Wave packets in AdS/CFT correspondence

Seiji Terashima*

Center for Gravitational Physics and Quantum Information, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

(Received 24 October 2023; accepted 9 April 2024; published 10 May 2024)

In this paper, we construct a general bulk wave packet in the AdS/CFT correspondence. This wave packet can be described both in bulk and conformal field theory (CFT) descriptions. Then, we compute the time evolution of the energy density of this wave-packet state on the vacuum in the CFT picture of AdS_3/CFT_2. We find that the energy density of the wave packet is localized at two points, which means that the bulk wave packet corresponds to two light-like particle-like objects in the CFT picture. Our result implies that the entanglement wedge reconstruction given by Almheiri *et al.* [Bulk locality and quantum error correction in AdS/CFT, J. High Energy Phys. 04 (2015) 163] is invalid.

DOI: 10.1103/PhysRevD.109.106012

I. INTRODUCTION AND SUMMARY

The AdS/CFT correspondence [1] is expected to be important to understanding quantum gravity. In particular, because the bulk spacetime should emerge from conformal field theory (CFT), we can, in principle, understand the bulk spacetime in quantum gravity from the AdS/CFT correspondence. For this purpose, several bulk spacetime probes in CFT are known, including correlation functions, Wilson loops, and entanglement entropy.

The important probes of bulk spacetime, which have not been studied intensively, are the wave packets in bulk spacetime. Such wave packets are fundamental objects for thought experiments in bulk spacetime. In particular, a local region in the bulk spacetime can be probed by the time-evolved wave packet. Then, the relationship between the local region of the bulk where the wave packet resides and the corresponding region of the boundary in the CFT picture will be the key to understanding how the bulk spacetime emerges from the CFT. Thus, it is important to understand how the bulk wave packets are described in the CFT picture. In [2,3], a special kind of bulk wave packets, for which only their direction is fixed, were considered in the AdS/CFT correspondence, although the general bulk wave packets have not been studied.

In this paper, we first construct a general bulk wave packet in the AdS/CFT correspondence. Here we take the large-N limit, which is the free limit in the bulk picture and the generalized free approximation in the CFT picture. Furthermore, we consider only the bulk scalar field and the corresponding scalar CFT operator, for simplicity. This wave packet can be described in both bulk and CFT descriptions.[1]

Then, we compute the energy density of this state of the wave packet on the vacuum in the CFT picture of AdS_3/CFT_2.[2] Note that the energy density does not vanish if the CFT state is excited by the local CFT operator there. Thus, if the distribution of the energy density is localized in some regions, the CFT state is localized in those regions. We find that the energy density of the wave packet is localized at two points, which are on the light cone, in the CFT picture, which means that the bulk wave packet corresponds to two light-like particle-like objects in the CFT picture, although these are not like the free particles. This result completely agrees with the result in [2,3] in which the entanglement wedge reconstruction given in [4] was shown to be invalid. Note that our results in this paper only use the Banks-Douglas-Horowitz-Matinec (BDHM) extrapolation relation [5], which is the basic AdS/CFT dictionary, like the Gubser-Klebanov-Polyakov-Witten relation [6,7], and the known three-point function in 2D CFT.

We also compute the vacuum expectation value (VEV) of the CFT primary scalar operator for the wave packet. The

*terasima@yukawa.kyoto-u.ac.jp

[1] In this paper, we consider the light-like wave packet. This is because we consider a generic Δ and the nonrelativistic wave packet can be considered only for $\Delta \gg l_{\rm AdS}$, where Δ is the conformal dimension of the CFT operator and $l_{\rm AdS}$ is the length scale of the AdS space, which we set to $l_{\rm AdS} = 1$.

[2] More precisely, we compute the energy density of this state of the wave packet represented by (2.13) in the CFT picture. The energy density depends on $1/N$ corrections of the wave-packet operator. Our operator is special because it is localized at a point of the boundary on a time slice.

distribution of this is completely different from that of the energy density. This is because there are infinitely many independent fields at a fixed time for the generalized free field.

II. WAVE PACKETS IN AdS/CFT

A. Bulk and CFT fields in AdS/CFT

Let us consider the global AdS_{d+1} and Ω represents coordinates of a $(d-1)$-dimensional sphere S^{d-1}. The coordinates τ and ρ run in the ranges $-\infty < \tau < \infty$ and $0 \le \rho < \pi/2$. In the coordinates, the metric takes the form

$$ds^2 = \frac{1}{\cos^2\rho}(-d\tau^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2). \quad (2.1)$$

For the Poincaré patch of AdS_{d+1}, the metric is

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + \delta_{ij}dx^i dx^j), \quad (2.2)$$

where $z > 0$ and $i, j = 1, 2, \ldots, d-1$.

Let us consider the canonical quantization of a free scalar field ϕ with mass m, which satisfies the equations of motion $(\Box - m^2)\phi = 0$. For the Poincaré AdS_{d+1}, the mode expansion of the scalar field is given by the Bessel functions as

$$\phi(t, z, x^i) = C\int_{\omega>\sqrt{k^2}} d\omega dk_i e^{i\omega t - ik_j x^j} z^{\frac{d}{2}} a^\dagger_{\omega,k} J_\nu\left(\sqrt{\omega^2 - k^2} z\right) + \mathrm{H.c.}, \quad (2.3)$$

where $\nu = \sqrt{m^2 + d^2/4} = \Delta - d/2$ and $k^2 \equiv k^j k_j$. Note that we included the normalizable modes only. Here the asymptotic behavior of the Bessel function is

$$J_\nu(\sqrt{\omega^2 - k^2} z) \to \sqrt{\omega^2 - k^2}^\nu z^\nu \quad \text{for } z \to 0, \quad (2.4)$$

$$J_\nu(\sqrt{\omega^2 - k^2} z) \to \sqrt{\frac{2}{\pi\sqrt{\omega^2 - k^2} z}} \times \cos\left(\sqrt{\omega^2 - k^2} z - \frac{2\nu + 1}{4}\pi\right) \quad \text{for } z \to \infty. \quad (2.5)$$

The overall constant C is usually chosen such that it satisfies the canonical commutator,

$$\left[\phi(t, z', x'^i), \frac{\partial}{\partial t}\phi(t, z, x^i)\right] = i\delta(z' - z)\delta(x'^i - x^i), \quad (2.6)$$

where we defined the creation operators as

$$[a^\dagger_{\omega',k'}, a_{\omega,k}] = \delta(\omega' - \omega)\delta(k'^i - k^i). \quad (2.7)$$

However, we make a different choice, as explained below.

The CFT primary operator $\mathcal{O}$ corresponding to the bulk scalar field ϕ is obtained by the BDHM relation $\mathcal{O}(t, x^i) = \lim_{z\to 0}\phi(t, z, x^i)/z^\Delta$ [5] as

$$\mathcal{O}(t, x^i) = C\int_{\omega>\sqrt{k^2}} d\omega dk_i e^{i\omega t - ik_j x^j} \sqrt{\omega^2 - k^2}^{\Delta - d/2} a^\dagger_{\omega,k} + \mathrm{H.c.}, \quad (2.8)$$

which is valid only for the large-N limit or the generalized free theory limit. We choose the normalization constant C such that the above BDHM relation holds with the standard normalization of the CFT primary field $\mathcal{O}(t, x^i)$. For $d = 2$, this means

$$\langle 0|\mathcal{O}(u_1, v_1)\mathcal{O}(u_2, v_2)|0\rangle = \frac{1}{(u_1 - u_2)^\Delta (v_1 - v_2)^\Delta}, \quad (2.9)$$

where $u = (t + x), v = (t - x)$.

B. Wave packets

In this paper, we consider essentially Gaussian wave packets because we study the generic properties of the wave packets. We mainly consider a one-particle state on the bulk side for simplicity. It is easy to generalize this state to the coherent state for the weak-coupling bulk theory, as done in Appendix A.

1. Minkowski spacetime

Let us remember the wave packets, at $t = \vec{x} = 0$, of a free scalar field in $d+1$-dimensional Minkowski spacetime:

$$\int d\vec{x} e^{-\frac{\vec{x}^2}{2a^2} + i\vec{p}\cdot\vec{x}}\phi(t=0, \vec{x})|0\rangle \propto \int d\vec{k} e^{-\frac{a^2(\vec{k}-\vec{p})^2}{2}} a^\dagger_{\vec{k}}|0\rangle, \quad (2.10)$$

where $\vec{p}$ is the momentum of the wave packet and $[a^\dagger_{\vec{k}}, a_{\vec{k}}] = \delta(\vec{k})$. Instead of this, we can consider another wave packet that is defined by a Gaussian integral for the time and x^i where $i = 2, \ldots, d$:

$$\int dt \prod_{i=2,\ldots,d} dx^i e^{-\frac{x^i x_i + t^2}{2a^2} + ip_i x^i + i\omega t}\phi(t, \vec{x})|_{x_1=0}|0\rangle \propto \int d\vec{k} e^{-\frac{a^2}{2}((k^i - p^i)(k_i - p_i) + (\sqrt{(k_1)^2 + k^i k_i} - \omega)^2)} a^\dagger_{\vec{k}}|0\rangle, \quad (2.11)$$

where i runs only for $2, \ldots, d$. Here we assume that $|ap^i| \gg 1$ and $a\omega \gg 1$, which are needed for the wave packet and we always assume these. With these, it is approximated as

$$\int d\vec{k} e^{-\frac{a^2}{2}((\delta k^i)^2 + (k_1 - p^1 + \frac{p^i \delta k_i}{p^1})^2)} a^\dagger_{\vec{k}}|0\rangle + \int d\vec{k} e^{-\frac{a^2}{2}((\delta k^i)^2 + (k_1 + p^1 + \frac{p^i \delta k_i}{p^1})^2)} a^\dagger_{\vec{k}}|0\rangle, \quad (2.12)$$

where $p^1 = \sqrt{\omega^2 - p^i p_i}$ and $\delta k^i = k^i - p^i$. These are the Gaussian integrals around $k^1 = \pm p^1$ and $k^i = p^i$. Thus, the wave packet (2.11) is essentially the same as the sum of the original wave packets (2.10) with opposite momenta. Note that if we change the Gaussian factor $e^{-\frac{x^i x_i + t^2}{2a^2}}$ in (2.11) to a general one $e^{-\frac{h_{ij}x^i x^j + h'_i x^i t + t^2}{2a^2}}$ with the appropriate constants h, h', the approximated Gaussian factor can be taken to be the same one as in (2.10). Furthermore, if we consider the theory on the half-space $x^1 > 0$ with a boundary condition on $x = 0$, only one wave packet can be obtained by (2.11). We consider such wave packets in AdS_{d+1} where x^1 corresponds to the radial direction z.

