Categorical-symmetry resolved entanglement in conformal field theory

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(Received 16 February 2024; accepted 19 April 2024; published 23 May 2024)

We propose a symmetry resolution of entanglement for categorical noninvertible symmetries (CaT-SREE) in $(1+1)$-dimensional conformal field theories. The definition parallels that of grouplike invertible symmetries, employing the concept of symmetric boundary states with respect to a categorical symmetry. Our examination extends to rational conformal field theories, where the behavior of CaT-SREE mirrors that of grouplike invertible symmetries. We find that CaT-SREE can be defined if there is no obstruction to gauging the categorical symmetry, as happens in the case of grouplike symmetries. We also provide instances of the breakdown of entanglement equipartition at the next-to-leading order in the cutoff expansion. Our findings shed light on how the interplay between conformal boundary conditions and categorical symmetries lead to specific patterns in the entanglement entropy.

DOI: 10.1103/PhysRevD.109.105026

I. INTRODUCTION

The notion of a global symmetry in quantum field theory (QFT) has been recently generalized in ways that go beyond those described by groups. Central to this is the idea that every symmetry can be associated to a topological operator [1]. The most striking of these generalizations are higher-form symmetries, related to the conservation of extended objects, and categorical or noninvertible symmetries, symmetries whose associated topological operators form a fusion category, that is, do not fuse according to a simple group law. The study of these noninvertible symmetries is providing new and deep insights into the characterization of universal properties of quantum systems of wide interest, spanning condensed matter and high-energy physics (see Refs. [2,3] for a comprehensive and pedagogical review of these developments.).

In $(1+1)$-dimensional conformal field theories (CFTs), on which we focus on this work, noninvertible symmetries implement dualities such as the Kramers-Wannier duality of the $(1+1)$D Ising model [4,5] and the duality between momentum and winding modes (T duality) of the free compactified boson [6–8]. In these theories, a finite categorical symmetry is defined through a fusion category $\mathcal{C}$ of one-dimensional topological defect line operators. In rational conformal field theories (RCFTs) with a diagonal modular invariant partition function [9,10], these topological defect line operators are known as Verlinde lines. Verlinde lines represent both invertible as well as noninvertible symmetries [11,12]. If the set of lines is denoted as $\{\mathcal{L}_i\}_{i\in\mathcal{V}}$ where $\mathcal{V}$ labels the operators $\mathcal{L}_i \in \mathcal{C}$, then the fusion algebra is given by

$$\mathcal{L}_i \times \mathcal{L}_j = \sum_{k\in\mathcal{V}} N_{ij}^k \mathcal{L}_k, \tag{1}$$

where $N_{ij}^k \in \mathbb{Z}_{\geq 0}$ are non-negative integer-valued fusion coefficients. Topological defect lines and particularly Verlinde lines $\mathcal{L}$, do not generically have an inverse $\mathcal{L}^{-1}$ such that $\mathcal{L}\times\mathcal{L}^{-1} = \mathbb{1}$.

Parallel to generalizing the concept of global symmetry, there has been a remarkable interest in understanding the relation between entanglement in QFT and symmetries. In systems with a global grouplike invertible symmetry, this has been carried out through the symmetry resolved entanglement entropy (SREE) [13–15] which intuitively quantifies the amount of entanglement for different charge sectors. Remarkably, it has been shown that at leading order in the UV cutoff expansion, the SRE entropies are equal for all the charge sectors, a result known as entanglement equipartition [15].

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Hitherto, the entanglement behavior in the presence of categorical symmetries in a QFT has been unknown. In this Letter, we establish the SREE for categorical symmetries (CaT-SREE), mirroring the case of grouplike invertible symmetries once the notion of symmetric boundary states with respect to a categorical symmetry is provided [16]. Our results shed light on how certain CFT boundary conditions preserve or enhance certain categorical symmetries, leading to specific patterns in the entanglement entropy. We illustrate the proposal with two RCFTs, the critical Ising model, where it is not possible to obtain a CaT-SREE and the tricritical Ising model, where it is possible, and the result at leading order shows entanglement equipartition.

II. SYMMETRY RESOLVED ENTANGLEMENT IN CFT

In extended quantum systems, the entanglement entropy (EE) measures the amount of quantum correlations between the degrees of freedom located within an arbitrary region A and those sited on its complement B. Assuming that the Hilbert space $\mathcal{H}$ of the system factorizes as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ contains the degrees of freedom in the region A and $\mathcal{H}_B$ the ones in B, for given a pure state $|\Psi\rangle \in \mathcal{H}$, the reduced density matrix of A is defined by tracing out the degrees of freedom corresponding to the complementary region B as $\rho_A = \mathrm{Tr}_{\mathcal{H}_B}|\Psi\rangle\langle\Psi|$.

