Time problem in primordial perturbations

Alice Boldrin,[1,*] Przemysław Małkiewicz,[1,†] and Patrick Peter[2,‡]
[1]*National Centre for Nuclear Research, Pasteura 7, 02-093 Warszawa, Poland*
[2]*𝒢ℝε℃𝒪—Institut d'Astrophysique de Paris, CNRS and Sorbonne Université, UMR 7095 98 bis boulevard Arago, 75014 Paris, France*

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We study the nonunitary relation between quantum gravitational models defined using different internal times. We show that, despite the nonunitarity, it is possible to provide a prescription for making unambiguous, though restricted, physical predictions independent of specific clocks. To illustrate this result, we employ a model of quantum gravitational waves in a quantum Friedmann universe.

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I. INTRODUCTION

Quantum gravity models suffer from the infamous time problem [1–4], as the external and absolute time on which nonrelativistic physics is based is absent in Einstein's theory of gravity. Therefore, one has to rely on largely arbitrary physical variables, known as internal time variables or internal clocks, to follow changes occurring in gravitational systems. By virtue of the principle of general relativity (time-reparametrization invariance), the free choice of internal time variable has no physical consequence in the classical theory. Upon passing to quantum theory, however, different choices of internal time variables are known to produce unitarily inequivalent quantum models [5–12]. The problem of finding the correct interpretation of these nonequivalent models is commonly referred to as the time problem.

In this article, we look for the most plausible interpretation of such nonequivalent clocks. Our analysis is based on the model of primordial gravitational waves propagating across the Friedmann universe. It is important to note that similar models were previously used for making predictions for the primordial amplitude spectrum of density perturbation, which are greatly constrained by observations (see, e.g., Refs. [13–15]). Remarkably, to the best of our knowledge, the time problem has never been studied for such models, so it is important to clarify the role and the interpretation of internal time variables in their dynamics. We expect that the ensuing conclusions should equally apply to all cosmological models.

The fact that the dynamics are unitarily inequivalent in different clocks is widely known and well documented with plenty of examples; see, e.g., Refs. [16–18]. In this context, it is sometimes emphasized that the only measurable quantities in quantum gravity are gauge-invariant variables that do not depend on the employed clock [19,20]. They are constants of motion. However, they are said to encode all the relational dynamics in spite of being nondynamical themselves. From this point of view, dynamical quantities are not fundamental and are ambiguously given by one-parameter families of gauge-invariant quantities, with each family representing the motion with respect to a specific internal time. The differences are seen as natural rather than inconsistencies that should be worried about. From our viewpoint, on the other hand, the dynamical variables can serve as fundamental variables, and the differences in their dynamics call for a careful interpretation, before allowing for physical predictions.

The cosmological system examined in this article exhibits, as expected, dynamical discrepancies when based on different clocks. The discrepancies concern both the background and perturbation variables. This leads us to ask a fundamental question: what are the dynamical predictions of quantum cosmological models that do not depend on the employed time variable?

We address the above question within the reduced phase space quantization. Namely, we solve the Hamiltonian constraint and choose the internal time variable prior to quantization. An alternative approach would be to first quantize and then solve the constraint quantum mechanically while promoting one of the variables as internal time. Both approaches lead to the same time problem and, therefore, using the technically less involved reduced phase approach is well justified (see, however, Ref. [21] for recent developments in the alternative approach). Most significantly, within the reduced phase space approach, there exists a theory of clock transformations, which is completely crucial for the purpose of this work [22]. Thanks to these precisely defined transformations, we are able to explore all possible clocks and quantize them with an

*Alice.Boldrin@ncbj.gov.pl
†Przemyslaw.Malkiewicz@ncbj.gov.pl
‡patrick.peter@iap.fr

assumption of fixed operator ordering. Hence, any quantum ambiguities found arise from the differences between clocks rather than the differences between quantization prescriptions.

The outline of this article is as follows. In Sec. II, we make a brief introduction to the theory of clock transformations in the reduced phase space of gravitational models. We explain how this theory allows one to remove irrelevant quantization ambiguities when passing to quantum theory based on different clocks and with different basic dynamical variables. In Sec. III, we formulate the reduced phase space description of the Friedmann universe with gravitational waves with respect to a fluid time and obtain the general clock transformation. In Sec. IV, we quantize our model and establish a convenient semiclassical approximation. Section V deals with concrete clock transformations applied to our model and makes comparisons between the resulting dynamics. We summarize our findings, discuss their plausible interpretation, and suggest some directions to move forward in Sec. VI.

II. CLOCK TRANSFORMATIONS IN TOTALLY CONSTRAINED SYSTEMS

One crucial characteristic feature of canonical relativity is the appearance of the Hamiltonian constraint; it is a consequence of the fact that the dynamics of three surfaces is generated by infinitesimal timelike diffeomorphisms, and the latter leave the full four-dimensional spacetime invariant. It by no means makes the dynamics of three surfaces spurious or redundant. Indeed, the Hamiltonian constraint dynamics is a feature of any canonical relativistic theory of gravity, be it Einstein's or any modified gravity theory, though their dynamics are different. The correct interpretation of canonical relativity assumes the lack of an absolute, external time in which three surfaces evolve, and replaces it with internal variables that serve as clocks in which the dynamics of three surfaces takes place. None of the internal clocks can play a privileged role as the principle of relativity states. This picture is certainly self-consistent in the classical theory. At the quantum level, no spacetime exists and, as we will see later, the principle of relativity takes a somewhat altered form. In order to study it, we need to extend the canonical formalism by including clock transformations that transform a canonical description from one internal clock to another; only then can we move to the quantum level where these new transformations become a key to unlock the principle of quantum relativity.

Let us consider a system consisting of a set of $N+1$ canonical variables $\{q_\alpha, p^\alpha\}_{\alpha=0,\ldots,N}$ and assume a Hamiltonian constraint taking the form

$$C(q_\alpha, p^\alpha) \approx 0,$$

where "$\approx$" is the weak equality in the Dirac sense [23]. Suppose that one of the positions, say q_0, varies monotonically with the evolution generated by the constraint, i.e., $\forall\ q_0, \{q_0, C\}_{\mathrm{PB}} \neq 0$. It is then possible to assign to q_0 the role of an internal clock in which the evolution of the remaining variables occurs. This evolution is then governed by a Hamiltonian that is not a constraint. At this stage, it may seem that the time variable is fixed once and for all, which would contradict the principle of relativity; we discuss below in what sense this is not the case.

The reduced Hamiltonian formalism is obtained from the initial symplectic form $\Sigma = \mathrm{d}q_\alpha \wedge \mathrm{d}p^\alpha$ (Einstein convention assumed), evaluated on the constraining surface, namely,

$$\begin{aligned}\Sigma|_{C=0} &= (\mathrm{d}q_I \wedge \mathrm{d}p^I + \mathrm{d}q_0 \wedge \mathrm{d}p^0)|_{C=0} \\ &= \mathrm{d}q_I \wedge \mathrm{d}p^I - \mathrm{d}t \wedge \mathrm{d}H,\end{aligned} \tag{1}$$

where $I = 1, \ldots, N$, and $H = H(q_0, q_I, p^I)$ is the nonvanishing reduced Hamiltonian such that $p_0 + H \approx 0$. Note that both q_0 (denoted by t from now on to emphasize its role as a time variable) and p^0 are removed from the phase space and the remaining dynamical variables are no longer constrained. Indeed, their dynamics reads

$$\frac{\mathrm{d}q_I}{\mathrm{d}t} = \frac{\partial H}{\partial p^I} \quad \text{and} \quad \frac{\mathrm{d}p^I}{\mathrm{d}t} = -\frac{\partial H}{\partial q_I},$$

which is entirely solved once an arbitrary initial condition $(q_I^{\mathrm{ini}}, p^I_{\mathrm{ini}}, q_0^{\mathrm{ini}})$ is provided.

In order to restore the principle of relativity, we need to allow for any clock, denoted by $\tilde{t}$, which monotonically varies with the evolution generated by the constraint $\{\tilde{t}, C\}_{\mathrm{PB}} \neq 0$. This new clock must be a function of the old clock and the old canonical variables, $\tilde{t} = \tilde{t}(q_I, p^I, t)$. Thus, it must satisfy

$$\frac{\mathrm{d}\tilde{t}}{\mathrm{d}t} = \frac{\partial \tilde{t}}{\partial t} + \underbrace{\frac{\partial \tilde{t}}{\partial q_I}\frac{\partial H}{\partial p^I} - \frac{\partial \tilde{t}}{\partial p^I}\frac{\partial H}{\partial q_I}}_{\{\tilde{t}, H\}_{\mathrm{PB}}} \neq 0. \tag{2}$$

The original symplectic form induced on the constraint surface $C = 0$ must read in some new canonical variables,

$$\Sigma|_{C=0} = \mathrm{d}\tilde{q}_I \wedge \mathrm{d}\tilde{p}^I - \mathrm{d}\tilde{t} \wedge \mathrm{d}\tilde{H},$$

so that the new reduced formalism is still canonical. This implies that there must exist an invertible map between the old and the new variables,

$$\tilde{t} = \tilde{t}(q_I, p^I, t), \quad \tilde{q}_I = \tilde{q}_I(q_J, p^J, t), \quad \tilde{p}^I = \tilde{p}^I(q_J, p^J, t), \tag{3}$$

and the natural question to ask is whether these transformations are canonical. In principle, and in all the relevant cases, they most certainly are not. It can be shown that clock transformations form a group of generally noncanonical transformations with canonical transformations as its normal subgroup [16]; finding them is, in general, a difficult task. However, for an integrable dynamical system, the problem can be reduced to that of solving a set of algebraic equations.

