Double space T dualization and coordinate dependent RR field

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In this article, we examine T dualization in the double space formalism of type II superstring theory in pure spinor formulation. All background fields are constant except the Ramond-Ramond field, which depends infinitesimally on bosonic coordinates x^μ. In double space, T dual transformations are represented as permutations of the starting x^μ and dual coordinates y_μ. Combining these two sets of coordinates into the double coordinate $Z^M = (x^\mu, y_\mu)$, while demanding that the T dual double coordinate has the same T dual transformation law as the initial ones, we obtain how background fields transform under T duality. Comparing these results with ones obtained using the Buscher T dualization procedure, we conclude that these two approaches are equivalent for the considered choice of the background fields.

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I. INTRODUCTION

One of the most peculiar properties of superstring theory is a net of dualities connecting all different types of superstrings [1–4]. Spanned by two types of duality transformations, called T duality and S duality, this network hints at existence of one unique theory lurking underneath it, M theory. While it is not yet exactly known what shape M theory will take, it is certain that by exploring properties of the string theory dualities, we will be able to gain some insights into this underlying theory. In this paper, we will focus just on exploring properties of T duality.

T duality is a phenomenon which does not appear in any form in the theories that deal with one-dimensional particles. This phenomenon is only experienced by extended objects, strings [3–8], and it connects theories which have radii of compactification R with ones where radii are $\propto \frac{1}{R}$. The basic mathematical framework for implementing T duality is given by the Buscher procedure [5,9], which relies on existence of the underlying global isometries in the theory under consideration. In short, in order to obtain T dual theory, we localize global isometry, usually the translational isometry, by introducing covariant derivatives. Covariant derivatives introduce the gauge fields into the theory. In order to keep the number of degrees of freedom during the T dualization procedure, we need to eliminate all the newly introduced degrees of freedom with Lagrange multipliers. By using the gauge freedom, we are able to fix starting coordinates, which leaves us with a theory only dependent on the gauge fields and Lagrange multipliers. Finding equations of motion for the gauge fields and inserting their solutions back into the theory, we obtain the T dual theory. While this procedure is applicable for many models, it breaks down when we have coordinate dependent background fields. In cases where background fields at least infinitesimally depend on coordinates, it is possible to generalize the Buscher procedure [10–13] by introducing invariant coordinates given as a line integral of covariant derivatives.

Even though the Buscher T dualization procedure can be considered as the definition of T duality, in order to gain a deeper understanding of duality, it is useful to consider alternative formulations. One of these alternative representations, called the double space formulation, casts T duality as a permutation of coordinates in space spanned by the initial coordinates and T dual ones. Double space formulation was first considered in papers [14–18]. Recently, this formalism has been related with $O(D,D)$ transformations [19–23], while in papers [14,24–30] T dualization along some directions has been represented as a permutation of these coordinates with the corresponding T dual ones.

In articles [27,28], it was shown that for a type II superstring theory, the Buscher procedure and double space formalism are equivalent. Analysis was conducted on a type II superstring in a pure spinor formulation where all background fields were constant. Here, we would like to repeat the procedure outlined in the aforementioned articles while allowing the Ramond-Ramond (R-R) field to be linearly dependent on bosonic coordinates.

The first reason why we chose such background fields is the assumption presented in the papers [31,32] that if we

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take the linearly coordinate dependent RR field strength, then we will get the anti-Poisson bracket of the fermionic coordinates proportional to the bosonic coordinates. Our intention in [33] was to check that conjecture using the generalized T dualization procedure. We showed that using this procedure and choosing a constant background with the exception of RR field strength, we could not prove that conjecture. The second reason was a completely practical one—model with such background fields we could treat analytically using the generalized T duality procedure.

In this article, we will begin by showing how we obtained our action and background fields from a general one. By integrating out fermionic momenta, we will obtain a theory which has a R-R field coupled with derivatives of bosonic coordinates. If this step was omitted, then the theory would have a R-R field coupled only to the fermionic degrees of freedom, which will not be dualized here. After obtaining the starting action, we will also give a short preview of the results obtained by the Buscher T dualization procedure. Transcribing T dual transformation laws in terms of the double coordinates, generalized metric, and generalized current, demanding that T duality does not alter the form of the transformation laws, we will be able to obtain the T dual generalized metric and T dual generalized current. By equating components of the starting and dual generalized metric and generalized current, we are able to show how background fields transform under T duality. Comparing these results with ones obtained from the Buscher procedure, we are able to see if these two approaches are consistent in the case of the coordinate dependent R-R field strength.

II. TYPE II SUPERSTRING THEORY WITH COORDINATE DEPENDENT R-R FIELD AND ITS T DUAL

In this section, our goal is to introduce action for a type II superstring in pure spinor formulation [34–37]. We will work with a theory whose all background fields are constant, except the Ramond-Ramond field. The Ramond-Ramond field depends only on the bosonic coordinates, where the tensor multiplying space-time coordinates is infinitesimal. Furthermore, we will also demand that R-R field is antisymmetric. Both constraints are necessary for practical (mathematical) reasons.