The entanglement between A and B is thus quantified through the Rényi and entanglement entropies

$$S_A^n = \frac{1}{1-n}\log \mathrm{Tr}\rho_A^n,$$
$$S_A = \lim_{n\to 1} S_A^n = -\mathrm{Tr}\rho_A \log \rho_A. \quad (2)$$

We consider now there is a local charge operator $\mathcal{Q} = \mathcal{Q}_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \mathcal{Q}_B$ that generates a global Abelian symmetry group G in our theory. When $|\Psi\rangle$ is an eigenstate of $\mathcal{Q}$, then $[\rho_A, \mathcal{Q}_A] = 0$ and ρ_A is block diagonal $\rho_A = \oplus_Q \Pi_Q \rho_A = \oplus_Q\, p_A[Q]\rho_A[Q]$, with $\sum_Q p_A[Q] = 1$ and $\mathrm{Tr}\rho_A[Q] = 1$, each block corresponding to a charge sector of $\mathcal{Q}_A$ where Q are eigenvalues of $\mathcal{Q}_A$, Π_Q is a projector to the eigenspace of Q and $p_A[Q] = \mathrm{Tr}[\Pi_Q \rho_A]$ is the probability of measuring the charge value Q in the region A. G being Abelian, the eigenvalues Q label the irreducible representations r of the group.

As a result, the entanglement between regions A and B, may be decomposed into the contributions of each charge sector [13–15] through the symmetry resolved Rényi entropy

$$S_A^n[Q] = \frac{1}{1-n}\log \mathrm{Tr}\rho_A^n[Q]. \quad (3)$$

Entanglement equipartition is the situation for which $\mathrm{Tr}\rho_A^n[Q]$ and thus $S_A[Q]$ do not depend on Q. With this, the fundamental object to compute the SREE is the replica partition function [17,18] at a fixed value of charge Q

$$Z_n[Q] = \mathrm{Tr}\Pi_Q \rho_A^n, \quad (4)$$

from which the SREE can be written as

$$S_A^n[Q] = \frac{1}{1-n}\log\frac{Z_n[Q]}{Z[Q]^n}, \qquad Z[Q] \equiv Z_1[Q]. \quad (5)$$

In a $(1+1)$-dimensional CFT, the factorization of the Hilbert space $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, requires imposing boundary conditions a, b that preserve conformal symmetry at the entangling surface ∂A. These boundary conditions have nontrivial consequences for the EE [19,20]. Specifying the region A to an interval of length ℓ, this is implemented by encircling the two entangling points at ∂A with two disks of radius $\varepsilon \ll 1$, acting as UV cutoffs at which the boundary conditions a and b are imposed (Fig. 1, upper panel). This manifold is mapped into an annulus of length $W = 2\log(\ell/\varepsilon) + \mathcal{O}(\varepsilon)$ and circumference 2π ($2\pi n$, after replicating) by a conformal transformation (Fig. 1, lower panel) where the space-time is periodic in one direction and the $|a\rangle$ and $|b\rangle$ states are defined at the ε boundaries. In this geometry, traces of ρ_A^n are evaluated in terms of BCFT partition functions as [21,22]

$$Z_n[q^n] = \mathrm{Tr}_{ab}[\rho_A^n] = \frac{Z_{ab}[q^n]}{Z_{ab}[q]^n},$$
$$Z_{ab}[q] = \mathrm{Tr}_{ab}\rho_A = \mathrm{Tr}_{ab}[q^{(L_0 - c/24)}], \quad (6)$$

with the Virasoro zero mode L_0 and the central charge c. Here, $\mathrm{Tr}_{ab} \equiv \mathrm{Tr}_{\mathcal{H}_A,ab}$ refers to a trace taking into account the nontrivial boundary conditions and $q = e^{2\pi i\tau}$ is the nome with the modular parameter $\tau = i\pi/W$. Therefore, $q = e^{-2\pi^2/W}$ and $\tilde{q} = e^{-2W}$, with $\tilde{q}$ obtained after a modular S transformation, $S\colon \tau \to \tilde{\tau} = -1/\tau$.