If a dynamical system is integrable, then we may find a complete set of canonical constants of motion, denoted by D_I. Let them be functions of the old internal time and old canonical variables, $D_I = D_I(q_J, p^J, t)$. Note that substituting back $t \to q_0$, they must commute with the original constraint, $\{D_I, C(q_\alpha, p^\alpha)\}_{\mathrm{PB}} = 0$. They are therefore genuine Dirac observables in the constrained system. The new internal time $\tilde{t} = \tilde{t}(q_I, p^I, t)$ and new canonical variables can then be found according to the algebraic relations

$$\tilde{t} = \tilde{t}(q_I, p^I, t), \qquad D_I(q_J, p^J, t) = D_I(\tilde{q}_J, \tilde{p}^J, \tilde{t}), \quad (4)$$

where we formally substitute the canonical variables in the expressions for Dirac observables D_I, i.e., we assume the same functional dependence of D_I in both sets of variables. The number of D_I is equal to the number of the new canonical variables $\tilde{q}_J$ and $\tilde{p}^J$, and thus, leaving aside singular cases, the above relations determine $\tilde{q}_J$ and $\tilde{p}^J$ completely. The result is a new canonical formalism based on a new internal clock. Let us note that, by virtue of Eq. (4), if a solution to the dynamics is known in one clock, i.e., $t \to [q_I(D_J, t), p^I(D_J, t)]$, then it is readily known for all other clocks and reads $\tilde{t} \to [\tilde{q}_I = q_I(D_J, \tilde{t}), \tilde{p}^I = p^I(D_J, \tilde{t})]$. This makes the choice of the new canonical variables $\tilde{q}_I$ and $\tilde{p}^I$ via Eq. (4) very convenient: the formal description of the system is the same in all clocks; only the physical meaning of the clock and basic variables changes, which is emphasized by the use of a tilde ($\tilde{\ }$) over the variable names.

The use of Dirac observables in the derivation of clock transformations gives an invaluable advantage when passing to quantum theory. Our goal is to make a comparison between quantum theories based on different internal clocks of a single physical system. Therefore, it is of uttermost importance to make sure that the theories are different only insofar as their clocks differ and not due to other quantization ambiguities, such as the well-known factor ordering. This state of affairs can be achieved by fixing a quantum representation of the Dirac observables and then defining basic and compound observables as functions of the quantum Dirac observables, both in the original

$$\hat{q}_I = q_I(\hat{D}_J, t), \qquad \hat{p}^I = p^I(\hat{D}_J, t),$$

and the new variables

$$\hat{\tilde{q}}_I = q_I(\hat{D}_J, \tilde{t}), \qquad \hat{\tilde{p}}^I = p^I(\hat{D}_J, \tilde{t}).$$

These definitions imply that q_I and p^I are promoted to the same operators as $\tilde{q}_I$ and $\tilde{p}^I$, respectively. We invert this reasoning and start by assuming the same operators for q_I and $\tilde{q}_I$ as well as p^I and $\tilde{p}^I$. This implies that the Dirac observables being the same functions in both sets of basic variables are promoted to the same operators irrespective of the choice of clock. Hence, the quantum descriptions in different clocks are formally the same; only the physical meaning of the basic operators changes from one clock to another, which is emphasized by the use of tilde. Obviously, a unique ordering prescription has to be used in all the above formulas. In principle, after this step, any physically interesting aspect of the quantum theories can be compared. In the following section, we introduce the model on which we discuss such comparisons.

III. CANONICAL COSMOLOGICAL MODEL

We consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe filled with radiation and perturbed by gravitational waves; the line element of the model reads (in units such that $c = 1$)

$$\mathrm{d}s^2 = -N^2(t)\mathrm{d}t^2 + a^2(t)[\delta_{ij} + h_{ij}(\boldsymbol{x}, t)]\mathrm{d}x^i\mathrm{d}x^j,$$

where h_{ij} represent the gravitational waves (tensor perturbations); it satisfies $h_{ij}\delta^{ij} = 0$ and $\partial^j h_{ij} = 0$. Finally, we assume a toroidal spatial topology with each comoving coordinate $x^i \in [0, 1)$. Setting $N \to a$ means one considers the conformal time; we shall henceforth denote it by η to agree with most of the cosmology literature.

A. Perturbative Hamiltonian

Let us now build the canonical description of these gravitational waves in an FLRW universe. The relevant canonical variables are the scale factor a and its conjugate momentum p_a to describe the background, while the tensor perturbations are represented by the gravitational wave amplitude $\mu^{(\lambda)} = ah^{(\lambda)}$ and its conjugate momentum $\pi^{(\lambda)}$, with $\lambda \in \{+, \times\}$ and $h_{ij} = \sum_\lambda h^{(\lambda)}\varepsilon_{ij(\lambda)}$ (see, e.g., Refs. [24,25] for details on the helicity expansion).

The matter component is assumed to be a radiation fluid with energy density p_0 conjugate to a timelike variable q_0. The gravitational constraint is expanded to second order through

$$H_{\mathrm{tot}} = H^{(\mathrm{b})} + \sum_{\boldsymbol{k}} H_{\boldsymbol{k}}^{(\mathrm{p})}$$

(recall the spatial sections are compact), with the background Hamiltonian given by

$$H^{(\mathrm{b})} = -\frac{1}{2}p_a^2 - p_0. \quad (5)$$

At this stage, one can identify the internal time q_0 with the conformal time η as it reduces the zeroth-order Hamiltonian into

$$\begin{aligned}\Sigma|_{H^{(0)}=0} &= (\mathrm{d}a \wedge \mathrm{d}p_a + \mathrm{d}q_0 \wedge \mathrm{d}p_0)|_{H^{(\mathrm{b})}=0} \\ &= \mathrm{d}a \wedge \mathrm{d}p_a - \mathrm{d}\eta \wedge \mathrm{d}\left(\frac{1}{2}p_a^2\right),\end{aligned} \tag{6}$$

leading to the physical zeroth-order Hamiltonian

$$H^{(0)} = \frac{1}{2}p_a^2, \tag{7}$$

while preserving the form of the perturbation Hamiltonian $H_{\boldsymbol{k}}^{(\mathrm{p})}$. The latter reads, at second order,

$$H_{\boldsymbol{k}}^{(\mathrm{p})} \to H_{\boldsymbol{k}}^{(2)} = -\sum_{\lambda=+,\times} H_{\boldsymbol{k},\lambda}^{(2)} \tag{8}$$

with

$$H_{\boldsymbol{k},\lambda}^{(2)} = \frac{1}{2}\left|\pi_{\boldsymbol{k}}^{(\lambda)}\right|^2 + \frac{1}{2}\left(k^2 - \frac{a''}{a}\right)\left|\mu_{\boldsymbol{k}}^{(\lambda)}\right|^2, \tag{9}$$

where a prime stands for a derivative with respect to the conformal time. Since the tensor perturbations are real, one has $\mu_{\boldsymbol{k}}^{(\lambda)*} = \mu_{-\boldsymbol{k}}^{(\lambda)}$. Moreover, since the background is isotropic, one can restrict attention to upward directed wave vectors $\boldsymbol{k}$ by merely canceling the factor $\frac{1}{2}$ in $H_{\boldsymbol{k},\lambda}^{(2)}$. This permits one to write the final second-order Hamiltonian as

$$H_{\boldsymbol{k},\lambda}^{(2)} = \pi_{\boldsymbol{k}}^{(\lambda)}\pi_{-\boldsymbol{k}}^{(\lambda)} + \left(k^2 - \frac{a''}{a}\right)\mu_{\boldsymbol{k}}^{(\lambda)}\mu_{-\boldsymbol{k}}^{(\lambda)}. \tag{10}$$

Note that, for the radiation fluid we are concerned with here, the Hamiltonian (7) yields as equations of motion $p_a = a'$ and $p_a' = 0$, thus leading to $a'' = 0$: the potential for producing gravitational waves is indeed classically vanishing if the universe is radiation dominated.

Determining the solution to the dynamics of gravitational waves is straightforward in the radiation case. While it is possible to consider a general fluid with an arbitrary barotropic index w (this case can be solved analytically in terms of Bessel functions, see, e.g., [26]), such a consideration is not relevant to the objectives of this work. We expect that the clock effects obtained below are not specific to any matter content but must be present whenever quantum uncertainties in the background geometry are taken into account. In fact, it can be argued that, since gravitational waves are affected by the choice of the equation of state only insofar as the background time development depends on it through Eq. (10), our results should qualitatively hold, if not quantitatively, for all physically relevant choices of w.