2. *Wave packet in* AdS/CFT

In anti–de Sitter (AdS) spacetime, wave packets are constructed as in flat spacetime. The wave packets should be very small in size compared with the AdS scale, where the AdS spacetime can be approximated as a Minkowski spacetime. Thus, the wave packets (2.10) and (2.11) in the flat spacetime can be regarded as the wave packets in AdS spacetime for $a \ll 1$. Furthermore, in AdS spacetime any wave packet will reach the asymptotic boundary by the time or backward time evolution. Thus, we need to prepare wave packets almost on the boundary only to represent a general wave packet. On the boundary, the bulk scalar field ϕ is identified as the CFT primary field $\mathcal{O}$ with an overall factor by the BDHM relation. This means that the bulk wave packet in AdS/CFT can be given by

$$\begin{aligned}|p,\bar{\omega}\rangle &= \lim_{z\to 0}\frac{1}{z^{\Delta}}\int dtdx^i e^{-\frac{x^i x_i + t^2}{2a^2} + ip_i x^i - i\bar{\omega}t}\phi(t,z,x^i)|0\rangle \\ &= \int dtdx^i e^{-\frac{x^i x_i + t^2}{2a^2} + ip_i x^i - i\bar{\omega}t}\mathcal{O}(t,x)|0\rangle \qquad (2.13)\end{aligned}$$

for the Poincaré AdS_{d+1} (2.2).[3] This can be regarded as the state in bulk and also the state in the CFT. Here we require that

$$a^2 p^2 \gg 1, \qquad a\bar{\omega} \gg 1, \qquad (2.16)$$

and then the wave packet has a definite orientation with the momentum p_i and energy $\bar{\omega}$.

Let us check the time evolution of this state in the bulk picture. The bulk localized (one-particle) state is

$$\begin{aligned}\phi(t,z,x^i)|0\rangle = C\int_{\omega>\sqrt{k^2}} d\omega dk_i e^{i\omega t - ik_j x^j} z^{\frac{d}{2}} J_{\nu} \\ \times \left(\sqrt{\omega^2 - k^2}z\right) a^{\dagger}_{\omega,k}|0\rangle, \qquad (2.17)\end{aligned}$$

which is not normalized.[4] In order to consider the bulk spatial distribution of the wave-packet state (2.13) at time t, we consider the following overlap:

$$\langle 0|\phi(t=0,z,x^i)e^{iHt}|p,\bar{\omega}\rangle = (a\sqrt{\pi})^d C^2 \int_{\omega>\sqrt{k^2}} d\omega dk_i e^{i\omega t + ik_j x^j} e^{-\frac{a^2}{2}((k_i - p_i)^2 + (\omega - \bar{\omega})^2)} \sqrt{\omega^2 - k^2}^{2\Delta - d} z^{\frac{d}{2}} J_{\nu}\left(\sqrt{\omega^2 - k^2}z\right). \qquad (2.18)$$

Because of the Gaussian factor, the integrals are dominated for the region near $k_i = p_i, \omega = \bar{\omega}$. Defining $\delta\omega \equiv \omega - \bar{\omega}$, $\delta k_i \equiv k_i - p_i$, and $p_z \equiv \sqrt{\bar{\omega}^2 - p^2}$, the overlap can be approximated as

$$\begin{aligned}\langle 0|\phi(t=0,z,x^i)e^{iHt}|p,\bar{\omega}\rangle &\sim (a\sqrt{\pi})^d C^2 \sqrt{2/\pi} z^{\frac{d-1}{2}} (p_z)^{2\Delta - d - 1/2} e^{i\bar{\omega}t + ip_j x^j} \\ &\times \int d\delta\omega d\delta k_i e^{i\delta\omega t + i\delta k_j x^j} e^{-\frac{a^2}{2}((\delta k)^2 + \delta\omega^2)} \cos\left(\frac{(p_z)^2 + \bar{\omega}\delta\omega - p^i \delta k_i}{p_z} z - \frac{2\nu + 1}{4}\pi\right). \qquad (2.19)\end{aligned}$$

The integral in this is proportional to

[3]Using (2.8), we can rewrite the state in momentum space,

$$|p,\bar{\omega}\rangle = (a\sqrt{\pi})^d C \int_{\omega>\sqrt{k^2}} d\omega dk_i e^{-a^2\frac{(k_i - p_i)^2 + (\omega - \bar{\omega})^2}{2}} \sqrt{\omega^2 - k^2}^{\Delta - d/2} a^{\dagger}_{\omega,k}|0\rangle, \qquad (2.14)$$

for the generalized free-field approximation. The norm of this state is

$$\mathcal{N}^2 = \langle p,\bar{\omega}|p,\bar{\omega}\rangle = (a\sqrt{\pi})^{2d} C^2 \int_{\omega>\sqrt{k^2}} d\omega dk_i e^{-a^2((k_i - p_i)^2 + (\omega - \bar{\omega})^2)} \sqrt{\omega^2 - k^2}^{2\Delta - d}, \qquad (2.15)$$

which is approximated as $\mathcal{N}^2 \simeq (a\sqrt{\pi^3})^d \sqrt{\bar{\omega}^2 - p^2}^{2\Delta - d} C^2$.

[4]This state cannot be normalized and we need to smear this state to eliminate the high-energy modes. Here we use this because we consider the overlap between this and the wave packet state, which is already smeared.

$$\int d\delta\omega d\delta k_i e^{i\delta\omega(t\pm\bar{\omega}z/p_z)+i\delta k_j(x^j\mp p^j z/p_z)} e^{-\frac{a^2}{2}((\delta k)^2+\delta\omega^2)} \simeq e^{-\frac{1}{2a^2}((t\pm\bar{\omega}z/p_z)^2+(x^j\mp p^j z/p_z)^2)}, \tag{2.20}$$

which is strongly suppressed by the Gaussian factor for

$$|t\pm\bar{\omega}z/p_z| \gg a, \quad \text{or} \quad |x^j \mp p^j z/p_z| \gg a. \tag{2.21}$$

Thus, at each time $t>0$, the wave packet is localized at $z=\frac{p_z}{\bar{\omega}}t, x^j=-\frac{p^j}{\bar{\omega}}t$ which is on the light-like trajectory from the boundary point at $t=0$ with the energy $\bar{\omega}$ and momentum p_z, p_i, as expected. This also implies that the size of the wave packet is $\mathcal{O}(a)$ for any time t in the coordinate z, x^i.

3. Remarks on the asymptotic AdS case

So far, we have considered the wave packets in AdS space. The wave-packet state (2.13) can also be regarded as the wave-packet state in the asymptotic AdS case. This is because the state is written by the bulk field on the asymptotic boundary or the CFT primary fields. Indeed, near the boundary, spacetime can be regarded as the AdS space and the state will represent the wave packet moving toward the inside of the asymptotic AdS space. The wave packet will be on the null geodesics of the asymptotic AdS space.

Another remark is that the wave-packet state (2.13) can be created from the vacuum or the semiclassical background by the source term in CFT because it is written by the CFT primary fields. Indeed, by adding the source term $e^{\int dt' dx' \epsilon J(t',x')\mathcal{O}(t',x')}$ with small ϵ, we can obtain the wave-packet state (2.13) at the subleading order in ϵ by setting the source term $J(t,x)$ as the Gaussian factor in (2.13). For general ϵ, the state becomes the coherent state given in Appendix A. In the bulk theory, the one-particle state is described by the free quantum field approximation around the AdS background and the coherent state can be considered as a free approximation of the classical field.

III. ENERGY DENSITY OF WAVE PACKET IN CFT PICTURE

We can consider the time evolution of the bulk wave packet (2.13) as a state in CFT. Here the bulk wave packet can be regarded as a basic probe of the bulk spacetime point. Thus, in order to understand how the bulk spacetime emerges from CFT, it is important to know what is the spacetime region of this state in the CFT picture.[5]

We compute the time evolution of the energy density of the bulk wave packet (2.13) in the CFT picture in order to investigate where the state (2.13) in the CFT picture is localized at each time. Another quantity that might behave like the energy density is the expectation value of the primary scalar operators $\mathcal{O}$. However, this quantity is not good for our purpose. Indeed, if we see it as representing the location of the state in the CFT picture, its time evolution violates causality, as we will see later. The reason for this is as follows. The generalized free field does not obey equations of motion and $\frac{\partial^n}{\partial t^n}\mathcal{O}$ with different n are independent at each time. The expectation values of $\frac{\partial^n}{\partial t^n}\mathcal{O}$ are independent quantities. Thus, if these are different, we cannot take one of them as a representative. Furthermore, because the number of these independent operators is infinite, it is difficult to obtain information on the location of the state in the CFT picture.