After the introduction of a type II superstring, we will present a theory, which is its T dual. This theory will have new background fields, and it has been showed that this theory is both a noncommutative and nonassociative one [33,38,39].

At the end of this section, we will transcribe both theories in the form that is more suitable for the double space formulation.

A. Type II superstring in pure spinor formulation

Action that describes the propagation of a type II superstring, in its most general form [31], is given as

$$S = S_0 + V_{SG}, \tag{2.1}$$

where the first term denotes action of the free superstring,

$$S_0 = \int_\Sigma d^2\xi \left(\frac{k}{2}\eta_{\mu\nu}\partial_m x^\mu \partial_n x^\nu \eta^{mn} - \pi_\alpha \partial_- \theta^\alpha + \partial_+ \bar{\theta}^\alpha \bar{\pi}_\alpha\right) + S_\lambda + S_{\bar{\lambda}}. \tag{2.2}$$

Here, integration is done on a world sheet Σ, parametrized by coordinates ξ^m, where the parameter m takes the values $m = 0, 1$ ($\xi^0 = \tau$, $\xi^1 = \sigma$). We will work in the light cone coordinates, which are given by $\xi^\pm = \frac{1}{2}(\tau \pm \sigma)$, while the light cone partial derivatives are given by $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Superspace is spanned by 10 bosonic coordinates x^μ ($\mu = 0, 1, \ldots, 9$) and fermionic ones θ^α and $\bar{\theta}^\alpha$, with 16 independent real components each ($\alpha = 1, 2, \ldots, 16$). Variables π_α and $\bar{\pi}_\alpha$ denote canonically conjugated momenta of fermionic coordinates. Terms S_λ and $S_{\bar{\lambda}}$ denote actions for pure spinors, λ^α and $\bar{\lambda}_\alpha$, and their canonically conjugated momenta, ω_α and $\bar{\omega}_\alpha$, where pure spinors satisfy pure spinor constraints,

$$\lambda^\alpha (\Gamma^\mu)_{\alpha\beta} \lambda^\beta = \bar{\lambda}^\alpha (\Gamma^\mu)_{\alpha\beta} \bar{\lambda}^\beta = 0. \tag{2.3}$$

The second term in Eq. (2.1) denotes all perturbations to the flat background. These perturbations are given by the integrated vertex operator for massless type II supergravity,

$$V_{SG} = \int_\Sigma d^2\xi (X^T)^M A_{MN} \bar{X}^N. \tag{2.4}$$

In the general case matrix, A_{MN} contains fields dependent on both bosonic and fermionic coordinates. We will work with following background fields:

$$A_{MN} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k\left(\frac{1}{2}g_{\mu\nu} + B_{\mu\nu}\right) & \bar{\Psi}^\beta_\mu & 0 \\ 0 & -\Psi^\alpha_\nu & \frac{2}{k}(f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.5}$$

where $g_{\mu\nu}$ is the symmetric tensor, $B_{\mu\nu}$ is Kalb-Ramond antisymmetric tensor, Ψ^α_μ and $\bar{\Psi}^\alpha_\mu$ are Majorana-Weyl gravitino fields, and $\frac{2}{k}(f^{\alpha\beta} + C^{\alpha\beta}_\rho x^\rho) = \frac{2}{k}F^{\alpha\beta}$ is the Ramond-Ramond field. The Ramond-Ramond field is composed of constant antisymmetric tensors $f^{\alpha\beta}$ and $C^{\alpha\beta}_\rho$, where $C^{\alpha\beta}_\rho$ is infinitesimal. We chose to work with an antisymmetric $F^{\alpha\beta}$ tensor in order to obtain transformation laws that can be easily recast in double space formulation. A dilaton field Φ is assumed to be constant, where the factor e^{Φ} has been incorporated into $f^{\alpha\beta}$ and $C^{\alpha\beta}_{\rho}$. Since we are only interested in classical analysis, we will not calculate the dilaton shift under T duality transformations. In the literature [31,32], it is assumed that this choice of background fields would produce anti-Poisson brackets between fermionic coordinates that are proportional to bosonic coordinates. This result would give credence to the assumption that our space-time could be obtained from some fermionic structure which lies underneath it. This choice of fields is accompanied with following constraints:

$$\gamma^{\mu}_{\alpha\beta}C^{\beta\gamma}_{\mu} = 0, \qquad \gamma^{\mu}_{\alpha\beta}C^{\gamma\beta}_{\mu} = 0. \tag{2.6}$$

In general, vectors X^M and $\bar{X}^M$ are given as columns containing partial derivatives of both fermionic and bosonic coordinates as well as containing fermionic momenta and pure spinor contribution. In order to simplify calculations, we will neglect all terms that are nonlinear in fermionic coordinates θ^α and $\bar{\theta}^\alpha$. This means the vectors X^M and $\bar{X}^M$ have the following form:

$$X^M = \begin{pmatrix} \partial_+\theta^\alpha \\ \partial_+x^\mu \\ \pi_\alpha \\ \frac{1}{2}N^{\mu\nu}_+ \end{pmatrix}, \qquad \bar{X}^M = \begin{pmatrix} \partial_-\bar{\theta}^\beta \\ \partial_-x^\mu \\ \bar{\pi}_\beta \\ \frac{1}{2}\bar{N}^{\mu\nu}_- \end{pmatrix}. \tag{2.7}$$