B. Dirac observables

Now we shall find the constants of motion that form canonical pairs. To this end, we need to solve the partial differential equations

$$\frac{\mathrm{d}D}{\mathrm{d}\eta} = \frac{\partial D}{\partial \eta} + \{D, H^{(0)} + H^{(2)}\}_{\mathrm{PB}} = 0. \tag{11}$$

At zeroth order, this is

$$\frac{\partial D}{\partial \eta} + p_a \frac{\partial D}{\partial a} = 0,$$

with solutions

$$D_1 = a - p_a \eta \quad \text{and} \quad D_2 = p_a. \tag{12}$$

At first order, Eq. (11) reads

$$\frac{\partial \delta D}{\partial \eta} + p_a \frac{\partial \delta D}{\partial a} = \pi_{\boldsymbol{k}}^{(\lambda)} \frac{\partial \delta D}{\partial \mu_{\boldsymbol{k}}^{(\lambda)}} - k^2 \mu_{\boldsymbol{k}}^{(\lambda)} \frac{\partial \delta D}{\partial \pi_{\boldsymbol{k}}^{(\lambda)}},$$

where we considered the classical solution $a'' = 0$. Since we are considering only first-order perturbations, we demand that δD be linear in the perturbation variables $\mu_{\boldsymbol{k}}^{(\lambda)}$ and $\pi_{\boldsymbol{k}}^{(\lambda)}$. The lhs of the above equation is greatly simplified if δD depends only on the variable $y = \eta + a/p_a$, so we look for a solution of the form $\delta D^{(\lambda)} = \mu_{\boldsymbol{k}}^{(\lambda)}\alpha(y) + \pi_{\boldsymbol{k}}^{(\lambda)}\beta(y)$, leading to

$$2\frac{\mathrm{d}\alpha}{\mathrm{d}y}\mu_{\boldsymbol{k}}^{(\lambda)} + 2\frac{\mathrm{d}\beta}{\mathrm{d}y}\pi_{\boldsymbol{k}}^{(\lambda)} = \alpha\pi_{\boldsymbol{k}}^{(\lambda)} - k^2\beta\mu_{\boldsymbol{k}}^{(\lambda)}.$$

Assuming independent variations of $\mu_{\boldsymbol{k}}^{(\lambda)}$ and $\pi_{\boldsymbol{k}}^{(\lambda)}$, one gets $2\mathrm{d}\alpha/\mathrm{d}y = -k^2\beta$ and $2\mathrm{d}\beta/\mathrm{d}y = \alpha$, and finally $4\mathrm{d}^2\alpha/\mathrm{d}y^2 = -k^2\alpha$, so that, setting

$$\Omega_k = \frac{k}{2}\left(\eta + \frac{a}{p_a}\right),$$

one gets two independent solutions for each polarization or, in other words, four first-order constants, reading

$$\begin{aligned}\delta D_{1,\boldsymbol{k}}^{(\lambda)} &= \sqrt{k}\sin\Omega_k \mu_{\boldsymbol{k}}^{(\lambda)} - \frac{\cos\Omega_k}{\sqrt{k}}\pi_{\boldsymbol{k}}^{(\lambda)}, \\ \delta D_{2,\boldsymbol{k}}^{(\lambda)} &= \sqrt{k}\cos\Omega_k \mu_{\boldsymbol{k}}^{(\lambda)} + \frac{\sin\Omega_k}{\sqrt{k}}\pi_{\boldsymbol{k}}^{(\lambda)}.\end{aligned} \tag{13}$$

In Eq. (13), the normalization has been chosen so as to ensure that all these Dirac observables indeed form canonical pairs, namely,

$$\{D_1, D_2\}_{\mathrm{PB}} = 1 \quad \text{and} \quad \{\delta D_{1,\boldsymbol{k}}^{(\lambda)}, \delta D_{2,\boldsymbol{k}}^{(\bar{\lambda})}\}_{\mathrm{PB}} = \delta_{\lambda\bar{\lambda}}.$$

From now on, we drop the index λ and consider just a single polarization mode $(\mu_{\boldsymbol{k}}, \pi_{\boldsymbol{k}})$.

C. Clock transformations

Having set the full model, and before moving on to its quantum counterpart, let us first consider a general clock transformation

$$\eta \to \tilde{\eta} = \eta + \Delta(a, p_a, \eta), \tag{14}$$

where Δ is a delay function that, in general, varies between the trajectories as well as along them. At the background level, implementing the recipe given by Eq. (3), i.e., $D_{1,2}(a, p_a, \eta) = D_{1,2}(\tilde{a}, \tilde{p}_a, \tilde{\eta})$, to the transformation (14) yields

$$a - p_a\eta = \tilde{a} - \tilde{p}_a\tilde{\eta} \quad \text{and} \quad p_a = \tilde{p}_a,$$

leading to

$$\tilde{a} = a + p_a\Delta \quad \text{and} \quad p_a = \tilde{p}_a. \tag{15}$$

In order that the clock transformation (14) actually defines a new and physically acceptable clock, the delay function Δ must be subject to two conditions. First, the new clock must run forward, that is,

$$\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}\eta} = 1 + \frac{\mathrm{d}\Delta}{\mathrm{d}\eta} = 1 + \frac{\partial\Delta}{\partial\eta} + p_a\frac{\partial\Delta}{\partial a} > 0, \tag{16}$$

where in the second equality we used the zeroth-order Hamiltonian $H^{(0)}$ given by Eq. (7) and the associated equations of motion.

The second condition that a clock transformation must satisfy is that the ranges of the basic variables a and p_a must be preserved, thereby preventing the appearance of non-trivial ranges that may induce new and potentially unsolvable quantization issues. This second condition implies

$$\lim_{p_a\to\pm\infty}\tilde{p}_a(a, p_a, \eta) = \pm\infty, \tag{17a}$$

$$\tilde{a}(a, p_a, \eta)|_{a=0} = 0. \tag{17b}$$

The first equality (17a) is trivially satisfied in the present case because of (15). For $\Delta = \Delta(a, p_a)$, to which we shall restrict attention in what follows, the second equality (17b) is identical to demanding that the delay function at vanishing scale factor should also vanish, $\Delta(0, p_a) = 0$. This condition also ensures that the slow-gauge clock is transformed into another slow-gauge clock, that is, the boundary is reached within a finite amount of time (see Ref. [17]). Such a condition (17b), although irrelevant in the classical theory, is crucial for the existence of a bounce at the quantum level, where the clock must smoothly connect contracting and expanding trajectories. Were (17) violated, the clock transformations would break the bouncing trajectories.

It turns out that the condition (16) is equivalent to the existence of a one-to-one map between the reduced phase spaces (a, p_a) and $(\tilde{a}, \tilde{p}_a)$, i.e., the determinant

$$\frac{\partial(\tilde{a}, \tilde{p}_a)}{\partial(a, p_a)} = \begin{vmatrix} \frac{\partial\tilde{a}}{\partial a} & \frac{\partial\tilde{a}}{\partial p_a} \\ \frac{\partial\tilde{p}_a}{\partial a} & \frac{\partial\tilde{p}_a}{\partial p_a} \end{vmatrix} > 0, \tag{18}$$

which is indeed Eq. (16) when $\partial\Delta/\partial\eta = 0$.

At first order, one must solve

$$\delta D_1(a, p_a, \mu_{\boldsymbol{k}}, \pi_{\boldsymbol{k}}) = \delta D_1(\tilde{a}, \tilde{p}_a, \tilde{\mu}_{\boldsymbol{k}}, \tilde{\pi}_{\boldsymbol{k}})$$

and

$$\delta D_2(a, p_a, \mu_{\boldsymbol{k}}, \pi_{\boldsymbol{k}}) = \delta D_2(\tilde{a}, \tilde{p}_a, \tilde{\mu}_{\boldsymbol{k}}, \tilde{\pi}_{\boldsymbol{k}})$$

in order to determine the clock-transformed perturbation variables. Explicitly, using (13), one gets

$$\begin{aligned} \sqrt{k}\sin\Omega_k\mu_{\boldsymbol{k}} - \frac{\cos\Omega_k}{\sqrt{k}}\pi_{\boldsymbol{k}} &= \sqrt{k}\sin\tilde{\Omega}_k\tilde{\mu}_{\boldsymbol{k}} - \frac{\cos\tilde{\Omega}_k}{\sqrt{k}}\tilde{\pi}_{\boldsymbol{k}}, \\ \sqrt{k}\cos\Omega_k\mu_{\boldsymbol{k}} + \frac{\sin\Omega_k}{\sqrt{k}}\pi_{\boldsymbol{k}} &= \sqrt{k}\cos\tilde{\Omega}_k\tilde{\mu}_{\boldsymbol{k}} + \frac{\sin\tilde{\Omega}_k}{\sqrt{k}}\tilde{\pi}_{\boldsymbol{k}}, \end{aligned} \tag{19}$$

where $\tilde{\Omega}_k = \frac{1}{2}k(\tilde{\eta} + \tilde{a}/\tilde{p}_a) = \Omega_k + k\Delta$. The above algebraic equations (19) can easily be inverted to yield the new canonical perturbation variables, namely,

$$\begin{pmatrix} \tilde{\mu}_{\boldsymbol{k}} \\ \frac{\tilde{\pi}_{\boldsymbol{k}}}{k} \end{pmatrix} = \begin{pmatrix} \cos k\Delta & -\sin k\Delta \\ \sin k\Delta & \cos k\Delta \end{pmatrix} \begin{pmatrix} \mu_{\boldsymbol{k}} \\ \frac{\pi_{\boldsymbol{k}}}{k} \end{pmatrix}. \tag{20}$$

It is important to note that the above are classical relations between canonical variables belonging to distinct canonical frameworks based on distinct internal clocks. Although they are canonically inequivalent, these two frameworks generate the same physical dynamics of the system, which is required by the principle of relativity.