The energy density for (2.13) is the three-point function of the two primary scalar operators and the energy-momentum tensor,

$$\begin{aligned}
&\langle p,\bar{\omega}|T_{00}(t=\bar{t},x^i=\bar{x}^i)|p,\bar{\omega}\rangle \\
&= \int dt_1 dx_1^i e^{-\frac{(x_1^i)^2+t_1^2}{2a^2}-ip_i x_1^i+i\bar{\omega}t_1} \int dt_2 dx_2^i e^{-\frac{(x_2^i)^2+t_2^2}{2a^2}+ip_i x_2^i-i\bar{\omega}t_2} \\
&\quad\times \langle 0|\mathcal{O}(t_1,x_1)T_{00}(t=\bar{t},x^i=\bar{x}^i)\mathcal{O}(t_2,x_2)|0\rangle,
\end{aligned} \tag{3.1}$$

in the Heisenberg picture. Because such a three-point function in CFT is known exactly, we can compute the energy density. It is important to note that this computation does not use the generalized free approximation, which is the leading order of the large-N expansion, and the result is valid for a large but finite N.

We also note that the operator ordering of this does not follow the time ordering. The ordering of the operators is fixed by the path of analytic continuation from the Euclidean correlation function. This can be implemented by slightly deforming the insertion points by a small imaginary time, as in the $i\epsilon$ prescription. For (3.1), we change $t_1 \to t_1 + i\epsilon_1$, $\bar{t} \to \bar{t} + i\epsilon_T$, and $t_2 \to t_2 + i\epsilon_2$, where $\epsilon_1 > \epsilon_T > \epsilon_2$. For example, we can take $\epsilon_1 = -\epsilon_2 = \epsilon > 0$ and $\epsilon_T = 0$.

Below, we will consider the energy density in the approximation $a \ll 1$ and $\bar{\omega}a \gg 1, (p_i)^2 a^2 \gg 1$. We will neglect the terms that become zero in this limit. We also assume $\Delta = \mathcal{O}(1)$, which implies the mass m is $\mathcal{O}(1)$ and the wave packet behaves like a massless particle because its the energy and momentum are much larger than the mass.[6] We will also consider the $d=2$ case only.

[5]A special kind of bulk wave packet and the corresponding CFT state was considered in [2,3], and we will see that the general bulk wave packet (2.13) also has the same property, as expected.

[6]If we take $\Delta \gg 1$ such that $ma \ll 1$, the Compton length $1/m$ will be larger than the size of the wave packet. In this case, the wave packet can correspond to the massive particle.

A. $d = 2$

For CFT_2 in the complex plane, the energy-momentum tensor is given by $T(z)$ and $\bar{T}(\bar{z})$. We need to compute $\langle 0|\mathcal{O}(z', \bar{z}')T_{00}(\xi, \bar{\xi})\mathcal{O}(z, \bar{z})|0\rangle$, where the energy density is $\frac{1}{2\pi}T_{00}(z,\bar{z}) = \frac{1}{2\pi}(T(z) + \bar{T}(\bar{z}))$. Using the conformal Ward identity, the three-point function[7] is evaluated (see, for example, [8]) as

$$\begin{aligned} \langle 0|T(\xi)\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)|0\rangle &= \sum_{i=1,2}\left(\frac{h_i}{(\xi - z_i)^2} + \frac{\partial_i}{\xi - z_i}\right)\langle 0|\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)|0\rangle \\ &= \sum_{i=1,2}\left(\frac{h_i}{(\xi - z_i)^2} + \frac{\partial_i}{\xi - z_i}\right)\frac{1}{(z_1 - z_2)^{\Delta_1+\Delta_2}(\bar{z}_1 - \bar{z}_2)^{\Delta_1+\Delta_2}}, \end{aligned} \tag{3.2}$$

where we used the normalization of the primary operator such that the two-point function is the standard one and h_i is the weight with $\Delta = h + \bar{h}$.

For Minkowski spacetime, we replace $z \to u = (t + x)$ and $\bar{z} \to -v = -(t - x)$, and then we obtain[8]

$$\begin{aligned} \langle 0|\mathcal{O}(t_1,x_1)T_{00}(t=\bar{t},x=\bar{x})\mathcal{O}(t_2,x_2)|0\rangle &= \langle 0|\mathcal{O}(u_1,v_1)(T(\bar{u}) + \bar{T}(\bar{v}))\mathcal{O}(u_2,v_2)|0\rangle \\ &= \sum_{i=1,2}\left(\frac{\Delta/2}{(\bar{u} - u_i)^2} + \frac{\partial_{u_i}}{\bar{u} - u_i} + \frac{\Delta/2}{(\bar{v} - v_i)^2} + \frac{\partial_{v_i}}{\bar{v} - v_i}\right)\frac{1}{(u_1 - u_2)^{\Delta}(v_2 - v_1)^{\Delta}} \\ &= \frac{\Delta}{2}\left(\frac{(u_1 - u_2)^2}{(\bar{u} - u_1)^2(\bar{u} - u_2)^2} + \frac{(v_1 - v_2)^2}{(\bar{v} - v_1)^2(\bar{v} - v_2)^2}\right)\frac{1}{(u_1 - u_2)^{\Delta}(v_2 - v_1)^{\Delta}}. \end{aligned} \tag{3.3}$$

Using this, the energy density for (2.13) is

$$\begin{aligned} \langle p,\bar{\omega}|T_{00}(t=\bar{t},x=\bar{x})|p,\bar{\omega}\rangle = &\int dt_1 dx_1 e^{-\frac{(x_1)^2+t_1^2}{2a^2} - ipx_1 + i\bar{\omega}t_1}\int dt_2 dx_2^i e^{-\frac{(x_2)^2+t_2^2}{2a^2} + ipx_2 - i\bar{\omega}t_2} \\ &\times \frac{\Delta}{2}\left(\frac{(u_1 - u_2)^2}{(\bar{u} - u_1)^2(\bar{u} - u_2)^2} + \frac{(v_1 - v_2)^2}{(\bar{v} - v_1)^2(\bar{v} - v_2)^2}\right)\frac{1}{(u_1 - u_2)^{\Delta}(v_2 - v_1)^{\Delta}}. \end{aligned} \tag{3.4}$$

We will evaluate this explicitly below. Because the calculation is not very technically simple, we state the results of the calculation first. The energy density $\mathcal{E}(t,x)$ of the wave-packet state is approximately given by

$$\mathcal{E}(t,x) \simeq \frac{1}{2\sqrt{2\pi}a}\left(e^{-\frac{(x+t)^2}{2a^2}}(\bar{\omega} - p) + e^{-\frac{(x-t)^2}{2a^2}}(\bar{\omega} + p)\right), \tag{3.5}$$

which is localized on the light cone $x = t$ or $x = -t$.

Before computing the energy density of the wave-packet state, let us consider the state $|p,\bar{\omega}\rangle$ with $p = \bar{\omega} = 0$ because it is simpler. This corresponds to the (Gaussian-smeared) local CFT operator insertion. We will see that the energy density of this state in CFT is localized on the light cone. For this, we have

$$\begin{aligned} \langle 0,0|T_{00}(t=\bar{t},x=\bar{x})|0,0\rangle = &\int dt_1 dx_1 \int dt_2 dx_2 e^{-\frac{(x_1)^2+(t_1)^2}{2a^2}} e^{-\frac{(x_2)^2+(t_2)^2}{2a^2}} \\ &\times \frac{\Delta}{2}\left(\frac{(u_1 - u_2)^2}{(\bar{u} - u_1)^2(\bar{u} - u_2)^2} + \frac{(v_1 - v_2)^2}{(\bar{v} - v_1)^2(\bar{v} - v_2)^2}\right)\frac{1}{(u_1 - u_2)^{\Delta}(v_2 - v_1)^{\Delta}}. \end{aligned} \tag{3.6}$$

The Gaussian integral approximately vanishes except for the region near $u_i = v_i = 0$. For $|\bar{u}| \gg a$ and $|\bar{v}| \gg a$, which implies that the energy-momentum tensor is inserted far from the light cone of the scalar operator insertion point, we can use the following expansion:

[7] Here we consider Euclidean, not Lorentzian, CFT and we do not care about the operator ordering as usual, although the operator ordering is fixed by the imaginary time.

[8] More precisely, we need to include the small imaginary part as the ϵ prescription according to the ordering of the operators. The definitions of $\frac{1}{(u_1-u_2)^{\Delta}}$ and similar terms should be given by the convergent sum expression in the Euclidean cylinder, as explained in, for example, [9]. In our case, although the overall phase factor depends on these definitions, it is indeed fixed by requiring that the energy is real and non-negative.

$$\langle 0,0|T_{00}(t=\bar{t},x=\bar{x})|0,0\rangle = \int dt_1 dx_1 dt_2 dx_2 e^{-\frac{(x_1)^2+t_1^2+(x_2)^2+t_2^2}{2a^2}} \frac{\Delta}{2}\left(\frac{(u_1-u_2)^2}{\bar{u}^4}+\frac{(v_1-v_2)^2}{\bar{v}^4}\right)\frac{1}{(u_1-u_2)^\Delta (v_2-v_1)^\Delta}+\cdots, \tag{3.7}$$

where $\cdots$ means $\mathcal{O}(\frac{1}{\bar{u}^5})$ and $\mathcal{O}(\frac{1}{\bar{v}^5})$ terms. This implies that $\langle 0,0|T_{00}(t=\bar{t},x^i=\bar{x}^i)|0,0\rangle \sim \mathcal{O}(\frac{1}{|\bar{x}\pm\bar{t}|^4})$ at $t=\bar{t}$ and the contribution of the region $|\bar{x}\pm\bar{t}| \gg a$ to the energy of the state at $t=\bar{t}$ is $\mathcal{O}(\frac{1}{|\bar{x}\pm\bar{t}|^3})$. Thus, the energy density is localized and only nonzero, for $a\to 0$, at $\bar{x}=\bar{t}$ or $\bar{x}=-\bar{t}$, which are on the light cone.

