Hierarchy of curvatures in exceptional geometry

Falk Hassler*

University of Wrocław, Faculty of Physics and Astronomy, Maksa Borna 9, 50-204 Wroclaw, Poland

Yuho Sakatani†

Department of Physics, Kyoto Prefectural University of Medicine, 1-5 Shimogamohangi-cho, Sakyo-ku, Kyoto 606-0823, Japan

(Received 18 December 2023; accepted 2 April 2024; published 1 May 2024)

Despite remarkable success in describing supergravity reductions and backgrounds, generalized geometry and exceptional field theory are still lacking a fundamental object of differential geometry, the Riemann tensor. We show that to construct it, a hierarchy of connections is required. They complement the spin connection with higher representations known from the tensor hierarchy. This approach allows to define generalized homogeneous spaces which underlie generalized U-duality, admit consistent truncations and provide a huge class of new flux backgrounds with nontrivial structure groups.

DOI: 10.1103/PhysRevD.109.106002

I. INTRODUCTION

Generalized geometry has evolved to a pivotal tool in dimensional reductions of supergravity with nonvanishing fluxes. Many of its key results have their root in differential geometry adapted to an extended tangent space incorporating forms in addition to a vector. We drop the prefix "generalized" in the following and assume it by default. Most mileage is gained when supersymmetry is at least partially preserved. Prominent examples include Scherk-Schwarz reductions to maximal gauged supergravities in two or more dimensions and special structure manifolds that provide the analogs of Calabi-Yau, Kähler, and hyper-Kähler manifolds in flux compactifications.

Despite all success, a central concept of differential geometry, Riemann curvature, is still lacking a complete understanding in generalized geometry. It is possible to define metric compatible connections, but imposing vanishing torsion does not fix all their components [1–4]. Moreover, the familiar expression for the Riemann tensor is not covariant anymore and has to be modified and projected [1–7]. Although sufficient to capture supergravity at the leading two-derivative level, the current construction is not completely satisfying. Therefore, we present a novel perspective on curvature in generalized geometry based on two key ideas:

(a) Torsion and curvature tensors have to transform covariantly under generalized diffeomorphisms *and* a structure group G_{S}.
(b) Reconciling internal and external diffeomorphisms for dimensional reductions is a crucial step in the construction of gauged supergravities, resulting in a hierarchy of gauge fields known as *tensor hierarchy* [8].

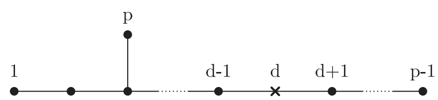
Both combined suggest to embed the d-dimensional space M_d, for which curvatures should be obtained, together with its structure group G_{S} into a larger space, called the megaspace M_p with $p = d + n$ and $n = \dim G_{\mathrm{S}}$. By construction, this new space is parallelizable and thus equipped with a globally defined frame which gives rise to a unique, curvature-free connection. Similar to Cartan geometry, its torsion decomposes into torsion and curvature on M_d. This approach can be understood as a direct construction of the tensor hierarchy where a tower of connection and their curvatures arises on M_d. At its bottom, we recover all quantities relevant for two-derivative supergravity. Beyond, we find new covariant tensors with more than two derivatives. In addition to conceptual insights, our approach provides an explicit construction for a new, large class of consistent truncations of supergravity and the intriguing web of dualities that relates them.

II. LEVEL DECOMPOSITION

Our starting point is a pair of two Lie groups $G_{\mathrm{D}} = \mathrm{E}_{d(d)}$ with $d \leq 6$, governing generalized diffeomorphisms on M_d, and $\mathrm{GL}(n)$, in which the structure group $G_{\mathrm{S}} \subset G_{\mathrm{D}}$ is embedded. They will not be studied separately but rather

*falk.hassler@uwr.edu.pl
†yuho@koto.kpu-m.ac.jp

are unified in $G_{\mathrm{M}} = \mathrm{E}_{p(p)}$ which governs diffeomorphisms on M_p. There are two different perspectives one can take on this setting: First, start with G_{M} and remove the dth root in its Dynkin diagram



(1)

to obtain the algebras G_{D} and $\mathrm{GL}(n) \supset G_{\mathrm{S}}$. Taking any irreducible representation (irrep) of G_{M} and branching it accordingly, results in a sum of irreps of $G_{\mathrm{D}} \times \mathrm{GL}(n)$. They can be organized according to a grading called level. For our purpose, this top-down approach is not ideal because G_{M} and the megaspace M_p are only book keeping tools. The protagonist is M_d which is controlled by G_{D} and G_{S}. Therefore, it is better to start with them and only step by step recover G_{M}.

Their generators will be called $K_{\mathbf{a}}$ and K_α^β for $\mathrm{GL}(n)$ with $\alpha, \beta, \ldots \in \{1, \ldots, n\}$ and $\mathbf{a}$ valued in the adjoint representation of G_{D}. They satisfy the nontrivial commutation relations

$$[K_{\mathbf{a}}, K_{\mathbf{b}}] = f_{\mathbf{ab}}{}^{\mathbf{c}} K_{\mathbf{c}},$$
$$[K_\alpha^\beta, K_\gamma^\delta] = \delta_\alpha^\delta K_\gamma^\beta - \delta_\gamma^\beta K_\alpha^\delta. \tag{2}$$

