Class $\mathcal{S}$ theories on S^2

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We study two-dimensional $\mathcal{N} = (0, 2)$ and $\mathcal{N} = (0, 4)$ theories derived from compactifying class $\mathcal{S}$ theories on S^2 with a topological twist. We present concise expressions for the elliptic genera of both classes of theories, revealing the topological quantum field theory structure on Riemann surfaces $C_{g,n}$. Furthermore, our study highlights the relationship between the left-moving sector of the (0,2) theory and the chiral algebra of the four-dimensional $\mathcal{N} = 2$ theory. Notably, we propose that the (0,2) elliptic genus of a theory of this class can be expressed as a linear combination of characters of the corresponding chiral algebra.

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I. INTRODUCTION

The study of quantum field theory (QFT) and string theory over the years has continually revealed deeper structures and interconnected webs of relationships. At the heart of this intricate web lies the six-dimensional (6D) $\mathcal{N} = (2, 0)$ superconformal field theory (SCFT), which describes the low-energy dynamics on the world volume of M5-branes in M-theory. The 6D $\mathcal{N} = (2, 0)$ SCFT stands out due to its maximal supersymmetry and the highest spacetime dimension that hosts a superconformal algebra. While directly handling the dynamics of 6D $\mathcal{N} = (2, 0)$ SCFT is challenging due to the lack of a Lagrangian description, it serves as a central hub from which a plethora of lower-dimensional theories can be derived through compactifications on various manifolds. When M5-branes wrap a certain manifold M with a suitable topological twist, it effectively gives rise to a lower-dimensional QFT $\mathcal{T}[M]$ associated with the manifold, leading to a rich interplay between geometry and QFT. Starting from Gaiotto's construction [1], subsequent development along this direction elucidated how various QFTs can be geometrically engineered from M5-branes, revealing a profound geometric structure underlying the space of QFTs.

A large family of four-dimensional (4D) $\mathcal{N} = 2$ SCFTs $\mathcal{T}[C]$ was constructed in [1] by considering M5-branes wrapping Riemann surfaces C with punctures. These theories are collectively known as *theories of class* $\mathcal{S}$. The complex moduli of the surfaces encode the gauge couplings of the SCFTs, and the punctures on the Riemann surface prescribing boundary conditions for the M5-brane determine the flavor symmetry and operator spectrum in the SCFTs. The class $\mathcal{S}$ construction furthers our understanding of M5-branes by offering a concrete, lower-dimensional perspective on the dynamics of M5-branes, and at the same time offers a geometric viewpoint on the resulting 4D $\mathcal{N} = 2$ theories.

Exact supersymmetric partition functions play an essential role in enhancing this geometric viewpoint. In particular, superconformal indices stand out as simple, yet powerful observables that count Bogomol'nyi-Prasad-Sommerfield (BPS) states (states that preserve a portion of supersymmetry). Since they are invariant under exactly marginal deformations, they encode crucial information about the Hilbert space even in the strong-coupling regime. A series of outstanding works [2–6] unveiled a topological quantum field theory (TQFT) structure underlying the superconformal indices of $\mathcal{T}[C]$ by identifying the indices with correlation functions on the Riemann surfaces C. The TQFT description maps various physical manipulations on the 4D theory $\mathcal{T}[C]$, such as gauging, Higgsing, and the insertion of nonlocal operators, to geometrical operations and objects on the corresponding Riemann surface C. In this geometric viewpoint, theories of class $\mathcal{S}$ and their indices can be built by gluing simple building blocks, and generalized S-duality becomes apparent.

A remarkable development in the study of 4D $\mathcal{N} = 2$ SCFTs is the deep connection to two-dimensional (2D)

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chiral algebra/vertex operator algebra (VOA) [7–10], referred to as an *SCFT/VOA correspondence*. In [7], it was shown that any 4D $\mathcal{N}=2$ SCFT $\mathcal{T}^{4\mathrm{D}}$ contains a protected subsector consisting of so-called Schur operators restricted on a 2D plane, which furnishes a 2D chiral algebra $\chi(\mathcal{T}^{4\mathrm{D}})$. Two notable examples of Schur operators are the $\mathrm{SU}(2)_R$ current and the flavor moment map operators, which represent the stress-energy tensor and Kac-Moody generators in $\chi(\mathcal{T}^{4\mathrm{D}})$. Through the correspondence, rich 4D physics is reincarnated in a 2D context, inspiring deeper understanding and a new construction of chiral algebras. In turn, by leveraging the rigidity of chiral algebras, we gain novel perspectives on 4D SCFTs [11–20]. In particular, the Schur index, which is a special limit of 4D $\mathcal{N}=2$ superconformal index, gets mapped to the vacuum character of $\chi(\mathcal{T}^{4\mathrm{D}})$. As discussed in [15], the fact that the stress-energy tensor in $\chi(\mathcal{T}^{4\mathrm{D}})$ is not a Higgs branch operator implies the existence of a certain null state (or descendant of null) in the Verma module of $\chi(\mathcal{T}^{4\mathrm{D}})$. Through Zhu's recursion formula [21,22], such a state translates to a modular differential equation that the Schur index should solve. In fact, additional null states may exist that lead to a set of flavored modular differential equations that all module characters of $\chi(\mathcal{T}^{4\mathrm{D}})$ must satisfy [23,24], putting stringent constraints on both the chiral algebra and the 4D physics.

In this paper, we push forward this research direction by further compactifying class $\mathcal{S}$ theories on S^2 with a topological twist. The 4D $\mathcal{N}=2$ SCFTs have an inherent $\mathrm{SU}(2)_R\times\mathrm{U}(1)_r$ R-symmetry. A topological twist on S^2 using $\mathrm{U}(1)_R\times\mathrm{U}(1)_r$ results in 2D $\mathcal{N}=(0,2)$ theories [25,26]. On the other hand, a twist with $\mathrm{U}(1)_r$ gives rise to 2D $\mathcal{N}=(0,4)$ theories [27]. In this work, we provide remarkably simple closed-form expressions of the (0,2) theories analogous to Lagrangian class $\mathcal{S}$ theories and those of the (0,4) theories of all class $\mathcal{S}$ theories of type A with genus $g>0$. Schematically, the elliptic genera for both classes take the form

$$\mathcal{I}^{2\mathrm{D}}_{g,n}=\mathcal{H}^{g-1}\prod_{i=1}^{n}\mathcal{I}_{\lambda_i}(b_i). \tag{1.1}$$

Here, $\mathcal{H}$ denotes the contribution from a handle on a Riemann surface $C_{g,n}$ and $\mathcal{I}_{\lambda_i}$ represents the contribution from the ith puncture. Surprisingly, each of these contributions, both $\mathcal{H}$ and $\mathcal{I}_{\lambda_i}$, can be expressed as a product of theta functions. This structure, as we will demonstrate, naturally unveils the explicit TQFT construction of elliptic genera on Riemann surfaces $C_{g,n}$ for both classes of 2D theories.

Moreover, in the IR, the left-moving sector of a (0,2) theory from compactifying $\mathcal{T}^{4\mathrm{D}}$ exhibits a connection to the associated chiral algebra $\chi(\mathcal{T}^{4\mathrm{D}})$. Concretely, we propose that the (0,2) elliptic genus is a linear combination of characters of the chiral algebra $\chi(\mathcal{T}^{4\mathrm{D}})$, which can be verified using flavored modular linear differential equations. This relation suggests that the (0,2) theory is endowed with the VOA $\chi(\mathcal{T}^{4\mathrm{D}})$ as the IR symmetry. Techniques for studying VOAs can be applied to gain insights into the Hilbert space and correlation functions of 2D $\mathcal{N}=(0,2)$ theories at the IR fixed point.