Now, let us go back to studying the wave-packet state $|p,\bar{\omega}\rangle$. For this, by defining

$$p_u = \bar{\omega}-p(\geq 0), \qquad p_v=\bar{\omega}+p(\geq 0), \tag{3.8}$$

we have

$$\begin{aligned}
\langle p,\bar{\omega}|T_{00}(t=\bar{t},x=\bar{x})|p,\bar{\omega}\rangle &= \int dt_1 dx_1 e^{-\frac{(x_1)^2+t_1^2}{2a^2}-ipx_1+i\bar{\omega}t_1}\int dt_2 dx_2 e^{-\frac{(x_2)^2+t_2^2}{2a^2}+ipx_2-i\bar{\omega}t_2}\\
&\quad\times\frac{\Delta}{2}\left(\frac{(u_1-u_2)^2}{(\bar{u}-u_1)^2(\bar{u}-u_2)^2}+\frac{(v_1-v_2)^2}{(\bar{v}-v_1)^2(\bar{v}-v_2)^2}\right)\frac{1}{(u_1-u_2)^\Delta(v_2-v_1)^\Delta}\\
&=\frac{1}{4}\int du_1 dv_1 du_2 dv_2 e^{-\frac{(u_1)^2+(v_1)^2+(u_2)^2+(v_2)^2}{4a^2}+i(p_u u_1+p_v v_1-p_u u_2-p_v v_2)/2}\\
&\quad\times\frac{\Delta}{2}\left(\frac{(u_1-u_2)^2}{(\bar{u}-u_1)^2(\bar{u}-u_2)^2}+\frac{(v_1-v_2)^2}{(\bar{v}-v_1)^2(\bar{v}-v_2)^2}\right)\frac{1}{(u_1-u_2)^\Delta(v_2-v_1)^\Delta}\\
&=\frac{1}{4}\int du_1 dv_1 du_2 dv_2 e^{-\frac{(u_1-ip_u a^2)^2+(v_1-ip_v a^2)^2+(u_2+ip_u a^2)^2+(v_2+ip_v a^2)^2}{4a^2}-\frac{a^2}{2}((p_u)^2+(p_v)^2)}\\
&\quad\times\frac{\Delta}{2}\left(\frac{(u_1-u_2)^2}{(\bar{u}-u_1)^2(\bar{u}-u_2)^2}+\frac{(v_1-v_2)^2}{(\bar{v}-v_1)^2(\bar{v}-v_2)^2}\right)\frac{1}{(u_1-u_2)^\Delta(v_2-v_1)^\Delta},
\end{aligned} \tag{3.9}$$

which is localized on the light cone $\bar{u}=0$ or $\bar{v}=0$, as we can easily see using the same argument above. We will evaluate this more explicitly below. Let us consider a part of it:

$$\begin{aligned}
A &\equiv \int du_1 dv_1 du_2 dv_2 e^{-\frac{(u_1-ip_u a^2)^2+(v_1-ip_v a^2)^2+(u_2+ip_u a^2)^2+(v_2+ip_v a^2)^2}{4a^2}-\frac{a^2}{2}((p_u)^2+(p_v)^2)}\frac{(u_1-u_2)^2}{(\bar{u}-u_1)^2(\bar{u}-u_2)^2}\frac{1}{(u_1-u_2)^\Delta(v_2-v_1)^\Delta}\\
&= e^{-\frac{a^2}{2}((p_v)^2)}\int dv_1 dv_2 e^{-\frac{(v_1-ip_v a^2)^2+(v_2+ip_v a^2)^2}{4a^2}}\frac{1}{(v_2-v_1)^\Delta}\\
&\quad\times e^{-\frac{a^2}{2}((p_u)^2)}\int du_1 du_2 e^{-\frac{(u_1-ip_u a^2)^2+(u_2+ip_u a^2)^2}{4a^2}}\frac{1}{(\bar{u}-u_1)^2(\bar{u}-u_2)^2}\frac{1}{(u_1-u_2)^{\Delta-2}}.
\end{aligned} \tag{3.10}$$

The other part is obtained by interchanging $\{\bar{u},p_u\}$ and $\{\bar{v},p_v\}$. Here the integration paths are taken as $u_i\in\mathbf{R}-i\epsilon_i$ and $v_i\in\mathbf{R}-i\epsilon_i$, with $\epsilon_2>0>\epsilon_1$ for the $i\epsilon$ prescription for the ordering of the operator. First, we perform the v_1 integration,

$$\begin{aligned}
&e^{-\frac{a^2}{2}((p_v)^2)}\int dv_1 dv_2 e^{-\frac{(v_1-ip_v a^2)^2+(v_2+ip_v a^2)^2}{4a^2}}\frac{1}{(v_2-v_1)^\Delta}\\
&= (-1)^\Delta e^{-\frac{a^2}{2}((p_v)^2)}\int dv_1 dv_2 \frac{1}{(v_1-v_2)^{\Delta-q}}\frac{\Gamma(\Delta-q)}{\Gamma(\Delta)}\frac{\partial^q}{\partial v_1^q}e^{-\frac{(v_1-ip_v a^2)^2+(v_2+ip_v a^2)^2}{4a^2}},
\end{aligned} \tag{3.11}$$

where q is an integer such that $1\geq\Delta-q>0$, and consider what happens if we move the path to $v_1\in\mathbf{R}+ia^2p_v$. Then, the integration does not depend on p_v and the Gaussian factor $e^{-\frac{a^2}{2}(p_v)^2}$, which is very small, cannot be canceled. Thus, this contribution can be neglected in our approximation. The remaining parts of this are contributions from the pole or the branching point at $v_1=v_2$ in the region between $\mathbf{R}-i\epsilon_1$ and $\mathbf{R}+ia^2p_v$.

For $\Delta \in \mathbf{Z}$, for which $q = \Delta - 1$, the contribution from the single pole at $v_1 = v_2$ is

$$(-1)^\Delta \frac{2\pi i}{\Gamma(\Delta)} e^{-\frac{a^2}{2}((p_v)^2)} \int dv_2 \frac{\partial^{\Delta-1}}{\partial v_1^{\Delta-1}} e^{-\frac{(v_1 - ip_v a^2)^2 + (v_2 + ip_v a^2)^2}{4a^2}} \Big|_{v_1=v_2}$$
$$\simeq \frac{2\pi i}{\Gamma(\Delta)} (-ip_v/2)^{\Delta-1} e^{-\frac{a^2}{2}((p_v)^2)} \int dv_2 e^{-\frac{(v_2)^2 - (p_v a^2)^2}{2a^2}}$$
$$= \frac{(2\pi)^{3/2}}{\Gamma(\Delta)} (-i)^\Delta a (p_v/2)^{\Delta-1}, \qquad (3.12)$$

where we have neglected the terms that are subleading in $1/(p_v a)$ expansion.[9]

For $\Delta \notin \mathbf{Z}$, there is a branching point $v_1 = v_2$. Here we can neglect the contribution around the branching point because $1 > \Delta - q$. Thus, the contribution from the cut from $v_1 = v_2$ is

$$(-1)^\Delta e^{-\frac{a^2}{2}((p_v)^2)} \int dv_2 (1 - e^{2\pi i (q-\Delta)}) i$$
$$\times \int_0^{p_v a^2} dy \frac{1}{(iy)^{\Delta-q}} \frac{\Gamma(\Delta - q)}{\Gamma(\Delta)} \frac{\partial^q}{\partial (iy)^q} e^{-\frac{(v_2 + iy - ip_v a^2)^2 + (v_2 + ip_v a^2)^2}{4a^2}}, \qquad (3.13)$$

where $v_1 = v_2 + iy$. Because of the Gaussian factor, which is almost zero for $y \gg a$, the y integration can be approximated as

$$\int_0^{p_v a^2} dy \frac{(-1)^\Delta}{(iy)^{\Delta-q}} \frac{\partial^q}{\partial (iy)^q} e^{-\frac{(v_2 + iy - ip_v a^2)^2}{4a^2}} \simeq \int_0^\infty dy \frac{(-1)^\Delta}{(iy)^{\Delta-q}} (ip_v/2)^q e^{-yp_v/2} e^{-\frac{(v_2 - ip_v a^2)^2}{4a^2}}$$
$$= \Gamma(1 - \Delta + q)(p_v/2)^{\Delta-1} e^{-\frac{(v_2 - ip_v a^2)^2}{4a^2}} (-1)^{\Delta-q} (-i)^\Delta. \qquad (3.14)$$

Then, we can easily check that this also gives (3.12) for $\Delta \notin \mathbf{Z}$.

