This is the interpretation of the causal set in CST: The partial order is the causal order, the past of an element $y$ are all elements $x$ such that $x \prec y$, the interval $\mathrm{int}(x, y)$ plays the role of the Alexandrov set (the intersection of the past of a point with the future of another) etc. It is in this sense the causal set encodes a causal structure and furnishes a discrete spacetime.

A tenet of CST is that the points (or "atoms") of the causal set spacetime are indistinguishable from one another, except through the partial order in which they are arranged. Mathematically, this idea is captured by order-isomorphism classes of causal sets, defined as follows. Two causets $\mathcal{C}$ and $\mathcal{C}'$ are *order isomorphic* if there exists a bijection $f\colon\ \mathcal{C}\to\mathcal{C}'$ that preserves the order relation; i.e., $f(x)\prec f(y)\Leftrightarrow x\prec y$ for all $x,y\in\mathcal{C}$ (where by standard abuse of notation, we write $\prec$ to denote both the partial order in $\mathcal{C}$ and in $\mathcal{C}'$). Order isomorphism is an equivalence relation and an *unlabeled causet* is an order-isomorphism equivalence class. An unlabeled causet is represented by an unlabeled Hasse diagram, and a causet representative of it can be constructed simply by labeling the vertices. In practice, working with labeled objects is simpler than working with unlabeled objects. Therefore, we will work with causets directly (not with their equivalence classes), bearing in mind that the choice of the underlying sets or of any labeling attached to the causet elements is pure gauge (just like coordinates are gauge in the continuum). For concreteness, we now specify the causets that we will work with. A causal set is *finite* if its underlying set is finite; i.e., $|\mathcal{C}|<\infty$. A finite causal set $\mathcal{C}$ is *naturally labeled* if its underlying set is $\{1,2,\ldots,|\mathcal{C}|\}$ and $x\prec y\Rightarrow x<y\ \ \forall\ \ x,y\in\mathcal{C}$. From here on, we will work with finite naturally labeled causal sets. If $\mathcal{C}$ and $\mathcal{C}'$ are naturally labeled causets that are order isomorphic to each other, we will say that $\mathcal{C}$ is a *relabeling* of $\mathcal{C}'$ (and vice versa), since the Hasse diagram of one can be obtained from the Hasse diagram of the other simply by relabeling the vertices. What we mean by saying that the labeling is pure gauge is that the physics remains unchanged under relabeling, i.e., under the interchange of $\mathcal{C}$ with $\mathcal{C}'$.

We will have use for the following causal set terminology. A subcauset $S\subseteq\mathcal{C}$ is called a *stem* if $x\in S\Rightarrow y\in S$ for all $y\prec x$ in $\mathcal{C}$. A causet $\mathcal{C}$ is a *chain* if it is totally ordered (i.e., $x\prec y$ or $y\prec x$ for all pairs $x,y\in\mathcal{C}$). A subcauset $S\subseteq\mathcal{C}$ is called a *path* if it is a chain and $y\prec\!\cdot x$ in $S\Rightarrow y\prec\!\cdot x$ in $\mathcal{C}$. An element $x\in\mathcal{C}$ is *minimal* (*maximal*) if there exists no $y\in\mathcal{C}$ such that $y\prec x$ ($y\succ x$). We say that an element $x\in\mathcal{C}$ is in level $l$ if the longest chain of which $x$ is the maximal element has cardinality $l$. Thus, level 1 of $\mathcal{C}$ comprises the minimal elements, level 2 comprises the minimal elements of what remains of $\mathcal{C}$ after the elements in level 1 are deleted, etc.

### B. Free quantum field theory on a causal set

We give a brief review of the Sorkin-Johnston construction for a free scalar field on a causal set [22–24,26,29]. This construction takes the retarded propagator as its starting point and uses it to write down covariant commutation relations for the field operators from which a distinguished vacuum state and its associated Fock representation can be derived. This approach is a covariant alternative to the canonical approach and is therefore suitable for our discrete setting. This is also the reason it is appropriate for curved (continuum) spacetimes to which it has been adapted in [31,32]. When applied to spacetimes with a timelike Killing vector, the Sorkin-Johnston ground state coincides with the ground state of the Hamiltonian [31].

A propagator is a real function of two spacetime points. On a (finite, naturally labeled) causet $\mathcal{C}$, we can represent a propagator as a matrix indexed by the elements of $\mathcal{C}$. The retarded propagator, denoted by $\Delta^R_{xy}$, is defined by the requirement that $\Delta^R_{xy}=0$ whenever $x\not\succ y$. Since $\mathcal{C}$ is naturally labeled, this constraint implies that $\Delta^R_{xy}$ is lower triangular. Various prescriptions can be found in the literature for fixing its nonvanishing entries. These prescriptions can be roughly split into two camps: those in which $\Delta^R_{xy}$ is defined as the inverse of a suitably discretised d'Alembertian [25,58] and those in which $\Delta^R_{xy}$ is defined in a way that guarantees that it approaches the continuum retarded propagator in an appropriate limit [22,29]. Depending on the exact prescription, $\Delta^R_{xy}$ may depend on parameters such as the dimension of the continuum spacetime that approximates $\mathcal{C}$, but once these parameters are fixed, we may regard $\Delta^R_{xy}$ as purely combinatorial in the sense that it can be obtained directly from $\mathcal{C}$ (e.g., via its adjacency matrix). What follows is independent of the choice of $\Delta^R_{xy}$.

Given the retarded propagators, we define the advanced propagator and the Pauli-Jordan (also known as the causal propagator), respectively, as

$$\begin{aligned}\Delta^A_{xy}&=\Delta^R_{yx},\\ \Delta_{xy}&=\Delta^R_{xy}-\Delta^A_{xy}.\end{aligned}\tag{2.1}$$

Next, associate a field operator $\phi(x)$ to each $x\in\mathcal{C}$ and impose the Peierls bracket,

$$[\phi(x),\phi(y)]=i\Delta_{xy}.\tag{2.2}$$

Note that Peierls brackets guarantees that $\phi(x)$ and $\phi(y)$ commute if $x$ and $y$ are spacelike separated (i.e., if $x\natural y$). Following from its definition, the matrix $i\Delta_{xy}$ is Hermitian and can be decomposed via the spectral theorem as

$$\begin{aligned}i\Delta_{xy}&=\sum_{\lambda}\lambda v^{(\lambda)}_x\bar{v}^{(\lambda)}_y\\ &=\sum_{\lambda>0}\lambda v^{(\lambda)}_x\bar{v}^{(\lambda)}_y-\overline{\sum_{\lambda>0}\lambda v^{(\lambda)}_x\bar{v}^{(\lambda)}_y},\end{aligned}\tag{2.3}$$

where the bars denote complex conjugation, and the $v^{(\lambda)}_x$ and the $\lambda$ are the eigenvectors and eigenvalues of $i\Delta_{xy}$, respectively,

$$i\Delta_{xy}v^{(\lambda)}_x=\lambda v^{(\lambda)}_x.\tag{2.4}$$

In the first line of (2.3), the sum is over all the eigenvalues $\lambda$. The second line (where the sums are restricted to the positive $\lambda$) is obtained by using the fact that the $\lambda$'s are real and that if $\lambda > 0$ is an eigenvalue with eigenvector $v^{(\lambda)}$, then $-\lambda$ is an eigenvalue with eigenvector $v^{(-\lambda)} = \bar{v}^{(\lambda)}$. The first term on the second line is called the *positive part* of $i\Delta$,

$$\mathrm{Pos}(i\Delta) = \sum_{\lambda>0} \lambda v_x^{(\lambda)} \bar{v}_y^{(\lambda)}, \tag{2.5}$$

while the second term is its complex conjugate. Another way to express $i\Delta$ as a difference between a $c$ number and its complex conjugate is to consider the expectation value of the Peierls bracket in some state of choice to obtain

$$\langle[\phi(x), \phi(y)]\rangle = W_{xy} - \bar{W}_{xy} = i\Delta_{xy}, \tag{2.6}$$

where $W_{xy} = \langle\phi(x)\phi(y)\rangle$ is the Wightman function in the chosen state. The insight of the Sorkin-Johnston formalism is that the eigenvectors of $i\Delta$ form a distinguished basis, which can be used to define a distinguished Gaussian vacuum state $|0\rangle$ by requiring that

$$\langle 0|\phi(x)\phi(y)|0\rangle = \mathrm{Pos}(i\Delta_{xy}). \tag{2.7}$$

We call $|0\rangle$ the Sorkin-Johnston (SJ) vacuum.

The SJ vacuum has a Fock representation where the positive eigenvalues and their associated eigenvectors play the role of positive frequencies and mode functions, respectively. For each $\lambda > 0$, introduce a pair of conjugate ladder operators $a_\lambda$ and $a_\lambda^\dagger$ and impose the commutation relations,

$$[a_\lambda, a_{\lambda'}^\dagger] = \delta_{\lambda,\lambda'}. \tag{2.8}$$

The SJ vacuum is the state annihilated by the $a_\lambda$, and the Fock representation is built by acting on it with the $a_\lambda^\dagger$. Expanding the fields as

$$\phi(x) = \sum_{\lambda>0} \sqrt{\lambda}(v_x^\lambda a_\lambda + \bar{v}_x^\lambda a_\lambda^\dagger), \tag{2.9}$$

one can verify that the Wightman function of the SJ state is given by (2.7).

### C. Causal ordering and the Feynman propagator

To complete our discussion of the free theory, we introduce the notion of *causal ordering*. Given some $x, y \in \mathcal{C}$, we say that the product $\phi(x)\phi(y)$ is causally ordered if $x \nprec y$. We write $C$ to denote the causal ordering operator whose action on a product of two fields is

$$C[\phi(x)\phi(y)] = \begin{cases} \phi(x)\phi(y) & \text{if } x \succ y \\ \phi(y)\phi(x) & \text{if } x \prec y, \end{cases} \tag{2.10}$$

where for a spacelike pair of points $x \natural y$, the associated field operators commute and the $C$ ordering is trivial: $C[\phi(x)\phi(y)] = \phi(x)\phi(y) = \phi(y)\phi(x)$. More generally, a product of field operators is causally ordered if no field operator is to the right of an operator that lives in its past. In a labeled causal set, ordering a product of operators by decreasing label from left to right always results in a causal ordering, e.g., $\phi(4)\phi(4)\phi(2)\phi(1)$.

The anticausal ordering operator $\bar{C}$ orders a product of field operators so that no field operator is to the left of an operator that lives in its past. In a labeled causal set, ordering a product of operators by increasing label from left to right always results in an anticausal ordering, e.g., $\phi(1)\phi(2)\phi(4)\phi(4)$.

