Partially celestial states and their scattering amplitudes

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We study representations of the Poincaré group that have a privileged transformation law along a p-dimensional hyperplane, and uncover their associated spinor-helicity variables in D spacetime dimensions. Our novel representations generalize the recently introduced celestial states and transform as conformal primaries of $SO(p,1)$, the symmetry group of the p-hyperplane. We will refer to our generalized states as "partially celestial." Following Wigner's method, we find the induced representations, including spin degrees of freedom. Defining generalized spinor-helicity variables for every D and p, we are able to construct the little group covariant part of partially celestial amplitudes. Finally, we briefly examine the application of the pairwise little group to partially celestial states with mutually nonlocal charges.

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I. INTRODUCTION

The classification of particles into representations of the Poincaré group is the basis of particle physics and quantum field theory (QFT): it allows for the definition of the S-matrix and scattering amplitudes and the easy identification of the propagating degrees of freedom of QFTs. The little group appearing in Wigner's method of induced representations is essential for building proper scattering amplitudes and forms the basis of modern scattering amplitude methods. While little group methods have by now become commonly used for particle scattering, it has not been widely applied to the description of the dynamics of branes. In this paper we initiate the first steps toward this direction. While we do not consider the seemingly formidable task of quantizing p-branes, we will consider a simpler situation where branelike objects appear and their general states can be constructed using Wigner's method. To achieve this we will define a new eigenbasis of ordinary quantum fields in D dimensions that have privileged transformation properties on a p-hyperplane. We call this state a *p-sheet* or a *p partially celestial* state, for reasons that will become obvious below. Though not quite a p-brane, the p-sheet does serve as an interesting toy model for p-branes, as it highlights the importance of $SO(D-p-1)$ transverse rotations, a feature that we expect to play a key role in a future "Wigner" quantization of p-branes.

Our starting point will be to look for states that (a) have well-defined $SO(p,1)$ transformation properties, reflecting the symmetry of a $p+1$-worldsheet, and (b) are not *zero-energy eigenstates*. In fact, these two requirements imply that our p-sheet states are not energy eigenstates at all. As we shall see in detail below, the $SO(p,1)$ covariance of our p-sheets makes them the analogs of the celestial states considered in [1–7], except only along p directions; hence they are "partially celestial." We will find the appropriate eigenbasis of these states, and also find the correct labels for characterizing p-sheet quantum states. With our knowledge of the little group and canonical Lorentz transformations we can use Wigner's method of induced representations to build up the full p-sheet Hilbert space. We are also able to present for the first time the generalized spinor-helicity variables in any spacetime dimension, which has applications far beyond those presented here and is the most far reaching result in this paper. These variables allow us to construct the most general three-point amplitudes for partially celestial states. We also briefly consider how the recently introduced pairwise little group [8–11] can be generalized to p-sheets. For the case of mutually nonlocal sheets of dimension p and $D-p-4$ we show that the pairwise little group is just a $U(1)$, providing a new example of pairwise helicity, which can be dynamically realized if $p+1$-form electrodynamics is electrically coupled to the p-sheet and magnetically coupled to the dual $D-p-4$ sheet.

The paper is organized as follows. First, we briefly review the celestial solutions [1–7] of the Klein-Gordon (KG) equation, which are solutions that transform covariantly with respect to $SO(D-1,1)$, viewed as the Euclidean conformal group. Using celestial solutions as an inspiration, we then present solutions of the d-dimensional KG equation which are $SO(p,1)$ covariant, reflecting the symmetry of a $p+1$-world volume. In their "rest" frame, these solutions are also manifestly $SO(D-p-1)$ rotationally invariant and R^{D-p-1} translationally invariant in the space orthogonal to the p-sheet. Next, we show how to interpret these $SO(p,1)$ covariant solutions of the KG equation as the *wave functions* of p partially celestial quantum states, thus constructing their Hilbert space. The generalization to spinning p-sheets is then achieved using Wigner's method of induced representations. We then construct spinor-helicity variables that allow us to write the most general three-point amplitudes. Finally, we present the pairwise little group of two parallel sheets and argue that for mutually nonlocal sheets the pairwise little group reduces to a $U(1)$ pairwise helicity.

II. PLANE WAVES AND CELESTIAL SCALARS

In preparation for presenting our p-sheet states we will first review the construction of the celestial scalars and their relation to plane waves. Consider first a massive classical scalar field $\phi(x)$ in D-dimensions. Its equation of motion is the KG equation (in a mostly plus signature as is commonly used in the celestial literature)

$$[-\partial_t^2 + \nabla^2 + m^2]\phi(x) = 0. \tag{1}$$

The most commonly used basis of solutions is the plane wave basis $\phi_p(x) = e^{\pm ip\cdot x}$. Each solution $\phi_p(x)$ in this basis is translationally invariant in $D-1$ directions $x^\mu \to x^\mu + \Delta x^\mu$ orthogonal to p^μ, $p\cdot\Delta x = 0$. One could instead look for solutions of (1) which are $SO(D-1,1)$ covariant —these are the massive celestial scalars [1,12] $\phi_\Delta(x;\vec{w})$. Instead of the p^μ labels, these solutions are labeled by a conformal dimension Δ and a vector $\vec{w}$ on R^d, where $d \equiv D-2$. Explicitly, they are given by

$$\phi_\Delta^{\pm,PS}(x;\vec{w}) = \frac{2^{\frac{d}{2}+1}\pi^{\frac{d}{2}}}{(im)^{\frac{d}{2}}}\frac{(is)^\alpha}{[-q(\vec{w})\cdot x \mp i\epsilon]^\Delta}K_\alpha(ms)$$
$$s = \sqrt{x\cdot x}, \qquad \alpha = \Delta - \frac{d}{2}. \tag{2}$$

The label PS here is to remind us that these are the celestial wave functions defined in [1]. Here

$$q^\mu(\vec{w}) = (1+|\vec{w}|^2, 2\vec{w}, 1-|\vec{w}|^2) \tag{3}$$

is a $D = d+2$ dimensional vector. A Lorentz transformation Λ acting on q^μ induces a nonlinear map $\Lambda: \vec{w} \to \vec{w}'$ via

$$q^\mu(\vec{w}') = \left|\frac{\partial\vec{w}'}{\partial\vec{w}}\right|^{1/d}\Lambda^\mu_\nu q^\nu(\vec{w}). \tag{4}$$

The map $\Lambda: \vec{w} \to \vec{w}'$ nonlinearly realizes $SO(d-1,1)$ as the Euclidean conformal group acting on $\vec{w} \in R^{d-2}$. By substituting (4) in (2), one can easily check that these solutions have the property that

$$\phi_\Delta(\Lambda^\mu_\nu x^\nu, \vec{w}'(\vec{w})) = \left|\frac{\partial\vec{w}'}{\partial\vec{w}}\right|^{-\frac{\Delta}{d}}\phi_\Delta(x^\nu,\vec{w}), \tag{5}$$

The solutions (2) form a complete eigenbasis for the KG equation for either $\Delta = \frac{d}{2} + i\mathbb{R}$ or $0 < \Delta < 1$, also called the *principal series* and *complementary series* representations of $SO(D-1,1)$, respectively.