This completes all level zero contributions in the decomposition of G_{M}'s adjoint representation. To proceed to levels ∓ 1, consider R_α^A, with the index A transforming in the R_1 representation of G_{D}, and its dual $\tilde{R}_B^\beta$. Their commutation relation with all Ks are easily fixed by representation theory. More complicated is the commutator

$$[\tilde{R}_A^\alpha, R_\beta^B] = \delta_A^B(\boldsymbol{\beta}\delta_\alpha^\beta L - K_\beta^\alpha) + \boldsymbol{\alpha}\delta_\beta^\alpha (t^{\mathbf{a}})_A^B K_{\mathbf{a}}, \tag{3}$$

where $(t_{\mathbf{a}})_B^C$ denotes the generators of G_{D} in the R_1 representation. Note that we raise and lower adjoint indices with the Killing metric of the duality group G_{D}. Moreover, $L = K_\alpha^\alpha$ is distinguished because its eigenvalues are the levels of the decomposition. Finally, we use the constants $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ from the definition of G_{D} generalized diffeomorphisms [9] whose values are listed in the Appendix. By a rescaling of $\tilde{R}_A^\alpha$ and R_β^B, it is always possible to fix the coefficient in front of K_α^β to minus one. The other two coefficients have to be as given to define the level 2 generators by

$$[R_\alpha^A, R_\beta^B] = \eta^{AB\bar{C}} R_{\alpha\beta\bar{C}} \quad \text{and}$$
$$[\tilde{R}_A^\alpha, \tilde{R}_B^\beta] = \eta_{AB\bar{C}} \tilde{R}^{\alpha\beta\bar{C}} \tag{4}$$

with the η-tensors representing the "square root" of G_{D}'s Y-tensor

$$Y_{CD}^{AB} = \eta^{AB\bar{E}}\eta_{CD\bar{E}} = -\boldsymbol{\alpha}(t^{\mathbf{a}})_D^A (t_{\mathbf{a}})_C^B + \boldsymbol{\beta}\delta_D^A\delta_C^B + \delta_C^A\delta_D^B. \tag{5}$$

This implies that the bared indices label the R_2 representation of G_{D}. All other new commutators at this level arise through the Jacobi identity from the lower levels. One can repeat this procedure level by level to obtain generators in higher representations of the tensor hierarchy algebra. However, already at level 2 one sees all relevant features of the construction. Therefore, we stop here and refer to appendix for level 3 or the companion article [10] for technical details.

For these generators, we introduce a representation building on the highest weight states $|^\alpha\rangle$. They are annihilated by all $\tilde{R}$ generators and by $K_{\mathbf{a}}$, while K_α^β acts as

$$K_\alpha^\beta |^\gamma\rangle = \delta_\alpha^\gamma |^\beta\rangle + \boldsymbol{\beta}' \delta_\alpha^\beta |^\gamma\rangle, \tag{6}$$

where $\boldsymbol{\beta}'$ is now taken with respect to G_{M} instead of G_{D}. Acting with any R generator(s) produces descendants. For our purpose, only

$$|^A\rangle = \frac{1}{n} R_\alpha^A |^\alpha\rangle, \tag{7}$$

and the dual states defined by

$$\langle_\alpha|^\beta\rangle = \delta_\alpha^\beta, \qquad \langle_A|^B\rangle = \delta_A^B \tag{8}$$

are needed. Using the commutator (3), one can easily show that

$$\langle_A| = -\frac{1}{n}\langle_\alpha|\tilde{R}_A^\alpha \tag{9}$$

has exactly the desired properties.

III. EXCEPTIONAL POLÁČEK-SIEGEL FORM

Next, we fix the form of the frame on M_p to compute the torsion of the flat derivative it induces. Because the lowest levels will be sufficient, it is convenient to suppress G_{M}-indices and instead just use the index-free form

$$\hat{E} = \tilde{M} N \tilde{V} \tag{10}$$

of the frame. Its splitting is inspired by results for the duality group $\mathrm{O}(d,d)$ [11] and will be motivated in the following. Finally, we need the generalized Lie derivative [12]

$$\mathbb{L}_{\langle U|}\langle V| = \langle U|\partial_V\rangle\langle V| + \langle V|\langle U|Z|\partial_U\rangle \tag{11}$$

that governs generalized diffeomorphisms on the megaspace M_p with $p \le 7$ through the Z-tensor

$$Z = -\boldsymbol{\alpha} K_{\mathbf{a}} \otimes K^{\mathbf{a}} + \boldsymbol{\beta} L \otimes L + \boldsymbol{\beta}' \mathbf{1} \otimes \mathbf{1} - K_\alpha^\beta \otimes K_\beta^\alpha + R_\alpha^A \otimes \tilde{R}_A^\alpha + \frac{1}{2} R_{\alpha_1\alpha_2\bar{A}} \otimes \tilde{R}^{\alpha_1\alpha_2\bar{A}} + \cdots + (R \leftrightarrow \tilde{R}). \tag{12}$$

With …, we denote suppressed levels higher than two. Our index-free notation in (11) assumes that ∂_V is the partial derivative acting on $\langle V|$ and the convention $\langle U_1|\langle U_2|A \otimes B|V_2\rangle|V_1\rangle = \langle U_2|B|V_2\rangle\langle U_1|A|V_1\rangle$.