This paper is structured as follows. In Sec. II we explore 2D $\mathcal{N}=(0,2)$ quiver gauge theories analogous to Lagrangian class $\mathcal{S}$ theories. Our primary objective is to unveil the duality between these theories and Landau-Ginzburg (LG) models while also exploring their connection to VOAs. In Sec. II A the focus is on establishing the relation between 2D (0,2) theories and 4D $\mathcal{N}=2$ SCFTs. We consider a twisted compactification of 4D $\mathcal{N}=2$ SCFTs on S^2, which leads to 2D (0,2) quiver gauge theories. We further discuss the connection between the 4D $\mathcal{N}=2$ Schur index and the (0,2) elliptic genera. Additionally, we put forth a conjecture that the (0,2) elliptic genus can be expressed as a linear combination of characters of the corresponding chiral algebra. In Sec. II B we examine 2D (0,2) $\mathrm{SU}(2)\times\mathrm{U}(1)$ quiver gauge theories. Here, we investigate our proposal on a case-by-case basis, demonstrating that these theories have LG duals. In specific cases, we identify the elliptic genus as a linear combination of characters of the associated VOA. When explicit characters are not available, we check that the elliptic genus solves the modular linear differential equations that constrain the VOA characters. In the end, we demonstrate that the (0,2) elliptic genera exhibit a TQFT structure on Riemann surfaces with a minimal number of U(1) gauge groups. In Sec. II C we extend the computation to $\mathrm{SU}(N)\times\mathrm{U}(1)$ gauge theories. Last, Sec. II D collects a few remarks on non-Lagrangian theories, providing a perspective on this particular area of study.

In Sec. III we study 2D $\mathcal{N}=(0,4)$ theories from another twisted compactification of A-type class $\mathcal{S}$ theories on S^2. Since theories in this class generally lack Lagrangian descriptions, we make use of the elliptic inversion formula to compute their elliptic genera in Secs. III B and III C. Given that such a (0,4) theory is characterized by a Riemann surface with punctures decorated by embedding $\mathrm{SU}(2)\hookrightarrow\mathrm{SU}(N)$, in Sec. III D we propose a Higgsing procedure to derive the contributions to the (0,4) elliptic genus from the puncture data. We show that the elliptic genera of all of these theories can be reorganized as simple products of theta functions, and they exhibit a TQFT structure under the cut-and-join operations on Riemann surfaces. We end this section by commenting on future directions and open problems.

Appendix A consolidates the notations and conventions, and introduces definitions of special functions and modular forms used throughout this paper. In Appendix B we revisit the definitions of Jeffrey-Kirwan (JK) residues, given their intricate nature and frequent reference in this paper. For readers interested in in-depth calculations, Appendix C offers detailed JK residue computations for (0,2) elliptic genera, while Appendix D provides those for (0,4) elliptic genera.

II. $\mathcal{N} = (0,2)$ GAUGE/LG DUALITY AND VOAs

In this section, we study 2D $\mathcal{N} = (0,2)$ quiver gauge theories analogous to 4D $\mathcal{N} = 2$ theories of class $\mathcal{S}$ [1] with Lagrangian descriptions. We construct these (0,2) quiver gauge theories by gauging a basic building block consisting of (0,2) chiral multiplets corresponding to a sphere with two maximal punctures and one minimal puncture. This family of 2D $\mathcal{N} = (0,2)$ theories includes a class of theories obtained via a particular twisted compactification on S^2, called Schur-like reductions [25], of class $\mathcal{S}$ theories with Lagrangian descriptions. For this subclass, we study the relation between the elliptic genus of a (0,2) theory and characters of the chiral algebra of the corresponding class $\mathcal{S}$ theory. Additionally, we demonstrate that, under certain conditions, the (0,2) quiver gauge theories are dual to LG models.

A. Relation between 2D (0,2) theories and 4D $\mathcal{N} = 2$ SCFTs

1. 4D $\mathcal{N} = 2$ SCFT and Schur index

A 4D $\mathcal{N} = 2$ superconformal theory has the symmetry algebra $\mathrm{SU}(2,2|2)$, which is generated by supercharges $(Q^I_\alpha, \tilde{Q}^I_{\dot\alpha})$, their superconformal partners $(S^\alpha_I, \tilde{S}^{\dot\alpha}_I)$, and other bosonic symmetry generators. The 4D $\mathcal{N} = 2$ superconformal index counts the 1/8-BPS states that are annihilated by one supercharge and its conformal partner, say $\tilde{Q}^1_{\dot -}$ and $\tilde{S}^{\dot -}_1$. In other words, it provides a measure of the $\tilde{Q}^1_{\dot -}$ cohomology, which consists of states that saturate the bound[1]

$$\tilde{\delta}_{1\dot -} \coloneqq \{\tilde{S}^{\dot -}_1, \tilde{Q}^1_{\dot -}\} = E - 2j_2 - 2R + r.$$

Here we use the Cartan generators (E, j_1, j_2, R, r) of $\mathrm{SU}(2,2|2)$. Note that $j_{1,2}$ represents the angular momentum of $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$, and (R, r) are quantum numbers associated with the $\mathcal{N} = 2$ superconformal R-symmetry $\mathrm{SU}(2)_R \times \mathrm{U}(1)_r$. We refer to Table I for charges of supercharges under these symmetry groups. Then, the 4D $\mathcal{N} = 2$ superconformal index, denoted as $\mathcal{I}^{\mathrm{4D}}(p, q, t)$, is defined as follows:

$$\mathcal{I}^{\mathrm{4D}}(p,q,t) = \mathrm{Tr}(-1)^F e^{-\beta\tilde{\delta}_{1\dot -}} p^{-j_1+j_2-r} q^{j_1+j_2-r} t^{R+r} \prod_a z_a^{f_a}. \tag{2.1}$$

The variables z_a correspond to flavor fugacities, and f_a represents flavor charges. Evaluating the 4D $\mathcal{N} = 2$ superconformal index can be done using single-letter indices [2–5].