Next, we compute

$$e^{-\frac{a^2}{2}((p_u)^2)} \int du_1 du_2 e^{-\frac{(u_1 - ip_u a^2)^2 + (u_2 + ip_u a^2)^2}{4a^2}} \frac{1}{(\bar{u} - u_1)^2 (\bar{u} - u_2)^2} \frac{1}{(u_1 - u_2)^{\Delta-2}}$$
$$= e^{-\frac{a^2}{2}((p_u)^2)} \int du_2 e^{-\frac{(u_2 + ip_u a^2)^2}{4a^2}} \frac{1}{(\bar{u} - u_2)^2} \int du_1 \frac{1}{(u_1 - \bar{u})} \frac{\partial}{\partial u_1} \left(e^{-\frac{(u_1 - ip_u a^2)^2}{4a^2}} \frac{1}{(u_1 - u_2)^{\Delta-2}} \right). \qquad (3.15)$$

As for the v_1 integration, we move the path of the u_1 integration to $u_1 \in \mathbf{R} + ia^2 p_u$ and take the residue at $u_1 = \bar{u}$.[10] Then, the result is

$$2\pi i e^{-\frac{a^2}{2}((p_u)^2)} e^{-\frac{(\bar{u} - ip_u a^2)^2}{4a^2}} \int du_2 e^{-\frac{(u_2 + ip_u a^2)^2}{4a^2}} \frac{1}{(\bar{u} - u_2)^\Delta} \left(ip_u/2 - (\Delta - 2) \frac{1}{(\bar{u} - u_2)} \right), \qquad (3.16)$$

where we have neglected the term proportional to $\bar{u}$ which is small because there is the Gaussian factor $e^{-\frac{(\bar{u})^2}{2a^2}}$ after the u_2 integration, as we will see below. For the u_2 integration, we move the path to $u_2 \in \mathbf{R} - ia^2 p_u$ and take the residue at $u_2 = \bar{u}$. Then, the result is

$$-(-1)^\Delta (2\pi)^2 e^{-\frac{(\bar{u})^2}{2a^2}} \frac{1}{\Gamma(\Delta)} (-ip_u/2)^\Delta \left(-1 + \frac{\Delta - 2}{\Delta} \right)$$
$$= (2\pi)^2 e^{-\frac{(\bar{u})^2}{2a^2}} \frac{1}{\Gamma(\Delta)} (ip_u/2)^\Delta \frac{2}{\Delta}. \qquad (3.17)$$

Thus, we obtain

$$A \simeq e^{-\frac{(\bar{u})^2}{2a^2}} (2\pi)^{5/2} \frac{1}{\Gamma(\Delta)^2} (p_v p_u/4)^{\Delta-1} \frac{p_u a}{\Delta}. \qquad (3.18)$$

We need to compute the normalization of the state,

[9]After the Gaussian v_2 integration, v_2 becomes a $\mathcal{O}(a)$ quantity.

[10]There is also the contribution from the singularities at $u_1 = u_2$. However, at this point, the Gaussian factor becomes $e^{-\frac{(u_2 + ip_u a^2)^2}{4a^2}} e^{-\frac{(u_2 - ip_u a^2)^2}{4a^2}} = e^{-\frac{(u_2)^2 - (p_u a^2)^2}{2a^2}}$ and the u_2 integration is p_u independent. Using this, we can easily see that this contribution is smaller than that from the pole at $u_1 = \bar{u}$.

$$\begin{aligned}\mathcal{N}^2 &= \langle p,\bar{\omega}|p,\bar{\omega}\rangle \\ &= \int dt_1 dx_1 e^{-\frac{(x_1)^2+t_1^2}{2a^2}-ipx_1+i\bar{\omega}t_1}\int dt_2 dx_2 e^{-\frac{(x_2)^2+t_2^2}{2a^2}+ipx_2-i\bar{\omega}t_2}\frac{1}{(u_1-u_2)^\Delta(v_2-v_1)^\Delta} \\ &= \frac{1}{4}\int du_1 dv_1 du_2 dv_2 e^{-\frac{(u_1-ip_u a^2)^2+(v_1-ip_v a^2)^2+(u_2+ip_u a^2)^2+(v_2+ip_v a^2)^2}{4a^2}-\frac{a^2}{2}((p_u)^2+(p_v)^2)}\frac{1}{(u_1-u_2)^\Delta(v_2-v_1)^\Delta},\end{aligned} \tag{3.19}$$

which is the same as the computation of (3.11) and (3.12). Hence, we obtain

$$\mathcal{N}^2 \simeq \frac{1}{4}\frac{(2\pi)^{3/2}}{\Gamma(\Delta)}(-i)^\Delta a(p_v/2)^{\Delta-1} \times \frac{(2\pi)^{3/2}}{\Gamma(\Delta)}(i)^\Delta a(p_u/2)^{\Delta-1}. \tag{3.20}$$

Finally, we have the energy density $\mathcal{E}(\bar{t},\bar{x})$ of the wave-packet state as

$$\begin{aligned}\mathcal{E}(\bar{t},\bar{x}) &= \frac{1}{\mathcal{N}^2}\langle p,\bar{\omega}|\frac{1}{2\pi}T_{00}(t=\bar{t},x=\bar{x})|p,\bar{\omega}\rangle \\ &\simeq \frac{1}{2\sqrt{2\pi}a}\left(e^{-\frac{(\bar{u})^2}{2a^2}}p_u + e^{-\frac{(\bar{v})^2}{2a^2}}p_v\right),\end{aligned} \tag{3.21}$$

which is localized on the light cone $\bar{u}=0$ or $\bar{v}=0$. The energies of the regions near $\bar{u}=0$ and $\bar{v}=0$ are $p_u/2$ and $p_v/2$, respectively, because $\frac{1}{\sqrt{2\pi}a}\int dx e^{-\frac{x^2}{2a^2}}=1$. Their sum is the correct energy $\bar{\omega}$ of the state.

In summary, the energy density of the wave-packet state (2.13) in AdS_3/CFT_2 at time t is localized on the light cone $x=\pm t$ for small a and small $1/(\bar{\omega}a)$. The energy localized near $x=t$, which is equivalent to $v=x-t=0$, is $(\bar{\omega}+p)/2$. The energy localized near $x=-t$, which is equivalent to $u=x+t=0$, is $(\bar{\omega}-p)/2$. Thus, the wave packet on the bulk corresponds to a pair of the excitations at $t>0$, which is given schematically by $(\tilde{\mathcal{O}}_v(x=t)+\tilde{\mathcal{O}}_u(x=-t))|0\rangle$, where $\tilde{\mathcal{O}}_v,\tilde{\mathcal{O}}_u$ are some local operators. For $\bar{\omega}=p$, the state is only at $x=t$, not a pair. Indeed, in the bulk picture also, the wave packet is on the boundary for $\bar{\omega}=p$, and then it is localized at $x=t$, $z=0$ in the bulk picture.

1. *Global* AdS/CFT

So far, we have considered the wave packets in the Poincaré AdS case. We can easily generalize this to the global AdS case by a conformal transformation. [More precisely, for the global AdS_3/CFT_2 case, the parameters in the wave-packet state (2.13) should be replaced by, for example, $z\to\pi/2-\rho$, $t\to\tau$ and $x\to\tanh(\theta)$, where $-\pi<\theta\le\pi$ is the coordinate for the S^1.] Instead of explicitly doing this, we can conclude that the above summary for the Poincaré AdS case is also true for the global AdS case. This is because the computations of the energy density essentially use the information on the singularities of the three-point function. Thus, the representation of the bulk wave packet from the perspective of the energy density in CFT is the same as the "simple bulk

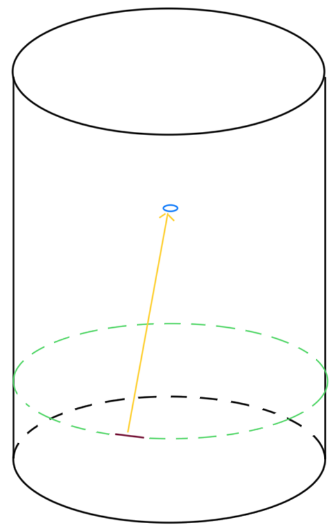

FIG. 1. An example of the bulk wave packet (moving toward the center).

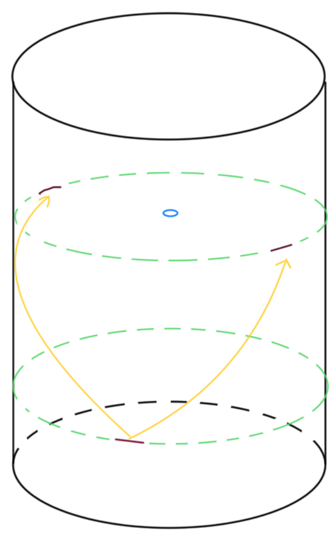

FIG. 2. The corresponding two "particles" in the CFT picture.

reconstruction" picture given in [2]. See Figs. 1 and 2. Note that only this is consistent with the causalities in both the bulk and the boundary theories because the bulk wave packet starting from the boundary at $t = 0$ will reach the boundary at $t = \pi$, and then the bulk local field can be regarded as the CFT local primary field at $t = 0$ and $t = \pi$.

One might think that our result is inconsistent with the Hamilton-Kabat-Lifschytz-Lowe (HKLL) bulk reconstruction formula [10] because the bulk local state corresponds to only two points on the light cone. However, as shown in [2], this is consistent with the HKLL bulk reconstruction formula because of the ambiguities of the smearing function in the formula. We summarize the discussion for this in Appendix B.