The megaspace's torsion is twisted by $\tilde{M}^{-1}$ to get rid of any dependence on the auxiliary coordinates that G_S introduces, resulting in

$$X_\mathcal{A} = \langle_\mathcal{A}|N|^\mathcal{B}\rangle\Theta_\mathcal{B} + \langle_\mathcal{A}|\Theta_\mathcal{B} Z N|^\mathcal{B}\rangle \tag{13}$$

with $X_\mathcal{A} = (X_\alpha \; X_A)$ and the corresponding Maurer-Cartan form

$$\Theta_\mathcal{A} = -\tilde{M}^{-1} D_\mathcal{A} \hat{E} \hat{E}^{-1} \tilde{M} \quad \text{where } \tilde{V}|\partial\rangle = |^\mathcal{A}\rangle D_\mathcal{A}. \tag{14}$$

To recover the tensor hierarchy, we require that $\hat{E}$ is only generated by generators with zero or negative levels. This renders it an element of a parabolic subgroup of G_M. $\tilde{M}$ is not further constrained, while N is unipotent and $\tilde{V}$ has only level 0 contributions. To see how to choose $\tilde{M}$ and $\tilde{V}$, we first single out the n generators [13]

$$t_\alpha = -X_{\alpha\beta}{}^\gamma K_\gamma^\beta - X_\alpha{}^{\mathbf{b}} K_{\mathbf{b}} \tag{15}$$

(with constant coefficients $X_{\alpha\beta}{}^\gamma = X_{[\alpha\beta]}{}^\gamma$ and $X_\alpha{}^{\mathbf{b}}$) of the parabolic subgroup and impose

$$X_\alpha = -t_\alpha. \tag{16}$$

This requires at least $\langle_\alpha|\Theta_B = 0$. Additionally, we restrict the discussion to the special case $X_{\alpha\beta}{}^\beta = 0$ for the sake of brevity and find that (16) further requires

$$\Theta_\alpha = X_\alpha - \frac{1}{2}(X_{\alpha\beta}{}^\gamma + S_{(\alpha\beta)}{}^\gamma) N K_\gamma^\beta N^{-1}. \tag{17}$$

The symmetric $S_{(\alpha\beta)}{}^\gamma$ will not contribute to X_A and leaves an antisymmetric $X_{\alpha\beta}{}^\gamma$. For $D_\alpha N = 0$ and $D_A \tilde{M} = 0$ it is possible to choose $\tilde{M}$, N and $\tilde{V}$ such that this relation follows from (14). In this case, we identify

$$\tilde{M}^{-1} D_\alpha \tilde{M} = t_\alpha \quad \text{and} \quad D_\alpha \tilde{V}\tilde{V}^{-1} = \frac{1}{2}(X_{\alpha\beta}{}^\gamma + S_{(\alpha\beta)}{}^\gamma) K_\gamma^\beta. \tag{18}$$

In principle one could try to make other identifications. But these two are distinguished because they can be integrated, if (and only if)

$$[t_\alpha, t_\beta] = X_{\alpha\beta}{}^\gamma t_\gamma \tag{19}$$

holds. An immediate consequence is that all t_α generate an n-dimensional Lie group, the structure group G_S. For consistency, we also verify that G_S's coordinates satisfy the section condition on the megaspace,

$$Y|^\alpha\rangle|^\beta\rangle D_\alpha \cdot D_\beta \cdot = 0, \tag{20}$$

where $Y = Z + \sigma$ and $\sigma|V_2\rangle|V_1\rangle = |V_1\rangle|V_2\rangle$.

To completely fix $\hat{E}$, we parametrize the unipotent part N in the decomposition (10) by [14]

$$N = \cdots \exp\left(\frac{1}{3!}\rho^{\alpha\beta\gamma\bar{\bar{D}}} R_{\alpha\beta\gamma\bar{\bar{D}}}\right) \exp\left(\frac{1}{2}\rho^{\alpha\beta\bar{C}} R_{\alpha\beta\bar{C}}\right) \times \exp(\Omega_A^\alpha R_\alpha^A), \tag{21}$$

where $\bar{\bar{D}}$ is for the R_3 representation, and impose

$$D_A \tilde{V}\tilde{V}^{-1} = E_A{}^I \partial_I E E^{-1} = -W_A \tag{22}$$

which implies with (18) that $\tilde{V}$ decomposes into $E \in G_\mathrm{D} \times \mathbb{R}^+$, the frame on M_d, and the left-invariant vector fields $\tilde{v}$ on G_S. The last two equations contain only fields restricted to M_d and thereby do not depend on the additional coordinates of M_p introduced by G_S. A frame of a similar form was first introduced in [15] for $\mathrm{O}(d,d)$ and later further refined to arbitrary structure groups [16]. Therefore, we refer to it as the *exceptional Poláček-Siegel form.*

IV. TORSIONS AND CURVATURES

In order to find covariant torsion and curvature tensors, we compute all contributions to X_A at levels less or equal to zero. First, we obtain

$$T_{AB}{}^C = \langle_B|X_A|^C\rangle = (W + \Omega)_{AB}{}^C + Z_{BD}^{CE}(W + \Omega)_{EA}{}^D, \tag{23}$$

after defining

$$W_{AB}{}^C = \langle_B|W_A|^C\rangle \quad \text{and} \quad \Omega_{AB}{}^C = \Omega_A^\alpha X_{\alpha B}{}^C \tag{24}$$

with $X_{\alpha B}{}^C = X_\alpha{}^{\mathbf{b}}(t_{\mathbf{b}})_B{}^C$. One can easily check that this is the torsion for the covariant derivative

$$\nabla_A E_B{}^I = E_A{}^J \partial_J E_B{}^I - \Omega_{AB}{}^C E_C{}^I + E_A{}^J \Gamma_{JK}{}^I E_B{}^K = 0. \tag{25}$$

Rewritten in curved indices,

$$T_{IJ}{}^K = \Gamma_{IJ}{}^K + Z_{JL}^{KM} \Gamma_{MI}{}^L, \tag{26}$$

it reproduces the known expression (3.5) in [6].