The contribution of the half-hypermultiplet with representation λ to the multiparticle index yields the elliptic gamma function (A4), given by

$$\begin{aligned}\mathcal{I}^{\mathrm{4D}}_{\frac{1}{2}H}(z;p,q,t) &= \prod_{w\in\lambda}\prod_{i,j=0}^{\infty} \frac{1 - z^{-w}p^{i+1}q^{j+1}/\sqrt{t}}{1 - z^{w}\sqrt{t}p^i q^j} \\ &= \prod_{w\in\lambda}\Gamma(z^w\sqrt{t}),\end{aligned} \tag{2.2}$$

where w runs over the weights of the representation λ. The 4D $\mathcal{N} = 2$ vector multiplet contributes as follows:

$$\begin{aligned}\mathcal{I}^{\mathrm{4D}}_{\mathrm{vec}}(z;p,q,t) &= \frac{\kappa^{\mathrm{rk}G}\Gamma(\frac{pq}{t})^{\mathrm{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}\frac{\Gamma(z^\alpha\frac{pq}{t})}{\Gamma(z^\alpha)}, \\ \kappa &= (p;p)(q;q),\end{aligned} \tag{2.3}$$

where Δ represents the set of roots associated with the gauge group G and $|W_G|$ is the order of the Weyl group of G. Then, the 4D $\mathcal{N} = 2$ superconformal index of a quiver gauge theory can be schematically expressed as the contour integral

$$\mathcal{I}^{\mathrm{4D}}(\boldsymbol{a};q) = \oint_{|z|=1}\prod_{\mathrm{gauge}}\frac{dz}{2\pi i z}\mathcal{I}^{\mathrm{4D}}_{\mathrm{vec}}(z;p,q,t)\prod_{\mathrm{matter}}\mathcal{I}^{\mathrm{4D}}_{\frac{1}{2}H}(z;p,q,t). \tag{2.4}$$

The 4D $\mathcal{N} = 2$ superconformal index has various specializations [4,5]. Among them, the Schur index can be obtained at the specialization of $t = q$, and it counts 1/4-BPS operators consisting of Higgs-branch operators annihilated by (Q^1_-, S^-_1) and $(\tilde{Q}^1_{\dot -}, \tilde{S}^{\dot -}_1)$. In the Schur limit, the hypermultiplet contribution is reduced to

$$\mathcal{I}^{\mathrm{4D}}_H = \Gamma(z^\pm\sqrt{t}) \coloneqq \Gamma(z\sqrt{t})\Gamma(z^{-1}\sqrt{t}) \xrightarrow{t\to q} \mathcal{I}^{\mathrm{Schur}}_H = \frac{\eta(q)}{\vartheta_4(z)}, \tag{2.5}$$

where $\eta(q)$ is the Dedekind eta function (A6) and the vector multiplet contribution is reduced to

$$\begin{aligned}\mathcal{I}^{\mathrm{4D}}_{\mathrm{vec}} &= \frac{\kappa^{\mathrm{rk}G}\Gamma(\frac{pq}{t})^{\mathrm{rk}G}}{|W_G|}\prod_{\alpha\in\Delta}\frac{\Gamma(z^\alpha\frac{pq}{t})}{\Gamma(z^\alpha)} \xrightarrow{t\to q} \mathcal{I}^{\mathrm{Schur}}_{\mathrm{vec}} \\ &= \frac{\eta(q)^{2\mathrm{rk}G}}{|W_G|}\prod_{\alpha\in\Delta} i\frac{\vartheta_1(z^\alpha)}{\eta(q)}.\end{aligned} \tag{2.6}$$

Leveraging the state/operator correspondence, a Schur state can be obtained from a Schur operator $\mathcal{O}(0)$ applied to

[1]Compared with the notation in [7], the supercharges $Q^1_-, \tilde{Q}^1_{\dot -}$ in this paper are identified with $Q^1_-, \tilde{Q}_{2\dot -}$ there.

the vacuum. The Schur operator at the origin (anti)commutes with the set of the four supercharges. While shifting the operator from this point generally disrupts the BPS condition, one can change the position of the operator across the $\mathbb{R}^2_{34} = \mathbb{C}_{z,\bar{z}}$ plane using the twisted translation proposed in [7]:

$$\mathcal{O}(z,\bar{z}) \coloneqq e^{-zL_{-1}-\bar{z}\hat{L}_{-1}}\mathcal{O}(0)e^{+zL_{-1}+\bar{z}\hat{L}_{-1}}a,$$

where

$$L_{-1} = P_{++}, \qquad \hat{L}_{-1} = P_{--} + R^2_1.$$

The twisted translated Schur operator $\mathcal{O}(z,\bar{z})$ is also annihilated by the two supercharges

$$\mathbb{Q}_1 \coloneqq Q^1_- + \tilde{S}^{\dot{-}}_1, \qquad \mathbb{Q}_2 \coloneqq \tilde{Q}^1_{\dot{-}} - S^-_1. \qquad (2.7)$$

Moreover, the $\bar{z}$ dependence of $\mathcal{O}(z,\bar{z})$ turns out to be $\mathbb{Q}_{1,2}$ exact. Consequently, at the level of $\mathbb{Q}_{1,2}$ cohomology, the cohomology class $\mathcal{O}(z) \coloneqq [\mathcal{O}(z,\bar{z})]$ depends on the location holomorphically in z. Furthermore, the operator product expansion coefficients of these Schur operators (as cohomology classes) are also holomorphic, forming a 2D VOA/chiral algebra on the plane $\mathbb{C}_{z,\bar{z}}$, as discussed in [7].

The space of Schur operators defines the space of states of the associated VOA, and thus the Schur limit of the superconformal index, which counts the Schur operators with signs, equals the vacuum character of the associated VOA. The associated chiral algebra and the vacuum character are interesting and powerful invariants of 4D $\mathcal{N} = 2$ SCFTs, which capture various aspects of 4D physics. Nonperturbative dynamics in four dimensions can be probed by surface defects [28]. To study the relation of the chiral algebra, one can introduce half-BPS surface operators in $\mathcal{T}$ preserving the same nilpotent supercharges $\mathbb{Q}_{1,2}$ where the support of the surface defect transversely intersects with the chiral algebra plane $\mathbb{C}_{z,\bar{z}}$ at the origin. From the perspective of the chiral algebra, such a surface defect introduces a nontrivial boundary condition at the origin, and the defect operators are acted on by the Schur operators in the 2D bulk through bulk-defect operator product expansion. Therefore, it is believed that such surface defects introduce nonvacuum modules of the associated chiral algebra.

Similar to the original Schur index, one can count defect operators in the $\mathbb{Q}_{1,2}$ cohomology to obtain the defect Schur index. In general, it is difficult to compute graded dimensions of such operators from first principles. However, a superconformal index in the presence of a surface defect can be evaluated with suitable manipulations [29,30]. A notable example of the manipulations involves vortex defects, which can be derived using the Higgsing procedure on a 4D $\mathcal{N} = 2$ SCFT [29]. The vortex defect index can be computed by an appropriate residue computation on the superconformal index of the theory $\mathcal{T}^{\mathrm{UV}}$. For example, a vortex defect with vorticity k in an A_1-type class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g,n}]$ can be computed by (up to some factors q)

$$\mathcal{I}^{\mathrm{def}}_{g,n}(b_1,\ldots,b_n) = \mathrm{Res}_{b_{n+1}\to q^{\frac{k+1}{2}}}\frac{\eta(\tau)^2}{b_{n+1}}\mathcal{I}_{g,n+1}(b_1,\ldots,b_{n+1}). \qquad (2.8)$$

Here b_i denote the SU(2) flavor fugacities that are manifest in the class $\mathcal{S}$ construction.

