Quantum transitions, detailed balance, black holes, and nothingness

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We consider vacuum transitions by bubble nucleation among 4D vacua with different values and signs of the cosmological constant Λ, including both up and down tunnelings. Following the Hamiltonian formalism, we explicitly compute the transition probabilities for all possible combinations of initial and final values of Λ and find that up tunneling is allowed starting not only from dS spacetime but also from AdS and Minkowski spacetimes. We trace the difference with the Euclidean approach, where these transitions are found to be forbidden, to the difference of treating the latter spacetimes as pure (vacuum) states rather than mixed states with correspondingly vanishing or infinite entropy. We point out that these transitions are best understood as limits of the corresponding transitions with black holes in the zero mass limit $M \to 0$. We find that detailed balance is satisfied provided we use the Hartle-Hawking sign of the wave function for nucleating spacetimes. In the formal limit $\Lambda \to -\infty$, the transition rates for anti–de Sitter (AdS) to dS agree with both the Hartle-Hawking and Vilenkin amplitudes for the creation of dS from nothing. This is consistent with a proposal of Brown and Dahlen to define "nothing" as AdS in this limit. For $M \neq 0$, the detailed balance is satisfied only in a range of mass values. We compute the bubble trajectory after nucleation and find that, contrary to the $M = 0$ case, the trajectory does not correspond to the open universe slicing of dS. We briefly discuss the relevance of our results to the string landscape.

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I. INTRODUCTION

Understanding the vacuum state of our Universe is of paramount importance. The vacuum energy determines in great part the geometry of spacetime once gravity is considered. Quantum transitions between states of different vacuum energies should provide fundamental information on how to understand the vacuum states in a fully fledged theory of quantum gravity. In particular it should help us to better understand the string theory landscape.[1]

Most of the progress in this direction has been made by using semiclassical techniques. In particular, the Euclidean approach pioneered by Coleman and collaborators [5] borrows results from nongravitational field theories [6,7] and makes a concrete proposal for the transition rate per unit volume of such a quantum transition. For the simplest setup of a scalar field theory with a potential energy with two minima A and B with different values of the vacuum energy $V_A > V_B$ the Coleman–De Luccia (CDL) prescription [5] for the transition rate is

$$\mathcal{P}_{A\to B} \propto e^{-[S(\mathrm{bounce})-S(A)]/\hbar}, \tag{1.1}$$

where $S(\text{bounce})$ refers to the Euclidean action evaluated at the instanton solution (bounce) that mediates between the two vacua, and $S(A)$ is the Euclidean action evaluated at the background spacetime A. The process amounts to the creation of a bubble of vacuum B in the background of vacuum A mediated by the bounce. The explicit calculation boils down to determining the bounce as a solution of the Euclidean equations and plugging it into Eq. (1.1). We emphasize that Eq. (1.1) is only a proposal and lacks an explicit derivation.

In principle the same instanton can mediate the up-tunneling transition, from B to A [8],

$$\mathcal{P}_{B\to A} \propto e^{-[S(\mathrm{bounce})-S(B)]/\hbar}, \tag{1.2}$$

[1]See [1] for a recent review of string cosmology including attempts to obtain de Sitter space in string theory. For alternatives such as quintessence see for instance [2–4].

and cancel in the ratio of transitions

$$\frac{\mathcal{P}_{B\to A}}{\mathcal{P}_{A\to B}} \propto e^{[S(B)-S(A)]/\hbar}. \tag{1.3}$$

However, it has been argued that only in the case in which both vacuum energies are positive [de Sitter (dS) to de Sitter, i.e., dS → dS transition] is up-tunneling allowed since otherwise the background contribution $S(B)$ is negative and infinite, implying $\mathcal{P}_{B\to A} \to 0$ [8]. In the allowed case

$$\frac{\mathcal{P}_{B\to A}}{\mathcal{P}_{A\to B}} \propto e^{[S(B)-S(A)]/\hbar} \propto e^{-[\mathcal{S}_B-\mathcal{S}_A]/\hbar}, \tag{1.4}$$

where $\mathcal{S}_A = \pi/H_A^2$, $\mathcal{S}_B = \pi/H_B^2$ are the entropies of the two de Sitter spaces and $H^2 = \Lambda/3$ determines the horizon radius $R_{\rm dS} = 1/H$ for a dS space [9]. Equation (1.4) is the statement of detailed balance: the ratio of probabilities is determined by the corresponding number of available states that is measured by the exponential of the entropies.

From this perspective, the fact that there are no up-tunneling vacuum transitions starting from Minkowski M or anti–de Sitter (AdS) spacetimes corresponds to the fact that, contrary to the case of de Sitter space, the volumes of both AdS and M are infinite. This implies that the corresponding background actions $S(B)$ diverge $S(B) \to -\infty$ and the transition rate vanishes.[2]

Note that, following the dS analogy, this is consistent with assigning an infinite entropy to both $\mathcal{M}$ and AdS.[3] However, Eq. (1.1) is only the result of an educated guess and it is not derived from a well-defined prescription in quantum gravity[4]; therefore we should be open to the possibility that it may not fully capture the relevant physics.

Note also that, contrary to dS, Minkowski and AdS do not have horizons and therefore assigning an entropy in terms of an area is not an option. In counting states, we know that the Hilbert space $\mathcal{H}$ in both cases is infinite which could indicate an infinite entropy. However, as vacuum states both cases would be pure states with vanishing entropies.[5] Therefore, thinking in terms of entropies we may ask two different questions: what is the transition rate between maximally mixed states in the Hilbert space or what is the transition rate only from the vacuum state (a pure state). In the latter, the entropy vanishes and the corresponding up-tunneling amplitude is allowed. We will see that this is what is obtained once we use the Hamiltonian approach rather than the Euclidean approach to compute vacuum transitions.

In the Hamiltonian approach developed by Fischler, Morgan, and Polchinski (FMP)[6] [17,18] (see also [16,19]), vacuum transitions are defined as generalizations of the standard WKB method in quantum mechanics. This method has the advantage that the transition rate is actually computed (not guessed) following the standard WKB approach. No Wick rotations to Euclidean spaces are needed, nor any assumption about the dominant instanton contribution needs to be made.

On the other hand, the Hamiltonian approach so far has not been developed beyond the brane/thin wall case. For example the more general setup in terms of a potential with two minima, studied by CDL, has not yet been addressed in this formalism. The Hamiltonian approach has only been developed for the simplest setup of two spacetimes of different cosmological constants Λ_A, Λ_B separated by a wall of tension κ that determines the bubble.

In any case, this is enough for our purposes in this paper. Precisely this simple setup is what allows the calculations to be explicit and reliable. This is because after solving the energy and momentum constraints the problem reduces to a quantum mechanics problem with a single degree of freedom: the location of the wall $\hat{R}$. The resulting equation of motion is simple: $\dot{\hat{R}}^2 + V(\hat{R}) = -1$, where $V(\hat{R})$ is a calculable function of $\hat{R}$ determined from the parameters of the metric and the wall tension [20]. Then the standard quantum mechanics rules apply in terms of a transition through a barrier for a potential energy $V(\hat{R})$.

We should emphasize at this point that in quantum gravity there is no *a priori* notion of time and it is not possible to discuss a transition rate as such. What we can do instead is to calculate the ratio or squares of wave functions for different configurations and interpret this relative probability as s transition probability. So in this approach

[2]Note that assuming that the partition function is approximated by one dS spacetime, the Euclidean action is $S_E = \beta F = \beta E - \mathcal{S}$. If $E = 0$ then $S_E = -\mathcal{S}$.

[3]It should be noted however that assigning infinite Euclidean action to Minkowski and AdS is a consequence of ignoring the infrared regulator that Gibbons and Hawking [9,10] included in their calculation of black hole entropy. To get the well-known formula for black hole entropy [in Minkowski (M) or AdS space] one needs this regulator which effectively implies that the Euclidean action and hence the entropy of empty M or AdS space is zero. For a recent review of these entropy calculations as well as a Hamiltonian version which gives the same result see [11].

[4]Also the assumption that the bounce solution in gravity inherits the $O(4)$ symmetry that was derived for flat space field theory is not justified.

[5]In principle we can assign vanishing entropy to a full dS spacetime also; however in practice we are interested in the regions that are accessible to a given observer which has no access to the region beyond the horizon. Tracing out the states beyond the horizon gives rise to a nonvanishing entropy.

[6]The FMP calculation was an attempt to understand from first principles in the Hamiltonian framework the singular instanton calculation of Farhi, Guth, and Guven (FGG) [12] whose validity was questioned even by the original authors. Further concerns about the validity of both FMP and FGG have been raised over the years [13–15]. See [16] for addressing the concerns in [13]. We will address the points raised by [15] in Sec. III D.

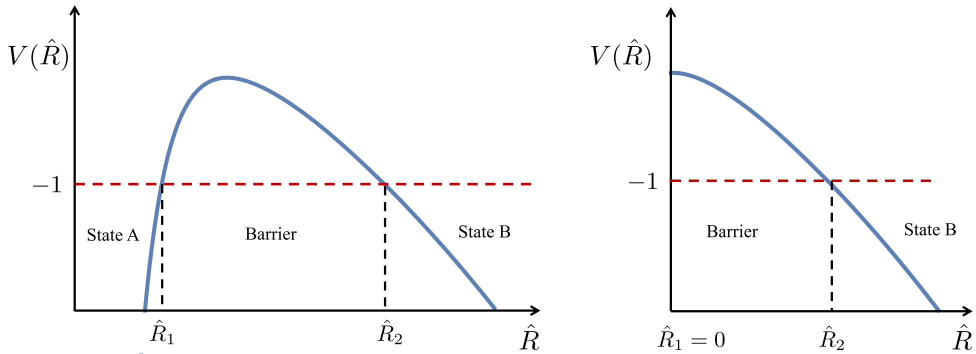

FIG. 1. Two realizations of the potential for the bubble wall $\hat{R}$. On the left, a bubble is materialized in region A and grows until it reaches the turning point $\hat{R}_1$ and classically bounces but quantum mechanically can tunnel to region B at the second turning point to continue expanding. The WKB approximation can be used in all the regions of the potential outside and inside the barrier. This is a typical situation for black hole geometries. On the right there is only one turning point (the first turning point has moved to zero) and the bubble materializes directly in state B. This is a typical potential for pure dS or AdS that can be obtained by setting the black hole mass to zero from the black hole geometry.

the transition probability from a state A to B is defined as the ratio of the two squares of the corresponding wave functions:

$$\mathcal{P}_{A\to B} = \frac{|\Psi(\hat{R}_2)|^2}{|\Psi(\hat{R}_1)|^2}, \tag{1.5}$$

where $\hat{R}_1$ and $\hat{R}_2$ are the two turning points that are determined from the potential barrier $V(\hat{R})$, see the left panel of Fig. 1. The classical picture is that the brane, which contains inside it spacetime B, is nucleated with $\hat{R} = 0$ in spacetime A, grows to a radius $\hat{R}_1$, is reflected back and collapses back to zero. There is also a classical trajectory where the brane comes in from infinity with infinite size (at least in a noncompact space such as a black hole space) hits the barrier at $\hat{R}_2$ and is reflected back. Quantum mechanically the brane can tunnel between these two classical configurations. The wave functions can be written in terms of the WKB approximation as in standard quantum mechanics

$$\Psi(\hat{R}) = ae^{I} + be^{-I}, \tag{1.6}$$

where $I = iS$ is i times the action evaluated at $\hat{R}$. For the case when one of the two exponentials dominate, the transition rate can be written in terms of a difference between two actions, similar to Eq. (1.1) but as we will see they differ in important ways.

In the next sections we will provide a short summary of this prescription and its application to explicitly compute the transition probabilities between vacuum states, we will also include transitions between spacetimes with black holes, and in the case that the true vacuum is de Sitter we will study its geometry.

At this point it behooves us to clarify a point which may cause some confusion. Some of our results pertaining to empty spaces (i.e., with no black hole) are obtained by starting from spaces which have a black hole and taking the limit of the black hole mass M going to zero. The reason for this is that an initial classical region exists only when the black hole mass is nonzero (see the left panel of Fig. 1). When the black hole mass is zero the initial classical region disappears as in the right panel of that figure. Thus it is only for a nonzero mass that the correspondence with standard quantum mechanical tunneling arguments apply. This limit is taken in well-defined expressions for the WKB factor (in the nonzero mass case) and there is no singularity anywhere in that function, so that the limit is well defined. There is no breakdown of the EFT since the expressions in question involve values of $\hat{R}$ that are far from the horizon. One might be concerned that the horizon curvature for small Planck sized black holes is at the Planck scale, since for the validity of the EFT $R_{\ldots\ldots}R^{\cdots\cdots} \sim (GM)^2/R^6 \sim R_s^2/R^6 < 1/l_P^4$ or $R^6 > l_P^4 R_s^2$ (where R_s is the horizon radius). The values of $\hat{R}$ (the positions of the brane at the turning points) that enter into the expressions for the tunneling exponent B in the black hole space times certainly satisfies this inequality and hence the limit $M \sim R_s \to 0$ is well defined.

Another point that needs some clarification is the meaning of nothing as used in this paper. We have followed [21] in identifying this as the limit of AdS space when the cosmological constant goes to negative infinity. In that reference however the spacetime is regarded as four-dimensional AdS times a compact space. Our discussion on the other hand is strictly in four dimensions so nothing for us is still four-dimensional spacetime but with no geometry. This is in fact the concept of nothing used in the original works on universe creation [22–26]. Of course our ultimate goal in this series of studies is to extend this work to full string theory in which case clearly we would have to generalize our definition of nothing. Indeed without actually giving some restrictions on nothing it would not be possible to give a meaningful definition of (relative) probabilities for the creation of a Universe such as ours.

Let us summarize the main results of this article:

(1) We explicitly compute the rates for transitions between any of dS and AdS states including both up and down tunneling and provide explicit expressions for each of the transition rates. The cases corresponding to up tunneling from AdS are new results whereas the others are known and wherever the nonvanishing result is known we agree with the previous results in the literature (see for instance [16,19]).

(2) We consider Minkowski spacetime M in three different limits. First, starting from pure dS with curvature $\Lambda > 0$ and taking the limit $\Lambda \to 0$: in this case we obtain vanishing up-tunneling transition as in the Euclidean case, as computed already in [16,19]. Second, we start with an AdS spacetime with curvature $\Lambda < 0$ and take the limit $\Lambda \to 0$. In this case we get a finite transition amplitude. We interpret the results by noticing that in the dS limit case, the entropy $\mathcal{S} \propto 1/\sqrt{\Lambda} \to \infty$ whereas in the AdS limit case the background contribution vanishes. This corresponds to a vanishing entropy for AdS that is inherited in the Minkowski limit. The third limit is the one taken by FMP corresponding to the $M \to 0$ of Schwarzschild spacetime. This coincides with the AdS limit and gives the same finite transition rate.

(3) Having computed up-tunneling transitions from AdS we also compute the limit $\Lambda \to -\infty$. In this case up tunneling to dS gives the well-known Vilenkin[7] and Hartle-Hawking [22–26] transitions from nothing to dS. This $\Lambda \to -\infty$ limit is consistent with the proposal of Brown and Dahlen [21] for a field theoretic definition of "nothing" precisely as AdS with infinite curvature, motivated by an interpretation of the bubble of nothing transition.[8] Therefore we find consistency between the two definitions of nothing (the decay to the bubble of nothing and the creation of dS from nothing). Furthermore we also explicitly compute transitions from nothing to AdS and $\mathcal{M}$, which were previously thought not to be allowed.

(4) A nontrivial check of our results is that we obtain detailed balance in all the transitions as long as $M \to 0$.

(5) A further nontrivial check of our results is to generalize the transitions to include mass M black hole backgrounds, which are the most general solutions with spherical symmetry which is the symmetry of the bubble/wall system. In this case we can compute the transitions and reproduce our previous results in the limit of $M \to 0$.

(6) In general, the black hole transitions depend on three regions for the values of M. Generalizing FMP we compute the bulk contribution to the transitions in all regimes. Contrary to the $M = 0$ limit some transitions are not allowed. However the Schwarzschild–de Sitter (SdS) transition SdS → SdS is allowed and reproduces previous proposals. In particular it is consistent with detailed balance for small black holes (to be defined below). This point has been recently questioned in [15]. A detailed discussion of the issues raised in this reference is given in Sec. III D.

(7) When the true vacuum is de Sitter we compute the trajectory of the wall. We will see that in the case when $M \neq 0$, contrary to that when $M = 0$, we find that it does not follow a geodesic that favors open universe slicing.

The order of the presentation is as expressed in the table of contents. In the next chapter we review the Hamiltonian approach to vacuum transitions of FMP and the well known dS to dS and AdS transitions. We then consider in detail transitions from AdS and Minkowski considering the different limits of obtaining Minkowski from $\Lambda \to 0$ starting from dS or AdS and discuss the differences. Finally, we consider the $|\Lambda| \to \infty$ limit of AdS transitions and discuss the interpretation in terms of bubbles of nothing where we reproduce the Hartle-Hawking and Vilenkin wave functions. Chapter 3 extends the results to the general case in which the geometry of the bubble is a black hole. All the different transitions are discussed in this case. The SdS → SdS case is technically more complicated due to the different horizons. It is done in full detail and in order to facilitate reading, we have moved it to an Appendix that includes all technical details.

In general, our results imply that populating the string landscape is straightforward and provide some hints on how this population may happen. In particular it incorporates the creation from nothing on the same footing as the other transitions.