Causal ordering is the causal set analog of the time ordering of the continuum, and we use it to define the Feynman propagator,

$$\Delta_{xy}^F = \langle C[\phi(x)\phi(y)]\rangle. \tag{2.11}$$

## III. INTERACTING FIELDS

In the interacting theory, the interaction picture fields carry the free evolution. In the causal set context, this means that the interaction picture fields satisfy the Peierls bracket (2.2) and have the mode expansion (2.9). We will denote the interaction picture fields simply by $\phi(x)$. We write $\mathcal{H}$ to denote the interacting Hamiltonian (density) in the interaction picture, and we restrict ourselves to local Hamiltonians, which are finite polynomials in the field, e.g., $\mathcal{H}(x) = \frac{g}{n!}\phi^n(x)$.

The Heisenberg picture is defined by analogy with the continuum, where the Heisenberg field $\phi^H(t, \mathbf{x})$ is related to the interaction picture field $\phi(t, \mathbf{x})$ via

$$\phi^H(t, \mathbf{x}) = U^\dagger(t, t_0)\phi(t, \mathbf{x})U(t, t_0), \tag{3.1}$$

where

$$U(t, t_0) = T\left[e^{-i\int_{t_0}^t H(t)dt}\right] \quad (t \geq t_0) \tag{3.2}$$

is the time-evolution operator and where $H$ is the interacting Hamiltonian in the interaction picture. The analogy suggests that we replace the time integral by a sum over causal set points and the time-ordering $T$ with the causal ordering $C$ (defined in Sec. II C). Under the action of $C$, all field commutators vanish, and we can express the exponential of a sum as a product of exponentials. This is the motivation for the following definitions.

Given a causet $\mathcal{C}$ and some $x, y \in \mathcal{C}$, we define the following family of evolution operators,

$$V_x = C\left[\prod_{z\prec x} e^{-i\mathcal{H}(z)}\right], \tag{3.3}$$

$$U_{x,y} = C\left[\prod_{y\le z<x} e^{-i\mathcal{H}(z)}\right] \quad \text{for } x > y, \tag{3.4}$$

$$U_x = \begin{cases} 1 & \text{if } x = 1, \\ U_{x,1} = C\left[\prod_{z<x} e^{-i\mathcal{H}(z)}\right] & \text{if } x > 1, \end{cases} \tag{3.5}$$

and note that they satisfy the following relations:

$$V_x^\dagger V_x = U_x^\dagger U_x = 1, \tag{3.6}$$

$$U_{x,y}U_{y,z} = U_{x,z}, \tag{3.7}$$

$$U_{x,z}U_{y,z}^\dagger = \begin{cases} 1 & \text{if } x = y \\ U_{x,y} & \text{if } x > y \\ U_{y,x}^\dagger & \text{if } x < y, \end{cases} \tag{3.8}$$

$$V_x^\dagger \mathcal{O}(x) V_x = U_x^\dagger \mathcal{O}(x) U_x \text{ for any local operator } \mathcal{O}(x). \tag{3.9}$$

We say that $V_x$ is a covariant operator because it relies only the partial order $\prec$, while $U_{x,y}$ and $U_x$ are label-dependent operators because they rely on the total order $<$ of the labeling. In other words, $V_x$ is a physical operator, and $U_{x,y}$ and $U_x$ are gauge-dependent operators.

We use the covariant operator $V_x$ to define the Heisenberg picture on a causal set: Given a local operator $\mathcal{O}(x)$ in the interaction picture, the Heisenberg picture operator $\mathcal{O}^H(x)$ is given by

$$\mathcal{O}^H(x) = V_x^\dagger \mathcal{O}(x) V_x. \tag{3.10}$$

In practice, working with gauge-dependent operators is simpler, and thanks to relation (3.9), we can express the Heisenberg operator in terms of $U_x$ as

$$\mathcal{O}^H(x) = U_x^\dagger \mathcal{O}(x) U_x. \tag{3.11}$$

(Note that our convention for the causal set evolution operators differs from that of [27,28], but the resulting Heisenberg operators are independent of convention.)

### A. Expanding the Heisenberg field

In the continuum, one solves the full equation of motion order by order in the interaction coupling to obtain a perturbative expansion for the Heisenberg field in terms of interaction picture fields. Alternatively, the same expansion can be extracted from the nested commutator expression for the Heisenberg field [40] (see also [38,39]),

$$\begin{aligned}\phi^H(t,\mathbf{x}) &= \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \ldots \int_{-\infty}^{t_{n-1}} dt_n [\ldots[[\phi(t,\mathbf{x}), H(t_1)], H(t_2)]\ldots, H(t_n)], \\ &= \sum_{n=0}^{\infty}(-i)^n \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t} dt_n [\ldots[[\phi(t,\mathbf{x}), H(t_1)], H(t_2)]\ldots, H(t_n)]\Theta(t_1,\ldots,t_n),\end{aligned} \tag{3.12}$$

where in the second line, the time integrals all share the same domain $(-\infty, t)$ thanks to the introduction of the generalized step function $\Theta(t_1,\ldots,t_n)$ defined as

$$\Theta(t_1,\ldots,t_n) = \begin{cases} 1 & t_1 \ge t_2 \ge \cdots \ge t_n \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Equivalently, $\Theta(t_1,\ldots,t_n)$ can be expressed as a product of step functions,

$$\Theta(t_1,\ldots,t_n) = \Theta(t_1-t_2)\Theta(t_2-t_3)\cdots\Theta(t_{n-1}-t_n), \tag{3.14}$$

where

$$\Theta(x-a) = \begin{cases} 1 & x \ge a \\ 0 & x < a. \end{cases} \tag{3.15}$$

(This latter form makes explicit the similarities with the retarded product of [59].)

The above formalism can be adapted to the causal set setting where it provides a route for expanding the Heisenberg field in the absence of an equation of motion. As we show in the Appendix, starting from expression (3.11) for the causal set Heisenberg field, one obtains the following expansion:

$$\phi^H(x) = \sum_{n=0}^{\infty}(-i)^n \sum_{z_1=1}^{x-1}\cdots\sum_{z_n=1}^{x-1}[\ldots[[\phi(x), \mathcal{H}(z_1)], \mathcal{H}(z_2)]\ldots, \mathcal{H}(z_n)]\Upsilon(z_1,\ldots,z_n), \tag{3.16}$$

where $\phi(x)$ is the interaction picture field, $\mathcal{H}(z)$ is the interacting Hamiltonian in the interaction picture and $\Upsilon(z_1, \ldots, z_n)$ is a modification of $\Theta(z_1, \ldots, z_n)$, which is sensitive the saturation of the inequality $z_1 \geq z_2 \geq \ldots \geq z_n$,

$$\Upsilon(z_1, \ldots, z_n) = \begin{cases} \frac{1}{l_1! \ldots l_k!} & z_1 \geq z_2 \geq \ldots \geq z_n \\ 0 & \text{otherwise,} \end{cases} \tag{3.17}$$

where $k$ denotes the number of strings of equal signs in $z_1 \geq z_2 \geq \ldots \geq z_n$, and $l_i$ is the length of the $i^{th}$ string of equal signs (i.e., the number of $z$ variables equated by the string). For example, $\Upsilon(8,7,5,2,1) = 1$, $\Upsilon(7,7,5,4,4,4) = \frac{1}{2!1!3!}$ and $\Upsilon(7,5,7,4,4,4) = 0$.

[Note that one could replace $\Theta$ with $\Upsilon$ in (3.12) since $\Theta$ and $\Upsilon$ differ only on sets of measure zero. On the other hand, in the discrete, there are no nonempty sets of measure zero, in particular, the cases when a number of variables take the same value contribute nonvanishingly to the sum and the definition of $\Upsilon$ ensures that these contributions are weighted correctly.]

To evaluate (3.16), we apply the result of [60] that expresses the commutator of functions of noncommuting operators as an expansion in powers of the commutators of these operators with each other. In our case, the operators are the fields $\phi(1), \ldots, \phi(x-1)$ that satisfy the commutation relations (2.2). Following [38] we define

$$\mathcal{F}_n(\phi(x), \phi(z_1), \ldots, \phi(z_n)) = [\ldots[[\phi(x), \mathcal{H}(z_1)], \mathcal{H}(z_2)] \ldots, \mathcal{H}(z_n)], \tag{3.18}$$

and note that $\mathcal{F}_n = [\mathcal{F}_{n-1}, \mathcal{H}(z_n)]$. This enables us to apply the result of [60] as

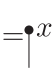
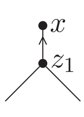
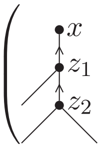
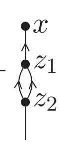
$$\mathcal{F}_n = -\underbrace{\sum_{k_1} \cdots \sum_{k_{n-1}} \sum_{k_x}}_{k \equiv \sum k_i \neq 0} \left( \frac{(-i\Delta_{x,z_n})^{k_x}}{k_x!} \prod_{i=1}^{n-1} \frac{(-i\Delta_{z_i,z_n})^{k_i}}{k_i!} \right) \times \partial^{k_1}_{\phi(z_1)} \cdots \partial^{k_{n-1}}_{\phi(z_{n-1})} \partial^{k_x}_{\phi(x)} \mathcal{F}_{n-1} \partial^{k}_{\phi(z_n)} \mathcal{H}(z_n), \tag{3.19}$$

where the sums are over all non-negative integers with the restriction that at least one of the $k_i$ is nonzero. This expansion has a useful diagrammatic representation that we give below.

The diagrammatic expansion for the nested commutator. Associate a vertex with each of the points $x, z_1 \ldots, z_n$. We call $x$ the external vertex and $z_i$ the internal vertices. The number of half-legs meeting at each vertex is equal to the number of fields at the associated point. To form the diagrams, connect the half-legs in all possible ways to form directed edges with the following properties: ($i$) every internal vertex is connected to the external vertex by at least one directed path, and ($ii$) every directed edge is of the form $z_i \to x$ or $z_i \to z_j$ with $i > j$. Property ($i$) corresponds to the restriction that at least one of the $k_i$ is nonzero. Property ($ii$) reflects the factors of $\Delta_{x,z_n}$ and $\Delta_{z_i,z_n}$ in (3.19), in particular, the fact that $z_n$ is always the second argument. Connecting the legs corresponds to the taking of derivatives, and each possible diagram corresponds to a different set of values for $k_1, \ldots, k_{n-1}, k_x$. The nested commutator in (3.16) is equal to the sum of these diagrams when each diagram is interpreted as follows:

(i) Each directed edge $a \to b$ gives a factor of $-i\Delta_{ba}$.
(ii) Each internal vertex contributes a factor of the coupling and any other constants appearing in $\mathcal{H}$.
(iii) Each uncontracted half-leg at a vertex contributes a field operator at the associated point; the product of these factors is ordered with $\phi(x)$ on the left and then factors of $\phi(z_i)$ with $i$ increasing from left to right.
(iv) Multiply by an overall factor coming from the different ways of connecting the half-legs to form the diagram.
(v) Multiply by an overall factor of $(-1)^n$.