In general, clock transformations involve modifying temporal relationships between events belonging also to different spacetimes. This aspect of clock transformations is not reflected in the lapse function $\tilde{N}$ of the new clock, which expresses the temporal relationship between points within a single spacetime. The clock transformations described in our framework, however, do preserve the foliation of cosmological spacetimes consisting of homogeneous spatial leaves with small perturbations. Given that the initial clock η corresponds to the conformal time, the new lapse function of the background foliation implied by the new clock $\tilde{\eta}$ reads

$$\tilde{N} = \frac{a}{1 + p_a\frac{\partial\Delta}{\partial a}} > 0,$$

where $\partial\Delta/\partial\eta = 0$ was assumed. Note that considering a delay function satisfying $1 + p_a\partial\Delta/\partial a = a$, one recovers the cosmic time with $\tilde{N} = 1$.

IV. QUANTIZATION

Having completed the classical treatment of our system, we now move to the investigation of the possible differences between the respective quantum dynamics obtained from the quantization of these two different frameworks.

A. Semiclassical background

Since, by definition, the scale factor is positive definite ($a > 0$), one needs to quantize our previous system on the half line. Although the position operator $\hat{Q} = a$ is self-adjoint on the half line, this is not the case for the momentum operator $\hat{P} = i\hbar\partial_a$, so we use instead the symmetric dilation operator,

$$\hat{D} = \{\hat{P}, \hat{Q}\} = \frac{1}{2}(\hat{P}\,\hat{Q} + \hat{Q}\,\hat{P}) = \frac{1}{2}i\hbar(a\partial_a + \partial_a a).$$

Classically the dilation variable is $d = ap_a$, so that the Hamiltonian, expressed in terms of d, is $H^{(0)} = \frac{1}{2}p_a^2 = \frac{1}{2}d^2/a^2$, and one can define its quantum counterpart as a symmetric ordering of $\frac{1}{2}\hat{Q}^{-2}\hat{D}^2$. Expanding on the basis $(\hat{Q}, \hat{P})$, this yields

$$\hat{H}^{(0)} = -\frac{1}{2}\frac{\partial^2}{\partial a^2} + \frac{\hbar^2 K}{a^2},$$

where the value of $K > 0$ depends on the ordering; fixing one ordering such that $K > \frac{3}{4}$ ensures $\hat{H}^{(0)}$ is self-adjoint on the half line [27].

We can find some approximate solutions to the Schrödinger equation with a family of coherent states (see, e.g., Refs. [28–30] for the specific case under study here). We choose the coherent states to read

$$|a(\eta), p_a(\eta)\rangle = \mathrm{e}^{ip_a(\eta)\hat{Q}/\hbar}\mathrm{e}^{-i\ln[a(\eta)]\hat{D}/\hbar}|\xi\rangle, \tag{21}$$

where $|\xi\rangle$ is such that the expectation values of $\hat{Q}$ and $\hat{P}$ in $|a(\eta), p_a(\eta)\rangle$ are, respectively, $a(\eta)$ and $p_a(\eta)$ and otherwise arbitrary (see, however, Ref. [31]).

The dynamics confined to the coherent states can be deduced from the quantum action

$$\mathcal{S}_{\mathrm{Q}} = \int \{a'(\eta)p_a(\eta) - H_{\mathrm{sem}}[a(\eta), p_a(\eta)]\}\mathrm{d}\eta, \tag{22}$$

with the semiclassical Hamiltonian given by

$$H_{\mathrm{sem}} = \langle a, p_a|\hat{H}^{(0)}|a, p_a\rangle, \tag{23}$$

from which one derives the ordinary Hamilton equations,

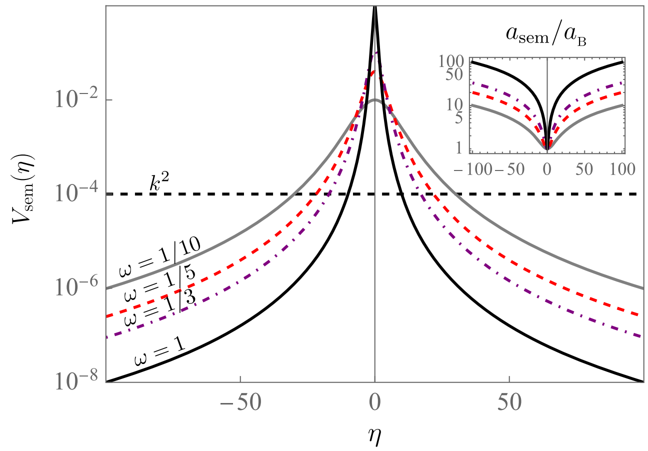

FIG. 1. The semiclassical potential V_{sem} given by Eq. (27) as a function of the conformal time η for various values of the inverse bounce duration ω. The potential has to be compared with the relevant value of k^2 ($k = 0.01$), indicated as a straight line. The corresponding scale factor time evolution is shown in the inset.

$$a' = \frac{\partial H_{\mathrm{sem}}}{\partial p_a} \quad \text{and} \quad p_a' = -\frac{\partial H_{\mathrm{sem}}}{\partial a}. \tag{24}$$

We find that the semiclassical background Hamiltonian reads [30]

$$H_{\mathrm{sem}} = \frac{1}{2}\left(p_a^2 + \frac{\hbar^2\mathfrak{K}}{a^2}\right), \tag{25}$$

where the new constant $\mathfrak{K}$ is positive ($\mathfrak{K} > 0$). Its specific value is related with both K and the fiducial state $|\xi\rangle$. We find the solution to (24) to read $a^2(\eta) = a_0 + a_1\eta + a_2\eta^2$, with $a_0a_2 - a_1^2/4 = \hbar^2\mathfrak{K} > 0$, so that the equation $a(\eta) = 0$ has no longer any real solution; the singularity is indeed quantum mechanically avoided. Choosing the origin of time such that $a' = 0$ for $\eta = 0$ permits us to rewrite this solution in full generality as

$$a(\eta) = a_{\mathrm{B}}\sqrt{1 + (\omega\eta)^2}, \tag{26a}$$

$$p_a(\eta) = \frac{a_{\mathrm{B}}\omega^2\eta}{\sqrt{1 + (\omega\eta)^2}}, \tag{26b}$$

where $a_{\mathrm{B}}^4\omega^2 = \hbar^2\mathfrak{K}$, which in turn implies $H_{\mathrm{sem}} = \frac{1}{2}a_{\mathrm{B}}^2\omega^2 = \frac{1}{2}\hbar\sqrt{\mathfrak{K}}\omega = \hbar^2\mathfrak{K}/(2a_{\mathrm{B}}^2)$; it is clear that the model contains one and only one free parameter, namely $\mathfrak{K}$. From now on, we assume that the background evolution is given by Eq. (26): this means the semiclassical potential

$$V_{\mathrm{sem}} = \frac{a''}{a} = \frac{\hbar^2\mathfrak{K}}{a^4} = \left[\frac{\omega}{1 + (\omega\eta)^2}\right]^2, \tag{27}$$

shown in Fig. 1, never vanishes except in the large scale factor limit ($a \gg 1 \Rightarrow \eta \gg \omega^{-1}$). This is appropriate as this is also the classical limit for which $a'' \to 0$. A classical radiation-dominated universe begins or ends with a singularity and produces no gravitational waves, whereas our quantum radiation-dominated universe naturally connects the contracting and expanding phases through a bounce, which is subsequently responsible for a nonvacuum spectrum of tensor perturbations, to which we now turn.

B. Quantum perturbations

For a given mode $\boldsymbol{k}$, the Hamiltonian $H_{\boldsymbol{k}}^{(2)}$, given by Eq. (10), is easily quantized using the usual prescriptions. We assume that the background follows the semiclassical approximation described above, so that the potential for the perturbation is given by $V_{\rm sem}$ [Eq. (27)]. The basic variables are replaced by a set of operators

$$\mu_{\boldsymbol{k}} \mapsto \hat{\mu}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}[\hat{a}_{\boldsymbol{k}}\mu_k^*(\eta) + \hat{a}_{-\boldsymbol{k}}^\dagger \mu_k(\eta)],$$
$$\pi_{\boldsymbol{k}} \mapsto \hat{\pi}_{\boldsymbol{k}} = \sqrt{\frac{\hbar}{2}}[\hat{a}_{\boldsymbol{k}}\mu_k^{*\prime}(\eta) + \hat{a}_{-\boldsymbol{k}}^\dagger \mu_k'(\eta)], \tag{28}$$

where we assume the Wronskian normalization condition $\mu_k'\mu_k^* - \mu_k\mu_k^{*\prime} = 2i$ for the complex mode functions μ_k. The creation $\hat{a}_{\boldsymbol{k}}^\dagger$ and annihilation $\hat{a}_{\boldsymbol{k}}$ operators satisfy the commutation relations $[\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{p}}^\dagger] = \delta_{\boldsymbol{k},\boldsymbol{p}}$ stemming from the canonical ones between the field operators $[\hat{\mu}_{\boldsymbol{k}}, \hat{\pi}_{-\boldsymbol{p}}] = i\hbar\delta_{\boldsymbol{k},\boldsymbol{p}}$.