II. HAMILTONIAN APPROACH TO VACUUM TRANSITIONS

Let us begin by reviewing vacuum transitions from the Hamiltonian approach as initiated by FMP [18]. Starting with the spherically symmetric metric

$$ds^2 = -N_t^2 dt^2 + L(r,t)^2 (dr + N_r dt)^2 + R(r,t)^2 d\Omega_2^2, \tag{2.1}$$

[7] Also known as tunneling or Vilenkin-Linde wave function.

[8] However, they argued that the up-tunneling transition is forbidden and questioned the validity of the Hartle-Hawking and Vilenkin setups. Our conclusion in this work is that not only are these transitions allowed but that they reproduce exactly the same results as Hartle-Hawking and Vilenkin wave functions.

in order to address the vacuum transition problem FMP considered the bulk-brane system with the brane (or wall) at $r = \hat{r}$ separating two regions with different cosmological constants $\Lambda_{\pm}$ and the following action:

$$S = S_{\text{bulk}} + S_{\text{brane}} - \int d^4x\sqrt{-g}\big(\Lambda_+\Theta(r-\hat{r}) + \Lambda_-\Theta(\hat{r}-r)\big), \quad (2.2)$$

with standard Einstein-Hilbert S_{bulk} and brane action S_{brane}, respectively, and with Θ the step function.

FMP reduced the vacuum transition problem to solving for the quantum mechanics of the brane (assumed spherically symmetric) with a wave function $\Psi(\hat{R})$ that solves the Wheeler-DeWitt equation. In the leading WKB approximation this implies solving the momentum and Hamiltonian constraints while satisfying the matching conditions at the brane [20]

$$\frac{R'(\hat{r}\pm\epsilon)}{\hat{L}} = \frac{1}{2\kappa\hat{R}}(\hat{A}_I - \hat{A}_O) \mp \frac{\kappa}{2}\hat{R}. \quad (2.3)$$

Here $\kappa = 4\pi G\sigma$, where σ is the tension of the wall and $\hat{A}_{I,O}$ are the static metric functions evaluated at $r = \hat{r}$. The indices I, O refer to interior and exterior of the wall:

$$A_\alpha = 1 - \frac{2GM_\alpha}{R} \mp H_\alpha^2 R^2, \qquad \alpha = I, O, \quad (2.4)$$

where $\pm H_\alpha^2 = \frac{8\pi G}{3}\Lambda_\alpha$, with upper sign corresponding to de Sitter and the lower one to anti–de Sitter, and M_α is the standard integration constant corresponding to a black hole mass. Using the explicit expression for π_L (in the gauge $N_r = 0$) one can write the (first integral of the) equation of motion for the brane:

$$\dot{\hat{R}}^2 + V = -1, \quad (2.5)$$

$$V = -\frac{1}{(2\kappa\hat{R})^2}((\hat{A}_I - \hat{A}_O) - \kappa^2\hat{R}^2)^2 + (\hat{A}_O - 1), \quad (2.6)$$

$$= -\frac{1}{(2\kappa\hat{R})^2}((\hat{A}_I - \hat{A}_O) + \kappa^2\hat{R}^2)^2 + (\hat{A}_I - 1), \quad (2.7)$$

which is unbounded from below with a local maximum and one or two turning points at $V = -1$ depending on the parameters H_α, M_α. So the equation $V = -1$ gives the (two) turning points for $\hat{R}$, $R_1 < R_2$ for the classical motion of the brane. At a turning point we see from Eq. (2.6) that $\hat{A}_O > 0$ and from Eq. (2.7) that $\hat{A}_I > 0$.

The classical turning points of the geometry correspond to vanishing conjugate momenta for L ($\pi_L = 0$, which also implies $\pi_R = 0$) [17], which gives

$$\frac{R'^2}{L^2} = A(R) = 1 - \frac{2MG}{R} \mp H^2R^2. \quad (2.8)$$

For $r = \hat{r}$ these correspond to the turning points of the potential (i.e., the solutions of $V = -1$).

The sign of $R'(\hat{r})$ plays an important role since R' is proportional to the extrinsic curvature $\hat{K}$ and indicates if the wall at $r = \hat{r}$ is bent towards the interior or exterior regions.

The transition probability[9] from the initial state $\mathcal{B}$ to the nucleated state $\mathcal{N}$ (including the two spacetimes and the wall) can be written as

$$\mathcal{P}(\mathcal{B}\to\mathcal{N}) = \frac{\|\Psi(\mathcal{N})\|^2}{\|\Psi(\mathcal{B})\|^2}, \quad (2.9)$$

where the numerator can be identified as the squared wave function at the turning point R_2 and the denominator with that at R_1, as in the previous section. These can be written in the WKB approximation as

$$\Psi = ae^I + be^{-I}, \quad (2.10)$$

where $I = iS$. For the numerator we will have a bulk contribution I_{B} and a boundary contribution I_{W}, which take the form

$$I_{\text{B}} = \frac{\eta}{G}\int_0^{\hat{r}-\epsilon} drR\left[\sqrt{A_IL^2 - R'^2} - R'\cos^{-1}\left(\frac{R'}{L\sqrt{A_I}}\right)\right] + \int_{\hat{r}+\epsilon}^{r_{\max}} dr[I\to O], \quad (2.11)$$

$$I_{\text{W}} = \frac{\eta}{G}\int \delta\hat{R}\,\hat{R}\cos^{-1}\left(\frac{R'}{L\sqrt{\hat{A}}}\right)\Bigg|_{\hat{r}-\epsilon}^{\hat{r}+\epsilon}. \quad (2.12)$$

Using Eq. (2.3) we can write explicitly

[9]Note that the Wheeler-DeWitt equation is like a time-independent Schrodinger equation in that there is no notion of time evolution. Consequently there is no obvious way to discuss the rate/lifetime of a state. In the case of nongravitational field theory too, in order to discuss a rate one needs to construct a wave packet. For a discussion of this and for a critique of the standard Coleman approach to tunneling in field theory, see [27,28].

$$I_{\mathrm{W}} = -\frac{\eta}{G}\int dRR\cos^{-1}\left(\frac{\frac{2G}{R}(M_O - M_I) + R^2(\pm H_O^2 \mp H_I^2 - \kappa^2)}{2\kappa R\sqrt{1 - \frac{2GM_O}{R} \mp H_O^2R^2}}\right)$$
$$+\frac{\eta}{G}\int dRR\cos^{-1}\left(\frac{\frac{2G}{R}(M_O - M_I) + R^2(\pm H_O^2 \mp H_I^2 + \kappa^2)}{2\kappa R\sqrt{1 - \frac{2GM_I}{R} \mp H_I^2R^2}}\right). \tag{2.13}$$

The total action is given by[10]

$$I_{\mathrm{tot}} = I_{\mathrm{B}} + I_{\mathrm{W}}, \tag{2.14}$$

Note that at the turning points $R'^2 = L^2A$ and therefore the term in the expression for I_{B} the square root term vanishes and the argument of $\cos^{-1}$ is ± 1. So the only nonvanishing contribution corresponds to $R'/L = -\sqrt{A} \leq 0$. This is a very strong constraint on the allowed values of the extrinsic curvatures. On the turning point geometry $\pi_L = 0$ the first term vanishes and the second term is nonzero only when the argument of the inverse cosine is -1. This means that the bulk integral is given by

$$S_{\mathrm{B}} = \frac{i\eta\pi}{G}\left[\int_0^{\hat{r}} dr\frac{dR}{dr}R\theta(-R') + \int_{\hat{r}}^{r_{\max}} dr\frac{dR}{dr}R\theta(-R')\right], \tag{2.15}$$

$$= \frac{i\eta\pi}{2G}[(\hat{R}^2 - R^2(0))\theta(-\hat{R}'_-) + (R^2(r_{\max}) - \hat{R}^2)\theta(-\hat{R}'_+)]. \tag{2.16}$$

Note that the sign of R' in the first line is determined by continuity and the sign of $\hat{R}'_\pm$, which is determined by the matching conditions and fixed by the geometry on either side of $\hat{r}$. It should also be noted that if the geometry on either or both sides has horizons then $R(0)$ may have to be replaced by the solution of $A_I = 0$ (i.e., the smallest horizon) and $R(R_{\max})$ by the solution of $A_O = 0$ since at a horizon the sign of R' will change and will no longer contribute to the integral.

A further condition is to guarantee that the bubble radius at the turning points is real. Both conditions play a role in the concrete cases we will study next.

A. dS to dS transitions

The simplest transitions to study are dS to dS for which both down and up tunneling are allowed. This correspond to the particular cases in which both black hole masses vanish, $M_I = M_O = 0$. In the case of an initial dS (with $H = H_O$) to a final dS (with $H = H_I$) we are in the limit $M_{O,I} \to 0$ had only one turning point so $R_1 \to 0, R_2 \equiv R_0$ with

$$R_0^2 = \frac{4\kappa^2}{(H_O^2 - H_I^2)^2 + 2\kappa^2(H_O^2 + H_I^2) + \kappa^4}. \tag{2.17}$$

Also

$$A_O = 1 - H_O^2R^2, \qquad A_I = 1 - H_I^2R^2, \tag{2.18}$$

$$V = -\frac{1}{4\kappa^2}\hat{R}^2[(H_O^2 - H_I^2)^2 + 2\kappa^2(H_O^2 + H_I^2) + \kappa^4], \tag{2.19}$$

and there is no initial turning point. Also the matching conditions are now

$$\frac{\hat{R}'_\pm}{L} = \frac{1}{2\kappa\hat{R}}(H_O^2 - H_I^2 \mp \kappa^2)\hat{R}^2 \equiv c_\pm\hat{R}. \tag{2.20}$$

In this case we were able to evaluate explicitly the wall term as well as the bulk term. For a general value of $\hat{R}$ the latter can be directly evaluated from Eq. (2.15) and gives

$$I_{\mathrm{B}}(\hat{R}) \equiv iS_{\mathrm{Bu}}(\hat{R})$$
$$= \frac{\eta\pi}{2G}[(\theta(-\hat{R}'_+) - \theta(-\hat{R}'_-)\hat{R}^2 + \theta(-\hat{R}'_-)H_I^{-2}], \tag{2.21}$$

where the subscript Bu refers to the bulk component. To get the tunneling factor we need to find the difference between the actions evaluated at the two turning points. However in the dS to dS case there is no initial turning point—so in effect it becomes $\hat{R} = 0$. To make these WKB tunneling calculations well defined in the absence of an explicit parametrization as in [16], one should really consider this as the limit of the case with an initial turning point before which there is a classical region. In other words this should be regarded as coming from the $M \to 0$ limit of the corresponding case with a black hole (see the Appendix) where we do have two turning points—R_1, R_2, with $R_1 < R_< < R_> < R_2$, where $R_<$, $R_>$ are given in Eq. (A13). In the limit where the black hole masses go to zero $R_1 < R_< \to 0$. Furthermore (see Fig. 5) we see that $\hat{R}'_-$ at $\hat{R} = R_1$ is positive (and so is $\hat{R}'_+$). Hence at this point

[10]Unfortunately it is possible to evaluate these integrals analytically only in the dS to dS case, as was done in our previous work [16]. In the general (or even in the black hole to dS case as in FMP) only the turning point integrals can be evaluated. It would be interesting to study these integrals numerically in the general case and we leave this to future work.

$\theta(-\hat{R}'_{\pm})$ are both zero. Hence with this definition of the initial turning point in the limit where the black hole masses are zero we have from Eq. (2.21), $I_{\rm B}(\hat{R}=R_1\to 0)=0$. Thus defining the difference of the two turning point actions I as $I(R_2)-I(R_1)\equiv I(R_2-R_1)$, we have (with $R_2=R_0, R_1=0$)

$$I_{\rm B}(R_0-0)\equiv iS_{\rm Bu}(R_0-0) = \frac{\eta\pi}{2G}[(\theta(-\hat{R}'_{+})-\theta(-\hat{R}'_{-})\hat{R}^2+\theta(-\hat{R}'_{-})H_{\rm I}^{-2}]. \tag{2.22}$$

Equation (2.21) is however not symmetrical between the outside and the inside of the spherical brane. This symmetry is a property of the full bulk (and wall) integrals [see Eq. A 6 in the Appendix] before imposing any turning point conditions. If we do impose this symmetry even at the turning point geometry (where $R'/L)=\pm\sqrt{A})$) then the above becomes

$$I_{\rm B}(\hat{R})\equiv iS_{\rm Bu}(\hat{R}) = \frac{\eta\pi}{2G}[(\theta(-\hat{R}'_{+})-\theta(-\hat{R}'_{-})\hat{R}^2+\theta(-\hat{R}'_{-})H_{\rm I}^{-2} + \theta(\hat{R}'_{+})H_O^{-2}]. \tag{2.23}$$

This formula explicitly displays the symmetry under $O\leftrightarrow I$ which implies $\hat{R}_{\pm}\leftrightarrow -\hat{R}_{\mp}$. This was also the result was obtained in [16] where an explicit parametrization was used to compute the action at a general point (i.e., neither the geometry nor the brane were at a turning point)[11] and hence it automatically had this symmetry. Since we will not have the luxury of such a parametrization in the subsequent analysis it behoves us to understand the results in terms of general properties of the integrals.

To find the bulk action at the initial point we have to put $\hat{R}=0$ in Eq. (2.23): However as argued below Eq. (2.21), $\theta(-\hat{R}'_{-})\to 0$ and $\theta(-\hat{R}'_{+})\to 0$, when $\hat{R}=R_1\to 0$.

$$I_{\rm B}(\hat{R}=0)=\frac{\eta\pi}{2G}H_O^{-2}, \tag{2.24}$$

In this case we have

$$I_{\rm B}(R_0-0)\equiv iS_{\rm B}(R_0-0) = \frac{\eta\pi}{2G}[(\theta(-\hat{R}'_{+})-\theta(-\hat{R}'_{-})\hat{R}^2+\theta(-\hat{R}'_{-})H_I^{-2} - \theta(-\hat{R}'_{+})H_O^{-2}]. \tag{2.25}$$

The expressions in Eqs. (2.22) and (2.25) differ when $\hat{R}'_{+}$ is negative.[12] As explained in detail in the Appendix this difference comes from the fact that the explicit parametrization used in [16] automatically includes the possibility of the brane being created behind the horizon (of an observer at $R=0$) in the outside dS space.

Also, following [16], the integrals appearing in the expression for $I_{\rm W}$ in Eq. (2.13) can be done analytically, which we reproduce here for further use:

$$I_{\rm W}|_{\rm tp} = -\frac{\eta}{G}\int_0^{R_0}\delta\hat{R}\,\hat{R}\left[\cos^{-1}\left(\frac{\hat{R}'_{+}}{L\sqrt{\hat{A}_O}}\right)-\cos^{-1}\left(\frac{\hat{R}'_{-}}{L\sqrt{\hat{A}_I}}\right)\right], = -\frac{\eta\pi}{4G}R_0^2\left[\frac{\epsilon(\hat{R}'_{+})}{1+|c_{+}|R_0}-\frac{\epsilon(\hat{R}'_{-})}{1+|c_{-}|R_0} + 2(\theta(-\hat{R}'_{+})-\theta(\hat{R}'_{-}))\right], \tag{2.26}$$

where $\epsilon(\hat{R}'_{\pm})$ refer to the sign of $\hat{R}'_{\pm}$.

Therefore, for general dS (and AdS) transitions the brane contribution being local is finite and can be computed explicitly:

$$I_{\rm W}(R_0-0)=\frac{\eta}{G}\left[\frac{\pi}{2}R_0^2(\Theta(-\hat{R}'_{-})-\Theta(-\hat{R}'_{+})) + \frac{\pi}{4H_I^2}\epsilon(\hat{R}'_{-})-\frac{\pi}{4H_O^2}\epsilon(\hat{R}'_{+}) + -\pi\frac{(H_O^2-H_I^2)^2+\kappa^2(H_O^2+H_I^2)}{8\kappa H_O^2H_I^2}R_0\right]. \tag{2.27}$$

The total turning point action difference between the turning point R_0 and 0 is then given by adding Eqs. (2.25) and (2.27).

The transition probability is then determined by $\mathcal{P}\propto e^B$ with $B=2([I_{\rm B}+I_{\rm W}](R_0-0))$, giving

$$B=-\frac{\eta\pi}{G}\left\{\frac{\{(H_O^2-H_I^2)^2+\kappa^2(H_O^2+H_I^2)\}R_0}{4\kappa H_O^2H_I^2} - \frac{1}{2}(H_I^{-2}-H_O^{-2})\right\}. \tag{2.28}$$

Note that R_0 and $I_{\rm B}$ and $I_{\rm W}$ are symmetric under the exchange $H_I\leftrightarrow H_O$ even though the transition probability

[11]In fact with the explicit parametrization used in that paper putting the geometry at a turning point implied that the brane was also at a turning point. In other words the equation above is valid only at the turning points. This is a consequence of the relation $1-H_{I,O}^2a^2=\frac{1-\hat{R}^2/R_0^2}{1-c_{-,+}^2\hat{R}^2}$ (with the c constants), which shows that when the geometry is at a turning point $a=H^{-1}$ then so is the brane, i.e., $\hat{R}=R_0$. Furthermore it shows that at $\hat{R}=0$ $a=0$, which is allowed since under the barrier dS space is a Euclidean four space.