For example, when $\mathcal{H} = \frac{g}{3!}\phi^3$, the Heisenberg field is given by

$\phi^H(x) =$  $- i \sum_{z_1=1}^{x-1}$  $+ (-i)^2 \sum_{z_1,z_2=1}^{x-1} \Bigg($  $+$  $\Bigg) \Upsilon(z_1, z_2) + \cdots$

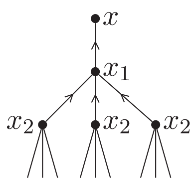
$$= \phi(x) + \frac{g}{2} \sum_{z_1=1}^{x-1} \Delta^R_{x,z_1} \phi(z_1)^2 + \frac{g^2}{2} \sum_{z_1,z_2=1}^{x-1} \Delta^R_{x,z_1} \Delta^R_{z_1,z_2} \left(\phi(z_1)\phi(z_2)^2 - 2i\Delta^R_{z_1,z_2}\phi(z_2)\right) \Upsilon(z_1, z_2) + \cdots,$$

where in the second line we used the relation,

$$\Delta_{z_i z_j} \Upsilon(z_1, \ldots, z_n) = \Delta^R_{z_i z_j} \Upsilon(z_1, \ldots, z_n) \tag{3.20}$$

to replace the Pauli-Jordan $\Delta$ by the retarded propagator $\Delta^R$. In this work, nested commutators always appear together with the ordering function $\Upsilon$, and therefore, we always interpret the directed edges as retarded propagators.

The field expansion (3.16) terminates at a finite order in the interaction coupling. This order increases with the order of the interaction Hamiltonian and with the level of $x$ (the length of longest chain $x$ is a maximal element of, see Sec. II A). This becomes particularly transparent when considering the diagrammatic representation for $\phi^H(x)$. Suppose $x$ is at level 3 with $x \succ x_1 \succ x_2$, and let $\mathcal{H} = \frac{g}{4!}\phi^4$. The diagram that contains the largest number of interaction vertices is




$$= \frac{g^4}{216}\Delta^R_{x,x_1}(\Delta^R_{x_1,x_2})^3\phi(x_2)^9, \quad (3.21)$$

where $z_1 = x_1$ and $z_2 = z_3 = z_4 = x_2$. Since $x_2$ is a minimal element, there is no way to add another internal vertex to produce a nonvanishing diagram. This illustrates the general pattern: When $\mathcal{H} \sim \phi^r$, the diagrams with the highest number of interaction vertices are those in which the directed edges form a tree with 1 interaction vertex as the root, $r-1$ interaction vertices directly below it, $(r-1)^2$ interaction vertices below them, etc., with the last layer containing $(r-1)^{l-2}$ vertices, where $l$ denotes the level of $x$. Hence, the expansion terminates at order $\sum_{k=0}^{l-2}(r-1)^k = \frac{(r-1)^{l-1}-1}{r-2}$ in the coupling constant [which is of order $(r-1)^{l-1}$ in the field, i.e. $\phi^{(r-1)^{l-1}}$].

### B. Properties of the field algebras

Here we list some properties of the Heisenberg and interaction picture fields on a causal set.

#### *1. Causality*

Heisenberg fields at spacelike separated points commute,

$$[\phi^H(x), \phi^H(y)] = 0 \quad \text{for all } x, \quad y \in \mathcal{C} \quad \text{with} \quad x \natural y. \quad (3.22)$$

This follows from the definition (3.11) of the Heisenberg field and the Peierls brackets (2.2).

#### *2. Polynomial property*

As we saw in Sec. III A, the Heisenberg field $\phi^H(x)$ can be written as

$$\phi^H(x) = \phi(x) + Q_x(\phi(y); y \prec x), \quad (3.23)$$

where $Q_x$ is a finite order polynomial in the interaction picture fields in the past of $x$. We can also invert this relationship and write

$$\phi(x) = \phi^H(x) + P_x(\phi^H(y); y \prec x), \quad (3.24)$$

where $P_x$ is a finite order polynomial in the Heisenberg fields in the past of $x$. To see this, suppose (3.24) is true for all $x$ in levels $0, 1, \ldots, l$ and consider $x$ at level $l+1$. Then $Q_x$ is a polynomial in fields $\phi(y)$ with $y$ at level $l$ or below. By the inductive assumption, these $\phi(y)$ can be written as a finite order polynomial in Heisenberg picture fields. Defining $P_x(\phi^H(y); y \prec x) = -Q_x(\phi(y); y \prec x)$ and rearranging (3.23) for $\phi_x$ gives the result.

#### *3. Observable algebras*

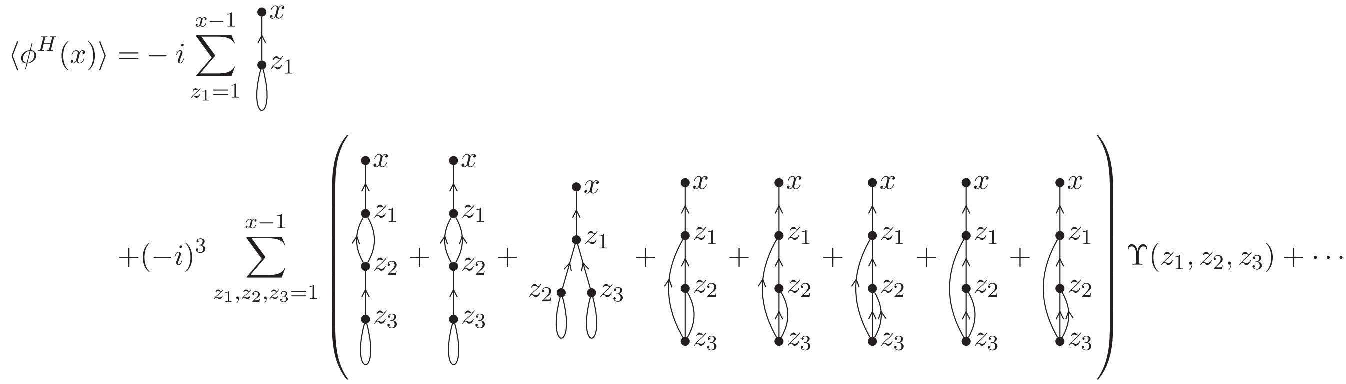
Given a causet $\mathcal{C}$ and a subcauset $\mathcal{R} \subseteq \mathcal{C}$, we write $\mathfrak{A}^H_{\mathcal{R}}$ and $\mathfrak{A}_{\mathcal{R}}$ denote the algebras generated by $\{\phi^H(x)\}_{x\in\mathcal{R}}$ and $\{\phi(x)\}_{x\in\mathcal{R}}$, respectively. Then it is a corollary of the polynomial property that $\mathfrak{A}^H_{\mathcal{R}} = \mathfrak{A}_{\mathcal{R}}$ if and only if $\mathcal{R}$ is a stem in $\mathcal{C}$.

## IV. THE IN-IN FORMALISM

Here we build on the results of Sec. III A to obtain a diagrammatic expansion for in-in correlators on a causal set. Throughout, we assume that the in-state is a Gaussian state so Wick's theorem applies.

### A. The expectation value of the field

We begin with the simplest in-in correlator: the expectation value of the field, $\langle\phi^H(x)\rangle$. Having expanded $\phi^H(x)$ in Sec. III A as a causally ordered polynomial in the interaction picture fields, we can apply Wick's theorem to obtain $\langle\phi^H(x)\rangle$. Diagrammatically, this corresponds to taking each diagram in the expansion of $\phi^H(x)$ and joining its half-legs to create undirected edges in all possible ways. Each undirected edge is a Feynman propagator, and each diagram is weighted by an additional Wick factor. Assuming $\langle\phi(x)\rangle = 0$, only the diagrams in which all half-legs are contracted contribute to the expectation value $\langle\phi^H(x)\rangle$. For example, when $\mathcal{H} = \frac{g}{3!}\phi^3$, we apply Wick's theorem to (3.20) and obtain



(4.1)

Note that thus far, all our diagrams are labeled—this was necessary in the expansion of the Heisenberg field because the vertex labels kept track of the order of the field operators in the expansion. However, in computing the scalar quantity $\langle \phi^H(x) \rangle$, this is no longer necessary, and we now proceed to obtain an expansion in terms of unlabeled diagrams akin to the Feynman diagrams of the continuum.

Consider some labeled diagram $G_n$ with $n$ internal vertices that appears in the expansion of $\langle \phi^H(x) \rangle$ and permute the labels of its internal vertices to produce another diagram $G_n'$. If $G_n'$ does not appear in the expansion of $\langle \phi^H(x) \rangle$, then it must contain a directed edge $z_i \to z_j$ (which we interpret as $-i\Delta^R_{z_j,z_i}$) for some $i < j$. We conclude that the product $G_n' \Upsilon(z_1, \ldots, z_n)$ vanishes, since $\Upsilon(z_1, \ldots, z_n)$ vanishes when $z_i < z_j$ and $\Delta^R_{z_j z_i}$ vanishes when $z_i \geq z_j$. Using this insight, we may write $\langle \phi^H(x) \rangle$ as

$$\langle \phi^H(x) \rangle = \sum_{n=0}^{\infty} \sum_{G_n} (-i)^n \sum_{z_1 \cdots z_n = 1}^{x_1 - 1} G_n \Upsilon(z_1 \cdots z_n), \quad (4.2)$$

where now $G_n$ denotes either an allowed labeled diagram or a diagram obtained from one via a permutation of the internal vertex labels. Now, note that the internal vertices are simply dummy indices, all of which are summed over the same domain. Therefore, for each isomorphism class $[G_n]$, choose some representative $G_n$, and for each diagram $G_n' \in [G_n]$, apply the necessary coordinate transformation $(z_1 \ldots z_n) \xrightarrow{G_n'} (z_1 \ldots z_n)$ to bring $G_n'$ into the labeling of $G_n$. Collecting the identical diagrams and taking into account the action of each coordinate transformation on $\Upsilon$, we obtain

$$\langle \phi^H(x) \rangle = \sum_{n=0}^{\infty} \sum_{[G_n]} (-i)^n \sum_{z_1 \cdots z_n = 1}^{x_1 - 1} \frac{G_n}{|\mathrm{Aut}(G_n)|} \sum_{\pi} \Upsilon(\pi), \quad (4.3)$$

where $[G_n]$ denotes an unlabeled diagram, $G_n$ denotes a labeled diagram representative of $[G_n]$, $\mathrm{Aut}(G_n)$ is the group of automorphisms of $G_n$, which keep the $x$ vertex fixed, $\pi = z_{i_1}, \ldots, z_{i_n}$ is a permutation of $z_1, \ldots, z_n$ and

$$\Upsilon(z_{i_1}, \ldots, z_{i_n}) = \begin{cases} \frac{1}{l_1! \ldots l_k!} & z_{i_1} \geq z_{i_2} \geq \ldots \geq z_{i_n} \\ 0 & \text{otherwise,} \end{cases} \quad (4.4)$$

where $k$ denotes the number of strings of equal signs in $z_{i_1} \geq z_{i_2} \geq \ldots \geq z_{i_n}$, and $l_i$ is the length of the $i^{th}$ string of equal signs. Noting that for any set of values for $z_1, \ldots, z_n$, we have $\sum_\pi \Upsilon(\pi) = 1$, expansion (4.3) simplifies to a sum over unlabeled diagrams $[G_n]$,

$$\langle \phi^H(x) \rangle = \sum_{n=0}^{\infty} \sum_{[G_n]} [G_n], \quad (4.5)$$

where each $[G_n]$ is assigned a value according to the diagrammatic rules summarized below.