In the Heisenberg picture, the equations of motion take the form

$$i\hbar\frac{\mathrm{d}\hat{\mu}_{\boldsymbol{k}}}{\mathrm{d}\eta} = [H_{\boldsymbol{k}}^{(2)}, \hat{\mu}_{\boldsymbol{k}}] \quad \text{and} \quad i\hbar\frac{\mathrm{d}\hat{\pi}_{\boldsymbol{k}}}{\mathrm{d}\eta} = [H_{\boldsymbol{k}}^{(2)}, \hat{\pi}_{\boldsymbol{k}}],$$

which imply that the mode function $\mu_k(\eta)$ satisfies

$$\frac{\mathrm{d}^2\mu_k}{\mathrm{d}\eta^2} + \left(k^2 - \frac{\hbar^2\mathfrak{K}}{a^4}\right)\mu_k = 0, \tag{29}$$

where $a(\eta)$ is given by the semiclassical solution (26a). Using (27), this transforms into

$$\frac{\mathrm{d}^2\mu_k}{\mathrm{d}\eta^2} + \left\{k^2 - \left[\frac{\omega}{1+(\omega\eta)^2}\right]^2\right\}\mu_k = 0, \tag{30}$$

which can be integrated numerically if initial conditions are provided: we assume that, far in the contracting branch, with $\eta_{\rm ini} < 0$ and $V_{\rm sem}(\eta_{\rm ini}) \ll k^2$, there was no gravitational wave, so the field was in a vacuum state. This implies the mode function satisfies $\mu_k(\eta_{\rm ini}) = \mathrm{e}^{-ik\eta_{\rm ini}}/\sqrt{2k}$ and $\mu_k'(\eta_{\rm ini}) = -i\sqrt{k/2}\mathrm{e}^{-ik\eta_{\rm ini}}$.

V. QUANTUM “CLOCKS”

In what follows, we study the effect of clocks on the quantum and semiclassical dynamics of selected dynamical variables. First, we obtain the dynamical trajectories in the reduced phase space (a, p, μ_k, π_k) that is associated with the initial clock η; note that, from that point on, since there is no risk of confusion, we shall replace what was previously denoted as p_a simply by p. Next, we choose a set of delay functions $\Delta(a, p)$ to define new clocks $\tilde{\eta}$ and obtain the new reduced phase spaces $(\tilde{a}, \tilde{p}, \tilde{\mu}_k, \tilde{\pi}_k)$ associated with the new clocks. Then, we make use of Eqs. (14), (15), and (20) to transport the dynamical trajectories to these new phase spaces. We assume that the latter admit a unique physical interpretation, and so the trajectories can be meaningfully compared in these new variables. In other words, there are many clocks denoted by η and only one denoted by $\tilde{\eta}$. Note that for $\Delta = 0$ the clocks η and $\tilde{\eta}$ coincide. For this case, we assume that η and (a, p, μ_k, π_k) are the variables of Sec. II, which sets the physical meaning of the phase space $(\tilde{a}, \tilde{p}, \tilde{\mu}_k, \tilde{\pi}_k)$ and the clock $\tilde{\eta}$.

A. Clock choices and background

In order to illustrate the clock choice issue, we consider a family of delay functions, namely,

$$\Delta(a, p) = A\frac{a^B}{(a+C)^D}\frac{\sin(Ep)}{p}, \tag{31}$$

where A, B, C, D, and E are arbitrary coefficients, whose values are limited to ensure that the conditions presented in Sec. III C hold. In the Appendix, we consider another set of acceptable delay functions to show that our conclusions are not restricted to the choice (31).

A few clocks corresponding to the delay function $\Delta(a, p)$ are represented along a semiclassical dynamical trajectory for different choices of the free parameters in Fig. 2. It shows that, contrary to the classical case where the condition (16) holds, the new clocks, in general, are no longer monotonic due to quantum corrections.

Applying the clock transformation of Fig. 2 to the background solution (26) yields Fig. 3 once mapped into the reduced phase space, with the original trajectory superimposed for comparison.

All the trajectories originate in the same classical regime at large $\tilde{a}$ and negative $\tilde{p}$, i.e., at a time at which the universe is large and contracting. Close to the $\tilde{a} = 0$ boundary, where the quantum behavior dominates, they all somehow bounce in the variables $\tilde{a}$ and $\tilde{p}$, diverging from one another and providing different accounts of the bounce. Finally, they reach the region of large $\tilde{a}$ and positive $\tilde{p}$ where they converge again to the unique classical behavior representing a large and expanding universe.

Possible differences between the trajectories include the values of $\tilde{a}$ and $\tilde{p}$ at which the bounce occurs, the level of

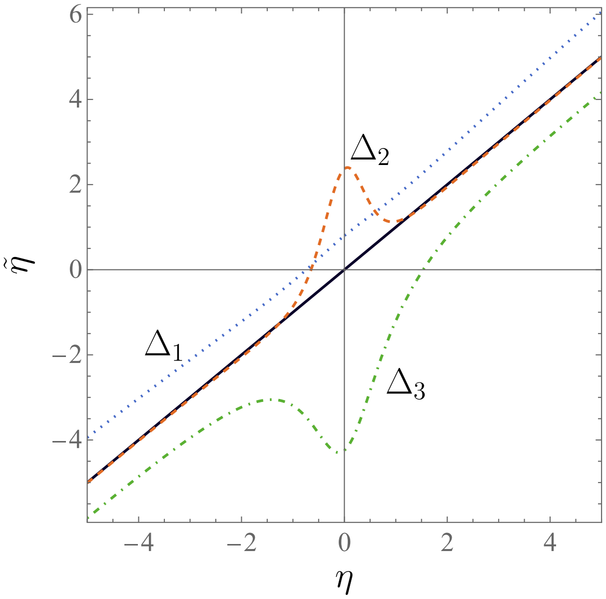

FIG. 2. The new time $\tilde{\eta}$ as a function of the original one η for three different shapes of delay functions Δ_1, Δ_2, and Δ_3 defined through Eq. (31) along the original fixed bouncing trajectory (26). The parameters are chosen as $A = B = D = 1$, $C = 4$, and $E = 2$ for Δ_1, while we set $A = 2$, $B = 0.2$, $C = 0.5$, $D = 3$, and $E = 4$ for Δ_2, and finally the set $A = -1$, $B = C = 1$, $D = 0.5$, and $E = 3$ defines Δ_3.

asymmetry between contracting and expanding branches, or even the number of bounces. These semiclassical trajectories illustrate the nonunitary relation between different clocks. Nevertheless, they all originate from a unique contracting classical universe and end toward a similarly unique expanding classical universe. Therefore, the semiclassical trajectories in different clocks yield the same outcome for large and classical universes. Notice that the trajectories' convergence before and after the bounce can be delayed as much as one wants by making use of appropriate delay functions, such as that discussed in the Appendix, i.e., Eq. (A1), whose effects on both background and perturbation trajectories can be seen in the Appendix.

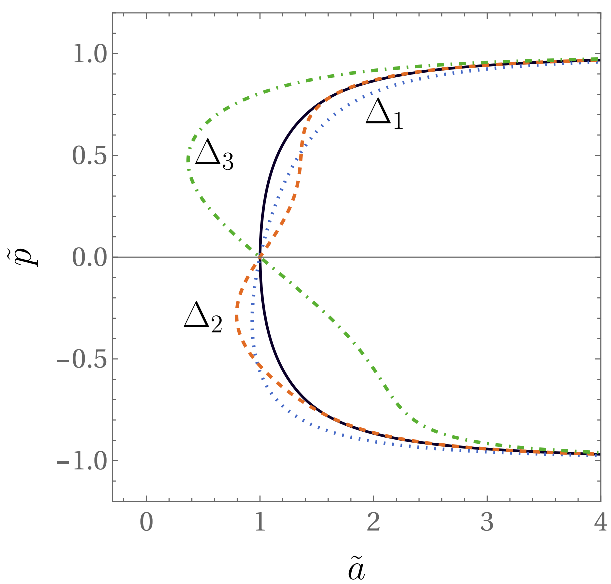

FIG. 3. Semiclassical trajectories obtained in different clocks and mapped into the initial reduced phase space $(\tilde{a}, \tilde{p})$ to compare with the original trajectory represented by the full black line.

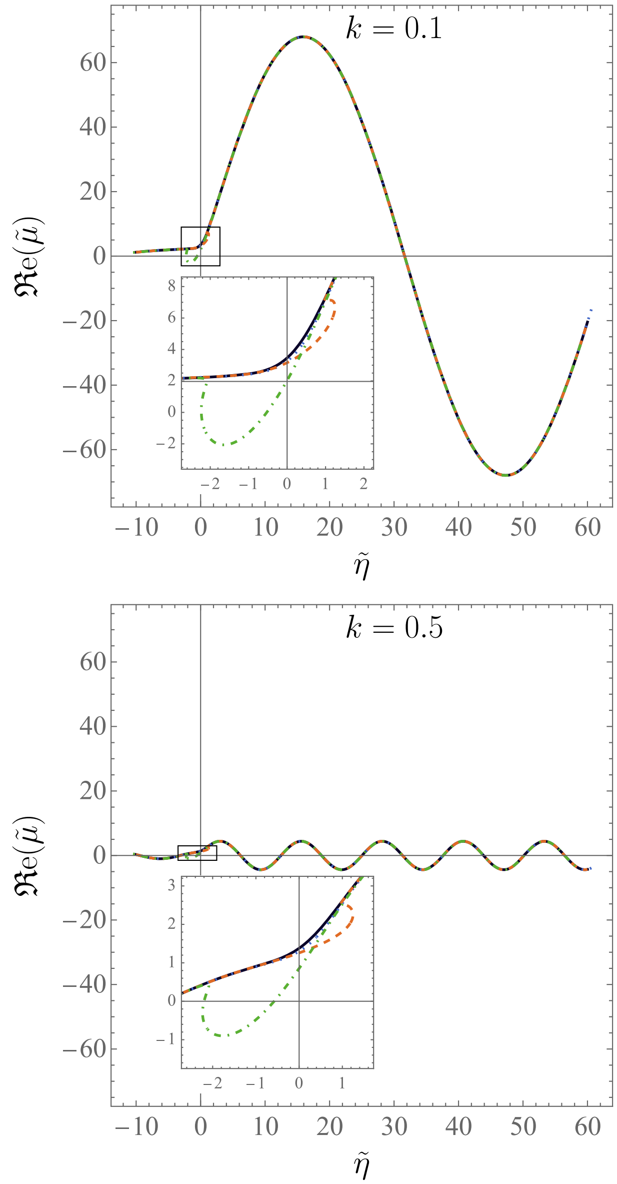

FIG. 4. Evolution of the primordial gravity wave $\mathfrak{Re}(\tilde{\mu})$ for two different wave numbers, $k = 0.1$ (top) and $k = 0.5$ (bottom), and for different clocks based on the first class of delay function, Δ_1, Δ_2, and Δ_3, represented by the dotted blue line, dashed red line, and dash-dotted green line, respectively. The original trajectory is represented by the full black line. In Fig. 9, the same plot for the second class of delay function is depicted to show how the choice of delay function affects the time of convergence.