After imposing the Hilbert space decomposition $\mathcal{H} = \mathcal{H}_{A,ab} \otimes \mathcal{H}_{B,ba}$, the remaining symmetry algebra in a CFT with a global symmetry is called $\mathcal{A}$ and

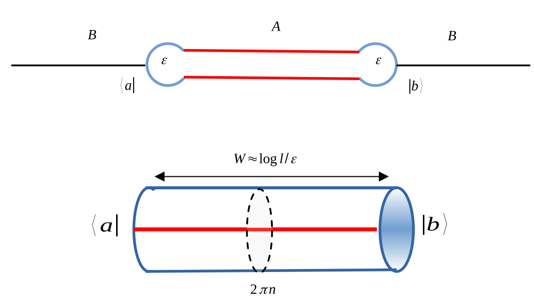


FIG. 1. The factorization ab imposes disks $\varepsilon \ll 1$ with boundary conditions a, b (upper panel). The resulting manifold is replicated and after tracing over $\mathcal{H}_{B,ba}$, a conformal transformation yields an annulus of width W and circumference $2\pi n$.

$\mathcal{H}_{A,ab} = \bigoplus_i \mathcal{H}_i^{n_{ab}^i}$ with i running over the allowed representations of $\mathcal{A}$ and the multiplicities n_{ab}^i depending on the boundary conditions a and b. Then, the replica BCFT partition functions can be written in terms of the characters $\chi_i(q) = \mathrm{Tr}_{\mathcal{H}_i}[q^{(L_0-c/24)}]$ for the representation i,

$$Z_{ab}[q^n] = \sum_i n_{ab}^i \chi_i(q^n) = \langle a|\tilde{q}^{\frac{1}{n}(L_0-c/24)}|b\rangle. \quad (7)$$

The last equality is obtained after a modular transformation to the S-dual channel where the boundary condition dependence explicitly appears in terms of Cardy conformal boundary states

$$|a\rangle = \sum_j \frac{S_{aj}}{\sqrt{S_{0j}}}|j\rangle\rangle, \quad (8)$$

with $|j\rangle\rangle$ being an Ishibashi state for the jth representation of $\mathcal{A}$ [23], and coefficients S_{aj} are elements of the modular matrix $\mathcal{S}$ of the CFT [21,24].

III. SREE FOR GROUP SYMMETRIES

The symmetry resolution of entanglement entropy of Abelian group symmetries has been well studied previously [25–28]. The projectors into different irreducible representations of a finite group G are given by

$$\Pi^r = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\,\widehat{\mathcal{L}}_g = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g) \xrightarrow{\quad\widehat{\mathcal{L}}_g\quad}, \quad (9)$$

where r labels the irreps of G and thus the different Q-charge sectors, d_r is the dimension of the irrep, $|G|$ is the order of the group, $\chi_r^*(g)$ is the character of the element of $g\in G$ in the irrep r and $\mathcal{L}_g$ is the topological operator implementing the action of g on states supported on the region A.

Using projectors (9) one may write the partition function associated to a charge sector labeled by r in Eq. (4) as

$$Z_{ab}[q^n, r] = \mathrm{Tr}_{ab}\Pi^r\rho_A^n = \frac{d_r}{|G|}\sum_{g\in G}\chi_r^*(g)\frac{Z_{ab}[q^n, g]}{Z_{ab}^n[q]},$$
$$Z_{ab}[q^n, g] = \mathrm{Tr}[\hat{\mathcal{L}}_g q^{n(L_0-c/24)}], \quad (10)$$

where the explicit action of the topological operator $\hat{\mathcal{L}}_g$ is encoded in the charged moment $Z_{ab}[q^n, g]$ (one for each element of the group). Here, we use $\mathcal{L}_g$ for a topological line in Euclidean spacetime, and $\hat{\mathcal{L}}_g$ for the corresponding operator acting on the Hilbert space. As before, one may express $Z_{ab}[q^n, g]$ in the S-dual channel in terms of boundary states $|a\rangle$ and $|b\rangle$ as

$$Z_{ab}[q^n, g] =_g \langle a|\tilde{q}^{\frac{1}{n}(L_0-c/24)}|b\rangle_g, \quad (11)$$

where the subindex g represents that the states belong to the Hilbert space generated by inserting the operator $\hat{\mathcal{L}}_g$ as a defect operator in the original theory, which is the defect or twisted Hilbert space $\mathcal{H}_{\mathcal{L}_g}$. Thus, in this approach, computing SREE reduces to find suitable boundary states $|a\rangle_g$ and $|b\rangle_g$. Namely, as $Z_{ab}[q^n, g]$ is defined through the insertion of $\mathcal{L}_g$ in the annulus partition function, it is required that $\mathcal{L}_g$ can end topologically on the boundary of the interval which imposes a constraint on the allowed boundary states in the dual S channel [16].