The diagrammatic expansion for $\langle \phi^H(x) \rangle$. The diagrams that contribute to the expansion at order $n$ contain a single external vertex labeled $x$ and $n$ unlabeled internal vertices. The valency of the internal vertices is given by the order of the Hamiltonian in the field and all half-legs are contracted to form legs. Each internal vertex is connected to $x$ by at least one directed path, and there are no closed directed cycles. The value of each diagram is obtained by assigning each internal vertex with a dummy index $z$ and following the rules:

(i) Each internal vertex $z$ gives a sum $i \sum_{z=1}^{x-1}$ and a factor of the coupling and any other constants appearing in the interaction Hamiltonian $\mathcal{H}$.

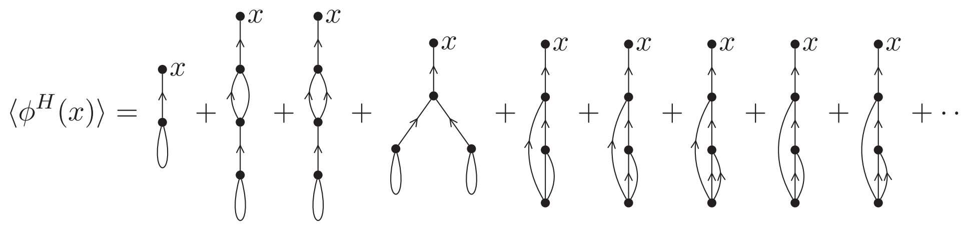
(ii) Each directed edge $a \to b$ gives a factor of $-i\Delta^R_{ba}$.

(iii) Each undirected edge $a — b$ gives a factor of $\Delta^F_{ab}$.

(iv) Multiply by an overall factor coming from the different ways of connecting the half-legs to form the diagram,

(v) Divide by $|\mathrm{Aut}(G_n)|$, the number of automorphisms which keep $x$ fixed.

For example, expansion (4.1) can be rewritten in terms of unlabeled diagrams as




### B. In-in correlators

In Sec. IV A we gave diagrammatic rules for computing the expectation value of the Heisenberg field, $\langle\phi^H(x)\rangle$. Here, we generalize these rules to in-in correlators of causally ordered products of local operators.

Given some causet $\mathcal{C}$ and an integer $1 \le k \le |\mathcal{C}|$, let $x_1, x_2, \ldots, x_k$ denote integers satisfying $|\mathcal{C}| \ge x_1 > x_2 \ldots > x_k \ge 1$. For each $i = 1, \ldots, k$ we write $\mathcal{O}^H(x_i)$ to denote a local Heisenberg operator at $x_i$. We allow for the dependence of $\mathcal{O}^H(x_i)$ on $\phi(x_i)$ to be different to the dependence of $\mathcal{O}^H(x_j)$ on $\phi(x_j)$, but we restrict ourselves to operators that are finite-order polynomials in the Heisenberg fields, e.g., $\mathcal{O}^H(x_2) = 3(\phi^H(x_2))^4$. We seek to compute the in-in correlator,

$$\langle\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k)\rangle. \tag{4.7}$$

We proceed in two stages. First, we expand the operator product $\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k)$ as a sum of locally ordered products of interaction picture fields. Second, we apply Wick's theorem to compute the expectation value of each term in the expansion.

Our strategy for obtaining the expansion of $\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k)$ is to express it as a nested product (cf. Lemma A.3),

$$\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k) = U^\dagger_{x_k}\mathcal{O}_{1\ldots k}U_{x_k}, \tag{4.8}$$

where $\mathcal{O}_{1\ldots k}$ is defined recursively via

$$\mathcal{O}_{1\ldots p} = \begin{cases} \mathcal{O}(x_1) & p = 1 \\ U^\dagger_{x_{p-1},x_p}\mathcal{O}_{1\ldots p-1}U_{x_{p-1},x_p}\mathcal{O}(x_p) & 1 < p \le k. \end{cases} \tag{4.9}$$

This enables us to obtain the product expansion by the recursive application of the relation (cf. Lemma A.2),

$$U^\dagger_{x,y}\mathcal{O}U_{x,y} = \sum_{n=0}^{\infty}(-i)^n \sum_{z_1,\ldots z_n=y}^{x-1}[\ldots[[\mathcal{O},\mathcal{H}(z_1)],\mathcal{H}(z_2)]\ldots,\mathcal{H}(z_n)]\Upsilon(z_1,\ldots,z_n), \tag{4.10}$$

where $\mathcal{O}$ is any (not necessarily local) operator, and the $n = 0$ term is understood to be equal to $\mathcal{O}$. We leave the result to the Appendix (cf. Corollary A.5), but note that inside the expectation value, the expansion simplifies to (cf. Lemma A.6)

$$\langle\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k)\rangle = \sum_{n=0}^{\infty}(-i)^n \sum_{z_1,\ldots,z_n=1}^{x-1}\langle[\ldots[[\mathcal{O}(x_1)\ldots\mathcal{O}(x_k),\mathcal{H}(z_1)],\mathcal{H}(z_2)]\ldots,\mathcal{H}(z_n)]\rangle\Upsilon(z_1,\ldots,z_n), \tag{4.11}$$

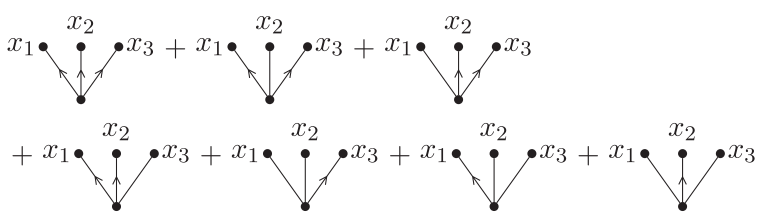
where on the rhs, the operators $\mathcal{O}(x_i)$ are in the interaction picture, and $\mathcal{H}$ is the interacting Hamiltonian in the interaction picture.

To evaluate the expectation value, we apply Wick's theorem diagrammatically. The diagrammatic expansion of the nested correlator in (4.11) is obtained by modifying the rules given in 3.1 so that each diagram has $k$ external vertices labeled $x_1, \ldots, x_k$, and each internal vertex is connected to at least one external vertex by at least one directed path. Additionally, the number of half-legs at each external vertex is equal to the number of fields at that point [e.g., three half-legs at $x_i$ if $\mathcal{O}(x_i) = (\phi^H(x_i))^3$], and there must be no outgoing edges from any of the external vertices. To apply Wick's theorem, we connect the remaining half-legs into undirected edges. The resulting diagrams are labeled, and we repeat the procedure outlined in Sec. IV A to obtain an expansion in terms of an unlabeled diagram. The rules are obtained from those given in Sec. IV A by allowing for multiple external vertices and accounting for any additional factors they may carry. For completeness, the rules are presented below.

The diagrammatic expansion for $\langle \mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k)\rangle$. The diagrams that contribute to the expansion at order $n$ contain $k$ external vertices labeled $x_1, \ldots, x_k$ and $n$ unlabeled internal vertices. The valency of the internal (external) vertices is given by the order of the Hamiltonian (local operator $\mathcal{O}$) in the field, and all half-legs are contracted to form legs. Each internal vertex is connected to at least one external vertex by at least one directed path. There are no closed directed cycles and no edges directed outward from an external vertex. The value of each diagram is obtained by assigning each internal vertex with a dummy index $z$ and following the rules:

(i) Each internal vertex $z$ gives a sum $i\sum_{z=1}^{x-1}$ and a factor of the coupling and any other constants appearing in the interaction Hamiltonian $\mathcal{H}$.
(ii) Each external vertex $x_i$ gives any constant factors appearing in $\mathcal{O}(x_i)$.
(iii) Each directed edge $a \to b$ gives a factor of $-i\Delta^R_{ba}$.
(iv) Each undirected edge $a$—$b$ gives a factor of $\Delta^F_{ab}$.
(v) Multiply by an overall factor coming from the different ways of connecting the half-legs to form the diagram.
(vi) Divide by $|\mathrm{Aut}(G_n)|$, the number of automorphisms which keep $x_1, \ldots, x_k$ fixed.

See [27,28] for an explicit computation of the two-point function in $\phi^4$ theory. In $\phi^3$, the connected part of $\langle \phi^H(x_1)\phi^H(x_2)\phi^H(x_3)\rangle$ to first order in the coupling is given by




(4.12)

## V. PATH INTEGRALS

The interacting quantum field theory on a causal set described above was first proposed in path integral or "histories" form as a decoherence functional by Rafael Sorkin in [26]. The decoherence functional, or double path integral of Schwinger-Keldysh form, is the basis for an alternative foundation for quantum theory in which histories and events, rather than operators and states, are fundamental (see, for example, [61]). In this section, we will not dwell on this foundational aspect of the path integral approach but instead briefly show how the interacting decoherence functional proposed by Sorkin encapsulates, mathematically, the time-ordered correlation functions in the in-in formalism calculated in Sec. IV. More details of the translation between the operator formalism and the path-integral formalism are given in [27] and expanded on in [28]. We use the decoherence functional to write down a generating functional for the in-in correlators of Sec. IV B. We also give the generating functional for in-out correlators.