The celestial wave functions for massless scalars were obtained in [1] by taking the massless limit of (2). In [13], an equivalent construction of the celestial states for massless particles in Four-dimensional (4D) was presented. The latter followed Wigner's method of induced representations, by starting from a reference quantum state whose little group is the "lower triangular" group $\{J_3, K_3, J_2-K_1, -J_1-K_2\}$. Inspired by this construction, we present a slightly modified derivation of the solution (2) using little group methods (see also a parallel discussion for the massless case in the very recent [14]), which will be easily generalized to our partially celestial p-sheet solutions. First, we redefine the expression in (2), as a function of q^μ rather than $\vec{w}$,

$$\phi_\Delta^\pm(x;q) = \frac{2^{\frac{d}{2}+1}\pi^{\frac{d}{2}}}{(im)^{\frac{d}{2}}}\frac{(is)^\alpha}{[-q\cdot x \mp i\epsilon]^\Delta}K_\alpha(ms)$$
$$s = \sqrt{x\cdot x}, \alpha = \Delta - \frac{d}{2}. \tag{6}$$

Note that this is a solution to the KG equation *only for null* q^μ. Naturally, we define a reference value of q^μ as

$$q^\mu_{\rm ref} \equiv (1,0,\ldots,0,1), \tag{7}$$

and note that this is a lightlike Lorentz D-vector as opposed to the $D-2$ vector $\vec{w}$. Accordingly, a "reference wave function" is

$$\phi_\Delta^{\pm,{\rm ref}}(x) \equiv \phi_\Delta^\pm(x;q_{\rm ref}). \tag{8}$$

Here $q^\mu_{\rm ref}$ is chosen so that $\phi_\Delta^{\pm,{\rm ref}}(x)$ is $SO(D-2)$ rotationally invariant, but it is also manifestly invariant under the linear combinations $M_{i,D-1} - M_{0,i}, i \in \{1,\ldots,D-1\}$ of $SO(D-1,1)$ generators. Overall, the little group under which $\phi_\Delta^{\pm,{\rm ref}}(x)$ is invariant is given by

$$LG^D = ISO(D-2), \tag{9}$$

where $\dim(LG^D) = \frac{(D-1)(D-2)}{2}$. Under general transformations in the Poincaré group $\mathcal{P}^D = \mathbb{R}^D \rtimes SO(D-1,1)$, the celestial scalar transforms as

$$\Omega = (\Lambda, v) \in \mathcal{P}^D:$$
$$\phi_\Delta^{\pm,{\rm ref}}(x) \to \phi_\Delta^{\pm,\Omega}(x) = \phi_\Delta^{\pm,{\rm ref}}(\Lambda(x+v)) = \phi_\Delta^\pm(x+v;\Lambda^{-1}q). \tag{10}$$

Without loss of generality, we perform the translation before the Lorentz transformation. Note also that we could further Taylor-expand the last line of (10) in v^μ, and see that the action of an internal translation P^μ shifts $\Delta \to \Delta + 1$ [15–17], but we will not do it explicitly in this paper. Importantly, not all Poincaré transformations actually lead to inequivalent solutions for ϕ—to label inequivalent solutions we have to mod out by the action of the little group. This is the same as finding canonical transformations $\mathcal{O} \in \mathcal{P}^D/LG^D$. From (10), we can see that the generic partially celestial scalar solution is parametrized by all possible q^μ that can be reached from $q^\mu_{\rm ref}$ by Lorentz transformations. These are one boost and $p-1$ rotations with angles α_k that take q^μ to a generic value $(\gamma, \gamma\vec{\beta})$. Hence, the canonical Lorentz transformations are

$$q^\mu = [L_q]^\mu{}_\nu q^\nu_{\rm ref},$$
$$L_q = \prod_{k=1}^{D-3} R_{k,k+1}(\alpha_k) \times B_p(\beta). \tag{11}$$

Here $R_{k,k+1}(\alpha_k)$ is a rotation by the angle α_k in the plane spanned by the k and $k+1$ directions, while $B_p(\beta)$ is a boost with velocity β along the p direction. Thus, the partially celestial scalar solution is parametrized by one boost, $D-2$ angles, and D translations, for a total of $2D-1$ parameters of the coset $\mathcal{P}^D/LG^D$. One can easily check that this is indeed the correct dimension of this coset. There is a one-to-one correspondence between the original $\phi^{\pm,PS}_\Delta$ and $\phi^{\pm,\mathcal{O}}_\Delta(x)$ for $\mathcal{O} \in \mathcal{P}^D/LG^D$. To see this, note that every q^μ in (11) uniquely defines a $\vec{w}$, and vice versa.

III. THE PARTIALLY CELESTIAL SCALAR SOLUTION

In this paper, we are interested in representations of the D-dimensional Poincaré group $\mathcal{P}^D = \mathbb{R}^D \rtimes SO(D-1,1)$ which transform covariantly under a $p+1$-dimensional Lorentz subgroup $SO(p,1) \subset \mathcal{P}^D$. These correspond to ordinary quantum fields on p-dimensional hypersurfaces in d dimensions, which we call p-sheets for short. As a first step, we would like to construct the most general $SO(p,1)$ covariant solutions to (1), drawing inspiration from the celestial solutions (2). These would be akin to celestial solutions in p "sheet-parallel" dimensions, while being translationally invariant in $D-p$ "external" directions. To this end, we define p *partially celestial* scalars as

$$\phi^\pm_{\Delta;p}(x;q,A) = \frac{2^{\frac{p-1}{2}+1}\pi^{\frac{p-1}{2}}}{(im)^{\frac{p-1}{2}}} \frac{(is)^\alpha}{[-q\cdot x \mp i\epsilon]^\Delta} K_\alpha(ms),$$
$$s = \sqrt{x_\mu A^{\mu\nu} x_\nu}, \qquad \alpha = \Delta - \frac{p-1}{2}, \tag{12}$$

where q^μ is a null D-dimensional vector and $A^{\mu\nu}$ is a D-dimensional two-index tensor. The latter is required to specify the embedding of a $p+1$-worldsheet in D-dimensional space. Similar to our little group construction of fully celestial scalars in the previous section, here we need to specify reference values for both $A^{\mu\nu}_{p,\rm ref}$ and $q^\mu_{p,\rm ref}$. We choose

$$\phi^{\pm,\rm ref}_{\Delta;p}(x) \equiv \phi^\pm_{\Delta;p}(x; q_{p,\rm ref}, A_{p,\rm ref}), \tag{13}$$

where

$$A^{\mu\nu}_{p,\rm ref} \equiv {\rm diag}(-1,1,\ldots,1,0,\ldots,0),$$
$$q^\mu_{p,\rm ref} \equiv (1,0,\ldots,0,1,0,\ldots,0), \tag{14}$$