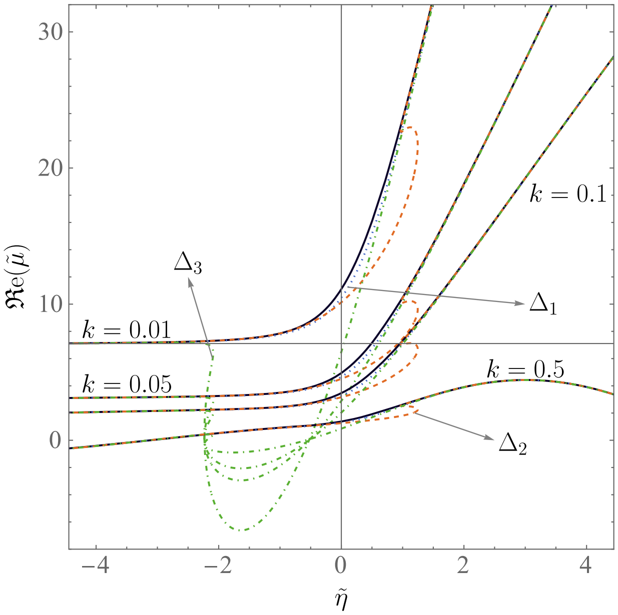

FIG. 5. Evolution of the primordial gravity wave $\Re\mathrm{e}(\tilde{\mu}_k)$ plotted for four different wave number k values. For each fixed k we changed the clock considering the family of delay functions Δ, whose value is the same as in Fig. 3.

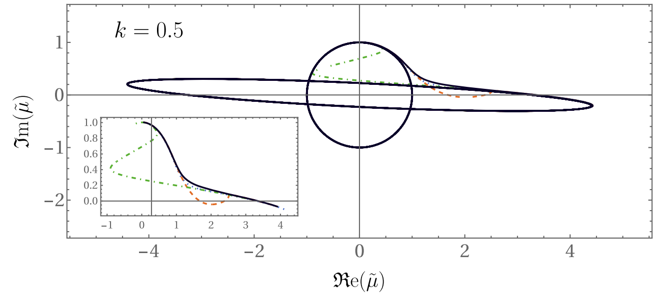

FIG. 6. Evolution of the real versus imaginary part of $\tilde{\mu}_k$ for a wave number $k = 0.5$ and a bounce parameter $\omega = 1$. The initial circle represents the initial vacuum state of the perturbation, while the ellipse shows the final squeezed state, which happens, in the case at hand, to have a slight phase shift with respect to the real axis. The transition between these two asymptotic cases differs for the different delay functions Δ_1, Δ_2, and Δ_3, whose trajectories are represented by the dotted blue line, dashed red line, and dash-dotted green line, respectively, the original trajectory being represented by the full black line.

Let us now move to the perturbation of these homogeneous solutions and compare the different evolution that can result from using different clocks.

B. Clocks and perturbations

In Fig. 4, we plot the dynamics of the real part of the perturbation variable $\tilde{\mu}_k$ against the delayed time $\tilde{\eta}$ for the three different functions of Eq. (31) displayed in Fig. 2 and for two values of the comoving wave number k. The figure illustrates our general finding that the absolute clock effect is more or less equally strong and lasting roughly equally long for all wavelength perturbations. This is shown more convincingly in Fig. 5, in which the evolution of four different modes is shown as a close-up in the quantum-dominated bouncing region. This means that the larger the wavelength of the perturbation, the larger the relative clock effect, and the longer it lasts in units of its oscillation period. Thus, the clock effect is more important for phenomena occurring at small timescales and over short distances.

Moreover, the evolving amplitude $\tilde{\mu}_k$, in general, is not a function of the clock $\tilde{\eta}$ due to quantum effects that disrupt the monotonicity relation between quantized clocks.

Given that both the background and the perturbation modes evolve in such a way as to reach a unique configuration, the primordial gravity-wave amplitude $\tilde{\mu}_k/\tilde{a}$, which is the quantity one expects to measure in practice [25], also converges to a unique solution, making the model predictive.

All the plots above illustrate the nonunitary relation between different clocks, as well as the spoiling of the clock monotonicity at the quantum level, which is illustrated in Fig. 2. Nevertheless, similar to the semiclassical background trajectories, the perturbation variable $\Re\mathrm{e}(\tilde{\mu})$ visibly converges to a unique classical solution from a well-defined asymptotic past initial condition to the asymptotic future. Therefore, one can safely extend the background conclusion to the perturbations: the time development of the mode $\Re\mathrm{e}(\tilde{\mu})$ using different clocks yields the same predictions in the large and classical universe regime. The delay of the convergence due to different choices of delay functions can be seen in Fig. 5.

As a final illustration of the perturbation behavior through the quantum bounce, we find it useful to inspect the phase space trajectories in the plane $[\Re\mathrm{e}(\tilde{\mu}_k), \Im\mathrm{m}(\tilde{\mu}_k)]$ as is displayed in Fig. 6. The initial vacuum state is represented by a circle that is squeezed into an ellipse during the contraction and bounce, squeezing that represents the amplification of the amplitude of the perturbation. From the point of view of the time problem, the initial circle and the final ellipse, respectively, represent the asymptotic past and future of the amplitude: from the point of view of physical prediction, the indeterminacy occurring near the bounce, as may develop through various different times, disappears in the asymptotic regimes, so that the existence of a classical approximation in our trajectory approach ensures the standard procedure of treating the perturbations leads to physically meaningful predictions.

VI. DISCUSSION AND PERSPECTIVES

In this work, we explored the time problem in the framework of quantum fields on quantum spacetimes.

We considered the specific example of primordial gravitational waves propagating through a bouncing quantum Friedmann universe. We pointed to several features that we believe to be universal for such models.

First, we showed that the dynamical variables, such as the scale factor or the amplitude of a gravitational wave, obtained from different internal clocks, evolve differently when compared in a clock-independent manner. Second, these expectation values (background evolution) and mode functions of operators (perturbations), irrespective of the clock chosen, converge to a unique evolution for large classically behaving universes. This is the phase space domain in which unambiguous predictions can be made. Third, for different clocks, the dynamics converges to the classical behavior at different times. In principle, there is no restriction on how far from the bounce the system must be in order to display the classical behavior. In practice, however, all the clocks considered were found to converge very quickly, allowing for unambiguous predictions shortly after the bounce.

Based on the above findings, we postulate that the physical predictions are only those predictions provided by any clock, which are not altered upon the clock's transformation. The fact that for large universes the semiclassical background dynamics and the quantum perturbation dynamics do not depend on the clock implies the following: Despite the fact that the dynamical variables are not Dirac observables, they provide physical predictions for large universes, which is precisely the regime in which we observe the actual Universe.

Note, however, that the word "large" is never precisely defined. One could expect that, at least in principle, some clocks require times larger than the present age of the Universe to converge to the classical behavior. This, however, poses no problem to our interpretation, as we simply exclude such clocks and retain only those that behave classically in the domain for which we make predictions. This may seem arbitrary and unjustified. We must, however, remember that, as a matter of fact, any semiclassical description of ordinary quantum mechanics is necessarily restricted to a limited set of observables, usually the simple ones, while more compound observables often display classically incompatible behavior (e.g., $\langle x\rangle^2 \neq \langle x^2\rangle$). For similar reasons, we are allowed to choose only those clocks in which the dynamics of the relevant observables is classically consistent.

On the one hand, we proved that the evolution of the expectation values of some observables constitute *physical* predictions of quantum cosmological models. On the other hand, the expectation values are not all that is measured in the large Universe. In other words, not all objects are classical in the large Universe. For instance, the position of an electron is a dynamical variable that can be measured in a laboratory. So, could the outcomes of such a measurement also be unambiguously predicted by a quantum cosmological model? The answer is affirmative. Note that the mode function μ_k, whose dynamics becomes unambiguous in a large universe, determines the evolution of the operator $\hat{\mu}_k$ via Eq. (28). This implies that the Heisenberg equation of motion encoded in Eq. (30) becomes unambiguous too. Obviously, the evolution of perturbation in the Schrödinger picture must consequently become unique as well. Hence, ordinary quantum mechanics of perturbation modes is recovered in a large universe. These conclusions must also apply to electrons and, in general, to all nongravitational degrees of freedom.