### A. The decoherence functional of the free theory

Consider a causal set $\mathcal{C}$ with $N$ elements. The space of histories of a real scalar quantum field theory on $\mathcal{C}$ is the space of real vectors $\mathbb{R}^N$. We denote a vector by $\xi$ with components $\xi_x$, $x \in \mathcal{C}$. The decoherence functional for the free theory in a particular state is given by [26]

$$D_0(\xi,\bar{\xi}) = \langle \delta(\phi_1 - \bar{\xi}_1)\delta(\phi_2 - \bar{\xi}_2)\ldots\delta(\phi_N - \bar{\xi}_N) \times \delta(\phi_N - \xi_N)\ldots\delta(\phi_2 - \xi_2)\delta(\phi_1 - \xi_1)\rangle, \quad (5.1)$$

where we write $\phi_x$ as a shorthand for the free field operator $\phi(x)$, and $\langle\cdot\rangle$ denotes expectation value in a Gaussian state, $\rho$, which may be the SJ state or another state:

$$\langle \hat{O}\rangle = \rho(\hat{O}) = \mathrm{Tr}[\hat{\rho}\,\hat{O}]. \quad (5.2)$$

The decoherence functional should have a label indicating its dependence on the state $\rho$, but we omit it for convenience.

Note that

(i) The decoherence functional is normalized

$$\int d^N\xi d^N\bar{\xi} D_0(\xi,\bar{\xi}) = 1. \quad (5.3)$$

(ii) The ordering of the operators in the expectation value (5.4) uses the label order of $\mathcal{C}$.
(iii) Since operators at spacelike points commute, we have

$$D_0(\xi,\bar{\xi}) = \langle \bar{C}[\delta(\phi_1-\bar{\xi}_1)\delta(\phi_2-\bar{\xi}_2)\ldots \times \delta(\phi_N-\bar{\xi}_N)]C[\delta(\phi_N-\xi_N)\ldots \times \delta(\phi_2-\xi_2)\delta(\phi_1-\xi_1)]\rangle, \tag{5.4}$$

where $C$ and $\bar{C}$ are the causal and anticausal ordering operators defined in Sec. II C, so the $\xi$ delta functions are causally ordered, and the $\bar{\xi}$ delta functions are anticausally ordered.

(iv) The delta functions ensure that the decoherence functional vanishes unless the field values $\xi_x = \bar{\xi}_x$ on every maximal element $x$ of $\mathcal{C}$.

(v) The decoherence functional (5.4) is evaluated in [26].

Let $F(\xi)$ and $G(\xi)$ be real functions on $\mathbb{R}^N$. If we integrate $F(\xi)G(\bar{\xi})$ over all *pairs* of field configurations, $(\xi,\bar{\xi})\in\mathbb{R}^N\times\mathbb{R}^N$ against a *measure*, which equals the decoherence functional, the delta functions in $D_0(\xi,\bar{\xi})$ act to causally/anticausally order the corresponding functions of the field operators:

$$\int_{\mathbb{R}^{2N}} d^N\bar{\xi}d^N\xi D_0(\xi,\bar{\xi})F(\xi)G(\bar{\xi}) = \langle \bar{C}[G(\phi)]C[F(\phi)]\rangle. \tag{5.5}$$

The simplest examples are the two-point correlators, and we have

$$\langle C[\phi_x\phi_y]\rangle = \int_{\mathbb{R}^{2N}} d^N\bar{\xi}d^N\xi D_0(\xi,\bar{\xi})\xi_x\xi_y, \tag{5.6}$$

$$\langle \bar{C}[\phi_x\phi_y]\rangle = \int_{\mathbb{R}^{2N}} d^N\bar{\xi}d^N\xi D_0(\xi,\bar{\xi})\bar{\xi}_x\bar{\xi}_y, \tag{5.7}$$

$$W_{xy} = \langle \phi_x\phi_y\rangle = \int_{\mathbb{R}^{2N}} d^N\bar{\xi}d^N\xi D_0(\xi,\bar{\xi})\bar{\xi}_x\xi_y. \tag{5.8}$$

Slightly more generally, when the functions are monomials in the field components, we have

$$\int_{\mathbb{R}^{2N}} d^N\bar{\xi}d^N\xi D_0(\xi,\bar{\xi})\xi_{x_1}\ldots\xi_{x_l}\bar{\xi}_{y_1}\ldots\bar{\xi}_{y_m} = \langle \bar{C}[\phi_{y_1}\ldots\phi_{y_m}]C[\phi_{x_1}\ldots\phi_{x_l}]\rangle. \tag{5.9}$$

The causal ordering results from the fact that in the integral, the factor $\xi_x$, say, is just a real variable and can be moved anywhere. When $\xi_x$ is moved next to the corresponding operator factor $\delta(\phi_x-\xi_x)$ in the decoherence functional, $\xi_x$ becomes the operator $\phi_x$ which, as an operator, is now in the causally ordered position of $\delta(\phi_x-\xi_x)$ in the operator product.

### B. The decoherence functional of the interacting theory

In the interacting theory with a $\phi^4$ interaction, Sorkin proposed the interacting decoherence functional,

$$D_g(\xi,\bar{\xi}) = D_0(\xi,\bar{\xi})e^{-i(\xi^4-\bar{\xi}^4)\cdot g}, \tag{5.10}$$

where

$$g = (g_1, g_2, g_3, \ldots g_N) \tag{5.11}$$

is a vector of $N$ coupling constants, one for each causet element, and

$$\mathcal{V}_{\rm int}(\xi) = \xi^4\cdot g := \sum_{x=1}^{N}(\xi_x)^4 g_x \tag{5.12}$$

is the self-interaction potential.

The interacting decoherence functional can be generalized to the case of any real polynomial self-interaction. In other words, the $\phi^4$ interaction potential can be replaced by

$$\sum_{x=1}^{N}(\xi_x)^4 g_x \longrightarrow \mathcal{V}_{\rm int}(\xi) = \sum_{x=1}^{N}\mathcal{P}_x(\xi_x), \tag{5.13}$$

where each local $\mathcal{P}_x$ is a real polynomial, which may vary from element to element. Then the interacting decoherence functional is

$$D_g(\xi,\bar{\xi}) = D_0(\xi,\bar{\xi})e^{-i\mathcal{V}_{\rm int}(\xi)+i\mathcal{V}_{\rm int}(\bar{\xi})}. \tag{5.14}$$

Note that

(i) The interacting decoherence functional is normalized; i.e., the integral of (5.14) over $\xi$ and $\bar{\xi}$ equals 1 [27,28].

(ii) The varying local polynomial allows for an interaction that is zero outside some interaction region of the causal set.

(iii) Another generalization is to $M$ scalar fields, $\phi^{(1)},\phi^{(2)},\ldots\phi^{(M)}$, with an interaction potential at each element that is a polynomial in the $M$ fields at that element.

(iv) In making the correspondence with the continuum, if the causal set is well approximated by a spacetime region $\mathcal{M}$ of dimension $d$ and spacetime volume $V$, then we have, for example, in the $\phi^4$ case with constant coupling $g_x = g$ for all $x$,

$$g\sum_{x=1}^{N}(\xi_x)^4 \leftrightarrow \lambda\int_{\mathcal{M}} d^dx\sqrt{-g}\xi(x)^4, \tag{5.15}$$

and so $g\leftrightarrow dV\lambda$, where $dV = V/N = l^d$, $l$ is the discreteness scale, and $\lambda$ is the coupling constant in the continuum.

The interacting causally ordered $n$-point function is given in path integral form as before, with the free decoherence functional measure replaced by the interacting decoherence functional (5.14),

$$\langle C[\phi^H_{x_1}\ldots\phi^H_{x_n}]\rangle = \int_{\mathbb{R}^{2N}} d^N\bar{\xi} d^N\xi D_g(\xi,\bar{\xi})\xi_{x_1}\ldots\xi_{x_n}. \quad (5.16)$$

It can be shown [27,28] that this equals the interacting in-in causally ordered $n$-point function of Sec. IV in a given state $\rho_{\rm in}$. That is, the Heisenberg field operators are given by (3.10) where $\mathcal{H}_x = \mathcal{P}_x(\phi_x)$ for each $x \in \mathcal{C}$, and $\phi_x$ are the interaction picture fields. $\rho_{\rm in}$ depends on the Gaussian state $\rho$ that defines the free decoherence functional (5.4) in the following way. Since the algebra of observables is generated by the interaction picture field operators (see Sec. III B), the in-state $\rho_{\rm in}$ is defined by its value on every product of pairs of interaction picture field operators, whose values are extended to all monomials of interaction picture operators (and thence by linearity to all polynomials) by Wick's theorem. So the state $\rho_{\rm in}$ is fully specified by the Wightman function of $\rho$ in the free theory:

$$\rho_{\rm in}(\phi_x\phi_y) = W_{xy}, \quad (5.17)$$

(plus the one-point functions if they are nonzero in $\rho$).

In the case that $\rho$ is a pure state $|\Psi\rangle$, then $\rho_{\rm in}$ is also a pure state $|\Psi_{\rm in}\rangle$.

### C. Generating functionals

The in-in generating functional is given in terms of the decoherence functional:

$$\mathcal{Z}^{\rm in-in}[J,\bar{J}] = \int d^N\xi d^N\bar{\xi} D_g(\xi,\bar{\xi}) e^{-iJ.\xi} e^{i\bar{J}.\bar{\xi}}, \quad (5.18)$$

where $J$ and $\bar{J}$ are two independent sources. For example, the in-in causally ordered two-point correlator is given by

$$i\frac{\partial}{\partial J_x} i\frac{\partial}{\partial J_y} \mathcal{Z}^{\rm in-in}[J,\bar{J}]|_{J=0,\bar{J}=0} = \langle C[\phi^H_x\phi^H_y]\rangle_{\rho_{\rm in}}, \quad (5.19)$$

with similar expressions with more derivatives for the causally ordered in-in $n$-point correlators. Derivatives with respect to $\bar{J}$ similarly result in anticausally ordered products of field operators.