For contributions from negative levels, we schematically write

$$X_A = \ldots + X^{\beta}_{AB} R^B_{\beta} + \frac{1}{2} X^{\beta_1\beta_2\bar{B}}_A R_{\beta_1\beta_2\bar{B}} + \ldots. \tag{27}$$

First, consider the -1 part, X^{β}_{AB}: From the Jacobi identity (19), it follows that the constants $X_{\alpha B}{}^C$ have to form a representation of the Lie algebra $\mathfrak{g}_S$ on the generalized tangent space of M_d. We impose that this representation is faithful, because otherwise it is not possible to interpret G_S as the structure group on M_d. Therefore, the tensor

$$R_{ABC}{}^D \coloneqq -X^{\beta}_{AB} X_{\beta C}{}^D \tag{28}$$

captures X^{α}_{AB} completely. Evaluating the latter gives rise to

$$R_{ABC}{}^D = 2D_{[A}\Omega_{B]C}{}^D - 2\Omega_{[A|C}{}^E\Omega_{|B]E}{}^D - \rho^{ED}_{AC;BE} - \left(2\Omega_{[AB]}{}^E + \frac{1}{2} Y^{FE}_{BG}\Omega_{FA}{}^G - T_{AB}{}^E\right)\Omega_{EC}{}^D \tag{29}$$

with

$$\rho^{AB}_{CD;EF} = \rho^{\alpha\beta\bar{C}} X_{\alpha C}{}^A X_{\beta D}{}^B \eta_{EF\bar{C}}. \tag{30}$$

This is the natural Riemann tensor in generalized geometry. Projecting it, such that $\rho^{ED}_{AC;BE}$ drops out, reproduce the existing results in the literature. Depending on the R_2 representation, there are two different common projection:

(i) $R_2 = \mathbf{1}$: G_D has an invariant metric, η_{AB}, which is applied to lower the last index of $R_{ABC}{}^D$. After symmetrized with respect to $(AB) \leftrightarrow (CD)$, the generalized Riemann tensor of double field theory [3,4] arises.

(ii) Otherwise: The indices B and D are contracted and the remaining indices are symmetrizes. Rewriting the result in curved indices with the affine connection recovers Eq. (5.16) of [7] or (5.4) of [6].

There is however no need for any projection because $R_{ABC}{}^D$ transforms covariantly under G_D and G_S after taking into account the correct transformation of ρ in (41). A similar pattern already holds at the level of the torsion (23). With only the frame, it is not covariant unless one introduces the spin-connection $\Omega_{AB}{}^C$ that transforms accordingly. Still, there are projections of $T_{AB}{}^C$ where the latter drops out. They give rise to the intrinsic torsion [17], used for example in consistent truncations. Here this pattern repeats after the substitutions $E \to \Omega \to \rho$.

Next, we need a field strength for $\rho^{AB}_{CD;EF}$. It follows from (27) along the same lines as (28), namely

$$R^{BC}_{ADE;FG} = -X^{\beta_1\beta_2\bar{B}}_A X_{\beta_1 D}{}^B X_{\beta_2 E}{}^C \eta_{FG\bar{B}}, \tag{31}$$

and evaluates to [18]

$$\begin{aligned} R^{BC}_{ADE;FG} = {} & D_A\rho^{BC}_{DE;FG} - \rho^{HBC}_{ADE;FGH} + 2[T_{A(F|}{}^H - \Omega_{A(F|}{}^H] \\ & \times \rho^{BC}_{DE;|G)H} - 2\Omega_{AD}{}^H\rho^{BC}_{HE;FG} + 2\Omega_{AH}{}^B\rho^{HC}_{DE;FG} \\ & + Y^{HI}_{FG}\left[R_{AHE}{}^C - \rho^{CJ}_{EA;HJ} - D_H\Omega_{AE}{}^C \right.\\ & \left. + \frac{1}{3}\Omega_{HA}{}^J\Omega_{JE}{}^C\right]\Omega_{ID}{}^B + \frac{1}{6}Y^{HI}_{JK}Y^{JL}_{FG}\Omega_{IA}{}^K \\ & \times \Omega_{HD}{}^B\Omega_{LE}{}^C - \left(\begin{smallmatrix}B\\D\end{smallmatrix}\leftrightarrow\begin{smallmatrix}C\\E\end{smallmatrix}\right). \end{aligned} \tag{32}$$

Here, we encounter the level 3 connection

$$\rho^{ABC}_{DEF;GHI} = \rho^{\alpha\beta\gamma\bar{\bar{D}}} X_{\alpha D}{}^A X_{\beta E}{}^B X_{\gamma F}{}^C \eta_{GHI\bar{\bar{D}}}, \tag{33}$$

which is required to render this new curvature covariant.