For invertible group symmetries the topological endability is equivalent to having G invariant boundary states. A natural definition in the S-dual channel for a (conformal) boundary a to be G symmetric is

$$\hat{\mathcal{L}}_h|a\rangle_g = |a\rangle_g, \quad \forall\ h\in G. \quad (12)$$

For finite groups the result at leading order in the limit when $\varepsilon \ll \ell$ (where $q \to 1$ and $\tilde{q} \to 0$) is quite simple. There, the main contribution comes from the untwisted sector [29], that is to say, the vacuum state propagation is the major contribution to the amplitude in the S-dual channel and the SREE reads as [25–28]

$$S_A[q, r] = \frac{c}{3}\log\frac{\ell}{\varepsilon} + \log\frac{d_r^2}{|G|} + \mathrm{g}_a + \mathrm{g}_b, \quad (13)$$

where $\mathrm{g}_a = \log\langle 0|a\rangle$, $\mathrm{g}_b = \log\langle 0|b\rangle$ are the Affleck-Ludwig boundary entropies [30], and d_r is the dimension of the irrep r. The term $\mathcal{O}(\log \ell/\varepsilon)$ captures the equipartition of EE among distinct charge sectors, primarily at the leading order. This equal distribution is broken by the term of order $\mathcal{O}((\ell/\varepsilon)^0)$ by the negative term $\log p_r$, with $p_r = d_r^2/|G|$ representing the probability of measuring the representation r within block A, a scenario denoted as weak entanglement equipartition to distinguish it from the strong equipartition. A parallel outcome was observed in the examination of Wess-Zumino-Witten models [31].

IV. CATEGORICAL-SYMMETRY RESOLVED ENTANGLEMENT ENTROPY

We propose the symmetry resolution of entanglement for CaT-SREE in analogy with the BCFT approach for grouplike invertible symmetries. For this, it is necessary to define topological endability and thus, symmetric boundary conditions, for the case of (categorical) noninvertible symmetries. These have been proposed in [16] through the notions of strongly symmetric and weakly symmetric boundary states. While these two concepts are equivalent for invertible grouplike symmetries, they diverge for categorylike noninvertible symmetries.

Recalling the fusion algebra in Eq. (1), we focus on the finite subset of boundary conditions $\{a\}_{a\in\mathcal{B}}$ with $\mathcal{B}$ labeling these boundaries, related by the action of a finite symmetry fusion category $\mathcal{C}$. The corresponding boundary states are denoted as $\{|a\rangle\}$. This is known as a module category and the action of $\mathcal{L}_i$ acting on such a class of boundary a is given by

$$\mathcal{L}_i \otimes a = \bigoplus_{b\in\mathcal{B}} \tilde{N}^b_{ia} b, \tag{14}$$

where $\tilde{N}^b_{ia} \in \mathbb{Z}_{\geq 0}$. With this, two notions of $\mathcal{C}$-symmetric boundary states can be established [16]: A conformal boundary condition a is $\mathcal{C}$-strongly symmetric if the corresponding boundary state $|a\rangle$ is an eigenstate under the action of $\mathcal{C}$ with eigenvalues given by the quantum dimensions $\langle\mathcal{L}_i\rangle$,

$$\hat{\mathcal{L}}_i|a\rangle = \langle\mathcal{L}_i\rangle|a\rangle \quad \forall\ \mathcal{L}_i \in \mathcal{C}. \tag{15}$$

This definition reduces to a G-symmetric boundary condition in the case of grouplike invertible symmetries. On the other hand it is considered that a conformal boundary condition a is $\mathcal{C}$-weakly symmetric if every topological line in $\mathcal{C}$ can end topologically on a. Operationally speaking this means that $\tilde{N}^a_{ia} \geq 1$ for every $\mathcal{L}_i$ in $\mathcal{C}$, which implies

$$\hat{\mathcal{L}}_i|a\rangle = |a\rangle \oplus \cdots \quad \forall\ \mathcal{L}_i \in \mathcal{C}. \tag{16}$$

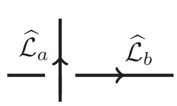
This second notion of $\mathcal{C}$-symmetric boundary condition relax enough the requirements for finding the appropriate boundary states needed to define SREE for fusion categorical noninvertible symmetries.