2. Subregion duality and entanglement-wedge reconstruction

If the subregion duality and the entanglement-wedge reconstruction are correct, if we take a region A in the CFT picture such that the bulk wave packet is in the bulk entanglement M_A wedge for A, the state should be supported only in the region A in the CFT picture.[11] This is not the case if the wave packet is the horizon-to-horizon type discussed in [2]. Here this statement takes into account the energy density from the perspective of CFT, which is one of the leading effects of bulk interactions. This invalidity of subregion duality and the entanglement-wedge reconstruction[12] can be seen from a simpler example. Let us consider a state that is obtained by acting the (smeared) bulk local operator ϕ at the center of the global AdS space, i.e., $\rho = 0$, on the vacuum. This can be written by the CFT operator using the HKLL bulk reconstruction formula [10]. Then, the energy density is obtained by an explicit calculation. Of course, by the symmetry of the state, the result is a uniform distribution on S^{d-1}. Then, if we take a region A in the CFT picture such that the bulk wave packet is in the bulk entanglement wedge M_A, the state should be supported only in the region A in the CFT picture according to the subregion duality and the entanglement-wedge reconstruction. This means that there exists an operator supported in the subregion A that produces a nonzero energy density outside A. It is obviously unphysical.

[11]Here the subregion duality and the entanglement-wedge reconstruction are those given in [4], in which the bulk Hilbert space was assumed to be a tensor product of the two Hilbert spaces for the subregions M_A and $M_{\bar{A}}$. This may be (approximately) realized by a gauge fixing, such as the Fefferman-Graham gauge. In particular, we claim that the global and Rindler HKLL bulk reconstructions of a bulk local operator in the overlap of the two entanglement wedges should be different in the leading order of the $1/N$ expansion.

[12]This invalidity may be due to the invalidity of the large-N expansion for the AdS/CFT correspondence for the subregion [11]. This is related to the brick wall in AdS/CFT [12,13] and the fuzzball conjecture [14,15].

Here it should be stressed that there exists a CFT operator $\mathcal{O}_A$ supported in the region A corresponding to the bulk operator ϕ_{M_A} supported in the region M_A such that $\mathcal{O}_A|\psi\rangle = \phi_{M_A}|\psi\rangle$, where $|\psi\rangle$ is an arbitrary low-energy state if the entanglement-wedge reconstruction is correct. This implies that

$$\mathcal{O}_A^2\mathcal{O}_A^1|\psi\rangle = \phi_{M_A}^2\phi_{M_A}^1|\psi\rangle \tag{3.22}$$

for $\mathcal{O}_A^i|\psi\rangle = \phi_{M_A}^i|\psi\rangle$ because, by writing $|\psi_1\rangle = \phi_{M_A}^1|\psi\rangle$, we find $\mathcal{O}_A^2|\psi_1\rangle = \phi_{M_A}^2|\psi_1\rangle$. Then, the Reeh-Schlieder theorem (and the mirror map of the thermofield double) is not useful for the entanglement-wedge reconstruction, although they can give a similar CFT operator such that $\mathcal{O}_A|0\rangle = \phi_{M_A}|0\rangle$, which does not satisfy (3.22). Furthermore, with the Reeh-Schlieder theorem, the vacuum acting by operators supported in any subregion can give any state. Thus, any small subregion can be dual to the whole space, and statements of the subregion duality and entanglement-wedge reconstruction will be meaningless using the Reeh-Schlieder theorem.

Below, we will show that the entanglement-wedge reconstruction is violated for the coherent state of the wave packet explicitly. First, we define

$$\phi_{p,\bar{\omega}} = \lim_{z\to 0}\frac{1}{z^\Delta}\int dt dx^i e^{-\frac{x^i x_i + t^2}{2a^2}} \times (e^{ip_i x^i - i\bar{\omega}t} + e^{-ip_i x^i + i\bar{\omega}t})\phi(t, z, x^i) \tag{3.23}$$

and consider $|\psi\rangle = e^{i\epsilon\phi_{p,\bar{\omega}}}|0\rangle$ represented by a CFT primary operator like (2.13). Note that $|p,\bar{\omega}\rangle \simeq \phi_{p,\bar{\omega}}|0\rangle$ for $\bar{\omega} \gg 1/a \gg 1$. We can show that $\langle\psi|T_{00}(t,x)|\psi\rangle = \langle p,\bar{\omega}|T_{00}(t,x)|p,\bar{\omega}\rangle + \mathcal{O}(\epsilon^3)$ because $\langle 0|T_{00}(t,x)\phi_{p,\bar{\omega}}^2|0\rangle \simeq 0$, which follows from the fact that the pole at $u_1 = 0$ or $u_2 = 0$ does not contribute to the u_1, u_2 integration in (3.15) for this ordering of the operators. Let us choose $p, \bar{\omega}$ such that at $t = \bar{t}$ the bulk wave packet is in M_A and the energy density in the CFT picture is nonzero at some points in A and $\bar{A}$. On the other hand, if $\phi_{p,\bar{\omega}}$ is reconstructed from the CFT operator on A, i.e., $\phi_{p,\bar{\omega}} = \phi_{p,\bar{\omega}}(\mathcal{O}_A)$, we can show that $\langle\psi|T_{00}(t=\bar{t}, x=\bar{x})|\psi\rangle = 0$ for $x \in \bar{A}$ because

$$\langle\psi|T_{00}(t=\bar{t}, x=\bar{x})|\psi\rangle = \langle 0|([T_{00}(t=\bar{t}, x=\bar{x}), i\epsilon\phi_{p,\bar{\omega}}(\mathcal{O}_A)] + \cdots)|0\rangle = 0, \tag{3.24}$$

because A and $t = \bar{t}, x = \bar{x}$ are spatially separated and the causality of the CFT implies that the commutators are zero. Thus, $\phi_{p,\bar{\omega}}$, which is supported on M_A, cannot be reconstructed from the CFT operator supported on A.

B. Overlap between the wave-packet state and CFT local state

Instead of the energy density, one might think that the overlap between the wave-packet state $|p,\bar{\omega}\rangle$ and CFT local state $\mathcal{O}(x,t)|0\rangle$ will give some information xabout where the state is localized or the spatial distribution of the state. This can be essentially regarded as the VEV of the scalar operator for the coherent state, which is discussed in Appendix A, because $\langle J|\mathcal{O}(x,t)|J\rangle \to \epsilon\langle 0|(\int dt'dx'\epsilon J(t',x')\mathcal{O}(t',x')\mathcal{O}(x,t) + \mathcal{O}(x,t)\int dt'dx'\epsilon J(t',x'))|0\rangle = \epsilon(\langle p,\bar{\omega}|\mathcal{O}(x,t)|0\rangle + \langle 0|\mathcal{O}(x,t)|p,\bar{\omega}\rangle)$ for small ϵ, where $|J\rangle = e^{\int dt'dx'\epsilon J(t',x')\mathcal{O}(t',x')}|0\rangle$ is the coherent state.[13] The VEV of the scalar operator is one of the most important calculable quantities in AdS/CFT, at least if it is time independent. Nevertheless, it is highly difficult to obtain any information on the properties of this state in CFT. This is because $\frac{\partial^n}{\partial t^n}\mathcal{O}(x,t)|0\rangle$ with any $n \geq 0$ is an independent state for fixed x, t in the generalized free approximation, which is the large-N limit we have taken. This means that there are infinitely independent states at each point in the large-N limit. Thus, even if we know some information about $\mathcal{O}(x,t)$ at fixed t, it is an infinitesimal piece of information.[14] With $\mathcal{O}(x,t)$ for all t, we can construct $\frac{\partial^n}{\partial t^n}\mathcal{O}(x,t)$. However, in order to do so, we need to know the nth-derivative coefficient precisely for arbitrary n.[15]

If we know information about $\mathcal{O}(x,t)$ for any x, t, we could recover, for example, the spatial distribution of the wave packet, in principle. However, this should be nonlocal in time. Furthermore, it is unclear how to recover it. Indeed, there may be ambiguities for it like in the computation of the mutual information for the generalized free field discussed in [17]. Therefore, we conclude that it is highly difficult to obtain any information about the properties of the wave-packet state in CFT using the overlap between it and the CFT local state or the VEV of the CFT operator. Below, we will explicitly see the difficulty for the global AdS_3 case.

Let us consider the global AdS_3. The CFT primary field in the large-N limit is written using the creation and annihilation operators [18] as

$$\mathcal{O}(\tau,\theta) \sim \sum_{n\in\mathbf{Z}_{\geq 0}, m\in\mathbf{Z}} e^{i(2n+|m|)\tau - im\theta} a^\dagger_{n,m} + \text{H.c.}, \quad (3.25)$$

where we took $\Delta = d/2$ for simplicity. Note that this is invariant under $\tau \to \tau + \pi$ and $\theta \to \theta + \pi$. We can easily extend this for general Δ. The bulk wave-packet state at $\tau = 0$, $\theta = 0$ with energy ω and momentum p is given by

$$|p,\bar{\omega}\rangle = \int d\tau d\theta e^{-\frac{\tau^2+\theta^2}{2a^2} + ip\theta - i\bar{\omega}\tau}\mathcal{O}(\tau,\theta)|0\rangle \sim a^2 \sum_{n\in\mathbf{Z}_{\geq 0}, m\in\mathbf{Z}} e^{-\frac{a^2}{2}(((2n+|m|)-\bar{\omega})^2 + (m-p)^2)} a^\dagger_{n,m}|0\rangle, \quad (3.26)$$

where we used the Gaussian factor $e^{-\frac{\theta^2}{2a^2}}$ instead of $e^{-\frac{\tan(\theta)^2}{2a^2}}$ because their difference is negligible for $a \ll 1$. Then, the overlap is