V. GAUGE TRANSFORMATIONS

To eventually prove covariance of the derived new torsion/curvatures, we need the transformation of the various connections under G_D-diffeomorphisms and G_S. They arise from distinguished G_M-diffeomorphisms, mediated by $\mathbb{L}_{\langle\hat{\xi}|}\hat{E}$, which preserves the Poláček-Siegel form. We twist infinitesimal variations of the frame by

$$\delta\mathrm{E} \coloneqq N^{-1}\tilde{M}^{-1}(\delta\hat{E})\tilde{V}^{-1} = \delta E E^{-1} + N^{-1}\delta N. \tag{34}$$

In this way, the right-hand side only contains contribution to the megaspace frame that can change without breaking the exceptional Poláček-Siegel form. Next, consider megaspace diffeomorphisms which are parametrized by

$$\langle\hat{\xi}| = \xi^{\alpha}\langle_{\alpha}|\tilde{V} + \xi^A\langle_A|\tilde{V}. \tag{35}$$

Note that both, ξ^{α} and ξ^A, are chosen such that they do not depend on the auxiliary coordinates of M_p to find the shift of the frame

$$\begin{aligned} \delta\mathrm{E} = {} & \xi^{A}(N^{-1}\tilde{M}^{-1}D_A\tilde{M}N + N^{-1}D_A N + D_A\tilde{V}\tilde{V}^{-1}) \\ & + \langle\xi|D_A\tilde{V}\tilde{V}^{-1}Z|^{\mathcal{A}}\rangle + D_A\xi^{\mathcal{B}}\langle_{\mathcal{B}}|Z|^{\mathcal{A}}\rangle. \end{aligned} \tag{36}$$

By comparing the left-hand side (lhs) of (36) with the second line of (34), we read off

$$\delta E_A{}^I = \mathbb{L}_{\xi}E_A{}^I + \xi_A{}^B E_B{}^I. \tag{37}$$

Here, the Lie derivative is restricted to M_D and has the parameter $\xi^I = \xi^A E_A{}^I$. For this transformation, we recognize ξ^A as the parameter of G_D-diffeomorphisms and $\xi_B{}^C = \xi^{\alpha}(t_{\alpha})_B{}^C$ as generator of G_S-transformations. Any covariant tensor should transform in the same way. We therefore define the anomalous part of the transformation as

$$\Delta_{\xi} = \delta - \mathbb{L}_{\xi} - \xi\cdot, \tag{38}$$

where we understand the last contribution as the action of ${\xi_A}^B$ on each index of the tensor under consideration.

In contrast to the frame, all other connections do not transform covariantly. Instead, we find

$$\Delta_\xi {\Omega_{AB}}^C = D_A {\xi_B}^C \tag{39}$$

after taking into account $\delta {X_{\alpha B}}^C = 0$. Again this result is in perfect agreement with the literature, because it implies (remember $[\Delta_\xi, \nabla_A] = 0$)

$$\Delta_\xi {\Gamma_{IJ}}^K = -Z^{KL}_{MJ} \partial_I \partial_L \xi^M \tag{40}$$

and thus matches (3.4) of [6]. On the next level, we obtain

$$\Delta_\xi \rho^{AB}_{CD;EF} = Y^{GH}_{EF} {\Omega_{GC}}^A D_H {\xi_D}^B - \left(\begin{smallmatrix}A\\C\end{smallmatrix} \leftrightarrow \begin{smallmatrix}B\\D\end{smallmatrix}\right). \tag{41}$$

One can now check that the expression for the Riemann tensor in (29) satisfies $\Delta_\xi R_{ABCD} = 0$ and therewith is covariant. In the same vein, one finds that the curvature of $\rho^{AB}_{CD;EF}$ in general is only covariant after introducing the level 3 curvature in (32). This pattern continues until the level decomposition of G_{M}'s adjoint has reached the "top" curvature that is covariant on its own.

A central element in the construction is the megaspace Lie derivative (11) that requires $p \leq 7$. Remarkably, one still finds that with the transformations given here the derived expression from above transform covariantly without any restrictions on p.

VI. EQUIVARIANT FRAMES

It is instructive to flip the perspective and ask: What are the constraints on an arbitrary megaspace frame to admit a Poláček-Siegel form after an appropriate transformation? Clearly, it has to contain n vectors $\langle k_\alpha|$, that generate the Lie algebra $\mathfrak{g}_{\mathrm{S}}$ through

$$\mathbb{L}_{\langle k_\alpha|} \langle k_\beta| = -{X_{\alpha\beta}}^\gamma \langle k_\gamma|. \tag{42}$$

Furthermore, the section condition has to be satisfied on the megaspace. It introduces a second level decomposition with respect to the pth root in (1) and break $\mathrm{E}_{p(p)}$ further down to $\mathrm{GL}(d(-1)) \times \mathrm{GL}(n)$ for M-theory (type IIB). We denote indices enumerating the fundamental of the first group in this product by lowercase Latin letters and the grading follows directly from their position: Each up/down index contributes with $+1/-1$. Considering this grading, a natural parametrization of the megaspace frame is

$$\hat{E} = L\tilde{E}U. \tag{43}$$

All negative elements produce the matrix L, followed by a diagonal $\tilde{E}$ that originates from all level 0 generators and finally an upper-triangular matrix U from the rest. Either L or U can be removed by acting with the maximal compact subgroup of G_{M} from the left. In the context of supergravity, usually L is eliminated while U captures all form-fields and $\tilde{E}$ the frame on M_p.