The generating functional for the causally ordered in-out correlators is not given in terms of the interacting decoherence functional but by a closely related expression,

$$\mathcal{Z}^{\rm in-out}[J] = \frac{\int d^N\xi d^N\bar{\xi} D_0(\xi,\bar{\xi}) e^{-i\mathcal{V}_{\rm int}(\xi)} e^{-iJ\cdot\xi}}{\int d^N\xi d^N\bar{\xi} D_0(\xi,\bar{\xi}) e^{-i\mathcal{V}_{\rm int}(\xi)}}, \quad (5.20)$$

where the case of the $\phi^4$ interaction is shown (again, this can be generalized to any polynomial local interaction). Now we have

$$i\frac{\partial}{\partial J_x} i\frac{\partial}{\partial J_y} \mathcal{Z}^{\rm in-out}[J]|_{J=0} = \frac{\langle \hat{S} C[\phi^H_x\phi^H_y]\rangle}{\langle \hat{S}\rangle}, \quad (5.21)$$

where the $S$ operator is given by

$$\hat{S} = C\left[\prod_{z\in\mathcal{C}} e^{-i\mathcal{H}(z)}\right], \quad (5.22)$$

and the $C$ operator applied to Heisenberg field acts as

$$C[\phi^H(x)\phi^H(y)] = \begin{cases} \phi^H(x)\phi^H(y) & \text{if } x\succ y \\ \phi^H(y)\phi^H(x) & \text{otherwise.} \end{cases} \quad (5.23)$$

To see this, note that the derivatives have no effect on the denominator of the generating functional so the denominator of (5.21) follows directly from (5.5). The numerator, after taking derivatives and setting $J=0$, becomes $\langle C[\prod_z e^{-i\mathcal{H}(z)}\phi_x\phi_y]\rangle$. When $x > y$, the causally ordered product can be expanded as

$$\begin{aligned} &\langle e^{-iH(N)}\ldots e^{-iH(x)} U_x\phi^H_x U^\dagger_x e^{-iH(x-1)}\ldots \\ &\quad\times e^{-iH(y)} U_y\phi^H_y U^\dagger_y e^{-iH(y-1)}\ldots e^{-iH(1)}\rangle \\ &= \langle \hat{S}\phi^H_x\phi^H_y\rangle, \end{aligned} \quad (5.24)$$

where on the lhs, we used relation (3.11) between the Heisenberg and interaction picture fields, and on the rhs, we simplified the products of exponentials. Similarly, when $x < y$, we obtain $\langle \hat{S}\phi^H_y\phi^H_x\rangle$, so overall, the numerator is given by $\langle \hat{S} C[\phi^H_x\phi^H_y]\rangle$.

In contrast to the in-in correlators, the expansion of the in-out correlators as a power series in the coupling(s) does not terminate at a finite order even if the causal set itself is finite. One can see this by expanding (5.21) diagrammatically: The diagrams are identical to those of the continuum, with $\langle \hat{S}\rangle$ given by the sum of vacuum bubble diagrams.

## VI. SCATTERING AMPLITUDES

In this section, we propose a definition for the $S$ matrix on a causal set. In curved spacetime, particles can be produced by a nonstatic metric, and one expects this also in QFT on a causal set that is a sprinkling into a nonstatic spacetime. On the causal set, there are many continuum structures and concepts that are used in continuum QFT that are missing. For example, there is no analog of a Cauchy surface in a causal set. We progress by adapting the concept of "scattering" to the technology we do have on the causal set: the SJ vacuum with its corresponding particle states, noting that in the continuum, in a spacetime with a timelike Killing vector, one can show formally that the SJ vacuum equals the usual canonical vacuum defined by the positive frequency modes [31]. To get as close as possible to the usual setup for scattering calculations, we assume that the interaction region is confined to a region between noninteracting "past" and "future" regions. This enables the definition of "asymptotic" states formally associated to these noninteracting past and future regions, whose states then provide a proposed definition for the causal set $S$ matrix.

To define the past and future regions, we will make use of the following terminology. A subcauset $\mathcal{S} \subseteq \mathcal{C}$ is called a *stem* (or *down-set*) if $x \in \mathcal{S}$ implies that $y \in \mathcal{S}$ for all $y \prec x$. A subcauset $\mathcal{S} \subseteq \mathcal{C}$ is called a *total stem* if it is a stem and for all $y \notin \mathcal{S}$ there exists some $x \in \mathcal{S}$ such that $x \prec y$. A subcauset $\mathcal{S} \subseteq \mathcal{C}$ is called an *up-set* if $x \in \mathcal{S}$ implies that $y \in \mathcal{S}$ for all $y \succ x$. A subcauset $\mathcal{S} \subseteq \mathcal{C}$ is called a *total up-set* if it is an up-set, and for all $y \notin \mathcal{S}$ there exists some $x \in \mathcal{S}$ such that $x \succ y$.

Consider a finite causet $\mathcal{C}$ with $N > 1$ elements. Let $\mathcal{P}$ and $\mathcal{F}$ denote a total down-set and a total up-set in $\mathcal{C}$, respectively, with $\mathcal{P} \cap \mathcal{F} = \emptyset$. We will refer to $\mathcal{P}$ and $\mathcal{F}$ as the past and future regions in $\mathcal{C}$ and require that there are no interactions in these regions; i.e., the interaction region is given by $\mathcal{C} \backslash \{\mathcal{P} \cup \mathcal{F}\}$. Recall that we require that $\mathcal{C}$ is naturally labeled, i.e., that if $x \prec y$ then $x < y$ (cf. Sec. II A). Therefore, regardless of which natural labeling one chooses, the element labeled 1 is always contained in the past region $\mathcal{P}$, and the element labeled $N$ is always contained in the future region $\mathcal{F}$. This will be the key to defining our "asymptotic" states.

To define particle states in the interacting theory, note that we can obtain a mode expansion of the Heisenberg field by substituting the free field mode expansion (2.9) into definition (3.11) of the Heisenberg field,

$$\phi^H(x) = \sum_{\lambda>0} \sqrt{\lambda}(v_x^\lambda b_\lambda(x) + \bar{v}_x^\lambda b_\lambda^\dagger(x)), \tag{6.1}$$

where we define $b_\lambda(x) = U_x^\dagger a_\lambda U_x$.

At $x = 1$, we recover the free theory ladder operators, $b_\lambda(1) = a_\lambda$, and the vacuum, denoted by $|0;1\rangle$ and defined by the requirement that $b_\lambda(1)|0;1\rangle = 0$ for all $\lambda$, is the SJ vacuum (cf. Sec. II).

At $x = N$, the vacuum, denoted by $|0;N\rangle$, is given by

$$b_\lambda(N)|0;N\rangle = U_N^\dagger a_\lambda U_N|0;x\rangle = 0 \Rightarrow U_N|0;x\rangle \propto |0\rangle \Rightarrow |0;N\rangle \propto U_N^\dagger|0\rangle, \tag{6.2}$$

where $\propto$ denotes equality up to an overall phase.

Finally, we define the number operator at $x = 1$ and $x = N$ as

$$\mathcal{N}(x) = \sum_\lambda b_\lambda^\dagger(x) b_\lambda(x). \tag{6.3}$$

We can now define an in-state as an eigenstate of $\mathcal{N}(1)$ and an out-state as an eigenstate of $\mathcal{N}(N)$. One can verify that the in-states are given by applications of $b_\lambda^\dagger(1)$ on $|0;1\rangle$, while the out-states are given by applications of $b_\lambda^\dagger(N)$ on $|0;N\rangle$. We choose our normalization to be such that a one-particle state is given by $|\lambda;x\rangle = \frac{1}{\sqrt{\lambda}} b_\lambda^\dagger(x)|0;x\rangle$ so that $\langle 0;x|\phi(x)|\lambda;x\rangle = v_x^\lambda$ for $x = 1, N$.

[Note that while it may seem that the notion of the number operator and the associated particle states could be extended to any $1 \leq x \leq N$, this is problematic because the "state at $x$" that one obtains in this way is generally label dependent unless $x = 1, N$. Replacing $U_x$ by the covariant $V_x$ (cf. Sec. III) results in label-independent states but does not allow for the notion of asymptotic regions, requiring instead that a state be defined at a point.]

Scattering amplitudes are given by the overlap of an in- and an out-state. Taking two-to-two scattering as an example, the associated amplitude is

$${}_{\text{out}}\langle\lambda_3,\lambda_4|\lambda_1,\lambda_2\rangle_{\text{in}} = \langle\lambda_3,\lambda_4;N|\lambda_1,\lambda_2;1\rangle = \langle\lambda_3,\lambda_4|\hat{S}|\lambda_1,\lambda_2\rangle, \tag{6.4}$$

where on the rhs, $|\lambda,\lambda'\rangle$ are the particle states of the free theory, and $\hat{S}$ is the $S$ operator given in (5.22), which reduces to

$$\hat{S} = C\left[\prod_z e^{-i\mathcal{H}(z)}\right], \tag{6.5}$$

where the product is over points $z$ in the interaction region.

To evaluate $S$ matrix elements, one follows the familiar continuum prescription: Expand $U_x$ order by order in the interaction coupling and apply Wick's theorem. A contraction of a pair of fields gives rise to a Feynman propagator (represented by an internal leg), while a contraction of a field with a one-particle state gives a mode function,

$$\overbrace{\phi(x)|\lambda}\rangle = v_x^\lambda|0\rangle, \tag{6.6}$$

and is represented by an external leg. From (6.4), we see that in the absence of interactions, the amplitude trivializes, and it does not capture any particle production due to a nonstatic metric. We will investigate this in future work.

## VII. CONCLUSION

### A. Summary

In the causal set approach to quantum gravity, spacetime is fundamentally discrete at the Planck scale and takes the form of a causal set. Establishing a framework for quantum field theory on causal sets is a valuable exercise: for describing matter on a causal set spacetime, for making phenomenological predictions under the assumption of spacetime discreteness and for exploring new avenues for regularizing the UV divergences of the continuum through numerical methods. The main result of this work extends the body of work on the in-in formalism on a causal set by proving that the work of [27,28] on two-point functions in

$\phi^4$ theory generalizes to all causally ordered in-in correlators of local operators in scalar theories. This approach complements the construction in [25] of algebraic quantum field theory on a causal set and the associated diagrammatic expansions of [62]. We leave understanding the relationship between the two approaches to future work.

Our approach adapted the continuum construction of [38] to a causal set background and recovered the same diagrammatic expansion. However, there are some differences between the continuum and discrete that stem from the difference in the interpretation of the diagrams. The fact that in the discrete the continuum integrals over the interaction vertices are replaced by sums means that several interaction vertices can live at the same spacetime point. Since the diagrams can be interpreted as decorated subcausal sets, when the interaction region is finite, there are only finitely many of them, and the series terminates at a finite order.

### B. Future directions

By sprinkling causal sets at large density $\rho$ into cosmological spacetimes, our framework can be applied to the computation of cosmological observables. Each diagram becomes a random variable, and understanding how their behavior depends on $\rho$ is of interest. In this scenario, the coupling constant should be scaled by $1/\rho$ (on dimensional grounds), e.g., $\mathcal{H} \sim \frac{g}{4!\rho}\phi^4$. There are many avenues one can explore here, for example, computing the mean of a particular diagram over a sample of sprinklings and the fluctuations around this average. A question of particular interest for comparison with the continuum is whether the mean tends to a (finite) limit when $\rho \to \infty$.

The similarities between the discrete in-in formalism and the continuum formalism of [38] make comparisons between the discrete and the continuum possible. For example, one can show that in $1+1$-dimensional Minkowski space with an interaction region confined to a finite causal diamond, the contribution from a diagram containing only retarded propagators (and no Feynman propagators) is equal to the $\rho \to \infty$ limit of the ensemble average of the same diagram in the discrete. The key that makes this calculation possible is that in $1+1$ dimensions, the retarded propagator is constant. Investigating whether this correspondence persists for diagrams that contain Feynman propagators/in higher dimensions is another direction for future research.