[12]Note however that if $\hat{R}'_{+}$ is negative then the last term of Eq. (2.23) is zero and the difference of the turning point actions is the same as before.

e^{-B} is not symmetric. Therefore the only difference between up and down tunneling comes from the action at $\hat{R}=0$, which is equal to the corresponding (η times the) entropy. Hence (choosing $\eta=+1$)

$$\mathcal{P}_{\rm up}=e^{(\mathcal{S}_{\rm fv}-\mathcal{S}_{\rm tv})}\mathcal{P}_{\rm down}, \tag{2.29}$$

where $\mathcal{S}_{\rm fv}$ and $\mathcal{S}_{\rm tv}$ refer to the entropies of the false vacuum (higher Λ) and true vacuum (smaller Λ), respectively [where the entropy for dS space is taken to be $\mathcal{S}=\pi/(GH^2)$]. So this is the statement of detailed balance and corresponds to the choice of Hartle-Hawking (HH) boundary conditions as discussed in [16]. If on the other hand we had used the tunneling boundary conditions of Vilenkin and Linde $\eta=-1$, we would have had to put (effectively) the coefficient a in Eq. (2.10) to zero. In this case we would have had the inverse of detailed balance. This is essentially because the wave function for tunneling to a dS space from nothing is $e^{-S/2}$ rather than $e^{+S/2}$ as we observed earlier. We speculate that this difference is due to the fact that the HH wave function is a superposition of outgoing and incoming waves, so it is more appropriate for analysis in terms of equilibrium thermodynamics than the tunneling wave function which is purely outgoing. For more details see [16].

B. dS to Minkowski

For a dS to dS transition corresponding to horizons H_O and H_I, respectively, we can take the limit of $H_I\to 0$ to obtain the dS to Minkowski transition. Expanding in powers of $\varepsilon=H_I/H_O$ the turning point radius from Eq. (2.17) is

$$R_0\simeq\frac{2\kappa}{H_O^2+\kappa^2}\left[1+\frac{H_O^2(H_O^2-\kappa^2)}{(H_O^2+\kappa^2)^2}\varepsilon^2\right]. \tag{2.30}$$

Plugging this into the equation for B, Eq. (2.28), and setting the limit $\varepsilon\to 0$ we get

$$B=-\frac{\eta\pi}{2GH_O^2}\left[\frac{\kappa^4}{(H_O^2+\kappa^2)^2}\right], \tag{2.31}$$

which is the well-known dS to Minkowski result [5,16,18,19]. Note however that this result is non-trivial since it required a cancellation between two divergent terms proportional to $1/\varepsilon^2$. In the up-tunneling case this cancellation does not occur and it can be seen as the source of the general belief that up tunneling from Minkowski is forbidden. We will identify the source of this divergence and challenge this belief later on.

C. dS to AdS

Transitions among AdS spaces have been studied in less detail in the Hamiltonian approach. They can be treated in a similar way as the dS transitions changing appropriately the signs of H^2 in the original Eqs. (2.17)–(2.20). Here we will study them in more detail with the goal of writing explicit expressions for the transition rates. Let us start with dS to AdS transitions.

Let us get back to the general formulas for dS to dS tunneling, Eq. (2.11), that were derived using the FMP formalism in De Alwis-Muia-Pasquarella-Quevedo (see also [19]).

On the turning point geometry $\pi_L=0$ the first term vanishes and the second term is nonzero only when the argument of the inverse cosine is -1. This means that the bulk integral is given by

$$S_{\rm B}=\frac{i\eta\pi}{G}\left[\int_0^{\hat{r}}dr\frac{dR}{dr}R\theta(-R')+\int_{\hat{r}}^{r_{\max}}dr\frac{dR}{dr}R\theta(-R')\right], \tag{2.32}$$

$$=\frac{i\eta\pi}{2G}[(\hat{R}^2-R^2(b))\theta(-\hat{R}'_-)+(R^2(c)-\hat{R}^2)\theta(-\hat{R}'_+)]. \tag{2.33}$$

Here we have defined $r=b$ to be the lowest value of the parameter for which R' is negative and c is the highest value of the parameter for which R' is negative. Note that the sign of R' in the first line (and hence its parametrization) is determined by continuity and the sign of $\hat{R}'_\pm$, which is determined by the matching conditions and fixed by the geometry on either side of $\hat{r}$. It should be also be noted that if the geometry on either or both sides has horizons then $R(b)$ should be replaced by the solution (horizon) to $A_I=0$ and $R(c)$ by the solution to $A_O=0$.

In the current case dS has a horizon $R_{\rm D}^2=H_{\rm dS}^{-2}$ while AdS has no horizon. Let us consider the transition $A\to B$ where A is de Sitter and B is AdS. Thus (recall that $H^2\equiv\frac{8\pi G\Lambda}{3}$ is positive for dS and negative for AdS), $A_O=1-H_A^2R^2$ and $A_I=1-H_B^2R^2=1+|H_B^2|R^2$ and $\hat{R}_\pm=\frac{1}{2\kappa}(H_A^2+|H_B|^2\mp\kappa^2)\hat{R}$. The last equation implies that $\hat{R}'_-/L>0$. Thus the first term in Eq. (2.16) is zero. Also the step function in the integral requires R to be a decreasing function of r to contribute, and there is internal horizon in empty dS (no black hole) $R(r_{\max})=0$. The latter region of course gives no contribution to the integral. Hence we have [note that from Eq. 4, $\hat{R}'_+$ is negative and remains negative in the limit $M\to 0$ for all $R\geq 0$]

$$S_{\rm B}(\hat{R})=\frac{i\eta\pi}{2G}(0-\hat{R}^2),\qquad S_{\rm B}(0)=\frac{i\eta\pi}{2G}(0). \tag{2.34}$$

Note that a potential divergence (since AdS is noncompact) is averted since the step function in the first term of Eq. (2.16) is zero. Subtracting the second equation from the first and adding the wall contribution we thus get

$$B^{\mathrm{dS}\to\mathrm{AdS}} = \frac{\eta\pi}{G}\left\{\frac{\{(H_A^2+|H_B^2|)^2+\kappa^2(H_A^2-|H_B^2|)\}R_{\mathrm{o}}}{4\kappa H_A^2|H_B^2|} - \frac{1}{2}(H_A^{-2}+|H_B^{-2}|)\right\}, \tag{2.35}$$

with R_0 given by Eq. (2.17) with the above substitution $H_B^2 \to -|H_B^2|$. Also in this case there is no constraint on the tension κ. Note that this formula is in agreement with that calculated by Brown and Tetelboim [29] using Euclidean methods. It is in fact exactly the same as the dS to dS formula with $H_B^2 \to -|H_B^2|$.

As in the case of dS to dS the configuration after the transition is actually the patching together of the original dS with an AdS space separated by a wall. The latter will however collapse if any matter is introduced as argued in [5]. After the collapse we will be left with a segment of dS space bounded by an end of the world brane. On the other side of the brane there is no geometry left and is equivalent to Witten's bubble of nothing [30].

D. AdS to AdS

The AdS to AdS transitions can be analyzed similarly to the previous ones. However for transitions between AdS states there is a constraint that needs to be satisfied to guarantee that the turning point radius R_0 is real. In this case $H_I^2 = -|H_I|^2 < 0$, $H_O^2 = -|H_O|^2 < 0$. Now we have, from Eq. (2.17),

$$\frac{1}{4}\left(\frac{1}{\kappa}(-|H_O^2|+|H_I^2|)-\kappa\right)^2 > |H_O^2|,$$

i.e.,

$$\kappa < \left|\sqrt{|H_I^2|}-\sqrt{|H_O^2|}\right| \quad \text{or} \quad \kappa > \left|\sqrt{|H_I^2|}+\sqrt{|H_O^2|}\right|. \tag{2.36}$$

In this case,

$$S_B[\hat{R}] = \frac{\eta\pi}{2G}[(\hat{R}^2-R^2(0))\theta(-\hat{R}'_-)+(R^2(\infty)-\hat{R}^2)\theta(-\hat{R}'_+)]. \tag{2.37}$$

Also from Eq. (2.20) (with $H^2 \to -|H^2|$) we see that for down tunneling $|H_I|^2 > H_O|^2$, $\hat{R}'_-$ and $\hat{R}'_+$ are both positive for small κ, so we get $S_B[\hat{R}] = S_B[0]$ and therefore the bulk contribution vanishes and the total rate comes from the wall contribution: $I_{\mathrm{tot}} = I_{\mathrm{W}}$.

For down tunneling we then have $B = 2I_{\mathrm{W}}$:

$$B = -\frac{\eta\pi}{2G}\left[\frac{(|H_I^2|-|H_O^2|)^2-\kappa^2(|H_I^2|+|H_O^2|)}{2\kappa|H_I^2||H_O^2|}R_0 - \left(\frac{1}{|H_O^2|}-\frac{1}{|H_I^2|}\right)\right]. \tag{2.38}$$

Even though this looks very similar to the dS to dS transition case note that the sign differences and the fact that the bulk contribution vanishes make a major difference. In particular note that for the up tunneling we have to change $|c_+|$ to $|c_-|$ in Eq. (2.27) but also the signs of $\hat{R}'_\pm$ are interchanged and therefore the amplitude does not change. This means that

$$\mathcal{P}_{\mathrm{up}}^{\mathrm{AdS}\to\mathrm{AdS}} = \mathcal{P}_{\mathrm{down}}^{\mathrm{AdS}\to\mathrm{AdS}}. \tag{2.39}$$

This is a new result and this relation is still trivially consistent with detailed balance if we assign zero entropy to AdS.

E. Minkowski to AdS

First let us look at the expression for the bubble radius Eq. (2.17). Taking the limit $H_0 \to 0$ we get after putting $H_I^2 \to -|H_I^2|$

$$R_0 = \frac{2\kappa}{|-|H_I^2|+\kappa^2|}\left(1-\frac{H_0^2(\kappa^2+|H_I^2|)}{(-|H_I^2|+\kappa^2)}+O(H_0^4)\right).$$

Now since $R_0 \geq 0$ one should take $R_0 = 2\kappa/|\kappa^2-|H_I|^2|+O(H_0^2)$, but FMP ruled out the case $\kappa > |H_I|$ (in this case it turns out that in the limit $H_0 \to 0$ the tunneling exponent B diverges), so let us focus on the case $|H_I| > \kappa$. Taking the limit $H_0 = H_A \to 0$ in (2.35) with $|H_B|^2 = |H_I|^2$ we get for the tunneling exponent,

$$B = 2(I_{\mathrm{tot}}|_{\mathrm{tp}}-\bar{I}) = -\frac{\eta\pi}{2G|H_I|^2}\left[\frac{2\kappa^4}{(|H_I|^2-\kappa^2)^2}\right], \tag{2.40}$$

in agreement with [5,18].

Let us look at this in stages, separating the bulk and boundary (wall) terms. First we note that, for $|H_I|^2 > \kappa^2$, $\hat{R}'_\pm > 0$. Thus we have $I_{\mathrm{B}} = \frac{\eta\pi}{2G}\frac{1}{H_0^2}$ and $\bar{I} = \frac{\eta\pi}{2G}\frac{1}{H_0^2}$ so that $I_{\mathrm{B}} - \bar{I} = 0$, hence the bulk contribution vanishes after background subtraction and the transition probability is fully determined by the wall contribution I_{W} which in this limit is

$$\frac{1}{2}B = I_{\mathrm{W}} = -\frac{\eta\pi}{4G|H_I|^2}\left[\frac{2\kappa^4}{(|H_I|^2-\kappa^2)^2}\right], \tag{2.41}$$

in agreement with Eq. (2.40) as expected.

Let us note the following issue regarding these transition probabilities. We have that $\mathcal{P}(\mathcal{M}\to\mathrm{AdS}) \sim e^{-|B|}$. This is the fact that the relative probability appears to tend to unity the deeper the AdS minimum is, i.e., in the limit $|H_I|^2 \to \infty$. Of course the effective field theory breaks down before this, i.e., for $|\Lambda|^{1/4} \lesssim M_P$ (or the string or

Kaluza-Klein scale if the theory is compactified string theory). However given our previous result for up and down tunneling between AdS spaces, and taking the initial AdS space to be in the Minkowski limit, there is still a dynamic equilibrium between Minkowski and AdS spaces, since the up-tunneling probability is still equal to the down-tunneling one.

F. AdS to dS/M

In order to avoid potential problems with the parametrization in this case, one needs to consider it as up-tunneling to dS (A) from an AdS black hole in the limit $M \to 0$ (B). The latter is essentially the same as that studied by FMP. Since both Minkowski and AdS have no horizon the calculation in the Appendix (which is a reproduction of the FMP one) for the small mass case (iii) applies and so from Eq. (A4) (setting $R_1 = R_s = M = 0$),

$$S_{\rm Bu}(R_2 = R_0) - S_{\rm Bu}(R_1 = 0) = -\frac{i\eta\pi}{G}(H_A^{-2}). \quad (2.42)$$

In this case there is no constraint on the tension κ and adding the wall term we get

$$B^{\rm AdS\to dS} = \frac{\eta\pi}{G}\left\{\frac{\{(|H_B^2| + H_A^2)^2 + \kappa^2(-|H_B^2| + H_A^2)\}R_{\rm o}}{4\kappa|H_B^2|H_{\rm A}^2} + \frac{1}{2}\left(\frac{1}{H_A^2} - \frac{1}{|H_B^2|}\right)\right\}, \quad (2.43)$$

with R_0 again given by Eq. (2.17) with the substitution $H_B^2 \to -|H_B^2|$.

For $|H_B^2| > |H_A|^2$ and small κ the factor in parentheses in the expression above for B is positive, so choosing $\eta = -1$ we get an exponentially suppressed tunneling probability and hence an exponentially enhanced lifetime and so gravitational collapse is exponentially more likely than tunneling to dS. However this depended on the choice of $\eta = -1$ which is not what one chose for the dS to dS case, where the issue was settled (as discussed in Sec. 3 of [16]), by arguing that this choice (which corresponds to the HH wave function rather than the tunneling one), gives the dominant contribution to the wave function (and indeed was consistent with detailed balance). Here we cannot make the same argument since that calculation depended crucially on the compactness of the spatial sections of dS.

On the other hand detailed balance holds (see below) as in the dS to dS case for the $\eta = 1$ case. In this case this quantum transition is exponentially more probable than the gravitational collapse of AdS. Then we have a situation where the AdS can tunnel to a configuration of AdS separated by a wall/brane from a dS space with the AdS eventually collapsing leaving behind a dS bounded by a end of the world brane.

However it should be noted that the "Minkowski" limit $H_A \to 0$ is in fact divergent $B^{\rm AdS\to dS} \to \frac{\eta\pi}{2G}\frac{1}{H_A^2} \to \pm\infty$. This is to be expected since the limit is taken from the amplitude for transition to a dS space whose horizon and hence entropy diverges as the dS radius goes to infinity. This is in contrast to the corresponding up tunneling from AdS to a Minkowski space that is the limit of the AdS radius going to infinity. Note that this limit has the same topology as M in contrast to the infinite radius limit of dS which still has the topology of a sphere.

G. From nothing and back?

In an interesting article, based on concrete cases of flux compactifications in 6D, Brown and Dahlen [21] have suggested interpreting "nothing" as the infinitely curved AdS space (to which their flux compactified 6D theory decays to). The argument is very intuitive and explicit since they consider how the minimum of the scalar potential gets reduced by reducing the quantized fluxes and the corresponding geometry gets closer and closer to the bubble of nothing geometry of Witten [30] until it reaches it in the limit of $-\infty$ cosmological constant. This interpretation is very appealing since it unifies the two concepts of nothing (the bubble of nothing and the creation from nothing). Also from the AdS/CFT interpretation, an infinite AdS curvature corresponds to a vanishing central charge which would imply zero degrees of freedom and fits well with the concept of nothing.

This interpretation is actually consistent with the minisuperspace "nothing" that was the starting point for the "no-boundary" wave function HH or the tunneling wave function of Vilenkin and Linde tunneling from nothing. However their argument that up tunneling from AdS to dS/M is prohibited (based on the noncompactness of the spatial sections of AdS)[13] is not necessarily valid since as we saw earlier the FMP bulk contribution is zero at the turning points so that the tunneling amplitude is actually finite. To see this let us take the limit $|H_B^2| \to \infty$ first in Eq. (2.17) (with $H_O^2 \to -|H_B^2|$, which gives $R_0 \to 2\kappa/|H_B^2|$) and then substituting in Eq. (2.43) we get

$$B^{\rm AdS\to dS} \to \frac{\eta\pi}{G}\left\{\frac{\{(|H_B^2|)^2\}2\kappa/|H_B^2|}{4\kappa|H_{\rm B}^2|H_A^2} + \frac{1}{2}\left(\frac{1}{H_A^2} + 0\right)\right\} = \frac{\eta\pi}{2G}\frac{1}{H_A^2}.$$

That is if we define as nothing the limit of AdS with $|H_B| \to \infty$. We get

$$B^{\rm Nothing\to dS} = \frac{\eta\pi}{2G}\frac{1}{H_A^2}. \quad (2.44)$$

[13]In any case the argument depended on not including the Gibbons-Hawking regulator term as in the Euclidean arguments mentioned earlier.

This is precisely the (log of the) Hartle-Hawking (for $\eta = +1$) or the Vilenkin-Linde (for $\eta = -1$) tunneling factor for creating a universe from nothing.