where the 1's are repeated p-times in $A^{\mu\nu}_{p,\rm ref}$, while in $q^\mu_{p,\rm ref}$ the 1 is in the $p+1$ entry. $A^{\mu\nu}_{p,\rm ref}$ is the projection tensor into the world volume of a p-hyperplane at rest, lying along the first p dimensions, while $q^\mu_{p,\rm ref}$ is chosen so that $\phi^{\pm,\rm ref}_{\Delta;p}(x)$ is $SO(p-1)$ rotationally invariant, but it is also manifestly invariant under $M_{ip} - M_{0i}, i \in \{1,\ldots,p-1\}$. It is also manifestly invariant under $\mathbb{R}^{D-p-1} \times SO(D-p-1)$ corresponding to "external" translations and rotations. Overall, the little group under which $\phi^{\pm,\rm ref}_\Delta(x)$ is invariant is given by

$$LG^D_p = \mathbb{R}^{D-p-1} \times ISO(p-1) \times SO(D-p-1), \tag{15}$$

where $\dim(LG^D_p) = D-p-1 + \frac{p(p-1)}{2} + \frac{(D-p-1)(D-p-2)}{2}$. For future reference, we also define for every $\Lambda \in SO(d-1,1)$ the Lorentz transformed tensor and vector,

$$q^\mu_{p,\Lambda^{-1}} \equiv [\Lambda^{-1}]^\mu{}_\nu q^\nu_{p,\rm ref},$$
$$A^{\mu\nu}_{p,\Lambda} \equiv \Lambda^\mu_\alpha \Lambda^\nu_\beta A^{\alpha\beta}_{p,\rm ref}. \tag{16}$$

Under Poincaré transformations, the partially celestial scalar transforms as

$$\Omega = (\Lambda, v) \in \mathcal{P}^D:$$
$$\phi^{\pm,\rm ref}_{\Delta;p}(x) \to \phi^{\pm,\Omega}_{\Delta;p}(x) = \phi^{\pm,\rm ref}_{\Delta;p}(\Lambda(x+v))$$
$$= \phi^\pm_{\Delta;p}(x+v; q_{p,\Lambda^{-1}}, A_{p,\Lambda}). \tag{17}$$

Without loss of generality, we do the translation first in the Poincaré transformation. Importantly, not all Poincaré transformations actually lead to inequivalent solutions ϕ—to label inequivalent solutions we have to mod out by the action of the little group. This is the same as finding canonical transformations $O \in \mathcal{P}^D/LG^D_p$. From (17), we can see that the generic partially celestial scalar solution is parametrized by all possible $(q^\mu, A^{\mu\nu})$ that can be reached from $(q^\mu_{p,\rm ref}, A^{\mu\nu}_{p,\rm ref})$ by Lorentz transformations.

To find the most generic $(q^\mu, A^{\mu\nu})$ we start from their reference values and perform a fixed set of independent

Lorentz transformations. We start with transformations that act on q^μ within the $p+1$ dimensional reference hyperplane along time and the first p spatial directions, while leaving $A^{\mu\nu}$ *invariant*. Since q^μ is a massless vector (even though our irreps are massive), we can get to a generic q^μ with one boost with velocity β and $p-1$ rotations with angles α_k. From now on, we can perform further transformations that act on $A^{\mu\nu}$, with q^μ going along for the ride. First, we can boost $A^{\mu\nu}$ by β' to give it velocity (nonzero first row/column) in the x_D direction. Without changing this velocity, we can perform rotations between all spatial directions orthogonal to x_D. However, out of these rotations, the ones outside the p-hyperplane leave the configuration invariant and are in fact part of the little group that leaves $p(D-p-2)$ rotation angles φ_{ij}. At this point both q^μ and $A^{\mu\nu}$ are aligned in arbitrary directions orthogonal to x_D and the velocity is in the x_D direction. Finally, we can rotate the entire configuration arbitrarily in $D-1$ spatial directions, with $D-2$ angles θ_i. In other words, the most general values for $(q^\mu, A^{\mu\nu})$ are $(q^\mu_{p,L_A L_q}, A^{\mu\nu}_{p,L_A})$ where

$$L_q = \prod_{k=1}^{p-2} R_{k,k+1}(\alpha_k) \times B_p(\beta).$$
$$L_A = \prod_{k=1}^{D-2} R_{k,k+1}(\theta_k) \times \prod_{i=1}^{p} \prod_{j=p+1}^{D-2} R_{i,j}(\varphi_{ij}) \times B_D(\beta'). \quad (18)$$

Hence, the partially celestial scalar solution is parametrized by $(D-p-1)(p+1)-3$ angles, two boosts, and $p+1$ translations, for a total of $p(D-p)+D$ parameters of the coset $\mathcal{P}^D/LG^D_p$. One can easily check that this is indeed the correct dimension of this coset. To summarize, p partially celestial scalar solutions are solutions of the D-dimensional KG equation that are invariant under the little group (15) and are labeled by $(D-p)(p+1)$ parameters of the coset $\mathcal{P}^D/LG^D_p$.

Finally, we note that in the massless limit, the dependence of p partially celestial scalars on the reference plane A drops out, as can be checked by explicit expansion of (13).

IV. PARTIALLY CELESTIAL SCALARS: EXPLICIT EXAMPLES

A. Fully celestial state in D dimensions

This is the case discussed in Sec. II. One can easily check that setting $p=D-1$ in (13)–(18), we have $A^{\mu\nu}=\eta^{\mu\nu}$ which is Lorentz invariant, and so the partially celestial solution coincides with the definitions in Sec. II.

B. Partially celestial line in D dimensions

A partially celestial line in D dimensions corresponds to a partially celestial solution (17) with $p=1$. Its little group is

$$LG^D_1 = R^2 \times SO(D-2), \quad (19)$$

whose dimension is $2+(D-2)(D-3)/2$. For $D=4$ we get $R^2\times SO(2)$. The coset $\mathcal{P}^D/LG^D_1$ has dimension $2D-1$, and it is parametrized by $D-3$ angles φ_{ij}, $D-2$ angles θ_k, two boost parameters β, β', one spatial translation, and one time translation. The most generic $\phi^{\rm line}$ is given by

$$\begin{aligned}
\phi^{\rm line}(x) &= \phi^\pm_{\Delta;p=1}(x+a;q,A),\\
a_\nu &= (a_0, a_1, 0, \ldots, 0),\\
q_\nu &= \sqrt{\frac{1+\beta}{1-\beta}} L_A q_{p=1,{\rm ref};\nu},\\
A^{\mu\nu} &= A^{\mu\nu}_{p=1,L_A},\\
L_A &= \prod_{k=1}^{D-2} R_{k,k+1}(\theta_k) \times \prod_{j=2}^{D-2} R_{1,j}(\varphi_{ij}) \times B_D(\beta'). \quad (20)
\end{aligned}$$

C. Massive particle in D dimensions

A massive particle in D dimensions corresponds to a partially celestial solution (17) with $p=0$. Since q^μ is ill-defined for $p=0$, the particle solution necessitates taking $\Delta=0$ rather than $\Delta=\frac{D-2}{2}+i\mathbb{R}$. In this case the little group (in Poincaré) is

$$LG^D_0 = R^{D-1} \times SO(D-1), \quad (21)$$

whose dimension is $D-1+(D-1)(D-2)/2$. For $D=4$ we get spatial translations R^3 times the usual $SO(3)\simeq SU(2)$ little group for massive particles in 4D. The coset $\mathcal{P}^D/LG^D_0$ has dimension $D-1$, and it is parametrized by $D-2$ angles θ_k, one boost parameter β', and one time translation. The reference wave function $\phi^{\rm particle}_{\rm ref}$ is given by

$$\phi^{\rm particle}_{\rm ref}(x) = \phi^\pm_{\Delta;0}(x;q_{0,{\rm ref}},A_{0,{\rm ref}}) = ie^{-imt}. \quad (22)$$