In particular, the techniques to evaluate the Schur index with a surface defect have been developed to study the relation to the chiral algebra [23,31,32]. The original Schur index of $\mathcal{T}$ can be viewed as a supersymmetric partition function on $S^3_{\varphi,\chi,\theta} \times S^1_t$ (with suitable background fields turned on),[2] and it localizes to a multivariate contour integral of an elliptic integrand $\mathcal{Z}(\mathsf{a}_i)$, where the integration variables a_i capture the holonomy of the dynamical gauge field along the temporal S^1 [33,34]. To define a surface defect, one can also specify a BPS singular boundary condition of the gauge fields at the defect plane $\mathbb{R}^2 \subset \mathbb{R}^4$ or, equivalently, at a particular T^2 in $S^3 \times S^1$. Such a singular background shifts the corresponding integration variables $\mathsf{a}_i \to \mathsf{a}_i + \lambda_i\tau$, where the λ_i reflect the singular boundary condition. As the values of λ_i vary, the shifted integration variables eventually cross the integral contour. Consequently, the Schur index with a surface defect is given by integration around a different contour, instead of the unit-circle integral like (2.4). The resulting defect Schur index is expected to be a linear combination of nontrivial characters of the associated chiral algebra.

2. 2D (0,2) theory and elliptic genus

2D $\mathcal{N} = (0,2)$ supersymmetric field theories have attracted considerable attention due to their importance in theoretical and mathematical physics. In a 2D $\mathcal{N} = (0,2)$ gauge theory, the matter content generically consists of chiral multiplets, Fermi multiplets, and vector multiplets. A chiral multiplet (ψ_+,ϕ) contains a complex scalar field and a right-moving Weyl fermion ψ_+, while a vector multiplet (A_μ,λ_-) contains a gauge field A_μ and gauginos $\lambda_-,\bar{\lambda}_-$. A Fermi multiplet $(\psi_-,E(\phi))$ consists of a left-moving Weyl fermion (ψ_-) and an E term which is a holomorphic function of some chiral multiplets. For a comprehensive explanation, we refer the reader to [35].

In the analysis of 2D supersymmetric theories, a fundamental tool is the elliptic genus [36], which counts BPS states protected under renormalization group flow. Conceptually, the elliptic genus can be understood as a partition function defined on a torus with a complex

[2]Here the S^3 is viewed as a $T^2_{\varphi,\chi}$ fibering over an interval $[0,\pi/2]_\theta$. The points with $\theta = 0$ and $\theta = \pi/2$ form two special tori.

structure parameter τ, where fermions exhibit periodic boundary conditions along the temporal circle. Moreover, the spatial circle allows for two distinct types of boundary conditions, Ramond and Neveu-Schwarz (NS), both applicable to the left- and right-moving sectors. For the sake of simplicity, we focus on the Ramond-Ramond and NS-NS sectors, referring to them as the Ramond and NS sectors, respectively.

In the context of $\mathcal{N}=(0,2)$ gauge theory, the elliptic genus in the Ramond and NS sectors can be defined, respectively, by

$$\mathcal{I}^{(0,2)_{\mathrm{R}}}(q,z)=\mathrm{Tr}_{\mathrm{R}}(-1)^F q^{H_L}\bar{q}^{H_R}\prod_a z_a^{f_a},$$
$$\mathcal{I}^{(0,2)_{\mathrm{NS}}}(q,z)=\mathrm{Tr}_{\mathrm{NS}}(-1)^F q^{H_L}\bar{q}^{(H_R-\frac{R}{2})}\prod_a z_a^{f_a}, \qquad (2.9)$$

where the left- and right-moving Hamiltonians are $2H_L=H+iP$ and $2H_R=H-iP$, respectively, in the Euclidean signature and R represents the $\mathrm{U}(1)_R$ R charge. In a superconformal theory, these operators correspond to the zero-mode generators L_0, $\bar{L}_0$, and $\bar{J}_0$ of the superconformal algebra.

Due to supersymmetry, only right-moving ground states ($H_R=0$) contribute to the elliptic genus in the Ramond sector, while right-moving chiral primary states ($H_R=\frac{R}{2}$) in the right-moving sector contribute to the elliptic genus in the NS sector. Consequently, the elliptic genera in both sectors are holomorphic functions of q.

For $\mathcal{N}=(0,2)$ theories described by a Lagrangian, the computation of the elliptic genus depends on the specific details of the gauge theory and its matter content, as outlined in [37,38]. Let us consider the contributions from different types of multiplets.

Chiral multiplet: The contribution of an $\mathcal{N}=(0,2)$ chiral multiplet in a representation λ of the gauge and flavor group is

$$\mathcal{I}_{\mathrm{chi}}^{(0,2)_{\mathrm{R}}}(q,z)=\prod_{w\in\lambda} i\frac{\eta(q)}{\vartheta_1(z^w)},$$
$$\mathcal{I}_{\mathrm{chi}}^{(0,2)_{\mathrm{NS}}}(\tau,u)=\prod_{w\in\lambda}\frac{\eta(q)}{\vartheta_4(q^{\frac{r-1}{2}}z^w)}. \qquad (2.10)$$

Fermi multiplet: The contribution of an $\mathcal{N}=(0,2)$ Fermi multiplet in a representation λ of the gauge and flavor group is given by

$$\mathcal{I}_{\mathrm{fer}}^{(0,2)_{\mathrm{R}}}(q,z)=\prod_{w\in\lambda} i\frac{\vartheta_1(z^w)}{\eta(q)},$$
$$\mathcal{I}_{\mathrm{fer}}^{(0,2)_{\mathrm{NS}}}(\tau,u)=\prod_{w\in\lambda}\frac{\vartheta_4(q^{\frac{r}{2}}z^w)}{\eta(q)}. \qquad (2.11)$$

Vector multiplet: The contribution of an $\mathcal{N}=(0,2)$ vector multiplet with gauge group G is

$$\mathcal{I}_{\mathrm{vec}}^{(0,2)_{\mathrm{R|NS}}}(q,z)=\frac{\eta(q)^{2\mathrm{rk}G}}{|W_G|}\prod_{\alpha\in\Delta} i\frac{\vartheta_1(z^\alpha)}{\eta(q)}. \qquad (2.12)$$

Note that the elliptic genera in the NS sector for chiral and Fermi multiplets depend on the R charge r of the multiplet.

Then, the elliptic genus of a quiver gauge theory can be schematically expressed as the Jeffrey-Kirwan (JK) residue integral [38–41]

$$\mathcal{I}^{(0,2)_{\mathrm{R|NS}}}=\int_{\mathrm{JK}}\prod_{\mathrm{gauge}}\frac{dz}{2\pi iz}\mathcal{I}_{\mathrm{vec}}^{(0,2)_{\mathrm{R|NS}}}(q,z)$$
$$\times\prod_{\mathrm{matter}}\mathcal{I}_{\mathrm{chi}}^{(0,2)_{\mathrm{R|NS}}}(q,z)\mathcal{I}_{\mathrm{fer}}^{(0,2)_{\mathrm{R|NS}}}(q,z). \qquad (2.13)$$

In this section, our computation focuses on NS elliptic genera to compare with characters of the associated VOA. Nevertheless, Ramond elliptic genera can be obtained from the NS ones simply by replacing ϑ_4 by ϑ_1.