$$\begin{aligned} \langle 0|\mathcal{O}(\tau,\theta)|p,\bar{\omega}\rangle &\sim a^2 \sum_{n\in\mathbf{Z}_{\geq 0}, m\in\mathbf{Z}} e^{-\frac{a^2}{2}(((2n+|m|)-\bar{\omega})^2+(m-p)^2)} e^{-i(2n+|m|)\tau + im\theta} \\ &\simeq a^2 e^{-i\bar{\omega}\tau + ip\theta} \sum_{n\in\mathbf{Z}, m\in\mathbf{Z}} e^{-\frac{a^2}{2}(((2n+|m|))^2+(m)^2)} e^{-i(2n+m)\tau + im\theta} \\ &\simeq a^4 e^{-i\bar{\omega}\tau + ip\theta}\delta(\tau + \pi\mathbf{Z})\delta(\theta - \tau + 2\pi\mathbf{Z}), \end{aligned} \quad (3.27)$$

where we assumed $p > 0$ other than $|p| \gg 1, \bar{\omega} \gg 1$, and we noted, for example, the Gaussian factor $e^{-\frac{\tau^2}{2a^2}}/a^2$ as $\delta(\tau)$, for notational simplicity. Thus, its distribution is localized on $\theta = 0$ at $\tau = 0 + 2\pi\mathbf{Z}$ and $\theta = \pi$ at $\tau = \pi + 2\pi\mathbf{Z}$. These spacetime points are when the bulk wave packet is at the boundary. For other τ, it almost vanishes. This means that the state is diffused to infinitely many states $\frac{\partial^n}{\partial t^n}\mathcal{O}(x,t)|0\rangle$ for it.

Note added. As this paper was being completed, we became aware of the preprint [19] in which a bulk wave packet similar to ours was constructed and its VEV of the CFT operator was discussed.

[13] Even for the case that ϵ is not small, a similar expression holds because the $\mathcal{O}(x,t)$ is linear in the creation and annihilation operators in the generalized free approximation.

[14] The statements here can be applied for the energy-momentum tensor. However, the energy-momentum tensor has special properties. The energy density will be non-negative and any local excitation will give a nonzero energy density. Thus, the energy density can be used to understand the spatial distribution of the wave-packet state.

[15] Even for time-dependent cases, it is possible to obtain some information from $\mathcal{O}(x,t)$ depending on the state, such as the state that represents a wave in AdS/boundary CFT [16].

ACKNOWLEDGMENTS

The author would like to thank N. Tanahashi for collaboration in the early stages of this work and helpful discussions. The author thanks S. Sugishita for the useful comments. This work was supported by JSPS KAKENHI Grant Number 17K05414. This work was supported by MEXT-JSPS Grant-in-Aid for Transformative Research Areas (A) "Extreme Universe," No. 21H05184.

APPENDIX A: COHERENT STATE

We denote the bulk local operator in the free limit as

$$\phi(X) = \sum_n \psi_n(X) a_n + \text{H.c.}, \tag{A1}$$

where X represents all coordinates of the bulk spacetime, n labels the mode expansion, and a_n represents the annihilation operator. The corresponding CFT primary operator in the free limit is given as

$$\mathcal{O}(x) = \sum_n \psi_n^{\text{CFT}}(x) a_n + \text{H.c.}, \tag{A2}$$

where x represents all coordinates of the CFT spacetime. Then, the (normalized) semiclassical state in the limit is given by the coherent state

$$|\alpha\rangle = e^{\sum_n(\alpha_n a_n^\dagger - \alpha_n^* a_n)}|0\rangle = e^{-\frac{1}{2}\sum_n |\alpha_n|^2} e^{\sum_n \alpha_n a_n^\dagger}|0\rangle, \tag{A3}$$

for which the time evolution is given by

$$|\alpha(t)\rangle = e^{iHt}|\alpha\rangle = e^{\sum_n(e^{iE_n t}\alpha_n a_n^\dagger - e^{-iE_n t}\alpha_n^* a_n)}. \tag{A4}$$

The VEVs of the bulk and CFT local operators, which are linear in the creation and annihilation operators, are given as

$$\langle\alpha|\phi(X)|\alpha\rangle = \sum_n \psi_n(X)\alpha_n + \text{c.c.},$$
$$\langle\alpha|\mathcal{O}(x)|\alpha\rangle = \sum_n \psi_n^{\text{CFT}}(x)\alpha_n + \text{c.c.} \tag{A5}$$

Let us rewrite the one-particle state for the bulk wave packet (2.13) as

$$|p, \bar{\omega}\rangle = (\phi^{wp})|0\rangle, \tag{A6}$$

where

$$\phi^{wp} = \sum_n \psi_n^{wp} a_n^\dagger = \int dt dx^i e^{-\frac{x^i x_i + t^2}{2a^2} + i p_i x^i - i\bar{\omega}t}\mathcal{O}^+(t, x), \tag{A7}$$

where $\mathcal{O}^+(t,x)$ is a part of $\mathcal{O}(t,x)$ which is linear in $a_n^\dagger$ and

$$\psi_n^{wp} = \int dt dx^i e^{-\frac{x^i x_i + t^2}{2a^2} + i p_i x^i - i\bar{\omega}t}(\psi_n^{\text{CFT}}(t, x^i))^*. \tag{A8}$$

Note that the overlaps between this and the bulk and CFT local states considered in this paper are given by

$$\langle 0|\phi(X)|p, \bar{\omega}\rangle = \sum_n (\psi_n(X))^* \psi_n^{wp},$$
$$\langle 0|\mathcal{O}(x)|p, \bar{\omega}\rangle = \sum_n (\psi_n^{\text{CFT}}(X))^* \psi_n^{wp}. \tag{A9}$$

Then, the corresponding coherent state representing the bulk wave packet is given by setting $\alpha_n = (\psi_n^{wp})^*$, i.e.,

$$|wp\rangle = e^{\sum_n((\psi^{wp})^* a_n^\dagger - \psi_n^{wp} a_n)}|0\rangle. \tag{A10}$$

The VEVs of the bulk and CFT local operators for this state are given as

$$\langle wp|\phi(X)|wp\rangle = \sum_n \psi_n(X)(\psi_n^{wp})^* + \text{c.c.}$$
$$= \langle 0|\phi(X)|p, \bar{\omega}\rangle + \text{c.c.},$$
$$\langle wp|\mathcal{O}(x)|wp\rangle = \sum_n \psi_n^{\text{CFT}}(x)(\psi_n^{wp})^* + \text{c.c.}$$
$$= \langle 0|\mathcal{O}(x)|p, \bar{\omega}\rangle + \text{c.c.}, \tag{A11}$$

which means that where the overlaps are distributed for the one-particle state is the same as where the VEVs are distributed for the coherent state. Thus, the corresponding coherent state represents the wave packet.

APPENDIX B: ON THE HKLL BULK RECONSTRUCTION FORMULA

The HKLL bulk reconstruction formula [10] is the formula representing the bulk local field as the spacetime integrals of the corresponding CFT primary operators in the generalized free limit, following the ideas in [5,20,21]. The explicit formula for the bulk local field at the center in global AdS space is given by

$$\phi(\rho = 0, \tau = 0) = \int_{-\frac{\pi}{2} \le \tau \le \frac{\pi}{2}} d\tau' d\Omega' K(\Omega', \tau')\mathcal{O}(\Omega', \tau'), \tag{B1}$$

where

$$K(\Omega, \tau) \sim \frac{1}{|\cos\tau|^{d-\Delta}}. \tag{B2}$$

First, we note that the CFT primary operator $\mathcal{O}(\Omega, \tau)$ is periodic, i.e., $\mathcal{O}(\Omega, \tau + 2\pi) = \mathcal{O}(\Omega, \tau)$, in the generalized free-field approximation. We also note that there are infinitely many choices of the smearing function $K(\Omega, \tau)$, as noted in [10], because of the Fourier transformation of the CFT primary operator $\mathcal{O}(\Omega, \tau)$ in the generalized free-field approximation. Indeed, they use this ambiguity to obtain this simple formula.

In [2,3], the dominant contributions in the τ integrals in (B1) were from $\tau = \pm\pi/2$. This means, for example, that the overlap $\langle\phi|\mathcal{O},\tau\rangle$ between the bulk local state $|\phi\rangle = \phi(\rho = 0, \tau = 0)|0\rangle$ and the spherically symmetric CFT state $|\mathcal{O},\tau\rangle = \int d\Omega' \mathcal{O}(\Omega',\tau)|0\rangle$ is zero for $\tau = \pi/2 + \pi\mathbf{Z}$. Note that the smearing function can be essentially taken to $K(\Omega,\tau) \sim \langle\phi|\mathcal{O},\tau\rangle$, which is zero for $\tau \neq \tau = \pi/2 + \pi\mathbf{Z}$ because for the generalized free field we can show that $\langle\mathcal{O},\tau|\mathcal{O},\tau'\rangle \sim \delta(\tau - \tau' + \pi\mathbf{Z})$.

The above statements are not precise because the bulk local operator and the CFT operator are ill defined by the UV divergences. More precisely, we can show that $\langle\phi;a|\mathcal{O},\tau;a\rangle \to 0$ for $a \to 0$. Here the smeared bulk local state at the center $|\phi;a\rangle$ is given by

$$|\phi;a\rangle = \frac{1}{\mathcal{N}_\phi{}^2}\int d\tau' e^{-\frac{(\tau')^2}{2a^2}}\phi(\rho = 0,\tau')|0\rangle, \qquad \text{(B3)}$$

where $\mathcal{N}_\phi{}^2$ is fixed by the normalization condition $\langle\phi,a|\phi,a\rangle = 1$ and the smeared spherically symmetric CFT state $|\mathcal{O},\tau;a\rangle$ is given by

$$|\mathcal{O},\tau;a\rangle = \frac{1}{\mathcal{N}_O{}^2}\int d\tau' e^{-\frac{(\tau'-\tau)^2}{2a^2}}\mathcal{O}(\tau')|0\rangle, \qquad \text{(B4)}$$

where $\mathcal{N}_O{}^2$ is fixed by the normalization condition $\langle\mathcal{O},\tau,a|\mathcal{O},\tau,a\rangle = 1$. The smearing by the Gaussian integral roughly corresponds to the energy cutoff with $1/a$.