This is simply a plane wave in the rest frame of the particle (up to an irrelevant constant phase). In any other frame, we have

$$\phi^{\rm particle}(x) = ie^{-im\gamma'(t+dt+\beta'\vec{x}\cdot\hat{n})},$$
$$\hat{n} = \prod_{k=1}^{D-2} R_{k,k+1}(\theta_k)(0,\ldots,0,1)^T. \quad (23)$$

V. FROM PARTIALLY CELESTIAL SCALARS TO PARTIALLY CELESTIAL QUANTUM STATES

The $SO(p,1)$ invariance of partially celestial scalars is suggestive of a new class of quantum states representing a p-*sheet*. To make this correspondence more concrete, we can interpret the p partially celestial solution $\phi(x)$ as the wave function of a single p partially celestial state. First, we set up some notation. We denote canonical Poincaré transformations by $(L,a)\equiv O\in\mathcal{P}^D$ so that $[O]\in\mathcal{P}^D/LG_p^D$. As shown above, each canonical Poincaré transformation is labeled by $(D-p)(p+1)$ parameters. For every canonical transformation there is a unique partially celestial scalar solution $\phi_O(x)=\phi_{\rm ref}(L(x+a))$. We can now identify

$$\phi_O(x)=\langle 0|a_O\Phi(x)|0\rangle \\ =\langle 0|\Phi(x)a_O^\dagger|0\rangle^*, \tag{24}$$

where $|0\rangle$ is the vacuum, $a_O^\dagger$ (a_O) is the creation (annihilation) operator for a p partially celestial scalar which is related to the reference scalar by the canonical Poincaré transformation O. $\Phi(x)$ is the field operator for a (real) scalar field, which we can expand as

$$\Phi(x)\equiv\int_{\mathcal{P}^D/LG_p^D} dO[\phi_O(x)a_O^\dagger+\text{H.c.}]. \tag{25}$$

As for the field operator for particles, we require the field operator transforms covariantly under the Poincaré group [18],

$$\Omega=(\Lambda,v^\mu)\in\mathcal{P}^D: \\ \Phi(x)\to U[\Omega]\Phi(x)U^{-1}[\Omega]=\Phi(\Lambda(x+v)). \tag{26}$$

In particular, $\Phi(x)$ is invariant under little group transformations $\Omega\in LG_p^D$. Similar to particles, this requirement fixes the Poincaré transformation properties of the creation and annihilation operators (see derivation in Appendix A),

$$a_O^\dagger\to U[\Omega]a_O^\dagger U^{-1}[\Omega]=a_{\Omega O}^\dagger, \\ a_O\to U[\Omega]a_O U^{-1}[\Omega]=a_{\Omega O}, \tag{27}$$

where the product ΩO is just the group product of the Poincaré group $\mathcal{P}^D$. In other words, a p partially celestial *scalar state* $|O\rangle\equiv a_O^\dagger|0\rangle$ transforms as

$$U[\Omega]|O\rangle=|\Omega O\rangle. \tag{28}$$

By construction, the reference state $|\text{ref}\rangle=|O=\text{identity}\rangle$ is invariant under $\Omega\in LG_p^D$. This concludes our definition of the quantum state of a single p partially celestial scalar.

VI. PARTIALLY CELESTIAL STATES WITH SPIN: WIGNER'S METHOD

In the previous section we defined p partially celestial scalar states by starting from a p partially celestial scalar solution and interpreting it as the wave function of a quantum state. Here we generalize our construction to p partially celestial states with spin. A direct generalization of our previous derivation would have been to start with a partially celestial solution with spin, $\phi(x)_{i_1,\ldots,i_l}$, and interpret it as the wave function for a spinning p partially celestial state (see, for example, the constructions of fully celestial spinors in [1,19,20]) and massless p-forms in [14]. Instead, we will follow a simpler route, using Wigner's method of induced representations. Similar to the scalar case, spinning states are labeled by $|O;\sigma\rangle$ where $O\in\mathcal{P}^D/LG_p^D$ and σ is a composite spin index. The reference state is defined as usual as $|\text{ref};\sigma\rangle=|O=\text{identity};\sigma\rangle$, and it is annihilated by all of the generators of the little group LG_p^D. Clearly, LG_p^D is generated by the Poincaré algebra generators $G_n=\{M_{0i},M_{ip},P_k,M_{kl}\}$ where $i\in[1,\ldots,p-1]$ and $k,l\in[p+1,\ldots,d]$, so that $G_n|\text{ref};\sigma\rangle=0$. Consequently, the reference p partially celestial state $|\text{ref}\rangle$ transforms in a representation of LG_p^D, i.e.

$$U[W]|\text{ref};\sigma\rangle=\mathcal{D}_{\sigma'\sigma}[W]|\text{ref};\sigma'\rangle, \tag{29}$$

for any $W\in LG_p^D$. Here $\mathcal{D}_{\sigma'\sigma}[W]$ is some representation matrix of LG_p^D and σ is a collective index denoting a "spin" label for LG_p^D representations. For particles, $p=0$, and $D=4$, the representation matrices reduce to the normal spin representation matrices.

As in Wigner's method for particles, for any $O\in\mathcal{P}^D/LG_p^D$ we can define the quantum state in a general frame as

$$|O;\sigma\rangle\equiv U[O]|\text{ref};\sigma\rangle. \tag{30}$$

This also serves as a definition of $U[O]$ that can be uniquely extended to all $U[\Omega],\Omega\in\mathcal{P}^D$ acting on generic states. But first, let us ask ourselves which generators annihilate $|O\rangle$. The answer is straightforward. Take $O=(L,a)\in\mathcal{P}^D/LG_p^D$ and define $M_{\mu\nu}^O$ and P_μ^O so that

$$M_{\mu\nu}={L_\mu}^\alpha{L_\nu}^\beta(M_{\alpha\beta}^O-a_{[\alpha}P_{\beta]}^O), \\ P_\mu={L_\mu}^\nu P_\nu^O. \tag{31}$$

From chapter 2 of Weinberg's QFT book [18], we have

$$M_{\mu\nu}=U^{-1}[O]M_{\mu\nu}^O U[O], \\ P_\mu=U^{-1}[O]P_\mu^O U[O]. \tag{32}$$

Then $G_n^O|O;\sigma\rangle=0$.