Since (0,2) theories are chiral theories, one must pay attention to anomalies. Let us consider a (0,2) theory with a global symmetry F described by a simple Lie algebra. The 't Hooft anomaly coefficient k_F associated with this symmetry can be determined by

$$\mathrm{Tr}\gamma^3 f^a f^b=k_F\delta^{ab}. \qquad (2.14)$$

Here, f^a represents the generators of F, γ^3 denotes the gamma matrix that quantifies chirality, and the trace is taken over Weyl fermions in the theory. For a global anomaly, the computation involves evaluating the difference between the sums over the sets of (0,2) chiral and Fermi multiplets:

$$k_F=\sum_{\Phi\in(0,2)\,\mathrm{chiral}}T(R_F^\Phi)-\sum_{\Psi\in(0,2)\,\mathrm{Fermi}}T(R_F^\Psi), \qquad (2.15)$$

where $T(R_F)$ represents the index of the representation R_F of F. Note that the (0,2) supermultiplet of the gauge-invariant field strength can be treated as a Fermi multiplet, and the gaugino contributes to the anomaly if it is charged under F. In this paper, we focus solely on $\mathrm{SU}(N)$ groups as instances of non-Abelian symmetries. For these groups, the indices are given by $T(\square)=\frac{1}{2}$ and $T(\mathbf{adj})=N$. In cases where the theory possesses two U(1) symmetries, $\mathrm{U}(1)_1$ and $\mathrm{U}(1)_2$, with charges f_1 and f_2, respectively, a mixed 't Hooft anomaly can emerge:

$$k_{12}=\mathrm{Tr}(\gamma^3 f_1 f_2). \qquad (2.16)$$

Certainly, any gauge anomaly must vanish for the theory to be well defined.

In particular, the anomaly associated with the $\mathrm{U}(1)_R$ symmetry is related to the right-moving central charge c_R as

$$c_R = 3\mathrm{Tr}(\gamma^3 R^2). \tag{2.17}$$

To determine $\mathrm{U}(1)_R$ charges of various fields, the c extremization is performed if the theory meets the following two assumptions [42,43]:

(1) The theory is bounded and the energy spectrum is bounded from below.

(2) The classical vacuum moduli space is normalizable.

These conditions imply several standard properties in CFTs, including the holomorphicity of the conserved currents and the existence of an operator with a lowest and positive dimension in a conformal family. Once the central charge c_R is determined, the left-moving central charge can be obtained from the gravitational anomaly, which is the difference between the number of chiral and Fermi multiplets,

$$c_R - c_L = \mathrm{Tr}(\gamma^3). \tag{2.18}$$

The presence of anomalies can be observed directly at the level of (0,2) elliptic genera [27]. Let us focus on an $\mathrm{SU}(N)$ global symmetry whose fugacities are denoted by $b_{1,\ldots,N}$, with $\prod_{i=1}^N b_i = 1$ in an elliptic genus. Then, the corresponding 't Hooft anomaly can be seen in the shift of the elliptic genus as

$$\mathcal{I}^{(0,2)}(\boldsymbol{b}) \to (qb_i/b_j)^{2k_F}\mathcal{I}^{(0,2)}(\boldsymbol{b}) \quad \text{as} \quad b_i \to qb_i, b_j \to b_j/q. \tag{2.19}$$

For a U(1) 't Hooft anomaly, the elliptic genus behaves as

$$\mathcal{I}^{(0,2)}(c) \to (-q^{1/2}c)^{k_F}\mathcal{I}^{(0,2)}(c) \quad \text{as} \quad c \to qc, \tag{2.20}$$

where c is the corresponding U(1) fugacity. In particular, for a theory to be gauge anomaly free, the integrand of its elliptic genus (2.13) must be invariant under shifts of the gauge fugacities $z_i \to qz_i$.

3. Twisted compactifications of 4D $\mathcal{N}=2$ SCFTs on S^2

A 2D $\mathcal{N}=(0,2)$ theory can be obtained from a 4D $\mathcal{N}=1$ gauge theory and, in particular, a 4D $\mathcal{N}=2$ Lagrangian SCFT by a compactification on S^2 with a certain topological twist. Such a reduction was referred to as a Schur-like reduction in [25]. (See also Sec. V of [26].)

The first explicit supersymmetric localization of 4D $\mathcal{N}=1$ theories on $T^2\times S^2$ was carried out in [44]. Subsequently, its relationship with the 2D $\mathcal{N}=(0,2)$ elliptic genera was explored around the same time [25–27,45,46]. In this context, let us summarize the key aspects involved in the twisted compactification of 4D $\mathcal{N}=1$ theories on S^2.

First of all, the generic holonomy group $\mathrm{Spin}(4)=\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$ reduces to $\mathrm{U}(1)_{T^2}\times\mathrm{U}(1)_{S^2}$ on the background $T^2\times S^2$, where $\mathrm{U}(1)_{T^2}$ [$\mathrm{U}(1)_{S^2}$] is the U(1) subgroup of the diagonal [antidiagonal] subgroup of $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_2$. In order to define covariant constant supercharges on S^2, we perform a topological twist of $\mathrm{U}(1)_{S^2}$ along with a global $\mathrm{U}(1)_{\mathfrak{R}}$ symmetry. Additionally, in a Lagrangian theory, the $\mathrm{U}(1)_{\mathfrak{R}}$ charge of a 4D $\mathcal{N}=1$ chiral multiplet must be an integer to ensure a well-defined compactification on S^2.

Under this compactification, a 4D $\mathcal{N}=1$ chiral multiplet with $\mathrm{U}(1)_{\mathfrak{R}}$ charge $\mathfrak{r}$ decomposes into $(1-\mathfrak{r})$ (0,2) chiral multiplets if $\mathfrak{r}<1$, or $(\mathfrak{r}-1)$ (0,2) Fermi multiplets if $\mathfrak{r}>1$. When $\mathfrak{r}=1$, the 4D chiral multiplet does not contribute to the 2D theory.

In general, the $T^2\times S^2$ partition function involves the summation of magnetic fluxes of gauge fields. Nevertheless, if the $\mathrm{U}(1)_{\mathfrak{R}}$ charges of all chiral multiplets are non-negative, then the contributions from all nonzero flux sectors vanish and only the zero flux contribution remains.

The R symmetry of 4D $\mathcal{N}=2$ superconformal theory is $\mathrm{SU}(2)_R\times\mathrm{U}(1)_r$. To perform the twisted compactification above, we treat the theory as an $\mathcal{N}=1$ theory by selecting a $\mathrm{U}(1)_{\mathfrak{R}}\subset\mathrm{SU}(2)_R\times\mathrm{U}(1)_r$. Let us consider the choice $\mathfrak{R}=R+\frac{r}{2}$. As in Table I, only the supercharges $Q^1_-, \tilde{Q}^2_{\dot{-}}$ are neutral under $\mathfrak{R}+2(j_1-j_2)$ and therefore survive under this twist. These two supercharges share the same $\mathrm{U}(1)_{T^2}$ charge $2(j_1+j_2)=-1$, and hence this twist leads to an $\mathcal{N}=(0,2)$ supersymmetry. The symmetry $\mathfrak{R}=R+\frac{r}{2}$ is referred to as the 2D (0,2) $\mathrm{U}(1)_R$ symmetry.

With the choice of $\mathfrak{R}$, the adjoint chiral Φ in an $\mathcal{N}=2$ vector multiplet has $\mathfrak{r}=1$, which does not contribute to the 2D theory. Consequently, an $\mathcal{N}=2$ vector multiplet simply reduces to a (0,2) vector multiplet. On the other hand, the hypermultiplet $(q,\tilde{q})$ is assigned a fractional charge $\mathfrak{r}=\frac{1}{2}$. In order for their $\mathfrak{R}$ charges to be integers, we further twist with the flavor symmetry $\mathrm{U}(1)_f$, which acts on the two chirals $(q,\tilde{q})$ with opposite charges. Under the resulting $\mathfrak{R}=R+\frac{1}{2}(r-f)$, the scalars q and $\tilde{q}$ acquire charges $\mathfrak{r}=0$ and $\mathfrak{r}=1$, respectively. Hence, an $\mathcal{N}=2$ hypermultiplet reduces to a (0,2) chiral multiplet in the representation of q (see Table I). In this paper, the above process of compactification on S^2 with the topological twist is referred to as the *(0,2) reduction*.