This seems to be impossible, in particular for $\tau = \pm\pi/2$ because $K(\Omega,\tau = \pm\pi/2) = 0$. However, it is indeed possible because of the choice of the ambiguity of the smearing function. The important point here is that the smearing function should be a periodic function; then, it is singular at $\tau = \pi/2 + \pi\mathbf{Z}$ because of the absolute value of $|\cos(\tau)|$. The local field contains an arbitrarily high-energy mode and the singularity gives the nontrivial contribution to the arbitrarily high-energy mode. Thus, the singular points give the dominant contribution to the reconstruction of the bulk local operator, as shown in [2].[16]

We can numerically check this. As an example, we take $d = 3$, $\Delta = 4.8$, and $a = 0.005$. Figure 3 is the plot of the overlap $\langle\phi;a|\mathcal{O},\tau;a\rangle$, where $\tau = \frac{2\pi}{100}m - \frac{\pi}{2}$ and the horizontal axis represents m. This shows the sharp peaks at $\tau = \pi/2 + \pi\mathbf{Z}$. Figure 4 is the same plot of the overlap $\langle\phi;a|\mathcal{O},\tau;a\rangle$, where $\tau = \frac{\pi}{2} + \frac{2\pi}{10000}(m - 50)$ and the horizontal axis represents m. This plot focuses near the $\tau = \pi/2$ region.

[16]The bulk local operator can be explicitly written using the CFT local operators only at $\tau = \pi/2$ with the time derivative [2] using the formulas given in [18,22].

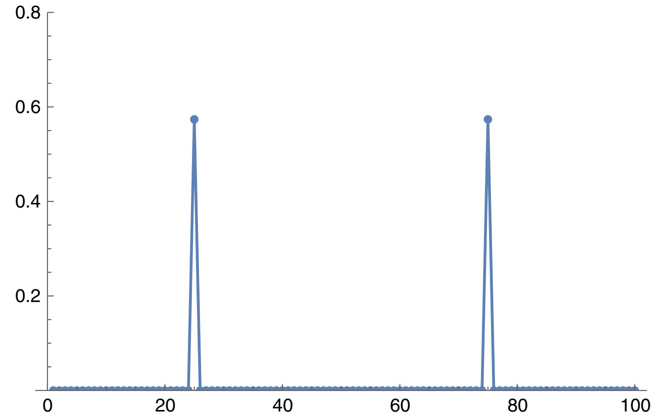

FIG. 3. Plot of the overlap $\langle\phi;a|\mathcal{O},\tau;a\rangle$ for $-\pi < \tau < \pi$.

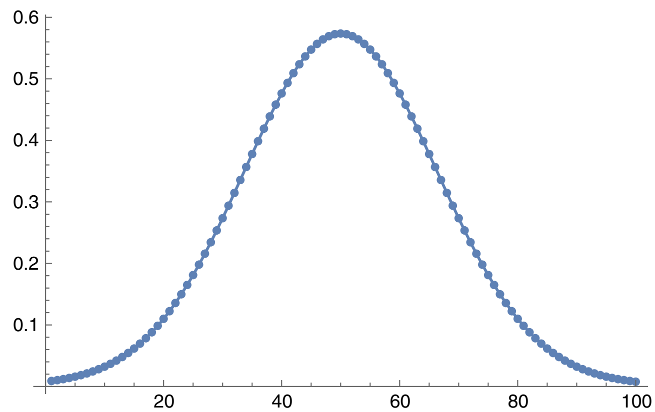

FIG. 4. Plot of the overlap $\langle\phi;a|\mathcal{O},\tau;a\rangle$ for $(\frac{1}{2} - \frac{1}{100})\pi < \tau < (\frac{1}{2} + \frac{1}{100})\pi$.

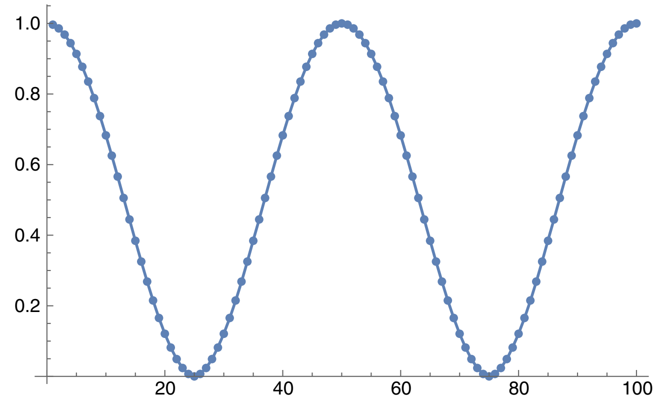

FIG. 5. Plot of the HKLL smearing function $\frac{1}{|\cos\tau|^{d-\Delta}}$.

For comparison, Fig. 5 is the plot of $\frac{1}{|\cos\tau|^{d-\Delta}}$, where $\tau = \frac{2\pi}{100}m - \frac{\pi}{2}$ and the horizontal axis represents m. We also show the plot of $\langle\mathcal{O},\tau;a|\mathcal{O},\tau' = 0;a\rangle$ in Fig. 6, where $\tau = \frac{2\pi}{100}m - \frac{\pi}{2}$ and the horizontal axis represents m. These clearly show the orthogonality.

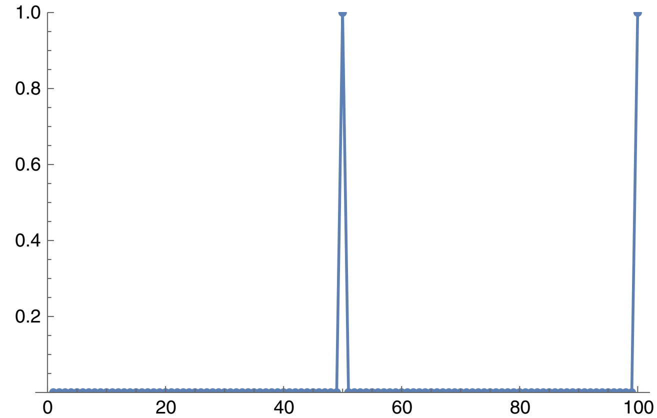

FIG. 6. The corresponding two "particles" in the CFT picture.

[1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] S. Terashima, Simple bulk reconstruction in anti-de Sitter/conformal field theory correspondence, Prog. Theor. Exp. Phys. **2023**, 053B02 (2023).
[3] S. Terashima, Bulk locality in the AdS/CFT correspondence, Phys. Rev. D **104**, 086014 (2021).
[4] A. Almheiri, X. Dong, and D. Harlow, Bulk locality and quantum error correction in AdS/CFT, J. High Energy Phys. 04 (2015) 163.
[5] T. Banks, M.R. Douglas, G.T. Horowitz, and E.J. Martinec, AdS dynamics from conformal field theory, arXiv:hep-th/9808016.
[6] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[7] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[8] A. Belin, D. M. Hofman, and G. Mathys, Einstein gravity from ANEC correlators, J. High Energy Phys. 08 (2019) 032.
[9] L. Nagano and S. Terashima, A note on commutation relation in conformal field theory, J. High Energy Phys. 09 (2021) 187.
[10] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, Holographic representation of local bulk operators, Phys. Rev. D **74**, 066009 (2006).
[11] S. Sugishita and S. Terashima, Rindler bulk reconstruction and subregion duality in AdS/CFT, J. High Energy Phys. 11 (2022) 041.
[12] N. Iizuka and S. Terashima, Brick walls for black holes in AdS/CFT, Nucl. Phys. **B895**, 1 (2015).
[13] G. 't Hooft, On the quantum structure of a black hole, Nucl. Phys. **B256**, 727 (1985).
[14] S. D. Mathur, The fuzzball proposal for black holes: An elementary review, Fortschr. Phys. **53**, 793 (2005).
[15] S. D. Mathur, The information paradox: A pedagogical introduction, Classical Quantum Gravity **26**, 224001 (2009).
[16] K. Izumi, T. Shiromizu, K. Suzuki, T. Takayanagi, and N. Tanahashi, Brane dynamics of holographic BCFTs, J. High Energy Phys. 10 (2022) 050.
[17] V. Benedetti, H. Casini, and P. J. Martinez, Mutual information of generalized free fields, Phys. Rev. D **107**, 046003 (2023).
[18] S. Terashima, AdS/CFT correspondence in operator formalism, J. High Energy Phys. 02 (2018) 019.
[19] S. Kinoshita, K. Murata, and D. Takeda, Shooting null geodesics into holographic spacetimes, J. High Energy Phys. 10 (2023) 074.
[20] I. Bena, On the construction of local fields in the bulk of AdS_5 and other spaces, Phys. Rev. D **62**, 066007 (2000).
[21] M. Duetsch and K.-H. Rehren, Generalized free fields and the AdS—CFT correspondence, Ann. Henri Poincare **4**, 613 (2003).
[22] S. Terashima, Classical limit of large N gauge theories with conformal symmetry, J. High Energy Phys. 02 (2020) 021.