Next, we can ask how the state $|O;\sigma\rangle$ transforms under a generic Poincaré transformation $\Omega\in\mathcal{P}^D$. This is uniquely defined using Wigner's method. Explicitly,

$$U[\Omega]|O;\sigma\rangle=U[\bar{O}]U[\bar{O}^{-1}\Omega O]|\text{ref};\sigma\rangle \\ =U[\bar{O}]U[W]|\text{ref};\sigma\rangle \\ =\mathcal{D}_{\sigma'\sigma}[W]|\bar{O};\sigma'\rangle, \tag{33}$$

where $\bar{O}\in\mathcal{P}^D/LG_p^D$ is unique canonical Lorentz transformation defined by

$$\bar{O}^\mu{}_\alpha\bar{O}^\nu{}_\beta A^{\alpha\beta}_{p,\mathrm{ref}} = A^{\mu\nu}_{p,\Omega O}, \\ \bar{O}^\mu{}_\alpha q^\alpha_{p,\mathrm{ref}} = q^\mu_{p,\Omega O}. \tag{34}$$

Note that generically ΩO is not a canonical Lorentz transformation in and of itself, and so $\bar{O}\neq\Omega O$. To conclude, in this section, we have straightforwardly applied Wigner's method of induced representations to single p partially celestial states with spin. The spin here is given by the representation $\mathcal{D}_{\sigma'\sigma}$ of the little group LG_p^D.

VII. SPINOR-HELICITY VARIABLES FOR p-SHEET SCATTERING

Once we have fixed the little group for p-sheets we can construct the generalizations of the spinor-helicity variables, which are the key for the construction of the scattering amplitudes. Consider the compact part of the little group (15):

$$cLG_p^D = SO(p-1)\times SO(D-p-1). \tag{35}$$

Our task is to define D-dimensional massive spinor-helicity variables that transform under cLG_p^D. We define two kinds of Minkowski spinors under the full D-dimensional Lorentz group, which also carries spinor indices under the (Euclidean) SO factors of the cLG_p^D. The first $|L_A\rangle_\alpha$ transforms with the little group $SO(D-p-1)$ spinor index, while the second $|L_a\rangle\!\rangle_\alpha$ transforms with an $SO(p-1)$ spinor index. For even D the spinor representation is chiral, and we also have spinors of the opposite chirality $|L^A]^{\dot\alpha}$ and $|L^a]\!]^{\dot\alpha}$. For even $D-p-1$ or $p-1$, we also have spinors with dotted little group indices. Undotted spinor indices are contracted in the northwest-southeast convention, while dotted ones are in the southwest-northeast convention. We begin with a definition of $|L_A\rangle_\alpha, [L_{\dot A}|_{\dot\alpha}$, where $A,\dot A$ are $SO(D-p-1)$ spinor indices and $\alpha,\dot\alpha$ are $SO(D-1,1)$ spinor indices. Defining for any N, $s_N = 2^{\lfloor N/2\rfloor -1}$, we have $A,\dot A=\{1,\ldots,s_{D-p-1}\}$ and $\alpha,\dot\alpha=\{1,\ldots,s_D\}$. The reference values for the single angle spinors are

$$|\mathrm{ref}_A\rangle_\alpha = \delta_{A+s,\alpha}, \qquad |\mathrm{ref}_{\dot A}\rangle_\alpha = \delta_{\dot A+s,\alpha}, \\ [\mathrm{ref}_{\dot A}|_{\dot\alpha} = \delta_{\dot A+s,\dot\alpha}, \qquad [\mathrm{ref}_A|_{\dot\alpha} = \delta_{A+s,\dot\alpha}, \tag{36}$$

where $s=s_D-s_{D-p-1}$. Note that the dotted Lorentz indices exist only for even dimensional D and the dotted little group indices only exist for even dimensional $D-p-1$. For odd D, we can now define Lorentzian D-dimensional $[\Gamma^\mu]_\alpha{}^\beta$ matrices, while for even D we have the corresponding $[\Sigma^\mu]_{\alpha\dot\beta}, [\bar\Sigma^\mu]^{\dot\alpha\beta}$ matrices. Similarly, for odd $D-p-1$ we have $D-p-1$ dimensional Euclidean $[\gamma_I]_A^B$ matrices, while for even $D-p-1$ we have the corresponding $[\sigma^I]_{A\dot B}, [\bar\sigma^I]^{\dot AB}$ matrices. We can always choose a basis so that the bottom right $s_{D-p-1}\times s_{D-p-1}$ block of the last $D-p-1$ Γ/Σ matrices is numerically identical to the γ/σ matrices. We can freely raise and lower the indices on the spinors via

$$\langle \mathrm{ref}^A|^\alpha = \varepsilon^{\alpha\beta}\epsilon^{AB}|\mathrm{ref}_B\rangle_\beta, \\ |\mathrm{ref}^{\dot A}]^{\dot\alpha} = [\mathrm{ref}_{\dot B}|_{\dot\beta}\epsilon^{\dot B\dot A}\varepsilon^{\dot\beta\dot\alpha}, \tag{37}$$

where $\varepsilon = i[\Gamma^{D-2}]$ for odd D and $\varepsilon = i[\Sigma^{D-2}]$ for even D and similarly for ϵ with γ and σ.

Now, we can use these gamma/sigma matrices to combine the spinors into

$$A^{\mu\nu}_{p,\mathrm{ref}} = \eta^{\mu\nu} - [v_{\mathrm{ref}}]_I^{\{\mu}[v_{\mathrm{ref}}]^{\nu\};I} \\ \text{odd } D, \qquad \text{odd } D-p-1: \\ [v_{\mathrm{ref}}]_I^\mu = \langle \mathrm{ref}^A|^\alpha[\Gamma^\mu]_\alpha{}^\beta[\gamma_I]_A{}^B|\mathrm{ref}_B\rangle_\beta \\ \text{even } D, \qquad \text{even } D-p-1: \\ [v_{\mathrm{ref}}]_I^\mu = \langle \mathrm{ref}^A|^\alpha[\Sigma^\mu]_{\alpha\dot\beta}[\sigma_I]_{A\dot B}|\mathrm{ref}^{\dot B}]^{\dot\beta}, \tag{38}$$

and similarly for mixed parity D and $D-p-1$. In analogy with Wigner's method, we define $|L_A\rangle_\alpha, [L_{\dot A}|_{\dot\alpha}$, in any other frame as

$$|L_A\rangle_\alpha = L_\alpha{}^\beta|\mathrm{ref}_A\rangle_\beta, \\ [L_{\dot A}|_{\dot\alpha} = [\mathrm{ref}_{\dot A}|_{\dot\beta}L^{\dot\beta}{}_{\dot\alpha}, \tag{39}$$

for every $L\in SO(D-1,1)/cLG_p^D$. One can readily check that

$$A^{\mu\nu}_{p,L} = \eta^{\mu\nu} - [v_L]_I^{\{\mu}[v_L]^{\nu\};I} \\ \text{odd } D, \qquad \text{odd} D-p-1: \\ [v_L]_I^\mu = \langle L^A|^\alpha[\Gamma^\mu]_\alpha{}^\beta[\gamma_I]_A{}^B|L_B\rangle_\beta \\ \text{even } D, \qquad \text{even } D-p-1: \\ [v_L]_I^\mu = \langle L^A|^\alpha[\Sigma^\mu]_{\alpha\dot\beta}[\sigma_I]_{A\dot B}|L^{\dot B}]^{\dot\beta}. \tag{40}$$

In fact, the first equation can also be thought of as the definition of the spinor-helicity variables. It is the generalization of the relation $p=|p\rangle[p|$ for the definition of the ordinary spinors.