Pure spinor contribution are given as

$$N^{\mu\nu}_+ = \frac{1}{2}\omega_\alpha(\Gamma^{[\mu\nu]})^\alpha{}_\beta\lambda^\beta, \qquad \bar{N}^{\mu\nu}_- = \frac{1}{2}\bar{\omega}_\alpha(\Gamma^{[\mu\nu]})^\alpha{}_\beta\bar{\lambda}^\beta. \tag{2.8}$$

From this point on, since pure spinors are decoupled from the rest of the action, we will be omitting them.

With these assumptions, we have that action (2.1) takes the following form:

$$S = k\int_\Sigma d^2\xi\Big[\Pi_{+\mu\nu}\partial_+x^\mu\partial_-x^\nu + \frac{1}{2}(\partial_+\bar{\theta}^\alpha + \partial_+x^\mu\bar{\Psi}^\alpha_\mu) \times (F^{-1}(x))_{\alpha\beta}(\partial_-\theta^\beta + \Psi^\beta_\nu\partial_-x^\nu)\Big], \tag{2.9}$$

where we have integrated out fermionic momenta π_α and $\bar{\pi}_\alpha$ as well as introduced the following tensors:

$$\Pi_{\pm\mu\nu} = B_{\mu\nu} \pm \frac{1}{2}G_{\mu\nu}, \tag{2.10}$$

$$F^{\alpha\beta}(x) = f^{\alpha\beta} + C^{\alpha\beta}_\mu x^\mu,$$
$$(F^{-1}(x))_{\alpha\beta} = (f^{-1})_{\alpha\beta} - (f^{-1})_{\alpha\alpha_1}C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta}. \tag{2.11}$$

Tensor $(F^{-1}(x))_{\alpha\beta}$ has inherited both properties of a Ramond-Ramond tensor, that is this new tensor is antisymmetric and the coordinate dependence is only tied to the infinitesimal tensor.

B. T dual theory

The T dualization procedure for the model presented above is detailed and described in papers [33,38,39]. Here, we mention just the final results and give a brief description of the T dualization procedure.

The procedure for obtaining a T dual theory is based on the extension of the standard Buscher procedure [5,9,11]. This procedure entails that dualization is to be carried out only along the isometry directions. Since we have that the Ramond-Ramond field is antisymmetric, we have that the action (2.9) is invariant to the translations along bosonic coordinates. The next step in the procedure is to localize this symmetry. This is accomplished by substituting partial derivatives with covariant ones. Because we are working with a theory that has coordinate dependent background fields, it is also required to introduce the invariant coordinates. The invariant coordinates are given as line integrals of the covariant derivatives. These substitutions bring into play the new gauge fields, which add the new degrees of freedom. In order to keep the number of the degrees of freedom in the T dual theory, we must introduce Lagrange multipliers. Utilizing gauge freedom, we can also fix the starting bosonic coordinates, which leaves us with theory that is described by only gauge fields and Lagrange multipliers. Finding equations of motion for the gauge field and inserting them into the gauge fixed action, we are left with T dual theory. Implementing the Buscher procedure, we obtain the following T dual action:

$${}^bS = \frac{k}{2}\int_\Sigma d^2\xi\Big[\frac{1}{2}\bar{\Theta}^{\mu\nu}_-\partial_+y_\mu\partial_-y_\nu + \partial_+\bar{\theta}^\alpha({}^bF^{-1}(V^{(0)}))_{\alpha\beta}\partial_-\theta^\beta + \partial_+y_\mu{}^b\bar{\Psi}^{\mu\alpha}({}^bF^{-1}(V^{(0)}))_{\alpha\beta}\partial_-\theta^\beta + \partial_+\bar{\theta}^\alpha({}^bF^{-1}(V^{(0)}))_{\alpha\beta}{}^b\Psi^{\nu\beta}\partial_-y_\nu\Big]. \tag{2.12}$$

Here, y_μ are dual coordinates, the left superscript ^{b} denotes bosonic T-duality, and V^0 represents the following integral:

$$\Delta V^{(0)\rho} = \frac{1}{2}\int_P d\xi^+ \breve{\Theta}_-^{\rho_1\rho}\left[\partial_+ y_{\rho_1} - \partial_+\bar\theta^\alpha (f^{-1})_{\alpha\beta}\Psi^\beta_{\rho_1}\right] - \frac{1}{2}\int_P d\xi^- \breve{\Theta}_-^{\rho\rho_1}\left[\partial_- y_{\rho_1} + \bar\Psi^\alpha_{\rho_1}(f^{-1})_{\alpha\beta}\partial_-\theta^\beta\right]. \quad (2.13)$$