A. CaT-SREE in RCFT

The simplest models to define the SREE for fusion CaT-SREE are two-dimensional RCFTs, for which there exists a correspondence between Verlinde lines and bulk primary operators [32]. Thus, each line representing a symmetry of the model is associated with one primary operator, and their fusion rules are those given by the operator product expansion coefficients of the corresponding primaries.

The first step to define CaT-SREE is to write a full set of projectors associated to the elements of fusion category $\mathcal{C}$ in terms of elements of the modular matrix $\mathcal{S}$ of the CFT [33]:

$$\Pi_a{}^c = \sum_b S_{0c}\bar{S}_{bc}$$

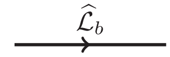


, (17)

where $\{\hat{\mathcal{L}}_a\}_{a\in\mathcal{V}} \in \mathcal{C}$. The lines pictorially represent the Verlinde lines of the RCFT inserted either along the time direction in the annulus (vertical ones), which twist the Hilbert space of the theory, or along the spatial direction, which amounts to charged operators acting over the states on the Hilbert space (horizontal ones) [11].

As we are interested in resolving EE on the original Hilbert space of the theory, we will consider only projectors of the form $\Pi^c_{\mathbb{1}}$. Here, we write these projectors in full analogy with the grouplike symmetry case Eq. (9) as

$$\Pi_{\mathbb{1}}{}^c := \Pi^c = \frac{d_c}{|\mathcal{C}|}\sum_{b\in\mathcal{C}} \chi^*_c(b)$$



, (18)

by defining $d_c = \frac{S_{0c}}{S_{00}}$ as the quantum dimension of the line $\hat{\mathcal{L}}_c$, the order of the category $|\mathcal{C}| = \sum_c d_c^2$, and the characters $\chi^*_c(b) = \frac{\bar{S}_{bc}}{S_{00}}$.

We note that the projectors in Eq. (18) are written for a simple element of the category $\mathcal{L}_c \in \mathcal{C}$. However, the element labeling an irrep of the category is not, in general, a simple object and may be described by nonsimple topological lines whose associated projectors can be written as (18) [34,35].

Thus, in analogy with grouplike invertible symmetries, we define the CaT-SREE in terms of the partition functions

$$Z_{c_1c_2}[q^n, a] = \mathrm{Tr}_{c_1c_2}[\Pi^a\rho_A^n] = \frac{d_a}{|\mathcal{C}|}\sum_{b\in\mathcal{C}}\chi^*_a(b)\frac{Z_{c_1c_2}[q^n, b]}{(Z_{c_1c_2}[q])^n}, \tag{19}$$

where the generalized charged moment in the S-dual channel is defined as

$$Z_{c_1c_2}[q^n, b] = {}_b\langle c_1|\tilde{q}^{\frac{1}{n}(L_0 - c/24)}|c_2\rangle_b, \tag{20}$$

and $|c_{1,2}\rangle_b$ are Cardy boundary states in the $\mathcal{C}$-weakly symmetric sense exposed above.

We illustrate our definition with two examples, the critical Ising model and the tricritical Ising model.

V. THE CRITICAL ISING MODEL

The critical Ising model is described by a $(1+1)$-dimensional RCFT with a central charge $c = \frac{1}{2}$. There are three primary operators in the model: the identity $\mathbb{1}$, the energy field ϵ, and the spin field σ. The symmetries of this model are described by three Verlinde lines: $\{\hat{\mathbb{1}}, \hat{\eta}\}$ which conform the usual $\mathbb{Z}_2$ symmetry of the Ising model, and $\hat{\mathcal{N}}$ that implements the Kramers-Wannier duality [4,5]. These lines follow the fusion rules of the Ising category

$$\eta\times\eta = \mathbb{1}, \qquad \mathcal{N}\times\mathcal{N} = \mathbb{1}+\eta, \qquad \eta\times\mathcal{N} = \mathcal{N} \tag{21}$$