By Wigner's method, $|L_A\rangle_\alpha, [L_{\dot A}|_{\dot\alpha}$ transform under a generic $\Lambda\in SO(D-1,1)$ as

$$\Lambda_\alpha{}^\beta|L_A\rangle_\beta = \bar{L}_\alpha^\beta W_\beta^\gamma|\mathrm{ref}_A\rangle_\gamma \\ [L_{\dot A}|_{\dot\beta}\Lambda^{\dot\beta}{}_{\dot\alpha} = [L_{\dot A}|_{\dot\gamma}W^{\dot\gamma}_{\dot\beta}\bar{L}^{\dot\beta}{}_{\dot\alpha}, \tag{41}$$

where $\bar{L} \in SO(D-1,1)/cLG_p^D$ is the unique canonical Lorentz transformation defined by $A_{p,\Lambda L} = \bar{L} A_{p,\mathrm{ref}} \bar{L}^T$, and $W = \bar{L}^{-1} \Lambda L \in SO(D-p-1)$ is a little group transformation. Now, by the definition (36), we have

$$W_\beta{}^\gamma |\mathrm{ref}_A\rangle_\gamma = W_A{}^B |\mathrm{ref}_B\rangle_\beta,$$
$$[\mathrm{ref}_{\dot{A}}|_{\dot{\gamma}} W^{\dot{\gamma}}{}_{\dot{\beta}} = [\mathrm{ref}_{\dot{B}}|_{\dot{\beta}} W^{\dot{B}}{}_{\dot{A}}. \quad (42)$$

In other words, when acting on the reference spinors with a spacetime-index little group transformation, it is the same as acting on them with the same transformation in the little group indices. This is the same thing that happens to massive spinors in 4D [21]. We conclude that

$$\Lambda_\alpha{}^\beta |L_A\rangle_\beta = W_A{}^B |\bar{L}_B\rangle_\alpha,$$
$$[L_{\dot{A}}|_{\dot{\beta}} \Lambda^{\dot{\beta}}{}_{\dot{\alpha}} = [\bar{L}_{\dot{B}}|_{\dot{\alpha}} W^{\dot{B}}{}_{\dot{A}}; \quad (43)$$

i.e. these spinors transform exactly with the correct $SO(D-p-1)$ little group factor. That makes them the right building blocks for p partially celestial amplitudes.

Similarly, we can define the spinors $|\mathrm{ref}^a\rangle\rangle_\alpha$, $[[\mathrm{ref}_a|_{\dot{\alpha}}$ where a is an $SO(p-1)$ little group index. Note that we do not dot the a index for reasons that will become apparent momentarily. These are defined as

$$|\mathrm{ref}_a\rangle\rangle_\alpha = \delta_{a\alpha},$$
$$[[\mathrm{ref}^a|_{\dot{\alpha}} = \delta^a_{\dot{\alpha}}. \quad (44)$$

They are defined so that

$$q^\mu_{p,\mathrm{ref}} = \langle\langle \mathrm{ref}^a|^\alpha [\Gamma^\mu]_\alpha{}^\beta |\mathrm{ref}_a\rangle\rangle_\beta, \quad (45)$$

for odd D and $p-1$, and

$$q^\mu_{p,\mathrm{ref}} = \langle\langle \mathrm{ref}^a|^\alpha [\Sigma^\mu]_{\alpha\dot{\beta}} |\mathrm{ref}_a]]^{\dot{\beta}}, \quad (46)$$

for even D and $p-1$. Similar to $|L\rangle, [L|$, the generic $|L\rangle\rangle$, $[[L|$ transform as

$$\Lambda_\alpha{}^\beta |L_a\rangle\rangle_\beta = W_a{}^b |\bar{L}'_b\rangle\rangle_\alpha,$$
$$[[L^a|_{\dot{\beta}} \Lambda^{\dot{\beta}}{}_{\dot{\alpha}} = [[\bar{L}'^b|_{\dot{\alpha}} W_b{}^a, \quad (47)$$

where $\bar{L}' \in SO(D-1,1)/cLG_p^D$ is the unique canonical Lorentz transformation defined by $q_{p,\Lambda L} = \bar{L}' q_{p,\mathrm{ref}}$, and $W = \bar{L}'^{-1} \Lambda L \in SO(p-1)$ is a little group transformation. One can readily check that

$$q^\mu_{p,L} = \langle\langle L^a|^\alpha [\Gamma^\mu]_\alpha{}^\beta |L_a\rangle\rangle_\beta, \quad (48)$$

for odd D and $p-1$, and

$$q^\mu_{p,L} = \langle\langle L^a|^\alpha [\Sigma^\mu]_{\alpha\dot{\beta}} |L_a]]^{\dot{\beta}}, \quad (49)$$

for even D and $p-1$, for any $L \in SO(D-1,1)/cLG_p^D$. Again these last two relations can be thought of as the definitions of double-line $SO(p-1)$ spinors. We then see that $|L\rangle\rangle$, $[[L|$ transform with the correct $SO(p-1)$ little group transformation and can be used to form p partially celestial amplitudes.

VIII. CONSTRUCTING PARTIALLY CELESTIAL AMPLITUDES

Using the spinor-helicity variables defined in the previous section, we can construct the little group-covariant part of any partially celestial amplitude, generalizing the 4D massive formalism of [21]. We take all external states to live in D-dimensional space and have "internal dimensions" p_n, with $n = 1, 2, 3, \ldots, N$ and representations $\mathcal{R}_n^{\mathrm{in}} \times \mathcal{R}_n^{\mathrm{out}}$ under the compact little group $cLG^D_{p_n} = SO(p_n - 1) \times SO(D - p_n - 1)$. To saturate the required little group transformation of the amplitude, we need to combine the little group indices of the spinor-helicity variables $|n_{A_n}\rangle_{\alpha_n}, [n_{\dot{A}_n}|_{\dot{\alpha}_n}, |n_{a_n}\rangle\rangle_{\alpha_n}$, and $[[n_{a_n}|_{\dot{\alpha}_n}$. This is achieved via contractions of the $cLG^D_{p_n}$ indices using the γ/σ matrices. Once the correct little group transformation is obtained, all Lorentz indices can be contracted via the little group invariants $\epsilon^{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}$, $[nk]^\beta{}_\alpha \equiv [n^I]_{\alpha\dot{\gamma}} [k_I]^{\dot{\gamma}\beta}$, and $[nk]_{\dot{\alpha}}{}^{\dot{\beta}} \equiv [n^I]_{\dot{\alpha}\gamma} [k_I]^{\gamma\dot{\beta}}$ where

$$[n^I]_{\alpha\dot{\beta}} = |n_A\rangle_\alpha \epsilon^{AB} \sigma^I_{B\dot{C}} \epsilon^{\dot{C}\dot{D}} [n_{\dot{D}}|_{\dot{\beta}}, \quad (50)$$

as well as their double angle/double square bracket counterparts.