For a Lagrangian theory, an interesting observation emerges when comparing the Schur limit (2.5) and (2.6) with the contributions of the (0,2) chirals (2.10) and the vector multiplet (2.12) to the elliptic genus. The integrand of the elliptic genus for the 2D (0,2) theory after the reduction coincides with that of the Schur index of the original 4D $\mathcal{N}=2$ SCFT, upon suitable shifts of U(1) flavor fugacities.

TABLE I. Symmetries of 4D $\mathcal{N}=2$ supercharges and fields. The 4D $\mathcal{N}=1$ chirals $(q,\tilde{q})$ constitute $\mathcal{N}=2$ hypermultiplets and Φ represents the $\mathcal{N}=1$ adjoint chiral in an $\mathcal{N}=2$ vector multiplet. The fifth column denotes the $\mathrm{U}(1)_f$ flavor symmetry that distinguishes q and $\tilde{q}$. The topological twist of $\mathrm{U}(1)_{S^2}$ with $\mathrm{U}(1)_{R+\frac{1}{2}(r-f)}$ results in the $\mathcal{N}=(0,2)$ supersymmetry, whereas the twist with $\mathrm{U}(1)_r$ yields the $\mathcal{N}=(0,4)$ supersymmetry.

	$\mathrm{SU}(2)_1$	$\mathrm{SU}(2)_2$	$\mathrm{SU}(2)_R$	$\mathrm{U}(1)_r$	$\mathrm{U}(1)_f$	$\mathrm{U}(1)_{T^2}$	$\mathrm{U}(1)_{S^2}$	$\mathrm{U}(1)^{(0,2)}$	$\mathrm{U}(1)^{(0,4)}$
Q^1_-	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	0	−1	−1	0	0
Q^1_+	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	1	1	2	2
Q^2_-	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	−1	−1	−1	0
Q^2_+	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	0	1	1	1	2
$\tilde{Q}^1_-$	0	$-\frac{1}{2}$	$\frac{1}{2}$	−1	0	−1	1	1	0
$\tilde{Q}^1_+$	0	$\frac{1}{2}$	$\frac{1}{2}$	−1	0	1	−1	−1	−2
$\tilde{Q}^2_-$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	−1	0	−1	1	0	0
$\tilde{Q}^2_+$	0	$\frac{1}{2}$	$-\frac{1}{2}$	−1	0	1	−1	−2	−2
q	0	0	$\frac{1}{2}$	0	1	0	0	0	0
$\tilde{q}$	0	0	$\frac{1}{2}$	0	−1	0	0	1	0
Φ	0	0	0	2	0	0	0	1	2

However, it is important to note that the computation of the (0,2) elliptic genus involves the JK residue integral, whereas the Schur index is evaluated by a contour integral along the unit circles. We provide a more detailed account of this comparison in the subsequent discussion.

4. Relation to VOAs

In the following analysis, we explore the 2D (0,2) quiver gauge theories obtained by gauging the free 2D (0,2) theories associated with the three-punctured spheres. This includes, as a subclass, the theories from the (0,2) reduction of Lagrangian class $\mathcal{S}$ theories. We demonstrate that under a certain condition, surprisingly, these quiver theories including the reduced class $\mathcal{S}$ theories are dual to LG models.

Let us discuss the central charges of the (0,2) reduction of Lagrangian class $\mathcal{S}$ theories. As illustrated below (2.17), if the two assumptions are satisfied, the central charges are determined by the c extremization. However, the vacuum moduli space of the (0,2) reduction of a class $\mathcal{S}$ theory is noncompact, so the second assumption of the c extremization is violated. Specifically, in such situations, a flavor current related to a noncompact direction is nonholomorphic so that it does not mix with the R-symmetry current [43,47]. Consequently, the naive application of the c extremization to a (0,2) theory of this class can yield negative central charges. Nonetheless, the left-moving central charge turns out to coincide with that of the VOA of the parent class $\mathcal{S}$ theory (see Sec. 5.2 of [26]),

$$c_L^{\text{naive}} = -12 c_{4D}, \tag{2.21}$$

if we assign the "wrong" $\mathrm{U}(1)_R$ charges to chiral multiplets as a result of the naive application of the c extremization

$$r_\Phi = 1. \tag{2.22}$$

To get the physical central charge, we must enforce no mixing with any nonholomorphic flavor current arising from noncompact directions in the moduli space. At a practical level, we assign zero $\mathrm{U}(1)_R$ charges,

$$r_\Phi = 0, \tag{2.23}$$

to the chiral multiplets which parametrize the noncompact directions. This procedure yields the correct positive central charges where the left-moving c_L is shifted from c_L^{naive} of the VOA associated with the 4D theory [26,48] by

$$c_L = c_L^{\text{naive}} + 3n_h = c_L^{\text{naive}} + 12(5c^{4d} - 4a^{4d}), \tag{2.24}$$

where n_h represents the number of hypermultiplets and a^{4d}, c^{4d} are the anomaly coefficients in the corresponding class $\mathcal{S}$ theory. This suggests that the (0,2) theory flows to an SCFT fixed point despite the noncompact moduli space.

Exploring 2D (0,2) theories at the IR fixed point through the viewpoint of associated VOA is of significant importance because the BPS operators in the (0,2) IR CFT constitute a VOA. As elucidated in [49], the VOA associated with a (0,2) theory of this type arises from a Bechi-Rouet-Stora-Tyutin reduction of the $bc\beta\gamma$ system at zero gauge coupling, where the $\beta\gamma$ systems in the corresponding gauge representations come from the free (0,2) chiral multiplets, and small bc ghosts in the adjoint of the gauge group come from the free vector multiplets. The conformal weights of an involved $\beta\gamma$ system are $(h_\beta, h_\gamma) = (1-\lambda, \lambda)$, where λ is related to the correct R-charge assignment of the IR CFT. The state space of the resulting VOA is independent of the parameter λ, but the parameter affects the stress-energy tensor T of the (0,2) IR CFT. In fact, the shift (2.24)

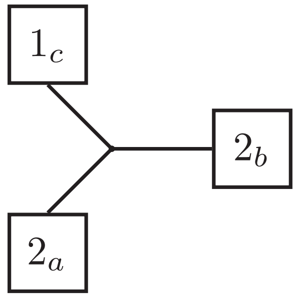

FIG. 1. Basic building block U_2 for SU(2) theory.

in central charge can be traced back to the difference between the stress-energy tensor T and T^χ of the VOA associated with the 4D theory by the derivative of the U(1) flavor current J of the $\beta\gamma$ system; schematically [48,49],

$$T = T^\chi + \left(\frac{1}{2} - \lambda\right)\partial J. \tag{2.25}$$

Therefore, the deviation from $\lambda = \frac{1}{2}$ reflects the noncompact nature of the vacuum moduli space, which invalidates the naive application of the c extremization.