To better understand the origin of the emergence of ordinary quantum mechanics, notice that any clock transformation (14) involves, by definition, only background variables. If the latter behave classically, the clock transformation is completely classical and amounts to a mere (in general, nonlinear) change of units of time. In Ref. [11], it was demonstrated that the relational dynamics of a quantum variable in a classical clock is unambiguous in the sense that switching to another classical clock does not induce any clock effect.

Let us put to test our approach and our result by addressing a set of questions that were proposed in Ref. [2] for assessing the completeness of any potential solution to the time problem.

(1) How should the notion of time be reintroduced into the quantum theory of gravity?

Our approach relies on evolving internal variables called clocks. We express the dynamics of the dynamical variables in terms of these clocks.

(2) In particular, should attempts to identify time be made at the classical level, i.e., before quantization, or should the theory be quantized first?

In our approach, we first reduce the Hamiltonian formalism based on a selected clock, then we quantize the reduced formalism as if the clock was an external and absolute time. However, it is neither external nor absolute. The instantaneous value of the clock determines the instantaneous physical state of the system. Switching to another clock entails a change in the physical interpretation of the clock and the entire state of the system.

(3) Can "time" still be regarded as a fundamental concept in a quantum theory of gravity, or is its status purely phenomenological?

In our approach, there is no fundamental time. The fundamental concept is "change" or "evolution," meaning we merely need to assume that the $3+1$ split of the underlying geometry imposes an ordered set of hypersurfaces. As we showed in this paper, extracting dynamical predictions from such a formalism is a subtle issue. The clocks serve as tools for deriving the predictions. Once a class of clocks converges to a unique dynamics, any one of them can be treated analogously and deserve the qualification of time, and any quantum dynamical variable becomes described in them by a unique Schrödinger equation. This is how ordinary quantum mechanics emerges.

(4) If time is only an approximate concept, how reliable is the rest of the quantum-mechanical formalism in those regimes where the normal notion of time is not applicable? In particular, how closely tied to the concept of time is the idea of probability?

The quantum-mechanical description in the regime where different clocks exhibit different dynamics is an essential part of our theory. It describes the deterministic evolution of the system. However, this regime does not seem to allow for any meaningful dynamical interpretation in terms of relational change. Although we have not explicitly addressed this question in the present work, our approach permits one to do it.

To conclude, one can mention that the chosen clock degrees of freedom, although perfectly acceptable as such in the classical framework of general relativity, are arguably not in the quantum regime. They do not qualify as actual clocks since, along the quantum trajectory, they yield a nonmonotonic change of time variable; in other words, they provide different hypersurface orderings. This might be cured by adding to the classical clock transformation (14) a quantum term that needs be identified. One may also argue that we are insisting upon using a trajectory to define the background evolution, while some might insist upon the fact that there is no such thing as a trajectory in quantum mechanics.

In any case, it is interesting to note that, whichever of the possibilities above happens to be valid, the critical point that is made here is that, even though the quantum-dominated phase is indeed ill-defined both at the background and perturbation levels from the point of view of time development, the asymptotic regimes end up being unique. As a result, setting well-motivated initial conditions in the classical past, one gets unambiguous physical predictions for the classical future in which we happen to perform the ensuing measurements. In other words, we have shown that the lack of predictability in the quantum regime does not exclude the fact that the theory permits meaningful physical predictions that can be tested with observations.

Finally, it is worth noting that there are alternative approaches that do not involve promoting internal variables to clock status, effectively avoiding the time problem. For instance, in Ref. [32], the Wentzel-Kramers-Brillouin approximation to the background wave function is made, and the resultant trajectory provides a well-defined cosmological background on which perturbations propagate, without ever introducing the physical inner product at the background level. An approach similar in spirit can be found in Ref. [33], which is based on coarse graining of the background wave function, thereby removing short timescale oscillations in the scale factor. The end result is similar to the previous case and allows for an unambiguous effective trajectory in the background variables along which the evolution of the perturbations occur. Although the time problem discussed in the present work is absent in these approaches, the cost is that of a limited physical interpretation of the background wave function, for which no notion of unitary dynamics is ever introduced. Consequently, the quantum uncertainties in the physical background variables are not well defined and thus their influence on the dynamics of the perturbations is assumed negligible. Choosing an internal time entails a prescription for calculating such uncertainties and permits one to incorporate them in the dynamics of perturbations. The resulting clock dependence of such a prescription leads to the questions addressed here.

A unitary approach to quantum cosmology that aims at a gauge-independent formulation was described in Ref. [34]. This interesting proposal offers important insights into the relation between the reduced phase space quantization and the Dirac-Wheeler-DeWitt superspace formalism. The author shows that, at least for some choices of internal clocks, both approaches are equivalent in a very well-defined sense. Specifically, the author discusses in detail the relation between physical and superspace propagators and inner products. However, the step of constructing real observables in a gauge-independent way is left out. It is not clear whether such a program can actually be achieved, which is the reason for the time problem studied here; a simple and general argument in favor of this position was given in Ref. [35]. Finally, another alternative approach would involve arguing in favor of a preferred clock. We are not aware of any widely recognized proposal of this type.

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APPENDIX

In this appendix, we consider the alternative choice of a family of two-parameter delay functions, namely,

$$\Delta'(a,p) = a^A e^{Bp}, \tag{A1}$$

which define a new set of clocks plotted in Fig. 7 for a few relevant values of the parameters A and B. Figure 8 depicts the trajectories with different clocks obtained

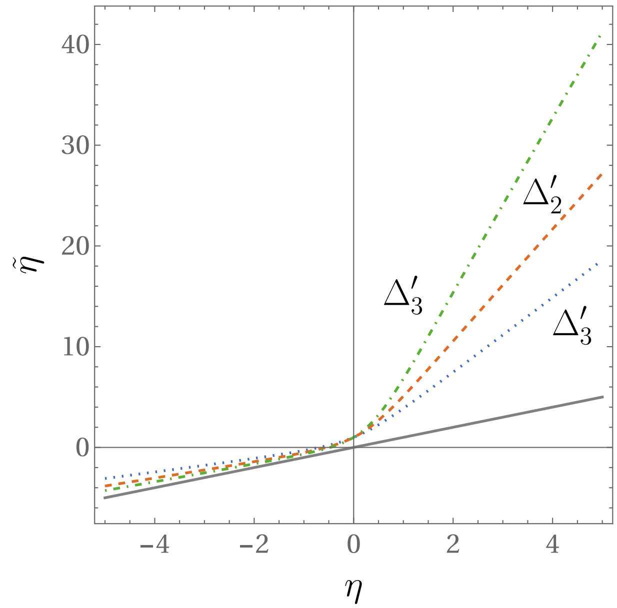

FIG. 7. Changes in the time variable η for the second family of delay functions Δ_1', Δ_2', and Δ_3' given by Eq. (A1) along a fixed bouncing trajectory, with parameters chosen such that $\Delta_1 = ae^{p_a}$, $\Delta_2 = ae^{3p_a/2}$, and $\Delta_3 = ae^{2p_a}$.

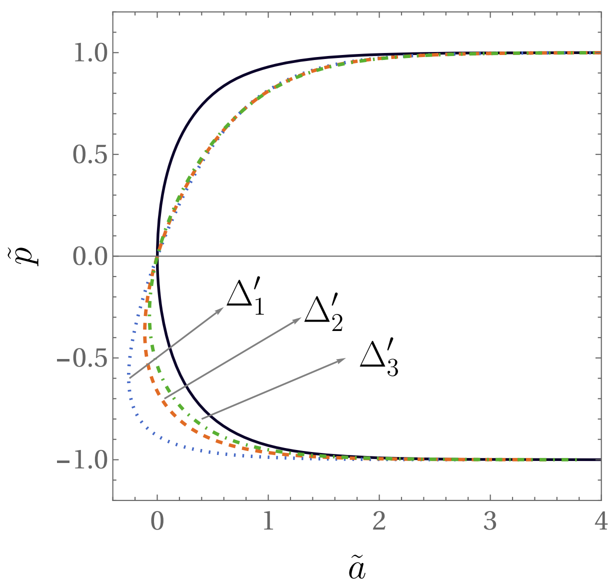

FIG. 8. Semiclassical trajectories mapped into the initial reduced phase space (a, p) for the second class of delay function (A1), with the same parameters as in Fig. 7.