As an example, consider the (little group covariant part of) the three-point amplitude for three 3-celestial amplitudes in 10D, transforming as the $(0, \mathbf{4})$, $(0, \bar{\mathbf{4}})$, and $(0, \mathbf{6})$ of $cLG_3^{10} = U(1) \times SO(6)$. This amplitude is given by

$$\mathcal{A}^{A_1, \dot{A}_2, I_3} = [3_{\dot{A}_3} 2^{\dot{A}_2}][\bar{\sigma}^{I_3}]^{\dot{A}_3 B_3} \langle 1^{A_1} 3_{B_3}\rangle, \quad (51)$$

where all the spinors are defined for $D = 10$ and $p = 3$. Another example is the (little group covariant part of) the three-point amplitude for a line (one partially celestial) state emitting a massive scalar particle in 4D. We consider the case in which the 2 one partially celestial legs have helicity $\pm\frac{1}{2}$ under $cLG_1^4 = SO(2) \simeq U(1)$ (in the all-incoming convention). The amplitude in this case is

$$\mathcal{A}^{A_1, A_2} = \langle 1^{A_1} 2^{A_2}\rangle, \quad (52)$$

where both the spinors are defined for $D = 4$ and $p = 1$, and the values of A_1, A_2 correspond to different choices of positive or negative helicity. We can get a direct analytical expression for this amplitude using the explicit values of the spinors given in Appendix B. As an illustration, consider a line at rest along the x-axis emitting a massive scalar particle, while remaining at rest and rotating by an angle φ. The amplitude for this process is

$$\mathcal{A}^{A_1,A_2} = \begin{pmatrix} 0 & e^{-\frac{i\varphi}{2}} \\ -e^{\frac{i\varphi}{2}} & 0 \end{pmatrix}. \tag{53}$$

Note that the amplitude is helicity conserving in the all-incoming convention.

Finally, note that translational invariance should pose additional constraints on partially celestial amplitudes. In fact, for particles we know that translational invariance (i.e. momentum conservation) dictates that three-point amplitudes are completely fixed by their little group transformations. In our case, similar to the case of fully celestial amplitudes [16,17], the generators for translation assume a nonlinear differential form when expressed in terms of (q, A, Δ). We leave the exploration of the constraints of translational invariance on partially celestial amplitudes for future work, including whether three-point partially celestial amplitudes are fixed by their little group transformations.

IX. PAIRWISE LITTLE GROUP

Consider a scalar p-sheet parallel to a scalar p' sheet in D dimensions. To be parallel, we require that $p + p' \leq D - 2$. By applying Poincaré transformations, we can always go to the "center of velocity" frame of the two sheets, in which their wave functions are given by

$$\begin{aligned}
\phi^{\pm,\mathrm{ref}}_{\Delta;p}(x) &= \phi^{\pm}_{\Delta;p}(x; q_{\mathrm{ref,p}}, B_{\mathrm{ref,p}}),\\
\phi^{\pm,\mathrm{ref}}_{\Delta';p'}(x) &= \phi^{\pm}_{\Delta';p'}(x; q_{\mathrm{ref,p'}}, B_{\mathrm{ref,p'}}),\\
B^{\mu\nu}_{\mathrm{ref,p}} &= \mathrm{diag}(0,1,\ldots,1,0,\ldots,0,0,\ldots,0) + M_\perp(\beta),\\
B^{\mu\nu}_{\mathrm{ref,p'}} &= \mathrm{diag}(0,0,\ldots,0,1,\ldots,1,0,\ldots,0) + M_\perp(-\beta),\\
M_\perp(\beta) &= -\begin{pmatrix} \gamma^2 & 0 & \ldots & 0 & \gamma^2\beta \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ \gamma^2\beta & 0 & \ldots & 0 & \gamma^2\beta^2 \end{pmatrix}.
\end{aligned} \tag{54}$$

In analogy with the pairwise little group for particles, we can ask which subgroup of $\mathcal{P}^D$ stabilizes both $\phi^{\pm,\mathrm{ref}}_{\Delta;p}(x)$ and $\phi^{\pm,\mathrm{ref}}_{\Delta';p'}(x)$. The answer is

$$\begin{aligned} pLG^D_{p,p'} &= ISO(p-1) \times ISO(p'-1) \\ &\quad \times R^{D-p-p'-2} \times SO(D-p-p'-2). \end{aligned} \tag{55}$$

For particles in 4D, $p = p' = 0$, and the pairwise little group reduces to $SO(2) \simeq U(1)$, consistent with [8–11] (see also [22] for a discussion of pairwise helicity in the context of 4D celestial amplitudes). In particular, we focus on the case in which $p' = D - p - 4$. This is the case where the sheets are *mutually nonlocal*, in the sense that they source p-form gauge fields that are Electromagnetic (EM)-dual to each other. In this case the p-form charges of the two sheets are constrained by Dirac quantization, as shown in [23],

$$q \equiv eg = \frac{n}{2}, \qquad p \neq D - p - 4, \tag{56}$$

where e and g are the charge of the p and $D - p - 4$ sheets, respectively. In the self-dual case, $p = \frac{D-4}{2}$, the sheets can be dyonic, and the Dirac quantization condition is generalized to [24]

$$q \equiv e_1 g_2 + (-1)^p e_2 g_1 = \frac{n}{2}, \qquad p = \frac{D-4}{2}. \tag{57}$$

In [8–11], the Dirac-quantized quantities q were shown to play the role of pairwise helicities labeling the representations of the $U(1)$ little group. Here the situation is similar; substituting $p' = D - p - 4$ in (55), we have

$$pLG^D_{p,D-p-4} = ISO(p-1) \times ISO(D-p-5)R^2 \times U(1), \tag{58}$$

and we see that indeed $pLG^D_{p,D-p-4}$ has a $U(1)$ factor. We can naturally identify the pairwise helicities labeling the representations of this factor of the pairwise little group with the q given in (56) and (57).

The presence of a $U(1)$ factor for the pairwise little group for mutually nonlocal partially celestial states hints that the entire structure exposed in [8–11] generalizes directly to the present case. This is reminiscent of a pair of mutually nonlocal branes, which source p-form and p'-form fields and thus carry extra angular momentum in these fields. This extra angular momentum modifies the selection rules for brane scattering, in the same way it modifies them for monopoles and charges in 4D.

X. OUTLOOK AND FUTURE WORK

In this paper we defined the quantum states for scalar and spinning p partially celestial states and, notably, the generalized LG^D_p-covariant spinor-helicity variables in D dimensions. These results allow us to find the little group covariant part of the most general three-point amplitudes for partially celestial states. Additionally we found the corresponding pairwise little group, which has a $U(1)$ factor for mutually nonlocal states. In a future little group construction for branes, the helicities under the pairwise little group should be identified with Dirac quantized products of charges by examining the Lorentz-transformation properties of soft-photon-dressed electric and magnetic states as in Ref. [11]; We expect that the same result can be shown for branes by considering their "soft-higher-gauge field" dressed multibrane states. The generalized spinor-helicity variables enable the bottom-up construction of (the little group covariant part of) scattering amplitudes for p partially celestial states. We give a procedure for constructing the little group covariant part of three-point functions for three partially celestial states in 10D and for two lines and a scalar particle in 4D. Unlike in the case of particles, we cannot be sure whether three-point amplitudes for partially celestial states are completely fixed by their little group transformation. We leave that question for future work, in which we will analyze in detail the constraints from translational invariance on partially celestial amplitudes.