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## APPENDIX: DERIVATION OF THE DIAGRAMMATIC EXPANSIONS

In the following, we denote an ordered list by bold letters; e.g., $\mathbf{a} = a_1 \ldots a_n$ is an ordered list of length $n$. We only consider lists whose entries take value in the natural numbers.

*Definition A.1* (The function $\Upsilon$). Given an ordered list $\mathbf{a} = a_1 a_2 \ldots a_n$, we define

$$\Upsilon(\mathbf{a}) = \begin{cases} \frac{1}{l_1! \ldots l_k!} & a_1 \geq a_2 \geq \ldots \geq z_n \\ 0 & \text{otherwise,} \end{cases} \tag{A1}$$

where $k$ denotes the number of strings of equal signs in $a_1 \geq a_2 \geq \ldots \geq a_n$, and $l_i$ is the length of the $i^{th}$ string of equal signs.

For example, if $\mathbf{a} = 1, 2, 2, 4, 5, 5, 5$ and $\mathbf{b} = 1, 4, 2, 4$, then $\Upsilon(\mathbf{a}) = \frac{1}{12}$ and $\Upsilon(\mathbf{b}) = 0$.

*Definition A.2* (Shuffle of ordered lists). Given a pair of ordered lists, $\mathbf{a}$ and $\mathbf{b}$, their shuffle, denoted by $\mathbf{a} \sqcup\!\sqcup \mathbf{b}$, is the set of ordered lists that can be constructed by putting together all of the entries of $\mathbf{a}$ and $\mathbf{b}$ into a single list $\mathbf{c}$ such that the order of elements in $\mathbf{c}$ is compatible with the order of elements in $\mathbf{a}$ and $\mathbf{b}$.

For example, if $\mathbf{a} = a_1 a_2$ and $\mathbf{b} = b_1$, then $\mathbf{a} \sqcup\!\sqcup \mathbf{b} = \{a_1 a_2 b_1, a_1 b_1 a_2, b_1 a_1 a_2\}$.

*Definition A.3* (Permutation of an ordered list). Given an ordered list $\mathbf{a} = a_1 a_2 \ldots a_n$, a permutation $\pi$ of $\mathbf{a}$ is an ordered list of length $n$ such that there exists a bijection $f\colon \mathbf{a} \to \pi$ with $f(a_i) = a_i$ for all $i$.

We will make use of the following properties of the $\Upsilon$ function. For any list $\mathbf{a}$,

$$\sum_{\pi} \Upsilon(\pi) = 1, \tag{A2}$$

where the sum is over all permutations $\pi$ of $\mathbf{a}$. For any pair of lists, $\mathbf{a}$ and $\mathbf{b}$,

$$\Upsilon(\mathbf{a})\Upsilon(\mathbf{b}) = \sum_{\mathbf{c} \in \mathbf{a} \sqcup\!\sqcup \mathbf{b}} \Upsilon(\mathbf{c}), \tag{A3}$$

where $\sqcup\!\sqcup$ denotes the shuffle product.

*Lemma A.1.*

$$U_{x,y} = \sum_{n=0}^{\infty} (-i)^n \sum_{z_1,\ldots,z_n=y}^{x-1} \mathcal{H}(z_1)\ldots\mathcal{H}(z_n)\Upsilon(\mathbf{z}^{(n)}), \tag{A4}$$

where $\mathbf{z}^{(n)} = z_1 z_2 \ldots z_n$ and the $n = 0$ terms is understood to equal 1.

*Proof.* Starting with definition (3.4) of $U_{x,y}$, expand the exponentials to obtain

$$U_{x,y} = \left(1 - i\mathcal{H}(x-1) + (-i)^2 \frac{\mathcal{H}(x-1)^2}{2}\cdots\right)\cdots\left(1 - i\mathcal{H}(y) + (-i)^2 \frac{\mathcal{H}(y)^2}{2} + \cdots\right). \tag{A5}$$

Note that every order $n$ term, which one can obtain from expanding the rhs of (A5) has the form,

$$(-i)^n \frac{\mathcal{H}(z_1)^{n_1}\cdots\mathcal{H}(z_m)^{n_m}}{n_1!\ldots n_m!}, \tag{A6}$$

with $x-1 \geq z_1 > z_2 \ldots > z_m \geq y$ and $n_1 + \cdots + n_m = n$ for some integer $1 \leq m \leq n$. The order $n$ term in $U_{x,y}$ is the sum of all terms of the form (A6), i.e., the sum of (A6) over the ordered partitions of $n$ and over the possible choices of $z_1, z_2 \ldots, z_m$:

$$\begin{aligned} &\sum_{m=1}^{n} \underbrace{\sum_{n_1>0}\cdots\sum_{n_m>0}}_{n_1+\cdots+n_m=n} \sum_{x-1\geq z_1>\ldots>z_n\geq y} (-i)^n \frac{\mathcal{H}(z_1)^{n_1}\cdots\mathcal{H}(z_m)^{n_m}}{n_1!\ldots n_m!} \\ &= (-i)^n \sum_{z_1,\ldots,z_n=y}^{x-1} H(z_1)\ldots\mathcal{H}(z_n)\Upsilon(\mathbf{z}^{(n)}), \end{aligned} \tag{A7}$$

where in the second line, $\Upsilon(\mathbf{z}^{(n)})$ enforces the nonstrict label ordering $z_1 \geq z_2 \ldots \geq z_n$ and provides the factorial factors in the denominator. ■

*Lemma A.2.* For any operator $\mathcal{O}$ and $x > y \in \mathcal{C}$,

$$U_{x,y}^{\dagger}\mathcal{O}U_{x,y} = \sum_{n=0}^{\infty}(-i)^n \sum_{z_1,\ldots,z_n=y}^{x-1} [\ldots[[\mathcal{O},\mathcal{H}(z_1)],\mathcal{H}(z_2)]\ldots,\mathcal{H}(z_n)]\Upsilon(\mathbf{z}^{(n)}), \tag{A8}$$

where $\mathbf{z}^{(n)} = z_1 z_2 \ldots z_n$ and the $n = 0$ term is understood to be $\mathcal{O}$.

*Proof.* By Lemma A.1 we have

$$\begin{aligned} U_{x,y}^{\dagger}\mathcal{O}U_{x,y} = &\left(\sum_{n_1=0}^{\infty}((-i)^{n_1})^* \sum_{z_1,\ldots,z_{n_1}=y}^{x-1} \mathcal{H}(z_{n_1})\ldots\mathcal{H}(z_1)\Upsilon(\mathbf{z}^{(n_1)})\right)\mathcal{O} \\ &\times\left(\sum_{n_2=0}^{\infty}(-i)^{n_2} \sum_{z_{n_1+1},\ldots,z_{n_1+n_2}=y}^{x-1} \mathcal{H}(z_{n_1+1})\ldots\mathcal{H}(z_{n_1+n_2})\Upsilon(\mathbf{z}^{(n_2)})\right), \end{aligned} \tag{A9}$$

with $\mathbf{z}^{(n_1)} = z_1 z_2 \ldots z_{n_1}$, $\mathbf{z}^{(n_2)} = z_{n_1+1} z_{n_1+2} \ldots z_{n_1+n_2}$ and the $n_1 = 0$ and $n_2 = 0$ terms understood to be equal to 1. The order $n > 0$ contribution to (A9) is

$$
\begin{aligned}
&(-i)^n \sum_{z_1,\ldots,z_n=y}^{x-1} \sum_{n_1+n_2=n} (-1)^{n_1} \mathcal{H}(z_{n_1})\ldots\mathcal{H}(z_1)\mathcal{O}\mathcal{H}(z_{n_1+1})\ldots\mathcal{H}(z_n)\Upsilon(\mathbf{z}^{(n_1)})\Upsilon(\mathbf{z}^{(n_2)}) \\
&= (-i)^n \sum_{z_1,\ldots,z_n=y}^{x-1} \sum_{n_1+n_2=n} (-1)^{n_1} \mathcal{H}(z_{n_1})\ldots\mathcal{H}(z_1)\mathcal{O}\mathcal{H}(z_{n_1+1})\ldots\mathcal{H}(z_n)\sum_{\mathbf{w}}\Upsilon(\mathbf{w}^{(n)}),
\end{aligned} \tag{A10}
$$

where in the second line, we applied Eq. (A3) to write the produce of the $\Upsilon$ as a sum over $\mathbf{w}^{(n)} \in \mathbf{z}^{(n_1)} \sqcup\!\sqcup\, \mathbf{z}^{(n_2)}$. Next, to each $\mathbf{w}^{(n)}$, we apply a coordinate transformation so that $\mathbf{w}^{(n)} = w_1 w_2 \ldots w_n \overset{\mathbf{w}^{(n)}}{\mapsto} \mathbf{z}^{(n)} = z_1 z_2 \ldots z_n$, i.e. $w_i \overset{\mathbf{w}^{(n)}}{\mapsto} z_i$. Writing $p(i)$ to denote the position of $z_i$ in $\mathbf{w}^{(n)}$, the transformation we need is $z_i = w_{p(i)} \overset{\mathbf{w}^{(n)}}{\mapsto} z_{p(i)}$. Plugging this back into (A10), we find

$$
(-i)^n \sum_{z_1,\ldots,z_n=y}^{x-1} \Upsilon(\mathbf{z}^{(n)}) \sum_{n_1+n_2=n} (-1)^{n_1} \sum_{\mathbf{w}} \mathcal{H}(z_{p(n_1)})\ldots\mathcal{H}(z_{p(1)})\mathcal{O}\mathcal{H}(z_{p(n_1+1)})\ldots\mathcal{H}(z_{p(n)}), \tag{A11}
$$

where the $p(i)$ labels are implicitly dependent on $\mathbf{w}^{(n)}$. By comparing (A11) with the order $n$ term in (A8), we find that to complete the proof, we must show that

$$
\begin{aligned}
&[\ldots[[\mathcal{O},\mathcal{H}(z_1)],\mathcal{H}(z_2)]\ldots,\mathcal{H}(z_n)] \\
&= \sum_{n_1+n_2=n} (-1)^{n_1} \sum_{\mathbf{w}} \mathcal{H}(z_{p(n_1)})\ldots\mathcal{H}(z_{p(1)})\mathcal{O}\mathcal{H}(z_{p(n_1+1)})\ldots\mathcal{H}(z_{p(n)}).
\end{aligned} \tag{A12}
$$

To prove (A12), we proceed with a counting argument. Consider the lhs of (A12) and note that

(1) It contains $2^n$ terms.
(2) Each term is of the form of $\mathcal{O}$ sandwiched between a product of Hamiltonians.
(3) Each of $\mathcal{H}(z_1),\ldots\mathcal{H}(z_n)$ appears in each term exactly once.
(4) A term has a positive sign if there is an even number of Hamiltonians to the left of $\mathcal{O}$ and negative otherwise.
(5) The product of Hamiltonians to the right (left) of $\mathcal{O}$ is ordered so that the labels of the $z$ coordinates increase (decrease) from left to right.