As discussed above, to obtain the CaT-SREE one must first compute the set of the $\mathcal{C}_{\text{Ising}}$-symmetric Cardy states through (8). In doing so, it is noticed that there are three simple boundary states in this model,

$$
\begin{aligned}
|\mathbb{1}\rangle &= \frac{1}{\sqrt{2}}|\mathbb{1}\rangle\!\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle\!\rangle + \frac{1}{2^{1/4}}|\sigma\rangle\!\rangle,\\
|\epsilon\rangle &= \frac{1}{\sqrt{2}}|\mathbb{1}\rangle\!\rangle + \frac{1}{\sqrt{2}}|\epsilon\rangle\!\rangle - \frac{1}{2^{1/4}}|\sigma\rangle\!\rangle,\\
|\sigma\rangle &= |\mathbb{1}\rangle\!\rangle - |\epsilon\rangle\!\rangle. \qquad (22)
\end{aligned}
$$

The boundary states $|\mathbb{1}\rangle$, $|\epsilon\rangle$, and $|\sigma\rangle$ conform the Ising, or more technically, the Tambara-Yamagami $\mathrm{TY}_+\mathbb{Z}_2$ fusion category, that is a regular module category. Therefore, there is a one-to-one correspondence between the boundary states (22) and the Verlinde lines such that $|\mathbb{1}\rangle \equiv \hat{\mathbb{1}}, |\epsilon\rangle \equiv \hat{\eta}, |\sigma\rangle \equiv \hat{\mathcal{N}}$ with

$$
\begin{aligned}
&\hat{\eta}|\mathbb{1}\rangle = |\epsilon\rangle, \qquad \hat{\eta}|\epsilon\rangle = |\mathbb{1}\rangle, \qquad \hat{\eta}|\sigma\rangle = |\sigma\rangle,\\
&\hat{\mathcal{N}}|\mathbb{1}\rangle = |\sigma\rangle, \qquad \hat{\mathcal{N}}|\epsilon\rangle = |\sigma\rangle, \qquad \hat{\mathcal{N}}|\sigma\rangle = |\mathbb{1}\rangle \oplus |\epsilon\rangle. \qquad (23)
\end{aligned}
$$

We note that only $|\sigma\rangle$ is invariant under the action of the group $\mathbb{Z}_2$. In this sense, $|\sigma\rangle$ is a $\mathbb{Z}_2$-symmetric state, and thus it is possible to use it to compute the SREE for the $\mathbb{Z}_2$ grouplike symmetry of the model [27]. However, none of the three boundary states are symmetric, neither in the strong nor weak sense, under the action of $\mathcal{N}$. As a result, it is not possible to define the CaT-SREE in the critical Ising model. A fusion category can only admit a strongly symmetric boundary if it is anomaly free, while it admits a weakly symmetric boundary if and only if it can be "gauged" in the generalized sense posed in [16]. Being $\mathcal{C}_{\mathrm{Ising}}$ anomalous, hence the impossibility of defining the CaT-SREE from the obstruction to gauging it, as just happens in the case of group-like symmetries [27,36].

VI. THE TRICRITICAL ISING MODEL

The tricritical Ising model is a RCFT with central charge $c = \frac{7}{10}$. The model is composed by six primary operators and six lines. In addition to the trivial line $\mathbb{1}$ and the $\mathbb{Z}_2$ invertible line η, there are four more simple lines, W, ηW, $\mathcal{N}$, and $W\mathcal{N}$. Nontrivial fusion rules for these lines are given by

$$
\begin{aligned}
&\eta \times \eta = \mathbb{1}, \qquad \mathcal{N} \times \mathcal{N} = \mathbb{1} + \eta,\\
&\eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N}, \qquad W \times W = \mathbb{1} + W. \qquad (24)
\end{aligned}
$$

From these relations we can identify $\{\mathbb{1}, \eta, \mathcal{N}\}$ as a $\mathrm{TY}_+\mathbb{Z}_2$ subcategory and $\{\mathbb{1}, W\}$ a Fibonacci subcategory $\mathcal{C}_{\mathrm{Fib}}$. Same as with critical Ising model, for the first group of lines we cannot find symmetric boundary conditions, neither strong nor weak. However, that is not the case for $\mathcal{C}_{\mathrm{Fib}}$ that is the simplest example of a category that can be gauged, and therefore admits a weakly symmetric boundary. Namely, there are three boundary states that are weakly symmetric under $\mathcal{C}_{\mathrm{Fib}}$:

$$
\begin{aligned}
&\hat{W}|W\rangle = |W\rangle \oplus |\mathbb{1}\rangle, \qquad \hat{W}|\eta W\rangle = |\eta W\rangle \oplus |\eta\rangle,\\
&\hat{W}|W\mathcal{N}\rangle = |W\mathcal{N}\rangle \oplus |\mathcal{N}\rangle. \qquad (25)
\end{aligned}
$$