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APPENDIX A: FROM FIELD OPERATOR TO LADDER OPERATORS

Here we show that the transformation properties of the field operator (26) lead to the transformation (27) of the creation operator. To see this, write for $\Omega = (\Lambda, v) \in \mathcal{P}^D$

$$\Phi(\Lambda(x+v)) = \int_{\mathcal{P}^D/LG_p^D} dO\{\phi_O(x)U[\Omega]a_O^\dagger U^{-1}[\Omega] + \phi_O^*(x)U[\Omega]a_O U^{-1}[\Omega]\}, \tag{A1}$$

or in other words

$$\int_{\mathcal{P}^D/LG_p^D} dO'\{\phi_{O'}(\Lambda(x+v))a_{O'}^\dagger + \phi_{O'}^*(\Lambda(x+v))a_{O'}\} = \int_{\mathcal{P}^D/LG_p^D} dO\{\phi_O(x)U[\Omega]a_O^\dagger U^{-1}[\Omega] + \phi_O^*(x)U[\Omega]a_O U^{-1}[\Omega]\}. \tag{A2}$$

Note that $\phi_{O'}(\Lambda(x+v)) = \phi_{\Omega^{-1}O'}(x)$, and we can change the integration variable on the left-hand side as $O' = \Omega O$,

$$\int_{\mathcal{P}^D/LG_p^D} dO\{\phi_O(x)a_{\Omega O}^\dagger + \phi_O^*(x)a_{\Omega O}\} = \int_{\mathcal{P}^D/LG_p^D} dO\{\phi_O(x)U[\Omega]a_O^\dagger U^{-1}[\Omega] + \phi_O^*(x)U[\Omega]a_O U^{-1}[\Omega]\}, \tag{A3}$$

from which (27) follows.

APPENDIX B: EXPLICIT PARAMETRIZATIONS

For completeness, we present here the generic $A^{\mu\nu}, \lambda, \tilde{\lambda}$ for a massive particle in 4D, and the generic $q^\mu, A^{\mu\nu}, \lambda, \tilde{\lambda}$ for a line partially celestial state in 4D.

1. Massive particle

As a special case of (20), the generic wave function for a massive particle in 4D depends on the translation $(a_0, 0, 0, 0)$, one boosts β' and two angles θ_1, θ_2. The most general $A^{\mu\nu}$ is then

$$A^{\mu\nu} = -u^\mu u^\nu, \tag{B1}$$

where u^μ is defined the same way as (B2), and is the four-velocity of the particle, given by

$$u^\mu = \gamma'(-1, \beta'\sin\theta_2\sin\theta_1, \beta'\sin\theta_2\cos\theta_1, \beta'\cos\theta_1). \tag{B2}$$

Finally, we define the spinors $|L\rangle, \langle L|$ corresponding to a massive particle in 4D. By the definitions in Sec. VII, they are given by

$$|L^A\rangle_\alpha = ([L^A|_{\dot{\alpha}})^* = \begin{pmatrix} e^{\frac{i\theta_1}{2}}a'_-\cos(\frac{\theta_2}{2}) & ie^{\frac{i\theta_1}{2}}a'_+\sin(\frac{\theta_2}{2}) \\ ie^{-\frac{i\theta_1}{2}}a'_-\sin(\frac{\theta_2}{2}) & e^{-\frac{i\theta_1}{2}}a'_+\cos(\frac{\theta_2}{2}) \end{pmatrix}, \tag{B3}$$

where $a'_\pm = \sqrt{\gamma'(1 \pm \beta')}$. Note that we do not have dotted little group indices since the little group is $SO(3) \simeq SU(2)$ whose $\mathbf{2}$ and $\bar{\mathbf{2}}$ are equivalent. One can readily check that (40) is satisfied.

2. Line

As a special case of (20), the generic wave function for a line partially celestial state in 4D depends on the translations $(a_0, a_1, 0, 0)$, two boosts β, β', and three angles $\theta_1, \theta_2, \varphi_{12}$. The most general $(A^{\mu\nu}, q^\mu)$ are then

$$A^{\mu\nu} = \xi^\mu\xi^\nu - u^\mu u^\nu, \qquad q^\mu = \sqrt{\frac{1+\beta}{1-\beta}}(u^\mu + \xi^\mu), \tag{B4}$$

where u^μ is the same four-velocity given in (B2), while ξ^μ is given by

$$\xi^\mu = (0, \cos\theta_1\cos\varphi_{12} - \sin\theta_1\cos\theta_2\sin\varphi_{12}, -\sin\theta_1\cos\varphi_{12} - \cos\theta_1\cos\theta_2\sin\varphi_{12}, \sin\theta_2\sin\varphi_{12}). \tag{B5}$$

Note that ξ^μ denotes the line's four-orientation, which is always transverse to the four-velocity, $u \cdot \xi = 0$. Finally, we define the spinors $|L\rangle, \langle L|$ corresponding to the generic line in 4D. By the definitions in Sec. VII, they are given by

$$
\begin{aligned}
|L^A\rangle_\alpha &= ([L^{\dot{A}}|_{\dot{\alpha}})^* \\
&= \begin{pmatrix} e^{\frac{i\theta_1}{2}}\left[a_+a'_-e^{\frac{i\varphi_{12}}{2}}\cos(\frac{\theta_2}{2}) - ia_-a'_+e^{-\frac{i\varphi_{12}}{2}}\sin(\frac{\theta_2}{2})\right] & e^{\frac{i\theta_1}{2}}\left[ia_+a'_+e^{-\frac{i\varphi_{12}}{2}}\sin(\frac{\theta_2}{2}) - a_-a'_-e^{\frac{i\varphi_{12}}{2}}\cos(\frac{\theta_2}{2})\right] \\ e^{-\frac{i\theta_1}{2}}\left[ia_+a'_-e^{\frac{i\varphi_{12}}{2}}\sin(\frac{\theta_2}{2}) - a_-a'_+e^{-\frac{i\varphi_{12}}{2}}\cos(\frac{\theta_2}{2})\right] & e^{-\frac{i\theta_1}{2}}\left[a_+a'_+e^{-\frac{i\varphi_{12}}{2}}\cos(\frac{\theta_2}{2}) - ia_-a'_-e^{\frac{i\varphi_{12}}{2}}\sin(\frac{\theta_2}{2})\right] \end{pmatrix}, \quad \text{(B6)}
\end{aligned}
$$

where $a_\pm = \sqrt{\frac{\gamma\pm1}{2}}$. Note that we do not have dotted little group indices; the little group is $SO(2) \simeq U(1)$, and so the two-component spinor representation is *reducible* and includes both $\pm\frac{1}{2}$ helicities under the $U(1)$. One can readily check that (40) is satisfied.

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