For frames in Poláček-Siegel form any contributions from positive level generators [with respect to the decomposition (1)] have to vanish. L is not affected by them and stays unconstrained. The frame e that governs $\tilde{E}$ has the components ${e_\alpha}^\mu$, ${e_\alpha}^i$, ${e_a}^\mu$, and ${e_a}^i$, but to avoid $\tilde{R}^\alpha_a$-contributions, ${e_\alpha}^i = 0$ has to hold. A consequence is that ${e_\alpha}^\mu$ is invertible. Finally, U has to satisfy

$$\langle_\alpha| U = 0. \tag{44}$$

In the M-theory section, U is parametrized in terms of a three-form $C^{(3)}$ and a six-form $C^{(6)}$ on M_p. Here (44) for example implies

$$\iota_\alpha C^{(3)} = 0 \quad \text{and} \quad \iota_\alpha C^{(6)} = 0, \tag{45}$$

where ι_α denotes the interior product with respect to the vector field ${e_\alpha}^\mu \partial_\mu$. This situation can be always achieved by an appropriate coordinate change and a form-field gauge transformations if (42) holds, resulting in $\langle k_\alpha| = \langle_\alpha|\hat{E} = \langle_\alpha|\tilde{E}$. Last but not least, one has to verify that the action of $\langle k_\alpha|$ extends to the full frame by

$$\mathbb{L}_{\langle k_\alpha|} \hat{E}\hat{E}^{-1} = t_\alpha \tag{46}$$

with a constant right-hand side. We call frames with this property *equivariant frames*.

VII. EXCEPTIONAL GENERALIZED COSETS

An important class of these frames arise from generalized parallelizations [19]. The latter are constructed on a coset $M_p = G\backslash H$ [20] and play a crucial role in the construction of maximally gauged supergravities by consistent truncations. In order to solve the section condition, H has to be a co-isotropic subgroup of G. By choosing a second isotropic subgroup $G_{\mathrm{S}} \subset G$, one obtains the scenario discussed above with $M_d = G_{\mathrm{S}}\backslash G/H$ being a double coset. Furthermore, the embedding tensor on the megaspace X_A is constant and invariant under the action of G_{S}. Hence, all torsions and curvatures, we computed are covariantly constant with respect to ∇_A.

According to Ambrose and Singer, structure compatible connections with covariantly constant torsion and curvature are in one-to-one correspondence with homogenous spaces [21]. Here, we encounter the lift of this idea to generalized geometry and thus call the cosets $M_d = G_{\mathrm{S}}\backslash G/H$ *exceptional generalized homogenous spaces*. They come with some remarkable properties:

(a) The G_{S}-action can have fixed points which result in singularities on M_d. They might hint toward additional localized objects which are well known in the context of supergravity as branes and monopoles.

(b) In general, there are different admissible co-isotropic subgroups H. While each of them leads to a different space and frame, they share the same torsion and curvatures. For the duality group $\mathrm{O}(d,d)$ this phenomena is known as generalized T-duality [22]. Here it becomes *generalized U-duality*.

(c) Their intrinsic torsion is constant and a singlet under the action of G_S. Therefore, they admit consistent truncations according to theorem 2 of [23].

(d) Because the frame on the megaspace is completely fixed by its embedding tensor, all components of the connections are determined. At the same time, all higher-level curvatures are in general nontrivial.

Items (a)–(c) makes them perfectly suited as backgrounds for dimensional reductions with various applications in flux compactifications and gauged supergravities. While (d) provides an ideal testing ground to address still open challenges in generalized geometry, like undetermined connection/curvature components and higher-derivative corrections. Later arise naturally in our framework at levels beyond two. For example, in the curvature (32) each term contains three derivatives. It is know that admissible higher-derivative corrections to supergravity are severely restricted by dualities (see for example [24] for a recent review). We leave it to future studies to see if these corrections can be captured in a geometric way similar to the two-derivative action, by using the presented construction or, most likely, an appropriate modification of it.

ACKNOWLEDGMENTS

We would like to thank Martin Cederwall, Achilles Gitsis, Ondřej Hulík, Eric Lescano, and Jakob Palmkvist for discussions and David Osten and Alex Swash for comments on the draft. F. H. is supported by the SONATA BIS Grant No. 2021/42/E/ST2/00304 from the National Science Centre (NCN), Poland and is grateful to the Mainz Institute for Theoretical Physics (MITP) of the DFG Cluster of Excellence PRISMA+ (Project ID 39083149), for its hospitality and its partial support during the completion of parts of this work. Y. S. is supported by JSPS KAKENHI Grant No. JP23K03391.

APPENDIX: REMAINING COMMUTATORS FOR LEVELS ≤ 3

For the reader's convenience Table I summarizes important constants and representation for duality groups relevant to this letter. Moreover, we provide for completeness all commutators that are required to compute the torsion and curvatures in the main text explicitly.

TABLE I. Relevant representations and constants.

	$\mathrm{O}(d,d)$	SL(5)	Spin(5, 5)	$\mathrm{E}_{6(6)}$	$\mathrm{E}_{7(7)}$
$\boldsymbol{\alpha}$	2	3	4	6	12
$\boldsymbol{\beta}$	0	1/5	1/4	1/3	1/2
$\boldsymbol{\gamma}$	$2d$	6	8	10	—
r (rank)	d	4	5	6	7
adj	$\mathbf{d(2d-1)}$	**24**	**45**	**78**	**133**
R_1	**2d**	$\overline{\mathbf{10}}$	$\mathbf{16}_\mathrm{c}$	**27**	**56**
R_2	**1**	**5**	**10**	$\overline{\mathbf{27}}$	**133**
R_3	–	$\bar{\mathbf{5}}$	$\mathbf{16}_\mathrm{s}$	**78**	**912**