Thus, we agree with the proposal of [21] to identify the two definitions of nothing, the limit of infinite curvature AdS as representing the bubble of nothing and the nothing of Vilenkin or Hartle-Hawking regarding the wave function of the Universe interpretation as creation from nothing. But contrary to the claim of [21] in which creation from nothing does not happen, we can reproduce the tunneling from nothing picture by interpreting nothing as deep AdS as they did. It is interesting to note that even though the bubble radius goes to zero in this limit (which normally would have been interpreted as signalling the absence of tunneling) the singularity in $B/2$ cancels resulting in a finite tunneling probability.

We may question the validity of taking the limit $|H_B| \to \infty$ since the EFT is only valid up to energies smaller than the Planck mass. But we can reproduce this result as the leading term in an expansion in powers of $\varepsilon^2 = H_A^2/|H_B|^2$ and $\delta^2 = \kappa^2/|H_B|^2$ with $\varepsilon, \delta \ll 1$ but still keeping $|H_B| \leq M_P$.

1. Detailed balance in dS/AdS transitions

The results of the above subsections shows that detailed balance holds for dS to and from AdS transitions provided we assign zero entropy to empty anti–de Sitter space (as one should expect given that empty AdS has no horizon). In this case we have

$$\frac{P^{\mathrm{AdS}\to\mathrm{dS}}}{P^{\mathrm{dS}\to\mathrm{AdS}}} = \frac{e^{B^{\mathrm{AdS}\to\mathrm{dS}}}}{e^{B^{\mathrm{dS}\to\mathrm{AdS}}}} = \frac{\exp\left(\frac{\eta\pi}{2G}\frac{1}{H_A^2}\right)}{\exp\left(-\frac{\eta\pi}{2G}\frac{1}{H_A^2}\right)} = e^{\eta(S_{\mathrm{dS}}-(S_{\mathrm{AdS}}=0))}, \tag{2.45}$$

which is the statement of detailed balance (taking $\eta = +1$). Note that the result holds also in the limit $H_B \to 0$ which is the Minkowski limit of AdS so that detailed balance holds for transitions between dS and Minkowski spacetimes provided of course that we assign zero entropy to empty Minkowski space.

2. Comparison to Euclidean methods

In the Euclidean calculations (Cdl and BT) it appears that up tunneling from AdS (or M) to dS is forbidden. However that is because in those calculations the background action is taken to be proportional to the volume of AdS space—which is infinite. However as we see from Eq. (2.43) in the WKB calculation the initial point is $\hat{R} = 0$ so we had $S_{\mathrm{B}}(0) = 0$ rather than infinity. This is also consistent with taking the limit of the black hole mass to zero in the Schwarzschild anti-de Sitter (SAdS) to dS calculation of Appendix A 2.

The difference in fact corresponds to the different assignments of entropy to AdS space. Our calculations (both the direct AdS to dS and the SAdS to dS) effectively assigned zero entropy to empty AdS space. The Euclidean calculation on the other hand effectively assigned infinite entropy to empty AdS.

III. TRANSITIONS FROM BLACK HOLE BACKGROUNDS

Let us now consider the most general situation allowed by spherical symmetry, namely that the solution to Einstein's equations include a mass parameter M that appears as an integration constant in the solution of the Hamiltonian constraints that corresponds to a black hole mass. Since the Schwarzschild solution implies the existence of a horizon considering $M \neq 0$ makes an important difference.[14]

A. SdS to SdS

As discussed above, the classical turning points for the geometry occur at $\pi_L = 0$, i.e.,

$$\frac{R'^2}{L^2} = A(R) = 1 - \frac{2MG}{R} - H^2R^2.$$

When $r = \hat{r}$ these are the turning points for the brane, i.e., the solutions of $V = -1$. For $3\sqrt{3}GM < H^{-1}$ the geometry has two horizons $R_b < R_c$. We may identify R_s as the black hole (Schwarzschild) horizon (becoming $2GM$ when $H \to 0$) while R_c is the cosmological horizon (becoming H^{-1} when $M \to 0$). We have

$$A = -\frac{H}{R}(R - R_-)(R - R_b)(R - R_c),$$
$$R_- < 0 < 2GM < R_b < 3GM < R_c.$$

We also see from the turning point equation $V = -1$ that

$$R_S < R_1 < R_2 < R_D. \tag{3.1}$$

For the turning point geometries the bulk action S_B simplifies with the first term in Eq. (2.11) giving zero and the second term contributes only when $\epsilon(R') = -1$, i.e., whenever $\cos^{-1}(\frac{R'}{L\sqrt{A_{\mathrm{I,O}}}}) = \pi$. Thus we have

$$iS_{\mathrm{B}}(\hat{R}) = -\frac{\eta\pi}{G}\left[\int_0^{\hat{r}} drR'R\theta(-R'_-) + \int_{\hat{r}}^{r_{\max}} drR'R\theta(-R'_+)\right],$$
$$= \frac{\eta\pi}{G}\left[\int_0^{\hat{R}} dRR\theta(-R'_-) + \int_{\hat{R}}^{R(r_{\max})} dRR\theta(-R'_+)\right]. \tag{3.2}$$

The general dS-black hole case is complicated. It is discussed in detail in the Appendix—see Appendix A 2.

[14] Note that in following the Hamiltonian approach, which at this point is amenable to computation only in the extreme thin wall (brane) approximation, we have nothing to add to the current debate regarding the impact of black holes in Higgs decay [31–34].

For the small black hole case the direct calculation in a static patch between the two horizons gives

$$I_{\rm Bu}[\hat{R}] = \frac{\eta\pi}{2G}[(\theta(-\hat{R}'_{+}) - \theta(-\hat{R}'_{-})\hat{R}^2 + \theta(-\hat{R}'_{-})(R^2_{I,D}) - \theta(-\hat{R}'_{+})(R^2_{O,S})]. \quad (3.3)$$

Hence we get after adding the wall contribution [see Eq. (A15)]

$$I^{AB}[R_2 - R_1] = \frac{\eta\pi}{2G}[(R^B_{\rm D})^2 - (R^A_{\rm S})^2] + I^{AB}_W[R_2 - R_1]. \quad (3.4)$$

Let us now consider the transitions $A \to B$ (with $S_A > S_B$) that we may call up tunneling and $B \to A$) we call down tunneling. Noting that the wall action I_W is symmetric under the interchange of A and B.[15] Hence we get

$$\frac{P_{\uparrow}}{P_{\downarrow}} = e^{\frac{\pi}{G}[(R^2_{B,D} - R^2_{A,S}) - (R^2_{A,D} - R^2_{B,S})]} = e^{\mathcal{S}_B - \mathcal{S}_A}. \quad (3.5)$$

Defining the total entropy of SdS space as $S \equiv \frac{\pi}{G}(R^2_D + R^2_S)$ (i.e., the sum of the cosmological and black hole horizon entropies). Thus we have nontrivially obtained detailed balance again.

In the limit $R_{A,BS} \to 0$ we recover the earlier results for dS to dS transitions. One may think that to get FMP/FGG we need to take $R_{A,S} \to \infty$. This would give $P_{\uparrow} \to 0$ in agreement with the corresponding limit in the dS to dS case. The FMP/FGG case however corresponds to subtracting the infinity in the horizon area term of dS/SdS when the dS radius goes to infinity to get the entropy of the black hole in asymptotically Minkowski space to be just the black hole entropy.

The difference comes from the fact that in the asymptotically Minkowski case Gibbons and Hawking [9] added a term (to the boundary term that is necessitated by Dirichlet boundary conditions) evaluated in flat space which cancels the otherwise infinite contribution of the Gibbons-Hawking-York (GHY) term in the asymptotic limit. This infrared subtraction is what gives the entropy of the black hole to be πr_s^2 and the entropy (as well as the Arnowitt-Deser-Misner energy) of flat Minkowski space to be zero.[16] This is a physical requirement. In other words the infinite radius limit of dS space is not flat Minkowski space any more than a topological three sphere of arbitrarily large radius is the same as R^3. The topology of dS is $R \times S_3$ and however large its radius, is not R^4. Thus the above formulas—although valid for SdS spaces of arbitrary radii, do not imply [from the vanishing of Eq. (3.5)] that they forbid up tunneling from asymptotically Minkowski space.

In fact it has been argued by many authors (perhaps the earliest were [17,36,37]) that the horizon entropy of dS space is the maximum entropy that this space can hold. With this interpretation then, in the limit of the horizon radius going infinity, the ensuing infinite entropy of Minkowski space, should also be interpreted as the maximum entropy that this space can contain, which of course is reasonable. It is not therefore the entropy of the Minkowski vacuum, which should indeed be zero.

As we will discuss soon, the same is true of SAdS to SdS (or dS) since the entropy of SAdS is simply the entropy of the black hole since AdS has no cosmological horizon.

However for large and intermediate black hole masses (as defined in Appendix A 2) we do not have detailed balance. This is also the situation for the FMP case reviewed in the Appendix.

B. Schwarzschild (A)dS to Schwarzschild (A)dS transitions

Let us now consider the generic case in which the mass parameters $M_{\rm O}$ and $M_{\rm I}$ are nonvanishing. We will then reconsider the forbidden uplifting configurations and illustrate that in the case $M_{\rm O} \neq 0$ some forbidden transitions become possible. The main reason for this change is the existence of a horizon and the fact that there are two rather than one turning points.

1. Black hole to dS—the FGG transition

We will start reviewing the original up-tunneling proposal of FGG [12] and reconsidered in [18] in which the background is asymptotically flat Schwarzschild black hole.

The brane tunnels through the potential and the probability of transition from the black hole state to the one with a dS space is given essentially by the relative probability for being at the turning point R_2 to that of being at the turning point R_1, i.e.,

$$\mathcal{P}(BH \to \mathcal{N}) = \frac{|\Psi(R_2)|^2}{|\Psi(R_1)|^2}, \quad (3.6)$$

with $R_{2,1}$ being the right and left turning points.

In the FGG/FMP process we have a transition from a Schwarzschild black hole to a de Sitter space (adjoined to the hole with a brane at the junction). The integrals in Eq. (2.13) for $H_{\rm O} = 0$, but $M_{\rm O} \neq 0$ cannot be done analytically but we can approximate them for a small

[15] While the bulk action is symmetric only for a general point and loses this symmetry at the turning points $R_{1,2}$ the wall action clearly has this symmetry even at these points as is seen from Eq. (A40).

[16] In [35], page 283, it is stated the GHY term for Minkowski is negative infinity and is zero in the absence of the GHY term. However it should be noted that the GHY term consists of two pieces and the second piece is explicitly included to make the Minkowski vacuum action (and hence its entropy) zero. A similar term should be included if the AdS vacuum is to be assigned zero entropy since it has no horizons. This means that the entropy of both a Minkowski black hole and an AdS black hole is just the horizon entropy of the black holes.

black hole mass. The important point is that now the background contribution is determined not by the entropy of dS but by the black hole entropy:

$$\bar{I} = S_{\rm BH} = \frac{\pi R_S^2}{G} = 4\pi M_{\rm O}^2. \tag{3.7}$$

Therefore for a small mass black hole $M_{\rm O} < M_D \equiv \frac{H^2-\kappa^2}{2GH^3} \simeq \frac{M_P^2}{H}$, we can neglect the $M_{\rm O}$ dependence in the integrals in Eq. (2.13) and with the background $\bar{I}$ given as above we have

$$\mathcal{P}({\rm BH} \to {\rm dS}) \simeq e^{\frac{\pi}{G}(R_{\rm dS}^2 - R_{\rm Schwarzchild}^2)+2I_{\rm brane}} \simeq e^{S_{\rm dS}-S_{\rm BH}} \simeq e^{S_{\rm dS}} \neq 0. \tag{3.8}$$

The important point is that, contrary to the limit $H_{\rm O} = 0$ for up tunneling dS to dS, which has a vanishing transition rate, now the transition rate from Schwarzschild black hole to dS is nonvanishing even for small values of $M_{\rm O}$.

Note here the entropy of the final state is much higher than the black hole entropy, so this is consistent with increasing entropy.

Let us contrast the situation here with that obtained earlier of dS to dS transitions. In [16] we have argued that the FGG/FMP starting configuration is a state with a definite energy (that of the black hole) and that the thermodynamic entropy of the initial state should be identified with the log of the degeneracy of the Hilbert space of the black hole and the background flat space being treated as the Minkowski vacuum identified as a unique state. In contrast to this in the dS case it has been argued that its entropy is maximal and indeed is supposed to correspond to the dimension of the Hilbert space that can be accommodated on the horizon. The latter obviously goes to infinity at infinite radius but this should not be confused with the entropy of empty Minkowski space.

C. Schwarzschild AdS to dS

For Schwarzschild AdS to dS we follow the same steps as FMP did for Schwarzschild to dS. The two turning points R_1, R_2 for which $V = -1$ are in between the dS radius $R_{\rm dS} = 1/H_I$ and the Schwarzschild radius $R_{\rm S}$ that is determined by solving the cubic equation:

$$H_{\rm O}^2 R_{\mathcal{S}}^3 + R_{\mathcal{S}} - 2GM = 0, \tag{3.9}$$

namely

$$R_{\mathcal{S}} \le R_1 < R_2 \le R_{\rm dS}. \tag{3.10}$$

This can be easily seen by the conditions coming from the expression of the potential at the turning points $\hat{A}_I = 1 - H_I^2 R^2 > 0$ implying $R_{2,1} \le R_{\rm dS}$ and $\hat{A}_O = 1 – 2GM/R + H_{\rm O}^2 R^2 > 0$, which by looking at the coefficients determining the single root of the cubic, it can seen that it implies $R_{\mathcal{S}} \le R_{1,2}$.

Also to define the domains for the integration parameter M we compute it first for the case $R'_+ = 0, V = -1$, which implies gives the value for M:

$$M = M_{\rm S} = \frac{H_{\rm O}^2 + H_{\rm I}^2 + \kappa^2}{2G(H_{\rm I}^2 + \kappa^2)^{3/2}}, \quad \text{for } R'_+ = 0, V = -1. \tag{3.11}$$

For the case $R'_- = 0, V = -1$ we get

$$M = M_{\rm D} = \frac{H_{\rm O}^2 + H_{\rm I}^2 - \kappa^2}{2GH_{\rm I}^3}, \quad \text{for } R'_- = 0, V = -1. \tag{3.12}$$

Note that for $H_{\rm O} = 0$ this reduces to the FMP results as it should. It is also easy to prove that, as in the FMP case, $M_{\rm D} \le M_{\rm S}$.

Therefore we have the same situation as in the Schwarzschild to dS transition in which the bulk contribution to the transition rate is determined by

$$I_{\rm B}|_{\rm tp} \equiv I_{\rm B}|_{R_1}^{R_2} = \begin{cases} \frac{\eta\pi}{2G}(R_2^2 - R_1^2), & M > M_{\rm S} \\ \frac{\eta\pi}{2G}(R_2^2 - R_{\mathcal{S}}^2), & M_{\rm S} > M > M_{\rm D} \\ \frac{\eta\pi}{2G}(R_{\rm dS}^2 - R_{\mathcal{S}}^2) & M_{\rm D} > M \end{cases}. \tag{3.13}$$

The relevant figure for this is the same as in the FMP case, i.e., Eq. 4 in the Appendix, since both Minkowski and AdS have no cosmological horizons.

As in FMP we are interested in the latest case $M_{\rm D} > M$ to take the small M limit. As in there we may take the limit $M \to 0$.[17]

So we have explicitly a nonzero transition rate from AdS black hole to dS, which is interesting by itself. The interesting questions to ask is how the whole transition rate depends on the values of the parameters $M, H_{\rm I}, H_{\rm O}, \kappa$. In particular if it prefers transitions to smaller or higher values of $H_{\rm I}$ for a fixed $H_{\rm O}$ or vice versa. Also to analyze the transition rate in the extreme cases $H_{\vec{m}{\rm I,O}} \to M_p$ from below and $M \to M_p$ from above. This should be done numerically combining the bulk and the wall contributions to the transition rate.

D. Vacuum transitions in quantum gravity and thermodynamics

In this subsection we address the interesting points made in [15] regarding the viability of the FGG/FMP process. Susskind argues that the FGG/FMP process should be forbidden since it violates what he calls the central dogma

[17]This limit is well defined as long as we stay away from the horizon. The squared curvature tensor is $\propto (GM)^2/R^6$, which goes to zero as $M \to 0$ unless $R = 2GM$.

of gravitational systems with horizons—specifically to de Sitter–like geometries.

As seen from a causal patch a cosmological spacetime can be described in terms of an isolated quantum system with Area$/4G$ degrees of freedom, which evolves unitarily under time evolution.

In the next paragraph of that paper the author assumes that an observer in a causal patch sees a world of finite entropy satisfying the second law of thermodynamics.

Now, while unitarity of the S matrix can be shown to lead to the second law for nongravitational systems (for a proof see for instance [38], chapter 3) it is far from clear (and interesting to explore) how to extend this to gravitational systems particularly transitions where spacetime itself changes. Furthermore there is the notorious problem of time in quantum gravity. So even though the first statement above holds referring to black holes in asymptotically flat spacetime (where the notion of a S matrix may be formulated) it is far from clear how these statements apply for say dS space or SdS space.

Nevertheless Susskind proceeds to conjecture that they do. Indeed as he says "The black hole version of this dogma is, with good reasons, widely accepted. Less is known about cosmological horizons, so we should consider the cosmological version to be a conjecture." Furthermore referring to the FGG process "Because this example involves following a causal patch as it falls behind a black hole horizon, there may be reasons to be less certain about the application of the cosmological dogma, but I will assume it is valid."