From our previous discussions, we observed that the integrand of the (0,2) elliptic genus of the reduction of 4D $\mathcal{N} = 2$ Lagrangian SCFT agrees with the integrand of the Schur index [upon appropriate redefinitions of U(1) flavor fugacities], albeit with distinct contour choices. As discussed at the end of Sec. II A 1, a different contour of the integrand is expected to provide the Schur index with a surface defect, which is intrinsically tied to nonvacuum characters of the corresponding VOA $\chi(\mathcal{T}^{\mathrm{4D}})$. Hence, we investigate the relationship between the (0,2) reduction of the class $\mathcal{S}$ theory and the corresponding VOA from this perspective.

There are two primary tools for us to investigate this relation: the elliptic genus [37,38] and modular (linear) differential equations [24,50,51]. Combining these tools, we present an intriguing conjecture proposing that the NS elliptic genus of the (0,2) reduction of a Lagrangian class $\mathcal{S}$ can be expressed as a linear combination of characters $\mathrm{ch}_{\lambda_i}^{\chi(\mathcal{T}_N[C_{g,n}])}$ of the corresponding VOA:

$$\mathcal{I}_{g,n}^{(0,2),N} = \sum_i a_i \mathrm{ch}_{\lambda_i}^{\chi(\mathcal{T}_N[C_{g,n}])}, \tag{2.26}$$

where $a_i \in \mathbb{Q}$ and λ_i represent the highest weights of representations of $\chi(\mathcal{T}_N[C_{g,n}])$. We remark that the U(1) flavor fugacities in the elliptic genus need to be appropriately redefined to precisely align with the characters of the VOA. Moreover, since the theory is dual to an LG model, the elliptic genus can be simply expressed as a product of theta and eta functions, which can be viewed as free field characters of suitable 2D $bc\beta\gamma$ systems. Consequently, it forms a Jacobi form with its index determined by the 't Hooft anomaly of the global symmetry.

B. SU(2) × U(1) gauge theories, LG duals, and VOAs

In this subsection, we consider 2D $\mathcal{N} = (0,2)$ gauge theories with SU(2) and U(1) gauge groups. Analogous to the 4D $\mathcal{N} = 2$ superconformal theories of class $\mathcal{S}$, the 2D theories we consider will have the building blocks depicted in Fig. 1.

The basic building block in class $\mathcal{S}$ is the theory corresponding to a three-punctured sphere $C_{0,3}$. In the case of type A_1, the 4D $\mathcal{N} = 2$ theory T_2 is a free theory of eight half-hypermultiplets with the flavor symmetry $\mathrm{SU}(2)_a \times \mathrm{SU}(2)_b \times \mathrm{SU}(2)_c$. (See the left side of Fig. 16.) We use $\mathrm{U}(1)_c \subset \mathrm{SU}(2)_c$ for the topological twist on S^2, and the (0,2) reduction of T_2 using this flavor symmetry leads to the theory of free (0,2) chiral multiplets in the representation $(\mathbf{2}, \mathbf{2}, 1)$ of $\mathrm{SU}(2)_a \times \mathrm{SU}(2)_b \times \mathrm{U}(1)_c$ flavor symmetry. This (0,2) theory, akin to T_2, serves as the fundamental building block and is denoted as U_2. In quiver notation, we depict this theory as a vertex with three external legs corresponding to $\mathrm{SU}(2)_a \times \mathrm{SU}(2)_b \times \mathrm{U}(1)_c$ flavor groups (see Fig. 1).

It follows from (2.10) that the NS elliptic genus of the U_2 theory is

$$\mathcal{I}_{U_2}^{(0,2)}(a, b; c) = \frac{\eta(q)}{\vartheta_4(q^{-\frac{1}{2}} a^{\pm 1} b^{\pm 1} \tilde{c})} = \frac{\eta(q)}{\vartheta_4(a^{\pm 1} b^{\pm 1} c)}. \tag{2.27}$$

As emphasized in (2.23), the $\mathrm{U}(1)_R$ charge of the (0,2) chiral multiplet is zero $r = 0$, but we shift the $\mathrm{U}(1)_c$ flavor fugacity by $c = q^{-\frac{1}{2}} \tilde{c}$. Note that the factors with a repeated sign $\pm$ in the arguments are all multiplied [see (A2)]. On the other hand, as seen in (2.5), the Schur limit of the superconformal index of the T_2 theory is given by [2]

$$\mathcal{I}_{T_2}^{\mathrm{4D}} = \Gamma(\sqrt{t} a^{\pm 1} b^{\pm 1} c^{\pm 1}) \xrightarrow{t \to q} \mathcal{I}_{T_2}^{\mathrm{Schur}} = \frac{\eta(q)}{\vartheta_4(a^{\pm 1} b^{\pm 1} c)}. \tag{2.28}$$

Therefore, the redefinition of the $\mathrm{U}(1)_c$ flavor fugacity in (2.27) ensures that the elliptic genus of the U_2 theory coincides with the Schur index of the T_2 theory. In this way, the elliptic genus can be decomposed into characters of the corresponding VOA.

Moreover, the contribution of a vector multiplet is the same in both the Schur index (2.6) and the elliptic genus (2.12). The SU(2) vector multiplet contribution is

$$\mathcal{I}_{\mathrm{vec}}^{(0,2)}(a) = -\frac{\vartheta_1(a^{\pm 2})}{2}, \tag{2.29}$$

and the SU(2) gauging leads to no gauge anomaly.

FIG. 2. U(1) gauging.

As a Riemann surface $C_{g,n}$ can be constructed by gluing pants, the corresponding 4D $\mathcal{N}=2$ theory $\mathcal{T}_2[C_{g,n}]$ can be obtained by gauging the T_2 theories. However, as observed, the distinction between the 4D T_2 theory and 2D U_2 theory lies in the global symmetry. To construct (0,2) quiver gauge theories using the U_2 building block, we outline the U(1) gauging method.

Given a $\mathrm{U}(1)_{c_1}\times\mathrm{U}(1)_{c_2}$ global symmetry, we gauge the antidiagonal part $\mathsf{a}=(\mathsf{c}_1-\mathsf{c}_2)/2$ with keeping the diagonal $\mathsf{d}=(\mathsf{c}_1+\mathsf{c}_2)/2$ as a global symmetry. See Fig. 2 for an illustration, where the circle denotes a gauge node. To cancel gauge anomaly during the gauging process, we include Fermi multiplets with $\mathrm{U}(1)_a$ gauge charges ± 2. The $\mathrm{U}(1)_R$ charge for these Fermi multiplets is taken to be $r=0$, a value determined by the c extremization. At the level of the elliptic genus, the gauging procedure is given by

$$\frac{\eta(q)}{\vartheta_4(c_1\cdots)}\frac{\eta(q)}{\vartheta_4(c_2\cdots)} \to \eta(q)^2\int_{\mathrm{JK}}\frac{da}{2\pi i a}\frac{\eta(q)}{\vartheta_4(da\cdots)}\frac{\eta(q)}{\vartheta_4(da^{-1}\cdots)}\frac{\vartheta_4(a^{\pm 2})}{\eta(q)^2}. \quad (2.30)$$

The details of U(1) gauging will be seen in examples in Secs. II B 4 and II B 6 and below.