Lets start with the level ± 3 generators

$$[[R^A_\alpha, R^B_\beta], R^C_\gamma] = \eta^{ABC\bar{\bar{D}}} R_{\alpha\beta\gamma\bar{\bar{D}}} \quad \text{and}$$
$$[[\tilde{R}^\alpha_A, \tilde{R}^\beta_B], \tilde{R}^\gamma_C] = \eta_{ABC\bar{\bar{D}}} \tilde{R}^{\alpha\beta\gamma\bar{\bar{D}}} \tag{A1}$$

with double bared indices in the R_3 representation. Taking into account the antisymmetry with respect to the indices α, β and γ, the Jacobi identity implies $\eta^{(AB)C\bar{\bar{D}}} = \eta^{ABC\bar{\bar{D}}}$ and $\eta^{(ABC)\bar{\bar{D}}} = 0$. Of course the same holds for the version with lowered indices. For $d \leq 5$ this observation is sufficient to completely fix $\eta^{ABC\bar{\bar{D}}}$ up to a factor because $(R_1 \otimes R_2) \cap (R_1^{\otimes 3})_\mathrm{sym} = R_3$ [9]. Finally the normalization is fixed by defining

$$Y^{ABC}_{DEF} = \eta^{ABC\bar{\bar{D}}} \eta_{DEF\bar{\bar{D}}} = (Y^{AB}_{DE}\delta^C_F - Y^{AB}_{GF} Y^{GC}_{DE}). \tag{A2}$$

The action of all level-zero generators follows directly from the representation theory of G_D and $\mathrm{GL}(n)$. It reads for the levels ± 1

$$[K_\mathbf{a}, R^B_\beta] = (t_\mathbf{a})_C{}^B R^C_\beta, \qquad [K^\beta_\alpha, R^C_\gamma] = -\delta^\beta_\gamma R^C_\alpha,$$
$$[K_\mathbf{a}, \tilde{R}^\beta_B] = -(t_\mathbf{a})_B{}^C \tilde{R}^\beta_C, \qquad [K^\beta_\alpha, \tilde{R}^\gamma_C] = \delta^\gamma_\alpha \tilde{R}^\beta_C, \tag{A3}$$

for the levels ± 2,

$$[K_\mathbf{a}, R_{\beta_1\beta_2\bar{B}}] = -(t_\mathbf{a})_{\bar{B}}{}^{\bar{C}} R_{\beta_1\beta_2\bar{C}},$$
$$[K_\mathbf{a}, \tilde{R}^{\beta_1\beta_2\bar{B}}] = (t_\mathbf{a})_{\bar{C}}{}^{\bar{B}} \tilde{R}^{\beta_1\beta_2\bar{C}},$$
$$[K^\beta_\alpha, R_{\gamma_1\gamma_2\bar{C}}] = 2\delta^\beta_{[\gamma_1} R_{\gamma_2]\alpha\bar{C}},$$
$$[K^\beta_\alpha, \tilde{R}^{\gamma_1\gamma_2\bar{C}}] = -2\delta^{[\gamma_1}_\alpha \tilde{R}^{\gamma_2]\beta\bar{C}}, \tag{A4}$$

and for the levels ± 3,

$$
\begin{aligned}
[K_{\mathbf{a}}, R_{\beta_1\beta_2\beta_3\bar{\bar{B}}}] &= -(t_{\mathbf{a}})^{\bar{\bar{C}}}_{\bar{\bar{B}}} R_{\beta_1\beta_2\beta_3\bar{\bar{C}}} \\
[K_{\mathbf{a}}, \tilde{R}^{\beta_1\beta_2\beta_3\bar{\bar{B}}}] &= (t_{\mathbf{a}})^{\bar{\bar{B}}}_{\bar{\bar{C}}} \tilde{R}^{\beta_1\beta_2\beta_3\bar{\bar{C}}}, \\
[K^{\beta}_{\alpha}, R_{\gamma_1\gamma_2\gamma_3\bar{\bar{C}}}] &= -3\delta^{\beta}_{[\gamma_1} R_{\gamma_2\gamma_3]\alpha\bar{\bar{C}}}, \\
[K^{\beta}_{\alpha}, \tilde{R}^{\gamma_1\gamma_2\gamma_3\bar{\bar{C}}}] &= 3\delta^{[\gamma_1}_{\alpha} \tilde{R}^{\gamma_2\gamma_3]\beta\bar{\bar{C}}}.
\end{aligned} \tag{A5}
$$

In particular, $(t_{\mathbf{a}})^{C}_{B}$ denotes the generators of the former in the R_1 representation, whereas $(t_{\mathbf{a}})^{\bar{C}}_{\bar{B}}$ and $(t_{\mathbf{a}})^{\bar{\bar{C}}}_{\bar{\bar{B}}}$ acts on the R_2 and R_3 representation respectively. They satisfy

$$
\begin{aligned}
(t_{\mathbf{a}})^{\bar{C}}_{\bar{D}} \eta^{AB\bar{D}} + 2(t_{\mathbf{a}})^{(A}_{D} \eta^{B)D\bar{C}} &= 0, \\
(t_{\mathbf{a}})^{\bar{\bar{D}}}_{\bar{\bar{E}}} \eta^{ABC\bar{\bar{E}}} + 2(t_{\mathbf{a}})^{(A}_{E} \eta^{B)EC\bar{\bar{D}}} + (t_{\mathbf{a}})^{C}_{E} \eta^{ABE\bar{\bar{D}}} &= 0.
\end{aligned} \tag{A6}
$$