Unlike FGG/FMP, who discuss an eternal black hole spacetime that nucleates a spherical wall/brane whose inside is a portion of a dS space, Susskind's version, as presented in Sec. IV, assumes a background dS space in which a black hole (of mass M) is nucleated as a thermal fluctuation with (Boltzmann) probability $P_{bh} \propto e^{-M/T_A} = e^{-2\pi M R_A}$. Here $T_A = 1/2\pi R_A$ is the horizon temperature of the background dS space. So Susskind's interpretation of FGG is that they predict an up-tunneling rate for the transition $A \to B$ [where B is a dS space with smaller horizon radius ($R_B < R_A$) and hence smaller entropy]

$$\bar{\Gamma}_{\rm up} = e^{-2\pi M R_A} \Gamma_{\rm FGG}. \tag{3.14}$$

On the other hand (according to Susskind) detailed balance requires the up-tunneling rate to be

$$\Gamma_{\rm up} = e^{\mathcal{S}_B - \mathcal{S}_A} \Gamma_{\rm CDL}, \tag{3.15}$$

where $\Gamma_{\rm CDL}$ is the Coleman-DeLuccia down-tunneling rate and $\mathcal{S}_A = \pi R_A^2/G$, $\mathcal{S}_B = \pi R_B^2/G$. Since $M, \mathcal{S}_B, \Gamma_{\rm FGG}$, and $\Gamma_{\rm CDL}$ are all independent of R_A for $R_A \to \infty$ it was argued that in this regime

$$\frac{\bar{\Gamma}_{\rm up}}{\Gamma_{\rm up}} \approx \frac{\Gamma_{\rm FGG}}{\Gamma_{\rm CDL} e^{\mathcal{S}_B}} e^{\mathcal{S}_A} \sim e^{\mathcal{S}_A}, \tag{3.16}$$

in violent disagreement with detailed balance.

Several comments are in order:

(1) The Hamiltonian calculation of $\mathrm{dS}_A \to \mathrm{dS}_B$ using the FMP [18] formulation gives exactly Eq. (3.15) [16] so this quantum mechanical calculation is not in conflict with detailed balance. In this sense we address the main criticism of Susskind for the process to occur.

(2) In the presence of a black hole the issue is more complicated. The original calculation of FGG and FMP discussed the nucleation of a brane with a dS space inside it and an asymptotically flat Schwarzschild space outside. Here there are three cases [18][18] as discussed in detail in the Appendix. The first two cases [see Eqs. (A2), (A3)] clearly do not satisfy detailed balance, but the third case $M < M_D < M_S$ gives Eq. (3.15) provided we identify $\mathcal{S}_A = \pi R_S^2/G$, i.e. the entropy of a black hole in flat space as the entropy of the initial state. All this assuming also that down tunneling is given by CDL, which is only true in the limit where the black hole mass tends to zero.

(3) Actually the relevant comparison to the FMP type calculation that should be made is to an SdS to SdS transition as discussed in detail in the Appendix. Here there are also three cases with the case of small black holes giving Eq. (3.15) with $\mathcal{S}_{A,B} = \frac{\pi}{G}(R_{A,B}^2 + (R_{A,B}^{\rm BH})^2)$ with the down tunneling rate modified from CDL to incorporate the black holes. In the limit where the black hole masses go to zero this is exactly Eq. (3.15).

(4) On the other hand as in the asymptotically flat case there are two other cases (see the Appendix) corresponding to different ranges of black hole masses, which do not seem to satisfy detailed balance. The interpretation of these regimes is not clear to us.

To summarize: the Hamiltonian calculation shows that transitions between spacetimes with horizons such as dS to dS, and SdS to SdS with small mass black holes, do satisfy detailed balance and may not be in conflict with Susskind's reasoning. However there are also cases with large or intermediate mass black holes (as defined in the Appendix), that do not satisfy Susskind's central dogma. Since the dogma in the case of tunneling of spacetimes through barriers, as he himself says, may not apply, this should not be surprising. A proper understanding of the

[18]Actually there is a discrepancy between FMP and FGG here. The relevant equation in the latter [12] is Eq. (5.34) which is not quite the same as the Eqs. (48) plus (49) of [18]. In particular for $M > M_S$ FGG gives $I = I_W$ whereas FMP give $I = I_W + \frac{\pi}{2G}(R_2^2 - R_1^2)$. See the Appendix for notation.

difference of the mass ranges regarding detailed balance would be interesting.

A last comment: from holography it is expected that the empty AdS has zero entropy. To see this let us consider the eternal AdS black hole [39]. Empty AdS can be obtained by taking the mass of the black hole to zero. On doing this the radius, the area and the entropy of the black hole vanish. If there is a conformal field theory (CFT), as in the eternal AdS case, measuring the number of degrees of freedom, of the system, then these will also decrease up to the point when there is no black hole and only one state is left which means vanishing entropy. It will be interesting to further explore this issue.

E. Brane trajectory after nucleation

In this section we will study the trajectory of the brane after the nucleation. We will focus on transitions where the true vacuum is dS. Following [20] the geometry of the brane can be studied by solving the equations for the brane position $\hat{R}$. For the readers' convenience we repeat the formulas for the brane motion. The metric on the brane is

$$ds^2 = -dt^2 + \hat{R}^2(t)d\Omega^2. \tag{3.17}$$

the first (energy) integral of the equation of motion for the brane is (from now on we will drop the hat on R in this section since we are just discussing the brane motion),

$$\dot{R}^2 + V = -1, \tag{3.18}$$

where the potential may be written as

$$V = -\frac{1}{(2\kappa R)^2}((A_I - A_O) + \kappa^2 R^2) + A_I - 1. \tag{3.19}$$

In the case of interest (i.e., AdS black hole to dS),

$$A_I = 1 - H_I^2 R^2, \qquad A_O = 1 - \frac{2GM}{R} + H_O^2 R^2,$$

where H_I is the inverse of the de Sitter radius and H_O is the inverse of the AdS radius. Substituting these into Eq. (3.19) we get

$$V = -\frac{R^2}{R^{*2}} + \frac{\alpha}{R} - \frac{\beta}{R^4}, \tag{3.20}$$

where

$$\alpha = 4(H_O^2 + H_I^2 - \kappa^2)\frac{GM}{(2\kappa)^2}, \qquad \beta = \frac{(2GM)^2}{(2\kappa)^2} > 0, \tag{3.21}$$

and

$$\frac{1}{R^{*2}} = \frac{(H_O^2 + H_I^2)^2 + 2\kappa^2(H_I^2 - H_O^2) + \kappa^4}{(2\kappa)^2}. \tag{3.22}$$

Note that the equation $dV/dR = 0$ has only one real root and that $V \to -\infty$ for both $R \to 0$ and $R \to \infty$, so the potential rises from negative infinity to a maximum below zero (recall that $V < 0$ for all R) and then falls back to negative infinity just as in the Blau-Guendelman-Guth (BGG) case [20]. Depending on the value of M we have the three cases discussed earlier.

To compute the motion of the brane in (say) the dS metric inside the bubble we choose (conformal) global coordinates

$$ds^2 = \frac{1}{H_I^2 \cos^2 T}(-dT^2 + dr^2 + \sin^2 r d\Omega^2),$$
$$T_0 \le T < \frac{\pi}{2}, \qquad r_0 \le r < \frac{\pi}{2}, \tag{3.23}$$

where T_0, r_0 are the coordinate time and radius at which the bubble is nucleated. Embedding the brane metric Eq. (3.17) in the above we get (denoting $\dot{X} \equiv \frac{dX}{dt} X' = \frac{dX}{dr}$),

$$T = T(t), \qquad r = r(t), \qquad R = \frac{\sin^2 r}{H_I^2 \cos^2 T}, \tag{3.24}$$

$$\dot{T} = \frac{H_I \cos T}{\sqrt{1 - r'^2}}, \qquad \dot{r} = \frac{H_I \cos T}{\sqrt{1 - r'^2}} r'. \tag{3.25}$$

We also have

$$\dot{R} = \frac{\partial R}{dT}\dot{T} + \frac{\partial R}{dr}\dot{r} = \frac{\dot{T}}{H_I}\left(\tan T \sec T \sin r + \frac{dr}{dT}\sec T \cos r\right),$$
$$= \frac{1}{\sqrt{1 - r'^2}}(\tan T \sin r + r' \cos r).$$

Hence using Eqs. (3.18), (3.20), and (3.24) we have

$$\frac{1}{\sqrt{1 - r'^2}}(\sin T \sin r + r' \cos r \cos T)$$
$$= \sqrt{\frac{\sin^2 r}{H_I^2 R^{*2}} - \alpha H_I \frac{\cos^3 T}{\sin r} + \beta H_I^4 \frac{\cos^6 T}{\sin^4 r} - \cos^2 T}. \tag{3.26}$$

The small black hole regime may be defined as $\alpha H_I \ll 1$, $\beta H_I^4 \ll 1$. In the limit $M \to 0$ this obviously tends to the formal AdS to dS result.

Note that as $T \to \pi/2$ (i.e., in the infinite future) we get $\sqrt{1 - r'^2} = H_I R^* \neq 0$; i.e., the brane speed does not reach the speed of light, as in the BGG case.

As is now well known [40], in the case of dS/dS transition, the wall after nucleation follows a geodesic in $SO(3,1)$. This has been to argued that the spacetime after the foliation is open since it fits naturally with the picture of the wall never crossing the horizon. The argument for the wall trajectory relied heavily in the fact that the tangent acceleration in the dS/dS case is always constant. Here we show that adding a mass to the exterior vacuum changes this pictures allowing for different trajectories of the wall. First, let us note that the acceleration of the wall can be computed from the junction conditions to be [20]

$$K_{\tau\tau} = -\frac{\ddot{R} - H^2 R}{\sqrt{1 - H^2 R^2 + \dot{R}^2}}, \tag{3.27}$$

whose value can be inferred by solving the potential equation for the junction conditions in Eq. (2.5). In the case of dS/dS, the solution can be obtained analytically, $R = R_0 \cosh(t/R_0)$ and the acceleration is constant. In general when there is a mass this does not hold, and we find that the acceleration is given by

$$-K_{\tau\tau} = \frac{\frac{(1-H_0^2 R_*^2)}{R_0^2} - \frac{\alpha}{2R(\tau)^3} - \frac{2\beta}{R(\tau)^6}}{\sqrt{\frac{(1-H_0^2 R_*^2)}{R_0^2} - \frac{\alpha}{R(\tau)^3} - \frac{\beta}{R(\tau)^6}}}, \tag{3.28}$$

where we have used the equation of motion. $\dot{R}^2 + V = -1$, Expanding for small αH_I and βH_I^4, we can write the lhs as

$$\frac{\ddot{R} - H^2 R}{\sqrt{1 - H^2 R^2 + \dot{R}^2}} = \frac{\sqrt{1 - H_0^2 R_*^2}}{R_*} - \frac{R_*}{\sqrt{1 - H_0 R_*^2}} \left(\frac{\alpha}{R^3} - \frac{5}{2}\frac{\beta}{R^6} + \mathcal{O}(R_0^3 \alpha^2 / R^6) \right). \tag{3.29}$$

From this we see that for small radius the wall initially accelerates/decelerates until $R(\tau)$ grows large enough such that the acceleration asymptotes to a constant. Notice that in the case that the mass in the outer region is zero both α and β are zero in which case the acceleration is always constant as we anticipated. Another feature is that this results do not depend on the outer region being SdS or AdS and hence this is a feature of the outside region having a mass.

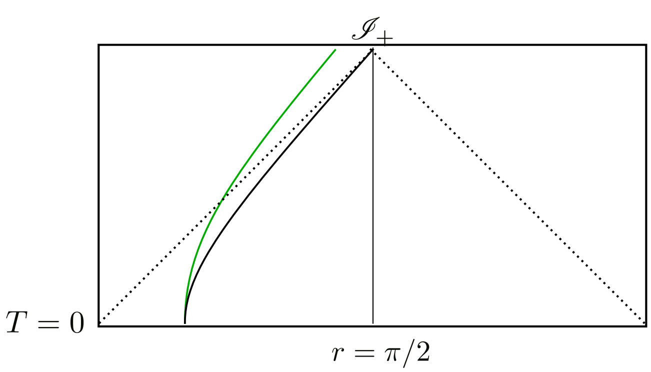

FIG. 2. Penrose diagram of the wall trajectory. The green line is the solution in the case of $\alpha = .11H$, $\beta = 0.0014H^4$, $R_* = 0.55H$. The black line corresponds to the case where $M = 0$ and $R_* = 0.55H$. Note that the green line crosses the dotted line meaning that the wall crosses the horizon of the de Sitter observer.

One of the consequences of the change of acceleration is that the wall can now enter into the comoving horizon of an observer on one of the hemispheres.

This was shown by writing the radius of the wall in global coordinates $R(T) = a(T)\sin(r(T)$ as above. We plot the solutions to Eq. (3.26), where we can see that for certain values of α the wall indeed crosses the light cone $r = T$ as can be seen from Fig. 2. Clearly this is due to the fact that the worldsheet of the trajectory only has $O(3)$ symmetry and hence it is not so restricted as in the dS/dS case. This implies that there might well be other trajectories which cross the horizon from the other hemisphere. Using Euclidean arguments [5] it was suggested that the interior region had an open de Sitter geometry. Even though it can be argued in the dS/dS case that this holds naturally since such foliation never crosses the horizon and thus the wall remains outside the light cone, we see that when including the mass this argument does not hold. From Lorentzian arguments there is no preferred foliation as the whole computation only assumes $O(3)$ symmetry (see also [40,41]). Thus in order to determine the curvature of the nucleated spacetime one would have to work out the consequences having a wall, for instance, the effects of a wall induced anisotropy.

A possible probe of the wall would be to compute the maximum correlation function that lies within the interior de Sitter spacetime. The correlation function on de Sitter for a massless field evaluated at equal times but at two different points $(\rho_1, \Omega_1, \rho_2, \Omega_2)$ is given by [42]

$$G(\rho_1, \Omega_1, \rho_2, \Omega_2) \sim H^2 \left(\frac{1}{1 - Z} - \log(1 - Z) \right), \tag{3.30}$$

where Z is the geodesic length between two points, $Z = H^2 \eta_{ab} X^a(r, \Omega) X^b(r', \Omega')$. Since we are interested in correlation functions with $O(3)$ symmetry we have that $\Omega = \Omega'$, and we get that

$$1 - Z = \sec(T)^2 (1 - \cos(r - r')). \tag{3.31}$$

From this we see that the log dominates over the first term in the correlation function. Now, when the wall lies behind the light cone the maximum value of $1 - Z$ is when $\rho - \rho' = \pi/2$ in which case the correlation function becomes

$$G(r) \sim H^2 \log(\sec(T)) \sim H^3 t, \tag{3.32}$$

where in the last line we have used that $\sec(T) = \cosh(Ht) \sim e^{Ht}$. Of course this is the usual covariance of

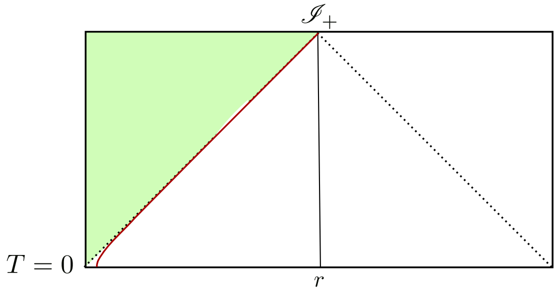

FIG. 3. Penrose diagram for the up-tunneling solution when $|H_0| \to \infty$. The light green region is the spacetime nucleated. The red line is the boundary of the space. Outside the red line there is nothing.

a massless field that grows linearly with time until the end of inflation t_e. Now in the presence of the wall, if we write $r - r' = \pi/2 - \Delta r$ for α small we get that the correlation function behaves as

$$G(r) \sim H^2 \log(\sec(T)(1-\cos r)) \sim H^2(Ht + \Delta r/2 + \mathcal{O}(\alpha^2)), \tag{3.33}$$

which means however that any effect of the wall is washed away by the expansion. Still, since inflation is not eternal there is a lower bound on which scales can be access by an observer without noticing the wall. For instance the largest distance on the cosmic microwave background can be estimated from the comoving distance to the cosmic microwave background dipole to be $\chi_L \sim .46$. This would put a lower bound on the maximum size of ρ without detecting a wall. Notice that there could still be effects from higher order correlation functions and/or on the analytical properties of the correlation function that we leave for future study.

On the other end we have the case when the false vacuum $|H_0| \to \infty$. This, we have identified as having the same transition rate that Hartle-Hawking (or Vilenkin-Linde) in Eq. (2.44). We can see from Eq. (3.22) that this limit corresponds to the turning point radius $R^* \to 0$. This happens because when $|H_0|$ grows the potential moves towards the origin becoming and narrower around $R = 0$ so that the barrier becomes infinitely thin in the limit. This explains why there is no exponential suppression in this limit.