For simplicity, we choose to work with the boundary condition $|W\mathcal{N}\rangle$. Through Eq. (8), this state can be written as

$$
|W\mathcal{N}\rangle = \frac{1}{\sqrt{N}}\left(|\mathbb{1}\rangle\!\rangle + \varphi^{-3/2}|\epsilon\rangle\!\rangle - \varphi^{-3/2}|\epsilon'\rangle\!\rangle - |\epsilon''\rangle\!\rangle\right), \qquad (26)
$$

with $\varphi = \frac{1+\sqrt{5}}{2}$ the golden ratio and $N = (\frac{10}{5+2\sqrt{5}})^{1/2}$. Thus, the charged moment associated to the untwisted sector is given in terms of the Virasoro characters by (see Supplemental Material [37]):

$$
\begin{aligned}
Z_{W\mathcal{N}}[q^n, \mathbb{1}] = \frac{1}{N}\Big[&\chi_0(\tilde{q}^{\frac{1}{n}}) + \varphi^{-3}\chi_{\frac{1}{10}}(\tilde{q}^{\frac{1}{n}})\\
&+ \varphi^{-3}\chi_{\frac{3}{5}}(\tilde{q}^{\frac{1}{n}}) + \chi_{\frac{3}{2}}(\tilde{q}^{\frac{1}{n}})\Big], \qquad (27)
\end{aligned}
$$

where, for notational convenience, we use $Z_{aa} \to Z_a$.

In order to compute the charged moment $Z_{W\mathcal{N}}[q^n, W]$, we need an explicit expression of the boundary state $|W\mathcal{N}\rangle$ twisted by the introduction of the Verlinde line $\hat{W}$ as a defect operator. In general, this new boundary state is given by a combination of twisted Ishibashi states, that is, conformal scalars on the W-twisted Hilbert space. The twisted W-Hilbert space contains nine primary operators; among them there are three scalars, ϵ_W, ϵ'_W, σ_W with conformal weights $\frac{1}{10}$, $\frac{3}{5}$, and $\frac{3}{80}$, respectively. This implies that the $\mathcal{C}_{\mathrm{Fib}}$-symmetric twisted Cardy state is a linear combination of the twisted Ishibashi states associated with these operators,

$$
|W\mathcal{N}\rangle_W = \alpha_1|\epsilon\rangle\!\rangle_W + \alpha_2|\epsilon'\rangle\!\rangle_W + \alpha_3|\sigma\rangle\!\rangle_W, \qquad (28)
$$

for some fixed coefficients α_i. With this state, the twisted charged moment is

$$
Z_{W\mathcal{N}}[q^n, W] = \alpha_1^2\chi_{\frac{1}{10}}(\tilde{q}^{\frac{1}{n}}) + \alpha_2^2\chi_{\frac{3}{5}}(\tilde{q}^{\frac{1}{n}}) + \alpha_3^2\chi_{\frac{3}{80}}(\tilde{q}^{\frac{1}{n}}). \qquad (29)
$$

Comparing the two contributions for the CaT-SREE of the $\mathcal{C}_{\mathrm{Fib}}$ subcategory of this model, one notices that $Z_{W\mathcal{N}}[q^n, \mathbb{1}]$ dominates over $Z_{W\mathcal{N}}[q^n, W]$ at leading order in the ε expansion (see the Supplemental Material [37]).

In order to find the CaT-SREE, we note that there are 2 irreps of $\mathcal{C}_{\mathrm{Fib}}$ that we label with $\mathrm{r}_{\mathcal{C}} = \{A, B\}$. With this, one may write the projectors associated to those representations as a combination of projectors of the form (18),

$$
\begin{aligned}
\Pi^A &= \Pi^{\mathbb{1}} + \Pi^{\eta} + \Pi^{\mathcal{N}},\\
\Pi^B &= \Pi^W + \Pi^{\eta W} + \Pi^{W\mathcal{N}}, \qquad (30)
\end{aligned}
$$

which explicitly read as

$$\Pi^{\mathrm{r}_{\mathcal{C}}} = \frac{d_{\mathrm{r}_{\mathcal{C}}}}{|\mathcal{C}|}\left(d_{\mathrm{r}_{\mathcal{C}}} \xrightarrow{\quad \hat{\mathbb{1}} \quad} + \chi^{*}_{\mathrm{r}_{\mathcal{C}}}(W) \xrightarrow{\quad \widehat{W} \quad} \right), \tag{31}$$

where $d_A = 1$ and $d_B = \varphi$ are the quantum dimensions of the Fibonacci anyons, the total quantum dimension is given by the usual formula $|\mathcal{C}| = d_A^2 + d_B^2 = 1 + \varphi^2$, and the characters are $\chi_A^*(W) = \varphi$ and $\chi_B^*(W) = -1$.