T dual tensors that appear in action have the following interpretation: $\bar\Theta^{\mu\nu}_\mp$ is the inverse tensor of $\bar\Pi_{\pm\mu\nu} = \Pi_{\pm\mu\nu} + \frac{1}{2}\bar\Psi^\alpha_\mu (F^{-1}(x))_{\alpha\beta}\Psi^\beta_\nu = \breve\Pi_{\pm\mu\nu} - \frac{1}{2}\bar\Psi^\alpha_\mu (f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho x^\rho (f^{-1})_{\beta_1\beta}\Psi^\beta_\nu$, defined as

$$\bar\Theta^{\mu\nu}_\mp \bar\Pi_{\pm\nu\rho} = \delta^\mu_\rho, \quad (2.14)$$

where

$$\bar\Theta^{\mu\nu}_\mp = \breve\Theta^{\mu\nu}_\mp + \frac{1}{2}\breve\Theta^{\mu\mu_1}_\mp \bar\Psi^\alpha_{\mu_1}(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\rho V^{(0)\rho}(f^{-1})_{\beta_1\beta}\Psi^\beta_{\nu_1}\breve\Theta^{\nu_1\nu}_\mp, \quad (2.15)$$

$$\breve\Theta^{\mu\nu}_\mp \breve\Pi_{\pm\nu\rho} = \delta^\mu_\rho, \qquad \breve\Theta^{\mu\nu}_\mp = \Theta^{\mu\nu}_\mp - \frac{1}{2}\Theta^{\mu\mu_1}_\mp \bar\Psi^\alpha_{\mu_1}(\bar f^{-1})_{\alpha\beta}\Psi^\beta_{\nu_1}\Theta^{\nu_1\nu}_\mp \quad (2.16)$$

$$\bar f^{\alpha\beta} = f^{\alpha\beta} + \frac{1}{2}\Psi^\alpha_\mu \Theta^{\mu\nu}_- \bar\Psi^\beta_\nu, \quad (2.17)$$

$$\Theta^{\mu\nu}_\mp \Pi_{\pm\nu\rho} = \delta^\mu_\rho, \qquad \Theta^{\mu\nu}_\mp = -4(G_E^{-1}\Pi_\mp G^{-1})^{\mu\nu}, \quad (2.18)$$

$$G_{E\mu\nu} = G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \quad (2.19)$$

$$\Pi_{+\mu\nu} = -\Pi_{-\nu\mu}, \quad \breve\Pi_{+\mu\nu} = -\breve\Pi_{-\nu\mu}, \quad \bar\Pi_{+\mu\nu} = -\bar\Pi_{-\nu\mu}, \quad (2.20)$$

$$\Theta^{\mu\nu}_+ = -\Theta^{\nu\mu}_-, \qquad \breve\Theta^{\mu\nu}_+ = -\breve\Theta^{\nu\mu}_-, \qquad \bar\Theta^{\mu\nu}_+ = -\bar\Theta^{\nu\mu}_-. \quad (2.21)$$

Tensor $({}^bF^{-1}(V^{(0)}))_{\alpha\beta}$ is T dual to $(F^{-1}(x))_{\alpha\beta}$, and it is given as

$$({}^bF^{-1}(V^{(0)}))_{\alpha\beta} = (F^{-1}(V^{(0)}))_{\alpha\beta} - \frac{1}{2}(F^{-1}(V^{(0)}))_{\alpha\alpha_1} \times \Psi^{\alpha_1}_\mu \bar\Theta^{\mu\nu}_- \bar\Psi^{\beta_1}_\nu (F^{-1}(V^{(0)}))_{\beta_1\beta}. \quad (2.22)$$

Finally, ${}^b\bar\Psi^{\mu\alpha}$ and ${}^b\Psi^{\nu\beta}$ are T-dual gravitino fields, given as

$${}^b\bar\Psi^{\mu\alpha} = \frac{1}{2}\Theta^{\mu\nu}_-\bar\Psi^\alpha_\nu, \qquad {}^b\Psi^{\nu\beta} = -\frac{1}{2}\Psi^\beta_\mu \Theta^{\mu\nu}_-. \quad (2.23)$$

Transformation laws that connect starting and T-dual coordinates are

$$\bar\Pi_{+\mu\nu}\partial_- x^\nu = -\frac{1}{2}\partial_- y_\mu - \frac{1}{2}\bar\Psi^\alpha_\mu (F^{-1}(x))_{\alpha\beta}\partial_-\theta^\beta - \beta^+_\mu(x), \quad (2.24)$$

$$\bar\Pi_{+\nu\mu}\partial_+ x^\nu = \frac{1}{2}\partial_+ y_\mu - \frac{1}{2}\partial_+\bar\theta^\alpha (F^{-1}(x))_{\alpha\beta}\Psi^\beta_\mu - \beta^-_\mu(x), \quad (2.25)$$

where β^+_μ and β^-_μ functions, which are obtained during the T dualization procedure from varying gauge fixed action with respect to gauge fields, are given as

$$\beta^+_\mu(x) = -\frac{1}{2}(\bar\theta^\alpha + x^{\nu_1}\bar\Psi^\alpha_{\nu_1})(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} \times (\partial_-\theta^\beta + \partial_- x^{\nu_2}\Psi^\beta_{\nu_2}), \quad (2.26)$$

$$\beta^-_\mu(x) = -\frac{1}{2}(\partial_+\bar\theta^\alpha + \partial_+ x^{\nu_1}\bar\Psi^\alpha_{\nu_1})(f^{-1})_{\alpha\alpha_1} \times C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta}(\theta^\beta + x^{\nu_2}\Psi^\beta_{\nu_2}). \quad (2.27)$$

Having introduced starting and T dual theory, we can now focus on combining them with double space formalism.