We mentioned before that the up-tunneling transition rate is similar to tunneling from nothing. However, from the previous discussion we can see that there is a key difference. As indicated in Fig. 3 the nucleated spacetime is not the whole de Sitter but only a portion that includes the whole causal light cone of an observer on the hemisphere of the true vacuum. There is also another small region outside the light cone where the spacetime ends. This starts at $r = M_{\rm UV}$, where r is the radial direction in global coordinates and $M_{\rm UV}$ is the cutoff of the theory. The reason why the spacetime starts there is because we are still demanding the spacetime to be a solution of the junction conditions, then in the limit we are considering, the trajectory has to satisfy the geodesic equation $\cos(r) = \sqrt{1 - H_0^2 R_0^2} \cos T$. Notice that when $R_0 = 0$ the equations are singular. This is addressed by noticing that R_0 becomes asymptotically small but we still require that $R_0 \geq 1/M_{\rm UV}$ in order for the EFT to still be valid.

This solution is actually similar to the Hawking-Turok instanton [43], which is an Euclidean solution that gives rise to an open universe (see however [44]), and whose wave function is the same as in Hartle-Hawking.[19]

Notice however that our solutions differ. Our method is Lorentzian and then we do not need to consider singular instantons, moreover the solution (which does not favor one or other slicing and so admits closed slicing too) can also be interpreted as the Vilenkin-Linde wave function, although in that case, as opposed to Hartle-Hawking, it does not satisfy detailed balance.

IV. CONCLUSIONS

In this paper we have continued our program (see [16,40,46]) of reformulating vacuum transitions in the Hamiltonian framework generalizing the seminal work of FMP [18]. FMP considered Schwarzschild to dS transitions in the limit $M \to 0$. The dS to dS transition was studied in detail in [16,19]. Here we have extended these works by including explicit expressions for the transition rates in all values and signs of the cosmological constants Λ_A, Λ_B and all values of the black hole mass M. Notice that the cosmological constants are input parameters of the theory whereas the black hole mass appears as an integration constant which should be included in gravitational configurations with spherical symmetry.

We find that for generic values of these parameters the transition rates are nonvanishing, including up tunneling from AdS to dS spacetimes. These nonvanishing transition rates can give rise to a huge network of vacua connected by quantum transitions among each other. This is relevant for understanding the structure and population of the string landscape.

In the general case, the contributions to the transition rate coming from the wall is only written in terms of integrals that are only possible to solve analytically in special limits (like $M \to 0$). However the ratio of up to down tunneling is such that these contributions cancel and it gives a very simple result. This ratio allows us to verify that detailed balance is satisfied in many of the cases. In particular, due to the nature of the Hamiltonian prescription, the transition rates do not determine the sign $\eta = \pm 1$. By comparison with the standard wave function of the Universe, we claim that this sign arbitrariness corresponds to the same sign

[19]The fact that up tunneling from an infinite AdS potential corresponds to the Turok-Hawking instanton has been discussed before, e.g., [21,45], although [21] concluded that this up-tunneling transition was not possible unlike our findings here.

arbitrariness that differentiates the Hartle-Hawking and Vilenkin probability amplitudes. In this sense we find detailed balance to work for the Hartle-Hawking but not Vilenkin. This may not be surprising since the initial conditions for the Vilenkin wave function select the expanding Universe branch of the wave function of the Universe that is clearly not and equilibrium situation so it is no surprise that detailed balance is not satisfied.

Having computed the general transition rates we were able to consider different limits for Λ_A, Λ_B, and M. We find some interesting results. In particular for studying a transition from Minkowski spacetime we may describe Minkowski starting from several directions. First, by setting $\Lambda_A = 0$ to start with and taking the transition probability in the $M \to 0$ limit as FMP did. This gives the finite transition rate obtained in FMP. Second, we may start with $M = 0$ and consider the dS to dS transition and then take the limit $\Lambda_A \to 0$. This gives a zero transition probability $e^{-3/\Lambda_A} \to 0$. Third, we can start with the AdS to dS transition and take the limit $\Lambda_A \to 0$, which in this case reproduces the nonvanishing FMP result, which is reassuring.

We argued that the $\Lambda_A \to 0$ limit is the correct definition of the Minkowski vacuum, if we start from AdS rather than from dS. In some sense AdS is closer to Minkowski since both are noncompact and horizonless. Actually the SdS to SdS transitions require a more detailed analysis as discussed in the Appendix, precisely because of the richer horizon structure. We offered several explanations for the difference among the limits. One is related to entropy: the dS entropy still needs to be better understood but it may be interpreted in terms of the fact that from the perspective of an observer dS is not a pure state since there are degrees of freedom that are inaccessible to the observer. On taking the $\Lambda_A \to 0$ limit the horizon moves to the surface at infinity. So it seems that taking the limit the two entangled static patches of dS then give rise to two disconnected but entangled "Minkowski" spacetimes (though they are actually the infinite radius limit of the two causal patches of dS), and therefore the apparent infinite entropy. Clearly this subject needs a better understanding.

Another interesting limit is the $\Lambda_A \to -\infty$. In this case our general expression for AdS to dS gives us a transition rate that is exactly the Hartle-Hawking and Vilenkin probabilities for the creation of dS from nothing. This result fits nicely with the Brown-Dahlen proposal for understanding nothing (of the bubble of nothing) in terms of this AdS limit. However they had concluded that the transition was not allowed and therefore questioned the Hartle-Hawking and Vilenkin results. Here we actually find perfect agreement between two independent calculations and provide then a strong case for this interpretation. We find this result remarkable. In general the limit $\Lambda_A \to -\infty$ should not be trusted since it takes us beyond the validity of effective field theory. The proper limit to take is to compare Λ_A with the other scales of the system, like the brane tension κ and the dS cosmological constant Λ_B and consider the regime $|\Lambda_A| \gg \Lambda_B, \kappa^2, M^2$. In this case we obtain again the Hartle-Hawking/Vilenkin probability at leading order plus corrections of order $\Lambda_B/|\Lambda_A| \ll 1$, $M^2/|\Lambda_A| \ll 1$ and $\kappa^2/|\Lambda_A| \ll 1$.

Note, however that our identification of the nucleated spacetime fits with the Hawking-Turok instanton (but not necessarily with their open universe interpretation) rather than the creation of a full closed spacetime as in the Hartle-Hawking and Vilenkin cases. Since the transition rate in both cases is the same, we may conjecture that also for these cases there is a nonvanishing probability to create AdS and Minkowski spacetimes if we assign vanishing entropies to these spaces. Clearly this needs to be better understood.

Finally we may consider the regime in which the black hole mass dominates ($M \gg \Lambda_A, \Lambda_B$). This is a much less understood limit that may need further study. In particular, unlike the small mass limit, the ratio of up and down transition rates does not give exponential of the entropies. Instead of entropies we find differences of areas that do not correspond to horizon areas. It may be tempting to speculate these are some generalized entropies but we do not yet have a proper interpretation.

Regarding the trajectory of the wall for the generic $M \neq 0$ case, we find it interesting that it does not correspond to a geodesic that favors the open slicing of dS, contrary to the $M = 0$ case. This leaves open the question if an open universe is a general implication of the bubble nucleation process.

Some approaches to the address the cosmological constant problem envisage the landscape transitions to Minkowski or near Minkowski (with positive but tiny cosmological constants) and with AdS being a terminal spacetime (for recent discussions see [47–49]). It would be interesting to explore what will be the implications that up tunneling from AdS could have to those proposals. There are many questions still open. Extending the Hamiltonian approach to include scalar field potentials is a totally open question. An AdS/CFT interpretation of our results may be interesting, following the lines of [50] that interpreted the CDL Minkowski to AdS transition holographically. Furthermore [46] proposed interesting holographic interpretations for the 2D transitions. Extensions of these ideas to the 4D systems considered here may be worth exploring.

Clearly our results leave plenty of issues still to be understood. But we hope that our work and the questions we raised may guide future progress in this area.

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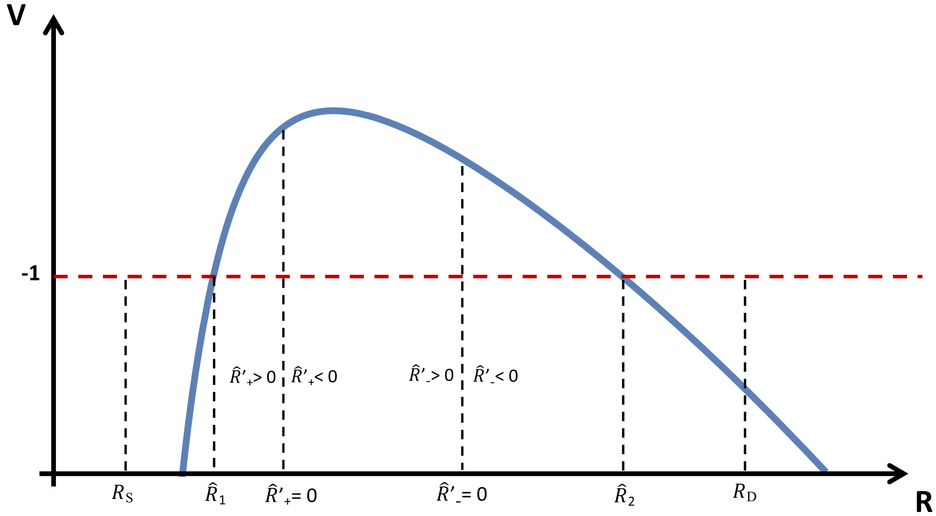

FIG. 4. The potential $V(R)$ for the Schwarzschild black hole to de Sitter transition. There are two regions outside the barrier and one below the barrier for $V(R) = -1$. The different regions separated by the values of $\hat{R}'$ are shown as well as the location of the turning points $\hat{R}_1, \hat{R}_2$ and the cosmological and black hole horizons R_D, R_S, respectively.

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APPENDIX: SCHWARZSCHILD BLACK HOLE TO DE SITTER

In this section we will give the details of the bulk term FMP calculation and extend it to the SdS case. FMP discuss the case studied by BGG-FGG [12,20], i.e., the nucleation of a dS space by tunneling from a Schwarzschild black hole. In this case, $A_I = 1–2GM/R$ and $A_O = 1 - H^2R^2$. As in the discussion after Eq. (2.5), at the turning point $V = -1$ $\hat{A}_O > 0, \hat{A}_I > 0$, but there are now two turning points $R_1 < R_2$ such that $R_{\rm b} < R_{1,2} < R_{\rm c}$, where $R_{\rm b} = 2GM$ is the black hole horizon (in this case the same as the Schwarzschild) while $R_{\rm D} = H^{-1}$ is the cosmological horizon (in this case the same as the dS).

$\hat{R}'_+$ vanishes at $\hat{R}^3 = \frac{R_{\rm b}}{H^2+\kappa^2} \equiv R^3_<$ and $\hat{R}'_-$ vanishes at $\hat{R}^3 = \frac{R_{\rm b}}{H^2-\kappa^2} \equiv R^3_>$. Note that if M (which is an integration constant) is such that $M = M_{\rm b} \equiv \frac{1}{2G\sqrt{H^2+\kappa^2}}$ the turning point $R_1 = R_< = R_{\rm S}$ and if $M = M_{\rm c} = \frac{H^2-\kappa^2}{2GH^3}$ (note that $M_{\rm b} > M_{\rm c}$), the turning point $R_2 = R_> = R_{\rm D}$.

From the matching condition it follows that R'_- is negative for $\hat{R} > R_>$, i.e., to the right of the point where R'_- vanishes while R'_+ is negative for $\hat{R} > R_<$. Thus we have for the bulk contribution at the turning point geometry,

$$S_{\rm Bu} = \frac{i\eta\pi}{G}\left[\int_0^{\hat{r}} dr \frac{dR}{dr} R\theta(-R') + \int_{\hat{r}}^{r_{\rm max}} dr \frac{dR}{dr} R\theta(-R')\right] = \frac{i\eta\pi}{2G}[(\hat{R}^2 - R_{\rm D}^2)\theta(\hat{R} - R_>) + (R_{\rm S}^2 - \hat{R}^2)\theta(\hat{R} - R_<)]. \quad (A1)$$

The wall contribution cannot be calculated analytically and FMP did not do so. All that matters for our present purposes is that it is finite. We also note that in the limit $M \to 0$ it was calculated in [16,19].

There are three cases to consider:

Case (i): For large black hole mass $M > M_{\rm b} > M_c$, $R_< < R_1 < R_2 < R_>$, we have $S_{\rm Bu}(R_2) = \frac{i\eta\pi}{2G}(R_{\rm S}^2 - R_2^2)$ since $R_2 > R_<$ and $S_{\rm Bu}(R_1) = \frac{i\eta\pi}{2G}(R_{\rm S}^2 - R_1^2)$. Hence

$$S_{\rm Bu}(R_2) - S_{\rm Bu}(R_1) = -\frac{i\eta\pi}{G}(R_2^2 - R_1^2). \quad (A2)$$

Case (ii): The second case is for $M_{\rm b} > M > M_{\rm c}$. Now $R_1 < R_< < R_2 < R_>$ and we get $S_B(R_2) = \frac{i\eta\pi}{2G}(R_{\rm S}^2 - R_2^2)$ and $S_B(R_1) = 0$. Hence

$$S_{\rm Bu}(R_2) - S_{\rm Bu}(R_1) = -\frac{i\eta\pi}{G}(R_2^2 - R_{\rm S}^2). \quad (A3)$$

Case (iii): The third case is for small black holes $M < M_{\rm c} < M_{\rm b}$ so that $R_{\rm S} < R_1 < R_2 < R_{\rm D}$. Here we have $S_{\rm Bu}(R_2) = \frac{i\eta\pi}{2G}(R_{\rm S}^2 - R_{\rm D}^2)$ and $S_B(R_1) = 0$ so that

$$S_{\rm Bu}(R_2) - S_{\rm Bu}(R_1) = -\frac{i\eta\pi}{G}(R_{\rm D}^2 - R_{\rm S}^2). \qquad (A4)$$

Thus we have given, for the reader's convenience, the derivation of Eq. (48) of FMP.

1. de Sitter to Schwarzschild black hole

In the down tunneling from dS to a Schwarzschild black hole, $A_I = 1–2GM/R$ and $A_O = 1 - H^2R^2$. The matching condition gives

$$\frac{\hat{R}'_\pm}{L} = \frac{1}{2\kappa\hat{R}}[-2GM + (H^2 \mp \kappa^2)\hat{R}^3]. \qquad (A5)$$

Note that if $M = 0$ $\hat{R}'_- > 0$ for all R. If one goes back for a moment to the dS to dS transitions for the bulk integral contribution we had a term $\theta(-R'_-)H_I^{-2}$, and in this case since we have Minkowski inside $H_I \to 0$ and this term is potentially divergent. However as we have just observed $\hat{R}'_- > 0$ so that $\theta(-R'_-) = 0$ and the potential divergence is avoided. This is why one gets a finite answer for down tunneling from dS to Minkowski even though there is no horizon to cut off the integral in Minkowski.

However in the present case as we see from Eq. (A5) there is a regime, namely $\hat{R}^3 < \frac{2GM}{H^2+\kappa^2} \equiv R_<^3$ where $\theta(-\hat{R}'_-) = 1$. So let us consider the same three cases that we analyzed in the previous subsection. It should be noted now that the region where $\hat{R}'_\pm$ is negative is to the right of (respectively) $R_<$ and $R_>$.

So for general $\hat{R}$ Inside the barrier, we have for the bulk integral

$$S_{\rm Bu}^{\rm down}(\hat{R}) = \frac{i\eta\pi}{2G}[(\hat{R}^2 - R_D^2)\theta(R_< - \hat{R}) + (R^2(r_{\rm max}) - \hat{R}^2)\theta(R_> - \hat{R})]. \qquad (A6)$$

Note that here we have introduced a cutoff in the first term R_c in the first term (which corresponds to the black hole region $r < \hat{r}$), which would have been there in the SdS case but for FMP case which corresponds to a asymptotically Minkowski black hole, must be sent to infinity. This however will make this transition ill defined in the last two cases as we will see below. For case (i), i.e., $M > M_{\rm b} > M_c$, $R_< < R_1 < R_2 < R_>$, we have $S_{\rm Bu}^{\rm down}(R_2) = \frac{i\eta\pi}{2G}(R^2(r_{\rm max}) - R_2^2)$ and $S_{\rm Bu}^{\rm down}(R_1) = \frac{i\eta\pi}{2G}(R^2(r_{\rm max}) - R_1^2)$. So the total contribution[20] is

$$S_{\rm Total}^{\rm down}(R_2 - R_1) = -\frac{i\eta\pi}{2G}(R_2^2 - R_1^2) + S_W(R_2 - R_1), \qquad (A7)$$

which is exactly the same as the total up-tunneling action difference, see Eq. (A2). This implies that the up- and down-transition rates from large black holes to dS is the same in the leading WKB approximation. In the other two cases however we have an ill-defined results for down tunneling.