One also needs the commutators between positive and negative level generators resulting in level ±1 and ±2 contributions. First, the levels ±2 give rise to

$$
\begin{aligned}
[\tilde{R}^{\alpha}_{A}, R_{\beta_1\beta_2\beta_3\bar{\bar{B}}}] &= -3\delta^{\alpha}_{[\beta_1} R_{\beta_2\beta_3]\bar{C}} Z_{A}{}^{\bar{C}}{}_{\bar{\bar{B}}}, \\
[\tilde{R}^{\alpha_1\alpha_2\alpha_3\bar{\bar{A}}}, R^{B}_{\beta}] &= 3\delta^{[\alpha_1}_{\beta} \tilde{R}^{\alpha_2\alpha_3]\bar{C}} Z^{B}{}_{\bar{C}}{}^{\bar{\bar{A}}}.
\end{aligned} \tag{A7}
$$

where we defined the intertwiners

$$
\begin{aligned}
Z_{A}{}^{\bar{B}}{}_{\bar{\bar{C}}} &= \frac{1}{\boldsymbol{\gamma}} \eta_{DEA\bar{\bar{C}}} \eta^{DE\bar{B}} \quad \text{and} \\
Z^{A}{}_{\bar{B}}{}^{\bar{\bar{C}}} &= \frac{1}{\boldsymbol{\gamma}} \eta^{DEA\bar{\bar{C}}} \eta_{DE\bar{B}}.
\end{aligned} \tag{A8}
$$

For the levels ±1, we obtain

$$
\begin{aligned}
[\tilde{R}^{\alpha_1\alpha_2\bar{A}}, R_{\beta_1\beta_2\beta_3\bar{\bar{B}}}] &= 6\delta^{\alpha_1}_{[\beta_1}\delta^{\alpha_2}_{\beta_2} R^{C}_{\beta_3]} Z_{C}{}^{\bar{A}}{}_{\bar{\bar{B}}}, \\
[\tilde{R}^{\alpha_1\alpha_2\alpha_3\bar{\bar{A}}}, R_{\beta_1\beta_2\bar{B}}] &= -6\delta^{[\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} \tilde{R}^{\alpha_3]}_{C} Z^{C}{}_{\bar{B}}{}^{\bar{\bar{A}}}, \\
[\tilde{R}^{\alpha}_{A}, R_{\beta_1\beta_2\bar{B}}] &= -2\delta^{\alpha}_{[\beta_1} R^{C}_{\beta_2]} \eta_{CA\bar{B}}, \\
[\tilde{R}^{\alpha_1\alpha_2\bar{A}}, R^{B}_{\beta}] &= 2\delta^{[\alpha_1}_{\beta} \tilde{R}^{\alpha_2]}_{C} \eta^{CB\bar{A}},
\end{aligned} \tag{A9}
$$

and finally at level 0,

$$
\begin{aligned}
[\tilde{R}^{\alpha_1\alpha_2\alpha_3\bar{\bar{A}}}, R_{\beta_1\beta_2\beta_3\bar{\bar{B}}}] &= -6\boldsymbol{\alpha}\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}(t^{\mathbf{a}})^{\bar{\bar{A}}}_{\bar{\bar{B}}} K_{\mathbf{a}} \\
&\quad + 18\Big(\boldsymbol{\beta}\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} L - \delta^{[\alpha_1}_{[\beta_1}\delta^{\alpha_2}_{\beta_2} K^{\alpha_3]}_{\beta_3]}\Big)\delta^{\bar{\bar{A}}}_{\bar{\bar{B}}}
\end{aligned} \tag{A10}
$$

and

$$
\begin{aligned}
[\tilde{R}^{\alpha_1\alpha_2\bar{A}}, R_{\beta_1\beta_2\bar{B}}] &= -2\boldsymbol{\alpha}\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}(t^{\mathbf{a}})^{\bar{A}}_{\bar{B}} K_{\mathbf{a}} \\
&\quad + 4(\boldsymbol{\beta}\delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} L - \delta^{[\alpha_1}_{[\beta_1} K^{\alpha_2]}_{\beta_2]})\delta^{\bar{A}}_{\bar{B}}.
\end{aligned} \tag{A11}
$$

All of them arise from already known commutators by the Jacobi identity.

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one-cocycles of G_S's Chevalley-Eilenberg complex $\Omega^\bullet(\mathfrak{g}_S, V_i)$ in appropriate representations V_i. If they are additionally coboundaries, they can be removed by a constant G_D-transformation. Therefore only nontrivial elements of the Lie algebra cohomology $H^1(\mathfrak{g}_S, V_i)$ are relevant. For semi-simple G_S, there are no such elements according to Whitehead's lemma. Therefore, we set them here to zero.

[14] this form, one can easily compute Maurer-Cartan forms and adjoint actions: Assume first $N = e^\nu$. Now one can use the identities

$$\mathrm{d}NN^{-1} = \sum_{m=0}^{\infty} \frac{[\nu, \mathrm{d}\nu]_m}{(m+1)!} \quad \text{and}$$
$$NXN^{-1} = \sum_{m=0}^{\infty} \frac{[\nu, X]_m}{m!}, \tag{A12}$$

where the bracket $[X, Y]_m$ is defined for $m \geq 0$ as

$$[X, Y]_m = [X, [X, Y]_{m-1}] \quad \text{with} \quad [X, Y]_0 = Y \tag{A13}$$

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