III. T DUALIZATION IN DOUBLE SPACE

The focus of this section is to show how the bosonic T duality of superstring with the coordinate dependent Ramond-Ramond field can be represented as permutation of coordinates in space that is spanned by both starting and T dual coordinates. Work done here mirrors work done in papers [27,28], where the same model was examined only with the constant background fields. While our transformation laws contain both bosonic and fermionic coordinates, it will be possible to separate the fermionic contributions into objects called "double currents".

A. T dual transformation laws in double space formulation

In order to show how permutations of coordinates can be interpreted as T dual transformations, we need to transcribe T dual transformation laws introducing more suitable notation. We begin by introducing the following substitutions:

$$\Psi^\alpha_\mu = \Psi^\alpha_{+\mu}, \qquad \bar\Psi^\alpha_\mu = \Psi^\alpha_{-\mu}, \qquad \theta^\alpha = \theta^\alpha_+, \qquad \bar\theta^\alpha = \theta^\alpha_-, \quad (3.1)$$

$$(F^{-1}(x))_{\alpha\beta} = (F^{-1}_+(x))_{\alpha\beta}, \qquad (F^{-1}(x))_{\beta\alpha} = (F^{-1}_-(x))_{\alpha\beta}, \quad (3.2)$$

$$(F^{-1}_+(x))_{\alpha\beta} = -(F^{-1}_-(x))_{\alpha\beta}, \quad (3.3)$$

$$(f^{-1})_{\alpha\alpha_1} C^{\alpha_1\beta_1}_\mu (f^{-1})_{\beta_1\beta} = C_{+\mu\alpha\beta}, \qquad (f^{-1})_{\beta\beta_1} C^{\beta_1\alpha_1}_\mu (f^{-1})_{\alpha_1\alpha} = C_{-\mu\alpha\beta}, \quad (3.4)$$

$$C_{+\mu\alpha\beta} = -C_{-\mu\alpha\beta}. \quad (3.5)$$

With this new notation, the transformation law and its inverse one take the following form:

$$\partial_\mp x^\nu = -\frac{1}{2}\hat{\Theta}^{\nu\mu}_\mp \partial_\mp y_\mu - \frac{1}{2}\hat{\Theta}^{\nu\mu}_\mp \Big[\Psi^\alpha_{\mp\mu}(F^{-1}_\pm(V^{(0)}))_{\alpha\beta} - (\theta^\alpha_\mp + V^{(0)\nu_1}\Psi^\alpha_{\mp\nu_1})C_{\pm\mu\alpha\beta}\Big]\partial_\mp\theta^\beta_\pm, \tag{3.6}$$

$$\partial_\mp y_\mu = -2\hat{\Pi}_{\pm\mu\nu}\partial_\mp x^\nu - \Big[\Psi^\alpha_{\mp\mu}(F^{-1}_\pm(x))_{\alpha\beta} - (\theta^\alpha_\mp + x^{\nu_1}\Psi^\alpha_{\mp\nu_1})C_{\pm\mu\alpha\beta}\Big]\partial_\mp\theta^\beta_\pm, \tag{3.7}$$

where $\beta^\pm_\mu$ functions have been expanded and separated into two parts, one containing the partial derivative of a bosonic coordinate and the other containing the partial derivative of a fermionic coordinate. The first part has been incorporated into two newly introduced tensors $\hat{\Pi}_{\pm\mu\nu}$ and $\hat{\Theta}^{\nu\mu}_\mp$,

$$\hat{\Pi}_{\pm\mu\nu} = \bar{\Pi}_{\pm\mu\nu} - \frac{1}{2}(\theta^\alpha_\mp + x^{\nu_1}\Psi^\alpha_{\mp\nu_1})C_{\pm\mu\alpha\beta}\Psi^\beta_{\pm\nu}, \tag{3.8}$$

$$\hat{\Theta}^{\nu\mu}_\mp = \bar{\Theta}^{\nu\mu_1}_\mp\left[\delta^\mu_{\mu_1} + \frac{1}{2}(\theta^\alpha_\mp + V^{(0)\nu_1}\Psi^\alpha_{\mp\nu_1})C_{\pm\mu_1\alpha\beta}\Psi^\beta_{\pm\nu_2}\breve{\Theta}^{\nu_2\mu}_\mp\right]. \tag{3.9}$$