In case (ii), i.e., $M_{\rm b} > M > M_{\rm c}, R_1 < R_< < R_2 < R_>$, we have $S_{\rm Bu}^{\rm down}(R_2) = \frac{i\eta\pi}{2G}(R^2(r_{\rm max}) - R_2^2)$ and $S_{\rm Bu}^{\rm down}(R_1) = \frac{i\eta\pi}{2G}(-R_D^2 + R^2(r_{\rm max}))$. Hence we have

$$S_{\rm Total}^{\rm down}(R_2 - R_1) = -\frac{i\eta\pi}{2G}(R_2^2 - R_D^2) + S_W(R_2 - R_1). \qquad (A8)$$

Finally for case (iii), i.e., $M < M_{\rm c} < M_{\rm b}$, $R_1 < R_< < R_> < R_2$, we have $S_{\rm Bu}^{\rm down}(R_2) = 0$ and $S_{\rm Bu}^{\rm down}(R_1) = \frac{i\eta\pi}{2G}(-R_D^2))$. So the total down tunneling amplitude is

$$S_{\rm Total}^{\rm down}(R_2 - R_1) = -\frac{i\eta\pi}{2G}(-R_c^2) + S_W(R_2 - R_1). \qquad (A9)$$

The classical actions in these last two down-tunneling cases is ill defined since for a Minkowskian black hole $R_D \to \infty$. The problem in Eqs. (A8) and (A9) lies in the fact that in cases (ii) and (iii) the junction condition in Eq. (A5) dictates that $R' < 0$ in the spacetime inside the bubble. Hence R_D corresponds to the maximum value of R for the Schwarzschild spacetime, which is infinity since there is no cosmologial horizon.[21] What is well defined for both up and down transitions involving black hole space times are transitions between SdS spaces. This we discuss in the next section.

2. Schwarzschild–de Sitter to Schwarzschild–de Sitter

Finally we observe that in none of these cases is detailed balance satisfied. Next we will follow these arguments to derive the corresponding equations for the rather more complicated case of transitions from SdS to SdS. In the general case,

$$A = 1 - \frac{2GM}{R} - H^2R^2 = -\frac{H^2}{R}(R - R_-)(R - R_{\rm S})(R - R_D). \qquad (A10)$$

The parameters are taken to be such that $3\sqrt{3}GM < H^{-1}$ in which case there is one negative real root and two positive real roots that satisfy

$$R_- < 0 < 2GM < R_{\rm S} < 3GM < R_D. \qquad (A11)$$

The smaller positive root R_b is identified with the horizon of the black hole and the larger one R_D with the cosmological horizon. We consider transitions from one SdS to another with parameters inside the bubble being denoted with the subscript I and those outside with O. Writing

[20]As before we just focus on one value of η in the numerator and denominator of the ratio of wave functions.

[21]Note that $R(r_{\rm max})$ correspond to the minimum value (as $R' < 0$) of R for the dS spacetime, which is 0.

$\Delta M \equiv M_O - M_I$ and $\Delta H^2 \equiv H_O^2 - H_I^2$, we have the matching condition

$$\frac{\hat{R}_\pm}{L} = \frac{1}{2\kappa\hat{R}}[2G\Delta M + (\Delta H^2 \mp \kappa^2)\hat{R}^3]. \quad \text{(A12)}$$

Let us simplify the discussion by choosing $\kappa^2 < |\Delta H^2|$ so that the sign of the second term in square brackets is determined by the sign of ΔH^2. Now as in the FMP/BGG discussion the sign of $\hat{R}_\pm$ changes at a zero but unlike in that case if ΔM and ΔH^2 have the same sign then $\hat{R}_\pm$ does not vanish anywhere. To remain close to the FMP discussion let us first choose these signs to be opposite, i.e., $\epsilon(\Delta M) = -\epsilon(\Delta H^2)$ with $\Delta M > 0$[22] so that $\hat{R}_+$ vanishes at $R_<$ and $\hat{R}_-$ vanishes at $R_>$, where

$$R_<^3 = \frac{2G\Delta M}{|\Delta H^2| + \kappa^2}, \qquad R_>^3 = \frac{2G\Delta M}{|\Delta H^2| - \kappa^2}. \quad \text{(A13)}$$

These points coincide with the turning points $R_{1,2}$ when the masses (which we recall are integration constants), are such that $R_< = R_S$ and $R_> = R_D$.

So let us consider the tunneling $A \to B$ so that we put $O = A$ and $I = B$. We will be interested mainly in what corresponds to case (iii) of FMP, i.e., the small black hole case.[23] Then we have $R_1 < R_< < R_> < R_2$. For the bulk action we have [see first line of Eq. (A1) and Fig. 5]

$$S_{\text{Bu}}(\hat{R}) = \frac{i\eta\pi}{2G}[(\hat{R}^2 - (R_{I,D})^2)\theta(\hat{R} - R_>) + ((R_{O,S})^2 - \hat{R}^2)\theta(\hat{R} - R_<)]. \quad \text{(A14)}$$

So at the turning points we have

$$\begin{aligned} S_{\text{Bu}}(\widehat{R_2}) &= \frac{i\eta\pi}{2G}[(\hat{R}_2^2 - (R_D^B)^2)\theta(\hat{R}_2 - R_>) \\ &\quad + ((R_S^A)^2 - \hat{R}_2^2)\theta(\hat{R}_2 - R_<)] \\ &= \frac{i\eta\pi}{2G}[-(R_D^B) + (R_S^A)^2], \\ S_{\text{Bu}}(\widehat{R_1}) &= \frac{i\eta\pi}{2G}[(\hat{R}_1^2 - (R_D^B)^2)\theta(\hat{R}_1 - R_>) \\ &\quad + ((R_S^A)^2 - \hat{R}_1^2)\theta(\hat{R}_1 - R_<)] \\ &= 0. \end{aligned}$$

Defining $F[R_2 - R_1] \equiv F[R_2] - F[R_1]$ we have for (i) times the total action (including the wall term)

$$I^{AB}[R_2 - R_1] = \frac{\eta\pi}{2G}[(R_D^B)^2 - (R_S^A)^2] + I_W^{AB}[R_2 - R_1]. \quad \text{(A15)}$$

Note that this is the same as FMP case (iii). Interchanging A and B we have

$$I^{BA}[R_2 - R_1] = \frac{\eta\pi}{2G}[(R_D^A)^2 - (R_S^B)^2] + I_W^{BA}[R_2 - R_1]. \quad \text{(A16)}$$

The wall term is clearly symmetric under the interchange, i.e., $I_W^{AB} = I_W^{BA}$. Hence we have for the ratio of the probabilities of going from A to B to the reverse,

$$\frac{P^{AB}}{P^{BA}} = \frac{\exp\{\frac{\eta\pi}{G}[(R_D^B)^2 - (R_S^A)^2] + 2I_W^{AB}[R_2 - R_1]\}}{\exp\{\frac{\eta\pi}{G}[(R_D^A)^2 - (R_S^B)^2] + 2I_W^{BA}[R_2 - R_1]\}} = e^{S^B - S^A}. \quad \text{(A17)}$$

Here we have defined $S^B = \frac{\eta\pi}{G}[(R_D^B)^2 + (R_S^B)^2]$ and $S^A = \frac{\eta\pi}{G}[(R_D^A)^2 + (R_S^A)^2]$, which are the sum of the horizon entropies of each SdS space. If we can interpret this sum as the total entropy of SdS space (we know of no direct demonstration of this), then indeed the relation Eq. (A17) is the statement of detailed balance.

3. Graphic display of the computation

We can give an intuitive understanding of the previous computations as described in this subsection.

First, note that the constraint in Eq. (2.8) dictates the sign of R', which can only changes at the horizons, i.e., where $R' = 0$. In a compound state with two spacetimes separated by a wall at $\hat{r}$, the sign of R' in the vicinity of the wall is determined by the junction conditions in Eq. (2.3). As can be observed in Fig. 5, the junction conditions imply that there are only three possible combinations of the signs of $\hat{R}'_\pm$, namely

$$R'_\pm = \{(+,+), (+,-), (-,-)\}. \quad \text{(A18)}$$

The three options in Eq. (A18) can be visualized as in Figs. 6–8. We consider the compound state to be given by the region of spacetime connected to the wall; i.e., it extends up to the closest horizon both in the inside and the outside spacetime regions. For instance, in Fig. 8 the dotted line is not part of the spacetime, as the closest horizon to the wall in the inside region is reached the cosmological horizon at $r_{\min}$.

The three cases (i), (ii), and (iii) in the previous section correspond to transitions between pairs of compound states like those displayed in Fig. 6–8. In particular

$$\begin{aligned} &\text{Case (i)} \quad \Leftrightarrow \quad (+,-) \to (+,-), \\ &\text{Case (ii)} \quad \Leftrightarrow \quad (+,+) \to (+,-), \\ &\text{Case (iii)} \quad \Leftrightarrow \quad (+,+) \to (-,-). \end{aligned}$$

For each configuration, it is easy to evaluate the contribution to the bulk integral: from Eq. (2.15) it is clear that only the regions where $R' < 0$ contribute with a factor R^2 evaluated at the extrema of these regions. For instance, for the configuration $R'_\pm = (-,-)$ in Fig. 8, the nonvanishing contribution comes from the region between the cosmological horizon of the interior spacetime at $r_{\min}$

[22] This choice goes over to the FMP case when $M_I \to 0$ and $H_O \to 0$.

[23] This is because ultimately we would like to send the black masses to zero.

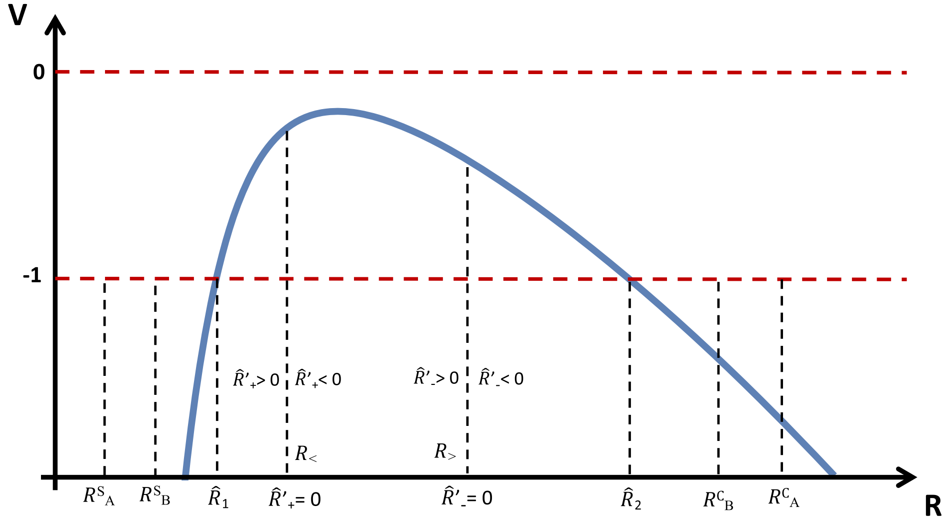

FIG. 5. The potential $V(R)$ for the Schwarzschild–de Sitter to Schwarzschild–de Sitter transition. The turning points, location of horizons and regions for different signs of $\hat{R}'$ are shown.

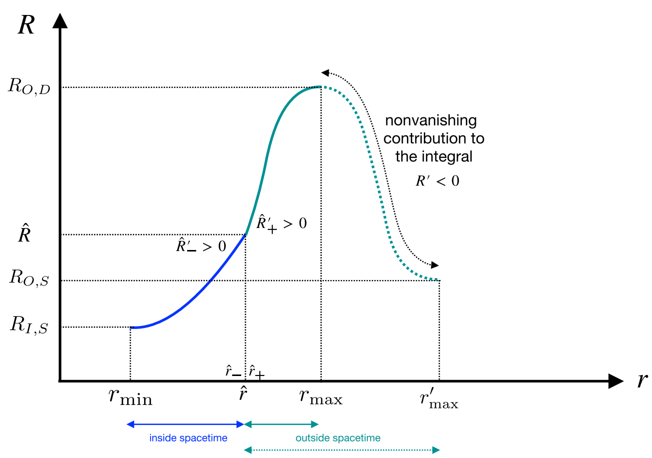

FIG. 6. Configuration with $R'_{\pm} = (+,+)$.

and the black hole horizon of the exterior spacetime at $r_{\max}$. The value of the action would then be

$$S_B = \frac{i\eta\pi}{G}\left[\int_{r_{\min}}^{\hat{r}} drR'R\theta(-R'_-) + \int_{\hat{r}}^{r_{\max}} drR'R\theta(-R'_+)\right] \tag{A19}$$

$$= \frac{i\eta\pi}{G}[(R_{O,S}^2 - \hat{R}^2) + (\hat{R}^2 - R_{O,D}^2)]$$

$$= \frac{i\eta\pi}{G}(R_{O,S}^2 - R_{O,D}^2). \tag{A20}$$

Therefore, defining as in FMP

$$F_I[\hat{R}] = \frac{i\eta\pi}{G}\int_{R(r_{\min})}^{R(\hat{r})} dRR\theta(-R'_-),$$

$$F_O[\hat{R}] = \frac{i\eta\pi}{G}\int_{R(\hat{r})}^{R(r_{\max})} dRR\theta(-R'_+), \tag{A21}$$

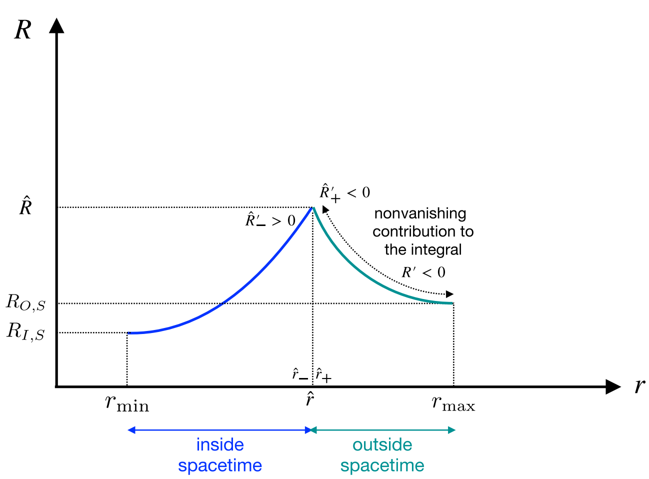

FIG. 7. Configuration with $R'_{\pm} = (+,-)$.

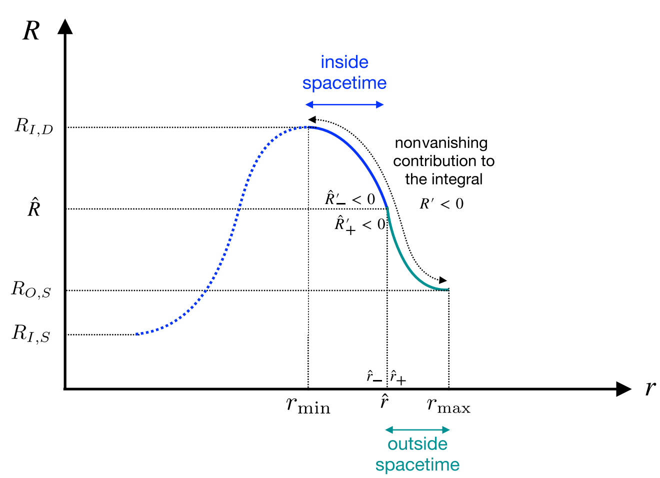

FIG. 8. Configuration with $R'_{\pm} = (-,-)$.

we can compute the bulk action case by case as in the previous section, using that

$$S_B \equiv F[\hat{R}_2 - \hat{R}_1] \equiv F_I[\hat{R}_2 - \hat{R}_1] + F_O[\hat{R}_2 - \hat{R}_1], \tag{A22}$$

and $F_i[\hat{R}_2 - \hat{R}_1] = F[\hat{R}_2] - F[\hat{R}_1]$ for $i = 1, 2$. Note that $\hat{R}_{1,2}$ are the turning points defined in the previous subsection.

(i) Case (i): In this case, both $\hat{R}_1$ and $\hat{R}_2$ fall in a region where $\hat{R}'_\pm = (+,-)$, corresponding to Fig. 7. Then

$$F_I[\hat{R}_2 - \hat{R}_1] = 0,$$
$$F_O[\hat{R}_2 - \hat{R}_1] = \frac{i\eta\pi}{2G}(\hat{R}_2^2 - \hat{R}_1^2), \tag{A23}$$

where F_I gives zero contribution because in the interior spacetime R' is always positive. This implies

$$S_B = \frac{i\eta\pi}{2G}(\hat{R}_2^2 - \hat{R}_1^2). \tag{A24}$$

Case (ii): In this case, $\hat{R}_1$ is in a region where $\hat{R}'_\pm = (+,+)$, while $\hat{R}_2$ is in a region where $\hat{R}'_\pm = (+,-)$ corresponding to Figs. 6 and 7, respectively. Then

$$F_I[\hat{R}_2 - \hat{R}_1] = 0,$$
$$F_O[\hat{R}_2 - \hat{R}_1] = \frac{i\eta\pi}{2G}(R_{O,S}^2 - \hat{R}_2^2), \tag{A25}$$

where F_I gives zero contribution because in the interior spacetime R' is always positive. This implies

$$S_B = \frac{i\eta\pi}{2G}(R_{O,S}^2 - \hat{R}_2^2). \tag{A26}$$

We observe that these two cases give exactly the same result as in FMP, i.e., (A2) and (A3).

Case (iii): In this case $\hat{R}_1$ is in the region with $R'_\pm = (+,+)$, while $\hat{R}_2$ is in the region $\hat{R}_\pm = (-,-)$, corresponding to Figs. 6 and 8, respectively. Then

$$F_I[\hat{R}_2 - \hat{R}_1] = \frac{i\eta\pi}{2G}(\hat{R}_2^2 - R_{I,D}^2),$$
$$F_O[\hat{R}_2 - \hat{R}_1] = \frac{i\eta\pi}{2G}(R_{O,S}^2 - \hat{R}_2^2), \tag{A27}$$

which implies

$$S_B = \frac{i\eta\pi}{2G}(R_{O,S}^2 - R_{I,D}^2), \tag{A28}$$

which coincides with the result found in the last subsection. The contribution of Fig. 6 however is clearly a disconnected term of sort discussed in the next subsection and cancels between the two turning points.