We observe that conditions (1)–(5) are also satisfied by the rhs. In particular, condition (1) follows from the fact that there are $\binom{n}{n_1}$ shuffles of two words of lengths $n_1$ and $n - n_1$ so the number of terms on the rhs is $\sum_{n_1}\binom{n}{n_1} = 2^n$, and condition (5) follows from the order-preserving properties of the shuffle. To complete the proof, we argue that there exists exactly $2^n$ terms that satisfy conditions (2)–(5), and the result follows.

To construct a term satisfying conditions (2)–(5), fix some $n_1$ and choose $n_1$ Hamiltonians to be on the left of $\mathcal{O}$ leaving the rest to the right. Now order them to satisfy condition (5), and note that there is a unique way to do that. Thus, there are $\binom{n}{n_1}$ ways to construct an appropriate term with $n_1$ Hamiltonians on the left. The total number of possible terms that can be constructed to satisfy conditions (2)–(5) is therefore given by $\sum_{n_1}\binom{n}{n_1} = 2^n$. ∎

For the remaining lemmas, consider a finite causal set $\mathcal{C}$ and $k$ integers satisfying $|\mathcal{C}| \geq x_1 > x_2 \ldots > x_k \geq 1$. Let $\mathcal{O}^H(x_i)$ denote a local Heisenberg picture operator living at $x_i$.

*Lemma A.3.*

$$
\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k) = U^\dagger_{x_k}\mathcal{O}_{1\ldots k}U_{x_k}, \tag{A13}
$$

where $\mathcal{O}_{1\ldots k}$ is defined recursively via

$$
\mathcal{O}_{1\ldots p} = \begin{cases} \mathcal{O}(x_1) & p = 1 \\ U^\dagger_{x_{p-1},x_p}\mathcal{O}_{1\ldots p-1}U_{x_{p-1},x_p}\mathcal{O}(x_p) & 1 < p \leq k. \end{cases} \tag{A14}
$$

*Proof.* This is a short proof by induction. Consider $k = 1, 2$ for the base case. Then assuming (A13) holds for all $k = 1, \ldots, s$, it follows from the composition properties of $U$ (3.7) and (3.8) that it also holds for $k = s + 1$. ∎

*Lemma A.4.*

$$U^{\dagger}_{x_{p-1},x_p}\mathcal{O}_{1\ldots p-1}U_{x_{p-1},x_p} = \sum_{n=0}^{\infty}(-i)^n \sum_{n_1+\cdots+n_{p-1}=n} \sum_{z_1,\ldots,z_{n_1}=x_2}^{x_1-1} \sum_{z_{n_1+1},\ldots,z_{n_1+n_2}=x_3}^{x_2-1} \cdots \sum_{z_{n_1+\cdots n_{p-2}+1},\ldots,z_n=x_p}^{x_{p-1}-1} [\cdots[\cdots[[\cdots[\mathcal{O}(x_1),\mathcal{H}(z_1)]\cdots,\mathcal{H}(z_{n_1})]\mathcal{O}(x_2),\mathcal{H}(z_{n_1+1})]\cdots\mathcal{O}(x_{p-1}),\mathcal{H}(z_{n_1+\cdots+n_{p-2}+1})]\cdots,\mathcal{H}(z_n)]\Upsilon(\mathbf{z}^{(n)}), \tag{A15}$$

where $\mathcal{O}_{1\ldots p-1}$ is as given in (A14).

*Proof.* We proceed by induction. Suppose true for $p=s$ and consider the case $p=s+1$. To compute

$$U^{\dagger}_{x_s,x_{s+1}}\mathcal{O}_{1\ldots s}U_{x_s,x_{s+1}}, \tag{A16}$$

we start by expanding $\mathcal{O}_{1\ldots s}$ by applying definition (A14) together with the inductive assumption,

$$\mathcal{O}_{1\ldots s} = \sum_{n=0}^{\infty}(-i)^n \sum_{n_1+\cdots+n_{s-1}=n} \sum_{z_1,\ldots,z_{n_1}=x_2}^{x_1-1} \sum_{z_{n_1+1},\ldots,z_{n_1+n_2}=x_3}^{x_2-1} \cdots \sum_{z_{n_1+\cdots n_{s-2}+1},\ldots,z_n=x_s}^{x_{s-1}-1} [\cdots[\cdots[\cdots[\mathcal{O}(x_1),\mathcal{H}(z_1)]\cdots\mathcal{O}(x_2),\mathcal{H}(z_{n_1+1})]\cdots\mathcal{O}(x_{s-1}),\mathcal{H}(z_{n_1+\cdots+n_{s-2}+1})]\cdots,\mathcal{H}(z_n)]\mathcal{O}(x_s)\Upsilon(\mathbf{z}^{(n)}). \tag{A17}$$

Now, we compute (A16) by substituting the rhs of (A17) for $\mathcal{O}_{1\ldots s}$ and applying Lemma A.2 to obtain

$$\sum_{m,n=0}^{\infty}(-i)^{m+n} \sum_{n_1+\cdots+n_{s-1}=n} \sum_{z_1,\ldots,z_{n_1}=x_2}^{x_1-1} \sum_{z_{n_1+1},\ldots,z_{n_1+n_2}=x_3}^{x_2-1} \cdots \sum_{z_{n_1+\cdots n_{s-2}+1},\ldots,z_n=x_s}^{x_{s-1}-1} \sum_{z_{n+1},\ldots,z_{n+m}=x_{s+1}}^{x_s-1} [\cdots[[\cdots[\cdots[\mathcal{O}(x_1),\mathcal{H}(z_1)]\cdots\mathcal{O}(x_{s-1}),\mathcal{H}(z_{n_1+\cdots+n_{s-2}+1})]\cdots,\mathcal{H}(z_n)]\mathcal{O}(x_s),\mathcal{H}(z_{n+1})]\cdots\mathcal{H}(z_{n+m})]\Upsilon(\mathbf{z}^{(n)})\Upsilon(\mathbf{z}^{(m)}), \tag{A18}$$

with $\mathbf{z}^{(m)} = z_n z_{n+1}\ldots z_m$. Now, apply (A3) to replace the product $\Upsilon(\mathbf{z}^{(n)})\Upsilon(\mathbf{z}^{(m)})$ with the sum $\sum_{\mathbf{z}}\Upsilon(\mathbf{z})$ with $\mathbf{z}\in\mathbf{z}^{(n)}\sqcup\mathbf{z}^{(m)}$ and note that the limits of the summations in (A18) mean that the only term that survives is $\Upsilon(\mathbf{z}^{(n+m)})$ where $\mathbf{z}^{(n+m)} = z_1\ldots z_{n+m}$. Finally to show that the lemma holds for $p=s+1$, relabel as $m+n\to n$ and $m\to n_s$. ■

*Corollary A.5.*

$$\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k) = \sum_{n=0}^{\infty}(-i)^n \sum_{n_1+\cdots+n_k=n} \sum_{z_1,\ldots,z_{n_1}=x_2}^{x_1-1} \sum_{z_{n_1+1},\ldots,z_{n_2}=x_3}^{x_2-1} \cdots \sum_{z_{n_1+\cdots+n_{k-1}+1},\ldots,z_n=1}^{x_k-1} [\cdots[\cdots[[\cdots[\mathcal{O}(x_1),\mathcal{H}(z_1)]\cdots,\mathcal{H}(z_{n_1})]\mathcal{O}(x_2),\mathcal{H}(z_{n_1+1})]\cdots\mathcal{O}(x_k),\mathcal{H}(z_{n_1+\cdots+n_{k-1}+1})]\cdots,\mathcal{H}(z_n)]\Upsilon(\mathbf{z}^{(n)}). \tag{A19}$$

*Proof.* It follows immediately from plugging (A15) into (A13). ■

*Lemma A.6.*

$$\langle\mathcal{O}^H(x_1)\ldots\mathcal{O}^H(x_k)\rangle = \sum_{n=0}^{\infty}(-i)^n \sum_{z_1,\ldots,z_n=1}^{x_1-1} \langle[\cdots[[\mathcal{O}(x_1)\ldots\mathcal{O}(x_k),\mathcal{H}(z_1)],\mathcal{H}(z_2)]\cdots,\mathcal{H}(z_n)]\rangle\Upsilon(\mathbf{z}^{(n)}). \tag{A20}$$

*Proof.* Begin by taking the expectation value of both sides of (A19). Then note that the value of the summand,

$$\langle[\cdots[\cdots[[\cdots[\mathcal{O}(x_1),\mathcal{H}(z_1)]\cdots,\mathcal{H}(z_{n_1})]\mathcal{O}(x_2),\mathcal{H}(z_{n_1+1})]\cdots\mathcal{O}(x_k),\mathcal{H}(z_{n_1+\cdots+n_{k-1}+1})]\cdots,\mathcal{H}(z_n)]\rangle\Upsilon(\mathbf{z}^{(n)}),$$

differs from the value of

$$\langle[\cdots[[\mathcal{O}(x_1)\ldots\mathcal{O}(x_k),\mathcal{H}(z_1)],\mathcal{H}(z_2)]\cdots,\mathcal{H}(z_n)]\rangle\Upsilon(\mathbf{z}^{(n)}),$$

only outside the domain of the nested sum, so replace the former by the latter. [One way to observe this equivalence is via the diagrammatic representations of the expectation value of the nested correlators (cf. Sec. IV). The latter expression is given by all diagrams with $n$ internal and $k$ external vertices such that each internal vertex is connected via a path of retarded propagators to at least one external vertex. The former expression is given by the subset of these diagrams in which only the last $n_k$ internal vertices may be connected by a path of retarded propagators to $x_k$, only the last $n_k + n_{k-1}$ internal vertices may be connected by a path of retarded propagators to $x_{k-1}$, etc. The diagrams that make up the difference between the two expressions vanish in the domain of the nested sum.]

To complete the proof, we note that $\Upsilon(\mathbf{z}^{(n)})$ vanishes everywhere outside the domain of the sum, which enables us to make the replacement,

$$\sum_{n_1+\cdots+n_k=n} \sum_{z_1,\ldots,z_{n_1}=x_2}^{x_1-1} \sum_{z_{n_1+1},\ldots,z_{n_2}=x_3}^{x_2-1} \cdots \sum_{z_{n_1+\cdots+n_{k-1}+1},\ldots,z_n=1}^{x_k-1} \longrightarrow \sum_{z_1,\ldots,z_n=1}^{x_1-1}. \tag{A21}$$

∎


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