These two newly introduced tensors are inverse to one another, $\hat{\Pi}_{\pm\mu\nu}\hat{\Theta}^{\nu\rho}_\mp = \delta^\rho_\mu$, and they can be decomposed in the following way:

$$\hat{\Pi}_{\pm\mu\nu} = \hat{B}_{\mu\nu} \pm \frac{1}{2}\hat{G}_{\mu\nu}, \qquad \hat{\Theta}^{\mu\nu}_\mp = -4(\hat{G}^{-1}_E\hat{\Pi}_\mp\hat{G}^{-1})^{\mu\nu}, \tag{3.10}$$

$$\hat{G}_{E\mu\nu} = \hat{G}_{\mu\nu} - 4\hat{B}_{\mu\mu_1}\hat{G}^{\mu_1\nu_1}\hat{B}_{\nu_1\nu}, \tag{3.11}$$

$$\hat{\Theta}^{\nu\mu}_\pm = -4\hat{G}^{\nu\nu_1}_E\hat{B}_{\nu_1\mu_1}\hat{G}^{\mu_1\mu} \mp 2(\hat{G}^{-1}_E)^{\nu\mu}. \tag{3.12}$$

Interpretation of these components is the following: $\hat{B}_{\mu\nu}$ and $\hat{G}_{\mu\nu}$ are antisymmetric and symmetric parts of tensor $\hat{\Pi}_{\pm\mu\nu}$ called the "improved Kalb-Ramond field" and the "improved metric tensor", respectively. Both improved tensors now have the additional bilinear form in NS-R fields Ψ^α_μ and $\bar{\Psi}^\alpha_\mu$. Tensor $\hat{G}_{E\mu\nu}$ is called the "improved effective metric". These decompositions allow us to rewrite transformation laws as

$$\pm\partial_\pm x^\mu = (\hat{G}^{-1})^{\mu\nu}\partial_\pm y_\nu + 2(\hat{G}^{-1})^{\mu\nu_1}\hat{B}_{\nu_1\nu}\partial_\pm x^\nu + (\hat{G}^{-1})^{\mu\nu}\Big[\Psi^\alpha_{\pm\nu}(F^{-1}_\mp(x))_{\alpha\beta} - (\theta^\alpha_\pm + x^{\nu_1}\Psi^\alpha_{\pm\nu_1})C_{\mp\nu\alpha\beta}\Big]\partial_\pm\theta^\beta_\mp, \tag{3.13}$$

$$\pm\partial_\pm y_\mu = \hat{G}_{E\mu\nu}\partial_\pm x^\nu - 2\hat{B}_{\mu\mu_1}\hat{G}^{\mu_1\nu}\partial_\pm y_\nu + \frac{1}{2}\hat{G}_{E\mu\nu}\hat{\Theta}^{\nu\mu_1}_\pm\Big[\Psi^\alpha_{\pm\mu_1}(F^{-1}_\mp(V^{(0)}))_{\alpha\beta} - (\theta^\alpha_\pm + V^{(0)\nu_1}\Psi^\alpha_{\pm\nu_1})C_{\mp\mu_1\alpha\beta}\Big]\partial_\pm\theta^\beta_\mp. \tag{3.14}$$

It should be noted that all tensors in Eq. (3.13) are functions of bosonic coordinates x^μ. On the other hand, all tensors in (3.14) are functions of the line integral $V^{(0)}$, which has been defined in (2.13).

Having transcribed transformation laws in a new notation, we are now free to introduce the double coordinates,

$$Z^\mu = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix}. \tag{3.15}$$

By utilizing double coordinates, transformation laws take a surprisingly simple form,

$$\pm\Omega_{MN}\partial_\pm Z^N = \breve{\mathcal{H}}_{MN}\partial_\pm Z^N + \breve{J}_{\pm M}, \tag{3.16}$$

where the generalized metric is given as

$$\breve{\mathcal{H}}_{MN} = \begin{pmatrix} \hat{G}_{E\mu\nu}(V) & -2\hat{B}_{\mu\mu_1}(\hat{G}^{-1})^{\mu_1\nu}(V) \\ 2(\hat{G}^{-1})^{\mu\nu_1}\hat{B}_{\nu_1\nu}(x) & (\hat{G}^{-1})^{\mu\nu}(x) \end{pmatrix}, \tag{3.17}$$

and the double current is

$$\breve{J}_{\pm M} = \begin{pmatrix} \frac{1}{2}\hat{G}_{\mu\nu_1}\hat{\Theta}^{\nu_1\nu}_\pm(V) \\ (\hat{G}^{-1})^{\mu\nu}(x) \end{pmatrix} J_{\pm\nu},$$
$$J_{\pm\nu} = \Big[\Psi^\alpha_{\pm\nu}(F^{-1}_\mp(x))_{\alpha\beta} - (\theta^\alpha_\pm + x^{\nu_1}\Psi^\alpha_{\pm\nu_1})C_{\mp\nu\alpha\beta}\Big]\partial_\pm\theta^\beta_\mp. \tag{3.18}$$

It should be noted that upper components all depend on the variable $V^{(0)}$, while lower components all depend on x.