4. Disconnected terms

Here we derive a general formula for the bulk contribution to the action for transitions from and to space times with two horizons. The formula seems to have been first written down in [19] [see Eq. (3.37)] but without a detailed derivation. Here for the readers convenience we include these details and also highlight some differences with Bachlechner's discussion [19].

For the turning point geometries the bulk action S_B simplifies with the first term in square brackets in (2.11) giving zero and the second term contributes only when $\epsilon(R') = -1$, i.e., whenever $\cos^{-1}(\frac{R'}{L\sqrt{A_{I,O}}}) = \pi$. Thus we have[24]

$$iS_{\mathrm{Bu}}(\hat{R}_r) = -\frac{\eta\pi}{G}\left[\int_0^{\hat{r}} drR'R\theta(-R'_-) + \int_{\hat{r}}^{\infty} drR'R\theta(-R'_+)\right]. \tag{A29}$$

Let us focus on the second integral.

$$\begin{aligned}\int_{\hat{r}}^{\infty} drR'R\theta(-R'_+) &= \int_{\hat{r}}^{\infty} dr\frac{1}{2}\left(\frac{dR^2}{dr}\right)\theta(-R'_+),\\ &= \int_{\hat{r}}^{\infty} dr\frac{1}{2}\left(\frac{d}{dr}R^2\theta(-R'_+)\right)\\ &\quad - \int_{\hat{r}}^{\infty} dr\frac{1}{2}R^2\frac{d}{dr}\theta(-R'_+).\end{aligned} \tag{A30}$$

Let us first consider the first integral above. Observe that at the turning points for the geometry $R'/L = \pm\sqrt{A_O(R)}$ (with $L > 0$) so the integrand becomes just a function of R. Also in the SdS case

$$A_O(R) = -\frac{H_O^2}{R}(R - R_-)(R - R_0^S)(R - R_0^D), \tag{A31}$$

where $R_- < 0, R_0^S < R_0^D$, and A_O is positive for $R_0^S < R < R_0^D$. Now working within one panel of the infinite strip of Penrose panels the one which contains the wall, the parametrization $R = R(r)$ is one valued we have for the first integral in (A30)

$$\begin{aligned}\int_{\hat{r}}^{\infty} dr\frac{1}{2}\left(\frac{d}{dr}R^2\theta(-R'_+)\right) &= \frac{1}{2}\int_{\hat{R}}^{R_\infty} d\left(R^2\theta(\sqrt{A_O(R)})\right)\\ &= \frac{1}{2}(R_0^{S2} - \hat{R}^2)\theta(-\hat{R}'_+).\end{aligned}$$

In the last step we have used the fact that R must decrease as one goes from the lower limit to the upper limit and that the integral gets cut off at the first zero of A_O as R decreases—i.e., at R_0^S, and the step function is the statement that in this regime the sign of R' must match the sign of $\hat{R}'_+$.

The second term in (A30) gives a delta function and so is disconnected from the wall. Let us evaluate it explicitly:

[24] We have taken the integration in r all the way up to infinity even though in practice it may be cut off at some finite value.

$$-\int_{\hat{r}}^{\infty} dr \frac{1}{2} R^2 \frac{d}{dr} \theta(-R'_+) = -N_O \int_{\hat{R}}^{R_\infty} dR \frac{1}{2} R^2 \frac{d}{dR} \theta(\sqrt{A_O(R)}) = -N_O \int_{\hat{R}}^{R_\infty} dR \frac{1}{2} R^2 \frac{d\sqrt{A_O}}{dR} \delta(\sqrt{A_O}),$$
$$= -N_O \int_{\hat{R}}^{R_\infty} dR \frac{1}{2} R^2 \frac{dA_O}{dR} \sum_{i=s,c} \frac{1}{|A_O(R_i)|} \delta(R - R_i),$$
$$= -N_O \int_{\hat{R}}^{R_\infty} dR \frac{1}{2} R^2 \sum_{i=S,D} \epsilon \left(\frac{dA_O}{dR} \bigg|_{R_0^i} \right) \delta(R - R_0^i). \quad \text{(A32)}$$

Here N_O is the number of panels included in the r integral. Using (A31) we have

$$\frac{dA_O}{dR}\bigg|_{R_0^D} = -\frac{H_O^2}{R_0^D}(R_0^D - R_0^S) < 0, \qquad \frac{dA_O}{dR}\bigg|_{R_0^S} = -\frac{H_O^2}{R_0^D}(R_0^S - R_0^D) > 0.$$

Hence we have

$$-\int_{\hat{r}}^{\infty} dr \frac{1}{2} R^2 \frac{d}{dr} \theta(-R'_+) = -N_O \sum_{i=S,D} \epsilon \left(\frac{dA_O}{dR} \bigg|_{R_0^i} \right) \frac{1}{2} R_0^{i2} \{ \theta(R_\infty - \hat{R}) \theta(R_h^i - \hat{R}) + \theta(\hat{R} - R_\infty) \theta(R_h^i - \hat{R}) \},$$
$$= N_O [R_0^{D2} \theta(R_0^D - \hat{R}) - R_0^{S2} \theta(R_0^S - \hat{R})] \theta(R_\infty - \hat{R})$$
$$+ N_O [R_0^{D2} \theta(\hat{R} - R_0^D) - R_0^{S2} \theta(\hat{R} - R_0^S)] \theta(\hat{R} - R_\infty).$$

For $\hat{R} = R_2$, since $R_0^S < R_2 < R_0^D$, the above gives

$$N_O \{ (R_0^{D2} - 0) \theta(R_\infty - \hat{R}) + (0 - R_0^{S2}) \theta(\hat{R} - R_\infty) \},$$

while for $\hat{R} = R_1$, since $R_0^S < R_1 < R_0^D$ we get exactly the same result since N_O is independent of $\hat{R}$. Thus when subtracting the two we get zero. In other words the disconnected contribution cancels between the two turning points.

5. dS horizon issues

Here we explain using the explicit parametrization used in [16] how the last term in Eq. (2.23) arises from including a term corresponding to the nucleation of the brane behind the horizon of the observer.

In that calculation, at the turning point geometry.[25] we had $R = H_I^{-1} \sin r_-$ (with $L = H_I^{-1}$) inside and $R = H_O^{-1} \sin r_+$ (with $L = H_O^{-1}$) outside. Since the physical radius R must be continuous at the wall $R = \hat{R} = H_I^{-1} \sin \hat{r}_- = H_O \sin \hat{r}_+$. Also $\frac{\hat{R}'_\pm}{L} = \cos \hat{r}_\pm$ and $A_{O/I} = 1 - H_{O/I}^2 R^2 = \cos^2 r_\pm$. Define the static patch identified as region III of the Penrose diagram as the one covered by the range $0 \le r_\pm < \pi/2$ and region I as the one covered by $\pi/2 < r_\pm < \pi$. Note that at $r = \pi/2$ the static patch coordinates [which have the factor $(1 - H^2 R^2)$] become singular.

On the other hand r is a global coordinate which may be extended beyond $\pi/2$. But going beyond this point is tantamount to going into region III from I, or the reverse, in the Penrose diagram. In fact since there are two values of r determining a given R [since $\sin r = \sin(\pi - r)$]—one corresponding to region I and one for region III.

So if we confine the calculation to region I, the outside integral [the second term in (A29)] becomes after putting $\sin r = s$, (note that $RR' = H_O^{-2} \sin r \cos r$)

$$-\frac{\eta}{G} H_O^{-2} \int_{\hat{r}_+}^{\pi} dr \sin r \cos r \cos^{-1} \left(\frac{\cos r}{|\cos r|} \right)$$
$$= -\frac{\eta}{G} H_O^{-2} \int_{\sin \hat{r}_+}^{0} ds s \pi \theta \left(\sin^{-1} s - \frac{\pi}{2} \right),$$
$$= \frac{\eta \pi}{2G} H_O^{-2} \sin^2 \hat{r}_+ \theta \left(\hat{r}_+ - \frac{\pi}{2} \right) = \frac{\eta \pi}{2G} \hat{R}^2 \theta(-\hat{R}'_+). \quad \text{(A33)}$$

For the inside integral [the first integral in (A29)] we have

$$-\frac{\eta}{G} H_I^{-2} \int_{0}^{\hat{r}_-} dr \sin r \cos r \cos^{-1} \left(\frac{\cos r}{|\cos r|} \right)$$
$$= -\frac{\eta \pi}{G} H_I^{-2} \int_{0}^{\sin \hat{r}_-} ds s \theta \left(\sin^{-1} s - \frac{\pi}{2} \right),$$
$$= -\frac{\eta \pi}{2G} H_I^{-2} \left[\sin^2 \hat{r} - \sin^2 \frac{\pi}{2} \right] \theta \left(\sin^{-1} \hat{s} - \frac{\pi}{2} \right),$$
$$= -\frac{\eta \pi}{2G} [\hat{R}^2 - H_I^{-2}] \theta(-\hat{R}'_-). \quad \text{(A34)}$$

Thus for the total bulk action we get

[25] As observed in footnote 11.

$$I(\hat{R}) = iS(\hat{R}) = \frac{\eta\pi}{2G}[(\theta(-\hat{R}') - \theta(-\hat{R}'_-))\hat{R}^2 + H_I^{-2}\theta(-\hat{R}'_-)]. \tag{A35}$$

This expression is exactly what is obtained in the limit $R_{O,S} \to 0$ of the SdS to SdS expression [i.e. (A14) for the bulk integral]—namely Eq. (2.22).

Comparing this expression to Eq. (2.23), we see that it is missing the term $\theta(\hat{R}'_+)H_O^2$ within the square brackets. Where did this come from?

In the calculation in [16] we used global coordinates to calculate the bulk integral. In terms of the static patch coordinates relevant to an observer in a given such patch, this involves integration through the de Sitter horizon. To see this in the above calculation we need to add a piece proportional to $\theta(\frac{\pi}{2} - \hat{r})$ corresponding to the brane being nucleated behind the horizon of the observer at $r = \pi$. Thus we add the term

$$\begin{aligned} &-\frac{\eta}{G}H_O^{-2}\int_{\sin\hat{r}_+}^{\sin\pi} ds s\pi\theta\left(\sin^{-1}s - \frac{\pi}{2}\right)\theta\left(\frac{\pi}{2} - \hat{r}\right) \\ &= -\frac{\eta\pi}{G}H_O^{-2}\int_{\sin\frac{\pi}{2}}^{\sin\pi} ds s\theta\left(\frac{\pi}{2} - \hat{r}\right) \\ &= \frac{\eta\pi}{2G}H_O^{-2}\theta(\hat{R}'_+). \end{aligned} \tag{A36}$$

Adding this to (A35) gives the result (2.23).

It should be noted that we could have done the integral in (A33) in a different way by writing (after identifying the integration variable as $t = \cos r$)

$$\begin{aligned} &-\frac{\eta}{G}H_O^{-2}\int_{\hat{r}_+}^{\pi} dr \sin r \cos r \cos^{-1}\left(\frac{\cos r}{|\cos r|}\right) \\ &= \frac{\eta}{G}H_O^{-2}\int_{\cos\hat{r}_+}^{\cos\pi} dt t\pi\theta(-t) \\ &= \frac{\eta\pi}{2G}H_O^{-2}[1 - (1 - \sin^2\hat{r}_+)\theta(-\cos\hat{r}_+)] \\ &= \frac{\eta\pi}{2G}[H_O^{-2}\theta(\hat{R}_+) + \hat{R}^2\theta(-\hat{R}'_+)]. \end{aligned}$$

However this way of calculating (essentially what was done in [16]) misses the fact that the extra term comes from going behind the horizon of the observer at $r = \pi$.

To belabor the point let us redo the calculation without using an explicit parametrization. This is all we can do in the general situation of SdS transitions where we do not have the luxury of such a parametrization. In this case the outside integral [the second term in (A29)] may written as (ignoring the delta function terms discussed in the previous subsection)

$$\begin{aligned} &= -\frac{\eta\pi}{G}\left[\int_{\hat{r}}^{r_{\max}} dr R' R\theta(-R'_+)\right] \\ &= -\frac{\eta\pi}{2G}\left[\int_{\hat{r}}^{r_{\max}} dr \frac{dR^2}{dr}\theta(-R'_+)\right] \\ &= -\frac{\eta\pi}{2G}\left[\int_{\hat{R}}^{0} dR^2\theta(-R'_+)\right], \\ &= \frac{\eta\pi}{2G}\hat{R}^2\theta(-R'_+). \end{aligned}$$

In agreement with (A33).

6. Reflection symmetry of tunneling action

In this subsection we will show explicitly the symmetry under the interchange of the outside and inside space times when the configuration is at a general point (i.e., not a turning point), for both the bulk and the brane actions.

Let us first look at (i times) the bulk action $I = iS$, with the wall/brane at some arbitrary point with radius $\hat{R} = R(\hat{r})$. We have for a transition from a state O outside to a state I inside,[26]

$$\begin{aligned} \frac{G}{\eta}I_B^{IO} = &\int_0^{\hat{r}-\epsilon} dr R\left[\sqrt{A_I L^2 - R'^2} - R'\cos^{-1}\left(\frac{R'}{LA_I}\right)\right] \\ &+ \int_{\hat{r}+\epsilon}^{r*} dr R\left[\sqrt{A_O L^2 - R'^2} - R'\cos^{-1}\left(\frac{R'}{LA_O}\right)\right], \end{aligned} \tag{A37}$$

$$\begin{aligned} \frac{G}{\eta}I_B^{OI} = &\int_0^{\hat{r}} dr R\left[\sqrt{A_I L^2 - R'^2} - R'\cos^{-1}\left(\frac{R'}{LA_I}\right)\right] \\ &+ \int_{\hat{r}}^{r*} dr R\left[\sqrt{A_O L^2 - R'^2} - R'\cos^{-1}\left(\frac{R'}{LA_O}\right)\right]. \end{aligned} \tag{A38}$$

In the first equation let us change the integration variable: put $r' = r* - r$. So we get

$$\begin{aligned} \frac{G}{\eta}I_B^{IO} = &\int_{r*}^{\hat{r}'+\epsilon} (-dr')R(r* - r')\left[\sqrt{A_I L^2 - R'^2} - \frac{dR}{d(r* - r')}\cos^{-1}\left(\frac{dR/d(r* - r')}{LA_I}\right)\right] \\ &+ \int_{\hat{r}-\epsilon}^{0} (-dr')R(r* - r')\left[\sqrt{A_O L^2 - R'^2} - \frac{dR}{d(r* - r')}\cos^{-1}\left(\frac{dR/d(r* - r')}{LA_O}\right)\right]. \end{aligned} \tag{A39}$$

[26]Note that we have replaced $r_{\max}$ by $r*$.

The above may be rewritten as

$$\begin{aligned}\frac{G}{\eta}I_B^{IO}[R(\hat{r})] &= \int_{\hat{r}'+\epsilon}^{r*}(-dr')R(r*-r')\left[\sqrt{A_I L^2 - R'^2} - \frac{dR}{dr}\bigg|_{r=r*-r'}\cos^{-1}\left(\frac{(dR/dr)|_{r=r*-r'}}{LA_I}\right)\right] \\ &\quad + \int_0^{\hat{r}-\epsilon} drR(r*-r')\left[\sqrt{A_O L^2 - R'^2} - \frac{dR}{dr}\bigg|_{r=r*-r'}\cos^{-1}\left(\frac{(dR/dr)|_{r=r*-r'}}{LA_O}\right)\right], \\ &= \frac{G}{\eta}I_B^{OI}[R(r*-\hat{r}')] = \frac{G}{\eta}I_B^{OI}[R(\hat{r})],\end{aligned}$$

which establishes the symmetry of the bulk action between the outside and inside spacetimes.

For the wall/brane action we have

$$-\frac{G}{\eta}I_W^{IO} = \int \delta\hat{R}\,\hat{R}\left[\cos^{-1}\left(\frac{\hat{R}_+}{L\sqrt{A_O}}\right) - \cos^{-1}\left(\frac{\hat{R}_-}{L\sqrt{A_I}}\right)\right]. \quad \text{(A40)}$$

Under the interchange $O \rightleftarrows I$ we have $\hat{R}_\pm \rightleftarrows -\hat{R}_\mp$. So since $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, we have

$$\begin{aligned}-\frac{G}{\eta}I_W^{OI} &= \int \delta\hat{R}\,\hat{R}\left[\left(\pi - \cos^{-1}\left(\frac{\hat{R}_-}{L\sqrt{A_I}}\right)\right) - \left(\pi - \cos^{-1}\left(\frac{\hat{R}_+}{L\sqrt{A_O}}\right)\right)\right] \\ &= \int \delta\hat{R}\,\hat{R}\left[\left(-\cos^{-1}\left(\frac{\hat{R}_-}{L\sqrt{A_I}}\right)\right) + \left(\cos^{-1}\left(\frac{\hat{R}_+}{L\sqrt{A_O}}\right)\right)\right] = -\frac{G}{\eta}I_W^{IO}.\end{aligned}$$

Hence the total action at $\hat{R}$ is invariant under the interchange of the outside and the inside.

It is important to note that this reflection symmetry is manifest only with at general points in field space—i.e., without imposing turning point values for either the geometry or the brane.

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