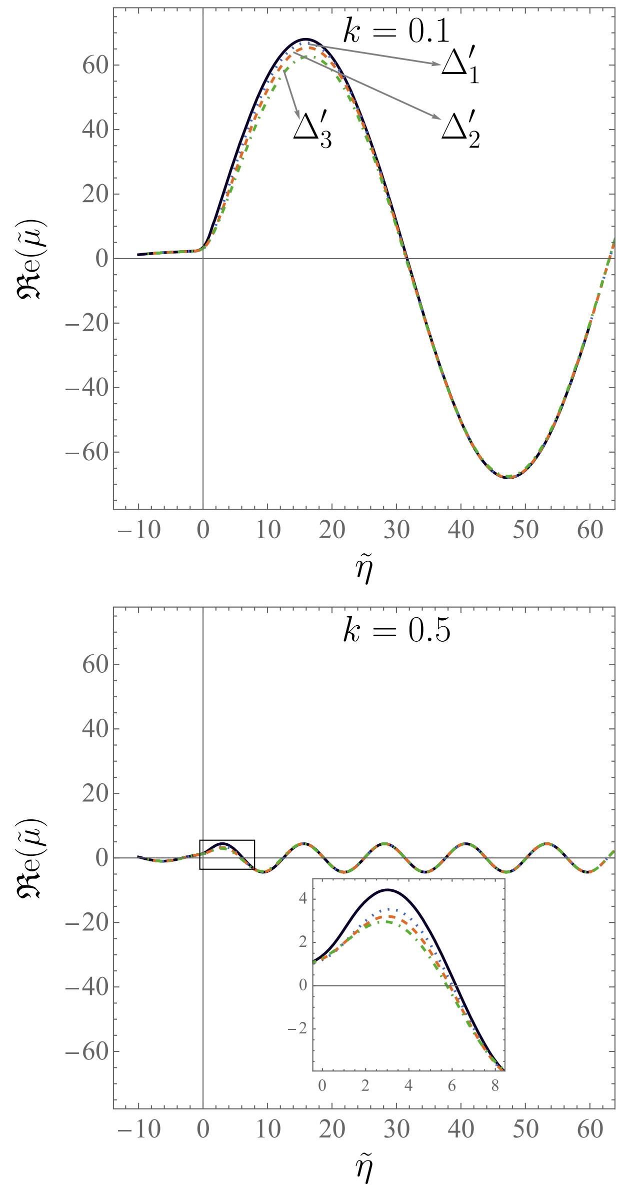

FIG. 9. Evolution of the real part of the primordial gravity wave $\Re\mathrm{e}(\tilde{\mu})$ for two different wave numbers, $k = 0.1$ and $k = 0.5$, and for different clocks for the second class of delay functions, Δ_1', Δ_2', and Δ_3', respectively, represented by the dotted blue line, dashed red line, and dash-dotted green line. The original trajectory is represented by the full black line.

from $\Delta'(a,p)$ for which the convergence happens much later than in the case discussed in the core of this paper, as can be seen by comparing with Fig. 3. The extent to which this delay can be increased, and how the matter content of the Universe can affect this limit, is not dealt with in the present article and will be the subject of a future work.

One can note that the delay functions (A1) tend to diverge in time from one another, all of them growing exponentially with the momentum; the phase space trajectories, however, do converge to the undelayed one, but at scales that are increasingly larger with the amplitude of the exponential behavior of the relevant delay function.

Moving to the perturbations, we performed the same analysis as in the core of this paper and show the time development of the real part of the mode function for different values of the wave number in Fig. 9, with a special emphasis at the near-bounce regime in Fig. 10. As for the other family of delay functions, we find that, whenever the classical approximation for the background holds, one recovers a unique prediction.

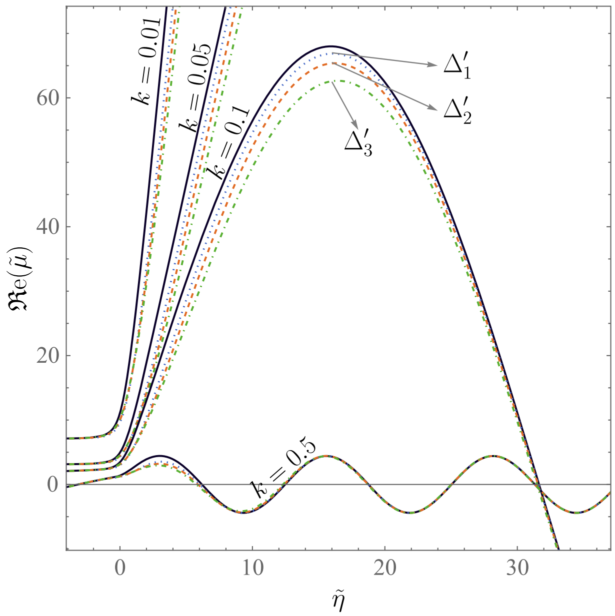

FIG. 10. Evolution of the primordial gravitational amplitude for different clocks obtained from the second class of delay function (A1). Convergence happens at a later time with respect to the first class of delay functions (31), as can be seen by comparison with Fig. 5.

[1] K. V. Kuchar, Time and interpretations of quantum gravity, Int. J. Mod. Phys. D **20**, 3 (2011).
[2] C. J. Isham, Canonical quantum gravity and the problem of time, NATO Sci. Ser. C **409**, 157 (1993).
[3] E. Anderson, The problem of time in quantum gravity, arXiv:1009.2157.
[4] C. Kiefer and P. Peter, Time in quantum cosmology, Universe **8**, 36 (2022).
[5] S. Deser, R. Arnowitt, and C. W. Misner, Heisenberg representation in classical general relativity, Nuovo Cimento **19**, 668 (1961).
[6] S. C. Beluardi and R. Ferraro, Extrinsic time in quantum cosmology, Phys. Rev. D **52**, 1963 (1995).
[7] P. Hajicek, Choice of gauge in quantum gravity, Nucl. Phys. B, Proc. Suppl. **80** (2000), https://inspirehep.net/literature/497294.
[8] G. Catren and R. Ferraro, Quantization of the Taub model with extrinsic time, Phys. Rev. D **63**, 023502 (2001).
[9] P. Hajicek and J. Kijowski, Spherically symmetric dust shell and the time problem in canonical relativity, Phys. Rev. D **62**, 044025 (2000).
[10] P. Małkiewicz, Multiple choices of time in quantum cosmology, Classical Quantum Gravity **32**, 135004 (2015).
[11] P. Małkiewicz and A. Miroszewski, Internal clock formulation of quantum mechanics, Phys. Rev. D **96**, 046003 (2017).
[12] M. Bojowald and T. Halnon, Time in quantum cosmology, Phys. Rev. D **98**, 066001 (2018).
[13] P. Peter, E. J. C. Pinho, and N. Pinto-Neto, Noninflationary model with scale invariant cosmological perturbations, Phys. Rev. D **75**, 023516 (2007).
[14] N. Pinto-Neto, Cosmological perturbations in quantum backgrounds and confrontation with inflationary predictions , Prog. Theor. Phys. Suppl. **172**, 178 (2008).
[15] P. Peter and N. Pinto-Neto, Cosmology without inflation , Phys. Rev. D **78**, 063506 (2008).
[16] P. Malkiewicz, Clocks and dynamics in quantum models of gravity, Classical Quantum Gravity **34**, 145012 (2017).
[17] P. Małkiewicz, P. Peter, and S. D. P. Vitenti, Quantum empty Bianchi I spacetime with internal time, Phys. Rev. D **101**, 046012 (2020).
[18] P. Małkiewicz, P. Peter, and S. D. P. Vitenti, Clocks and trajectories in quantum cosmology, Universe **8**, 71 (2022).
[19] C. Rovelli, Partial observables, Phys. Rev. D **65**, 124013 (2002).
[20] B. Dittrich, Partial and complete observables for Hamiltonian constrained systems, Gen. Relativ. Gravit. **39**, 1891 (2007).
[21] M. Bojowald and A. Tsobanjan, Algebraic approach to the frozen formalism problem of time, Phys. Rev. D **107** (2023).
[22] P. Małkiewicz, What is dynamics in quantum gravity?, Classical Quantum Gravity **34**, 205001 (2017).

[23] P. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science. Monographs Series (Belfer Graduate School of Science, Yeshiva University, 1964), https://books.google.fr/books?id=oQNRAAAAMAAJ.

[24] P. Peter and J.-P. Uzan, *Primordial Cosmology*, Oxford Graduate Texts (Oxford University Press, Oxford, 2013), ISBN 978-0-19-966515-0, 978-0-19-920991-0.

[25] A. Micheli and P. Peter, *Handbook of Quantum Gravity* (Springer, Singapore, 2024).

[26] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Theory of cosmological perturbations, Phys. Rep. **215**, 203 (1992).

[27] A. Vilenkin, Quantum cosmology and the initial state of the Universe, Phys. Rev. D **37**, 888 (1988).

[28] J. R. Klauder, *Enhanced Quantization: Particles, Fields and Gravity* (World Scientific, Hackensack, 2015), ISBN 978-981-4644-62-4, 978-981-4644-64-8.

[29] J. R. Klauder, Enhanced quantization: A primer, J. Phys. A **45**, 285304 (2012).

[30] J. d. C. Martin, P. Małkiewicz, and P. Peter, Unitarily inequivalent quantum cosmological bouncing models, Phys. Rev. D **105**, 023522 (2022).

[31] H. Bergeron, J.-P. Gazeau, P. Małkiewicz, and P. Peter, New class of exact coherent states: Enhanced quantization of motion on the half line, Phys. Rev. D **109**, 023516 (2024).

[32] D. Brizuela, C. Kiefer, and M. Krämer, Quantum-gravitational effects on gauge-invariant scalar and tensor perturbations during inflation: The slow-roll approximation, Phys. Rev. D **94** (2016).

[33] A. Tronconi, G. P. Vacca, and G. Venturi, Inflaton and time in the matter-gravity system, Phys. Rev. D **67**, 063517 (2003).

[34] A. O. Barvinsky, Unitarity approach to quantum cosmology, Phys. Rep. **230**, 237 (1993).

[35] P. Hajicek, Choice of gauge in quantum gravity, arXiv:gr-qc/9903089.