The matrix,

$$\Omega_{MN} = \begin{pmatrix} 0 & I_D \\ I_D & 0 \end{pmatrix}, \tag{3.19}$$

is an invariant $SO(D,D)$ metric where I_D denotes a unity matrix in D dimensions.

Since the generalized metric contains both an improved Kalb-Ramond field and improved metric tensor, it does not come in the standard form. However, it still satisfies the following relations:

$$\breve{\mathcal{H}}^T\Omega\breve{\mathcal{H}} = \Omega, \qquad (\Omega\breve{\mathcal{H}})^2 = I, \qquad \Omega^2 = I, \qquad \det(\breve{\mathcal{H}}) = 1, \tag{3.20}$$

which means that $\breve{\mathcal{H}} \in SO(D,D)$ [14,19].

B. T duality in double space

To obtain full T duality let us introduce a permutation matrix,

$$T^M{}_N = \begin{pmatrix} 0 & I_D \\ I_D & 0 \end{pmatrix}. \tag{3.21}$$

Now we can define the T dual double coordinate as

$${}^bZ^M = T^M{}_N Z^N. \tag{3.22}$$

Transformation laws (3.16) must have the same form for both the dual and starting coordinates, that is

$$\pm\Omega_{MN}\partial_\pm {}^bZ^N = {}^b\check{\mathcal{H}}_{MN}\partial_\pm {}^bZ^N + {}^bJ_{\pm M}. \tag{3.23}$$

From (3.16) and (3.23), we can deduce how both a generalized metric and double current transform under permutations,

$${}^b\check{\mathcal{H}}_{MN} = T_M{}^P\check{\mathcal{H}}_{PQ}T^Q{}_N, \qquad {}^b\check{J}_{\pm M} = T_M{}^N\check{J}_{\pm N}. \tag{3.24}$$

Expanding the first equation, we get

$${}^b\check{\mathcal{H}}_{MN} = \begin{pmatrix} {}^b\hat{G}_E^{\mu\nu}(V) & -2{}^b\hat{B}^{\mu\mu_1}({}^b\hat{G}^{-1})_{\mu_1\nu}(V) \\ 2({}^b\hat{G}^{-1})_{\mu\nu_1}{}^b\hat{B}^{\nu_1\nu}(x) & ({}^b\hat{G}^{-1})_{\mu\nu}(x) \end{pmatrix} \tag{3.25}$$

$$= \begin{pmatrix} (\hat{G}^{-1})^{\mu\nu}(x) & 2(\hat{G}^{-1})^{\mu\nu_1}\hat{B}_{\nu_1\nu}(x) \\ -2\hat{B}_{\mu\mu_1}(\hat{G}^{-1})^{\mu_1\nu}(V) & \hat{G}_{E\mu\nu}(V) \end{pmatrix}. \tag{3.26}$$

Let us notice that the variables $V^{(0)}$ and x also exchange places.

Equating block components (2, 2) and (2, 1), we obtain the following equations:

$$({}^b\hat{G}^{-1})_{\mu\nu}(x) = \hat{G}_{E\mu\nu}(V), \to {}^b\hat{G}_{\mu\nu}(x) = (\hat{G}_E^{-1})_{\mu\nu}(V), \tag{3.27}$$

$$\begin{aligned}({}^b\hat{G}^{-1})_{\mu\nu_1}{}^b\hat{B}^{\nu_1\nu}(x) = -\hat{B}_{\mu\mu_1}(\hat{G}^{-1})^{\mu_1\nu}(V), &\to {}^b\hat{B}^{\mu\nu}(x) \\ &= -(\hat{G}_E^{-1})^{\mu\nu_1}\hat{B}_{\nu_1\mu_1}(\hat{G}^{-1})^{\mu_1\nu}(V).\end{aligned} \tag{3.28}$$

Using these two results, we obtain

$${}^b\hat{\Pi}_{\pm\mu\nu}(x) = {}^b\hat{B}_{\mu\nu}(x) \pm \frac{1}{2}{}^b\hat{G}_{\mu\nu}(x) = \frac{1}{4}\hat{\Theta}_{\mp\mu\nu}. \tag{3.29}$$

This result coincides with the result obtained from the Buscher procedure. Equating the other two components of the matrix, we obtain the same information.

Expanding equation for dual current, we obtain

$${}^b\check{J}_{\pm M} = \begin{pmatrix} \frac{1}{2}{}^b\hat{G}_{\mu\nu_1}{}^b\hat{\Theta}_\pm^{\nu_1\nu}(V) \\ ({}^b\hat{G}^{-1})^{\mu\nu}(x) \end{pmatrix} \quad J_{\pm\nu} = \begin{pmatrix} (\hat{G}^{-1})^{\mu\nu}(x) \\ \frac{1}{2}\hat{G}_{\mu\nu_1}\hat{\Theta}_\pm^{\nu_1\nu}(V) \end{pmatrix} J_{\pm\nu}, \tag{3.30}$$

where T dual current has the same factor $J_{\pm\nu}$, while the vector components are switched.

Comparing our results to ones obtained in the paper [28], we notice that the generalized metric, double current, and double space transformation laws have the same form. However, all fields that emerge in the formalism are now modified. These modifications all stem from the fact that the starting theory had a coordinate dependent RR field.