These projectors coincide with those characterizing a theory with Fibonacci anyons, those describing the low-energy physics of the fractional quantum Hall effect at filling factor $\nu = 5/2$ [38].

As there are two simple anyons in this category, there are two projectors of the type of Eq. (18). This ensures that the projectors (31) successfully project into the irreps of the Fibonacci subcategory $\mathcal{C}_{\mathrm{Fib}}$.

With this, the partition functions associated to each charged sector (10) are given by

$$Z[q^n, \mathrm{r}_{\mathcal{C}}] = \frac{d_{\mathrm{r}_{\mathcal{C}}}}{|\mathcal{C}|}\left(d_{\mathrm{r}_{\mathcal{C}}} \frac{Z[q^n, \mathbb{1}]}{(Z^n[q]} + \chi^*_{\mathrm{r}_{\mathcal{C}}}(W) \frac{Z[q^n, W]}{Z^n[q])} \right), \tag{32}$$

where the subscript $W\mathcal{N}$ has been suppressed for convenience. From these, the CaT-SREE at leading order for both irreps $\mathrm{r}_{\mathcal{C}} = \{A, B\}$ reads as

$$S[q, \mathrm{r}_{\mathcal{C}}] = \frac{c}{3}\log\frac{\ell}{\varepsilon} + \log\frac{d^2_{\mathrm{r}_{\mathcal{C}}}}{|\mathcal{C}|} + 2\mathrm{g}_{W\mathcal{N}}, \tag{33}$$

with $\mathrm{g}_{W\mathcal{N}} = \log\langle\langle \mathbb{1}|W\mathcal{N}\rangle$ the Affleck-Ludwig boundary entropy. This is the main result in this work. Formally, it is analogous to the one obtained for finite groups (13), that is, the CaT-SREE at leading order in the UV cutoff is equally distributed among the different (Fibonacci anyon) charge sectors A and B. Similarly to invertible grouplike symmetries, the entanglement equipartition is broken by constant terms related to the quantum dimension $d_{\mathrm{r}_{\mathcal{C}}}$ and the boundary entropy.

We note that the same entanglement resolution can be obtained for the tetracritical Ising model, whose topological lines generate the nonanomalous symmetry $\mathrm{Rep}(S_3)$ that possesses weakly symmetric boundary states under the subcategory $\mathcal{C}_{\mathrm{Fib}}$. Finally, we remark that, in some cases, the CaT-SREE can be obtained in terms of strongly symmetric states, as, for instance, the double Ising CFT, which admits a strongly symmetric boundary condition with respect to the categorical symmetry $\mathrm{Rep}(H_8)$ [16].

VII. CONCLUSIONS

We have shown how categorical symmetries shape the entanglement structure in CFTs by proposing a symmetry resolution of entanglement for these symmetries. Our proposal has then provided new insights into the relationship between EE and boundary conditions. As the EE of a topological phase reflects the fusion rules of anyons and their braiding statistics, a deeper understanding of the imprint of categorical symmetries on entanglement provides a future route to characterize topological order. It is also interesting to understand in future works, the microscropic description of CaT-SREE by exploring the entanglement properties of noninvertible operators on the lattice [39–42].

ACKNOWLEDGMENTS

We thank Nabil Iqbal, Sridip Pal, Thomas Bartsch, Matthew Bullimore, Karan Bhatia, Tudor Dimofte, Sašo Grozdanov, Giorgio Frangi, Joan Simon Soler, Naveen Balaji Umasankar, and Mile Vrbica for insightful discussions on this project. The work of P. S.-B. is supported by Fundación Séneca de la Región de Murcia, Grant No. 21609/FPI/21. A. D. is supported by the STFC Consolidated Grant No. ST/T000600/1—"Particle Theory at the Higgs Centre." J. M.-V. thanks Programa Recualificación del Sistema Universitario Español 2021–2023. P. S.-B. and J. M.-V. thank the financial support of Spanish Ministerio de Ciencia e Innovación PID2021-125700NAC22. G.-S. acknowledges funding from Grant No. PID2021-127726NB-I00.

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