IV. CONCLUSION

The aim of this article was to investigate alternative method for obtaining T dual theories, namely the double space method. This investigation was carried out on a type II superstring theory with a specific choice of background fields. Our choice of the model was motivated by the fact that this exact model and its T dual was examined in great detail in papers [33,38,39], where T duality was obtained with the Buscher procedure. This work provided us with the reference point on which we can compare our results.

At the beginning of our examination, we described how we obtained our action from a more general one. This was done by demanding that all background fields, except the Ramond-Ramond field, are constant. The Ramond-Ramond field was chosen to have an infinitesimal linear dependence on the bosonic coordinates x^μ. Furthermore, we also demanded that the RR field is totally antisymmetric. Terms that were nonlinear in fermionic coordinates have been neglected, and all fermionic momenta have been integrated out of the action. These assumptions were necessary in order to obtain simple transformation laws between starting and dual coordinates (for the case where RR field is not antisymmetric. See Ref. [38]). After this. we presented T dual theory, T dual fields, and T dual transformation laws, which were obtained with the Buscher procedure. While the starting theory is assumed to be commutative, the T dual theory exhibits both noncommutative and nonassociative properties. These properties are a consequence of $\beta^\pm_\mu$ functions that show up in transformation laws.

Section III was dedicated to the double space formulation of T duality. We began by transcribing the T dual transformation laws and background fields in a more suitable notation. By splitting and recasting old background fields into "improved" fields, we were able to gain a more clearer picture of underlying connections between starting and dual coordinates. Combining the space of the starting theory, spanned by x^μ, with the space of the T dual theory, spanned with y_μ, we obtain the double space formulation. Double space is now spanned by coordinates $Z^M = (x^\mu, y_\mu)$. Rewriting the T dual transformation laws into double space coordinates, we define two new objects, the generalized metric $\check{\mathcal{H}}_{MN}$ and double current $\check{J}_{\pm M}$. Their components are expressed through an improved Kalb-Ramond field, improved metric tensor, and improved effective metric, containing additional terms bilinear in a NS-R background Ψ_μ^α and $\bar{\Psi}_\mu^\alpha$. Furthermore, it should be noted that these components are not constant; the upper row depends on the coordinate x^μ and the lower row on its T dual image V^μ.

In the context of double space formalism, T duality is given by a simple permutation of coordinates. By demanding that the double space coordinates Z^M and T dual coordinates ${}^bZ^M = T^M{}_N Z^N$ possess the same transformation laws, we are able to find a T dual generalized metric ${}^b\check{\mathcal{H}}_{MN}$ and T dual double current ${}^b\check{J}_{\pm M}$. Since a T dual generalized metric has the same form as the starting one, by comparing components, we are able to deduce expressions for T dual background fields as functions of starting fields. It should also be noted that the permutation of coordinates also permutes arguments of background fields. This means that a T dual generalized metric and T dual double current now have an upper row that depends on V^μ and a lower row that depends on x^μ. Doing the same analysis for the T dual double current, we obtain relations that connect its components to background fields of starting theory.

Comparing results obtained for T dual fields by means of double space coordinate permutation with ones obtained with the Buscher procedure, it is evident that both methods produce the same result. Additionally, it should be noted that comparing results from this paper with results from [28], where all background fields were constant, we notice that double space transformation laws have the same form, but individual components of a generalized metric and double current are now coordinate dependents.

At the end, we should give a comparison between our work and work done in papers [40–45]. In the given papers, the double space formulation of string theory as well as the T dual transformation laws for fields were obtained by starting from a double field formulation of the NS-NS sector, which was invariant to $O(D,D)$ group. Utilizing the fact that in this case, we can locally define two set of vielbeins, we can construct a pair of Lorentz groups $\mathrm{spin}(1,9)\times\mathrm{spin}(9,1)$. These two Lorentz frames are associated with left-moving and right-moving sectors of the world sheet. All fermions as well as the RR field lie in a spinor representation of $O(D,D)$ group, and their transformations are governed by a $\mathrm{spin}(1,9)\times\mathrm{spin}(9,1)$ group. Finding a Hermitian element of a $\mathrm{spin}(1,9)\times\mathrm{spin}(9,1)$ group whose projection to $O(D,D)$ produces generalized metric, we can construct action that contains both IIA and IIB theories, where different types of type II theories are given as solutions to this action. Comparing this approach with ours, we can note that, while the method for obtaining double space theory and transformation laws in these papers is more general, it requires of us to *a priori* start with a NS-NS sector already given in double space formulation. Our approach starts from either type IIA or type IIB theory, and then we find its dual and combine transformation laws to obtain a generalized metric tensor, the same metric tensor that is given in preceding papers for the NS-NS sector. Since we are focused only on the bosonic T duality, we did not have any need for the introduction of spin representation of $O(D,D)$ group. All our field transformations were obtained from the Buscher T dualization procedure, and we only arranged them into one encompassing formalism, eliminating any need for examining how certain fields transform under certain groups.

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