Through U(1) gauging, one can construct a family of (0,2) quiver gauge theories. However, when a U(1) gauge group is involved, a (0,2) theory is no longer the (0,2) reduction of a class $\mathcal{S}$ theory in general. Nonetheless, as we will demonstrate, (0,2) theories of genus $g>0$ with $(g-1)$ U(1) gauge groups are dual to each other, making them frame independent. We further propose that the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g>0,n}]$ of type A_1 on S^2 is closely related to the corresponding (0,2) quiver theory of genus g with $(g-1)$ U(1) gauge groups. Notably, if one replaces $\vartheta_4(a^{\pm 2})$ by $\vartheta_1(a^{\pm 2})$ in the U(1) gauging, then the integrand of the elliptic genus is the same as that of the corresponding Schur index (up to a factor of 2^{g-1}), though their integration contours differ; the elliptic genus uses the JK prescription, while the Schur index uses the maximal tori of the gauge groups. Additionally, the (0,2) theory of this class turns out to be dual to an LG model.

Furthermore, we explore generalized quiver gauge theories by gluing the U_2 theories, extending the (0,2) reduction of the class $\mathcal{S}$ theories. Specifically, we demonstrate that a quiver gauge theory with g loops and g U(1) gauge nodes is dual to an LG model.

1. SU(2) supersymmetric QCD

First, let us consider the SU(2) gauge theory with four fundamental chiral multiplets, whose quiver diagram is presented in Fig. 3. As described in [47], the naive application of the c extremization leads to the "wrong" central charges of the theory,

$$c_L^{\mathrm{naive}}=-14, \qquad c_R^{\mathrm{naive}}=-9, \quad (2.31)$$

while the enforcement of the zero R charge to the chiral multiplets results in the "correct" positive central charges,

$$c_L=10, \qquad c_R=15. \quad (2.32)$$

Nevertheless, the "wrong" left-moving central charge c_L^{naive} is equal to the central charge of the corresponding VOA $\mathfrak{so}(8)_{-2}$

This theory was first studied in [27] and its elliptic genus was evaluated there as

$$\begin{aligned}\mathcal{I}_{0,4}^{(0,2),2} &= \int_{JK}\frac{da}{2\pi i a}\mathcal{I}_{U_2}^{(0,2)}(b_1,a;c_1)\mathcal{I}_{\mathrm{vec}}^{(0,2)}(a)\mathcal{I}_{U_2}^{(0,2)}(b_2,a;c_2)\\ &= \frac{\eta(q)^5\vartheta_1(c_1^2c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1c_2b_1^{\pm}b_2^{\pm})}. \end{aligned} \quad (2.33)$$

As pointed out in [25,27,47,49], the theory exhibits an LG dual description. This dual description involves two chiral multiplets $\Phi_{1,2}$, and one chiral meson multiplet $\tilde{\Phi}_{i,j=1,2}$ with $\mathrm{U}(1)_R$ charge of 0, as well as a Fermi multiplet Ψ with $\mathrm{U}(1)_R$ charge of 1, and they form a J-type superpotential

$$W=\Psi(\Phi_1\Phi_2+\det\tilde{\Phi}).$$

The six chiral multiplets Φ can be regarded as the $\wedge^2\,\mathbf{4}=\mathbf{6}$ representation of SU(4) so that the superpotential can be

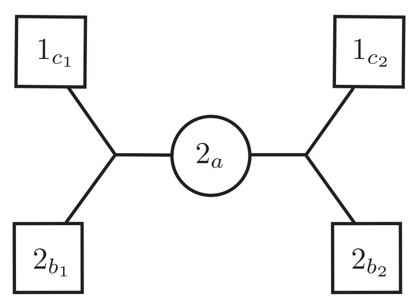

FIG. 3. Quiver diagram for SU(2) SQCD with four fundamental chirals. In the diagram, a circle node represents a gauge group and square nodes denote flavor groups. The number N within a node means $\mathrm{SU}(N)$ for $N>1$ and U(1) for $N=1$.

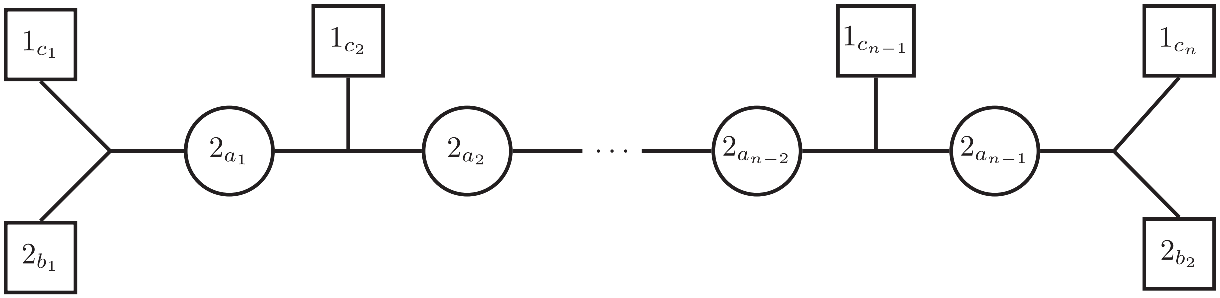

FIG. 4. SU(2) linear quiver.

expressed as $W = \Psi Pf(\Phi)$. Then, the expression (2.33) can be naturally understood as the elliptic genus of the LG model.

More remarkably, it was revealed in [48,49] that the space of BPS states in the SU(2) supersymmetric QCD (SQCD) has a relation to the VOA $\mathfrak{so}(8)_{-2}$, which is the VOA $\chi(\mathcal{T}_2[C_{0,4}])$ of the corresponding 4D $\mathcal{N}=2$ theory [7]. Concretely, it was found [48] that the elliptic genus (2.33) can be written as a linear combination of characters of $\mathfrak{so}(8)_{-2}$,

$$\mathcal{I}^{(0,2),2}_{0,4}(q,b,c) = \mathrm{ch}_0^{\mathfrak{so}(8)_{-2}}(q,b,c) - \mathrm{ch}_{-2\omega_4}^{\mathfrak{so}(8)_{-2}}(q,b,c), \tag{2.34}$$

where ch_0 is the vacuum character and $\mathrm{ch}_{-2\omega_4}$ is one of the three nonvacuum characters of $\mathfrak{so}(8)_{-2}$. Therefore, the space of BPS states furnishes a nontrivial representation of the associated chiral algebra $\chi(\mathcal{T}_2[C_{0,4}])$.

The VOA $\mathfrak{so}(8)_{-2}$ contains many null states in its vacuum module. For example, the Sugawara condition $T - T_{\mathrm{Sug}} = 0$ is a trivial null state, simply stating that the stress-energy tensor T is given by the Sugawara stress-energy tensor. The Sugawara condition is part of the so-called Joseph relations

$$(J^A J^B)|_{\mathfrak{R}} = 0, \qquad (J^A J^B)_{\mathbf{1}} \sim T, \tag{2.35}$$

which correspond to more null states or descendants of null states. Using Zhu's recursion relations [21], the null states (or their descendants) that are uncharged under the Cartan of $\mathfrak{so}(8)$ may lead to flavored modular differential equations that any $\mathfrak{so}(8)_{-2}$ character must satisfy. Concretely, there are ten equations of weight-two, four equations of weight-three and one equation of weight-four that together constrain the characters of $\mathfrak{so}(8)_{-2}$ [23,24]. In particular, the above elliptic genus $\mathcal{I}^{(0,2),2}_{0,4}(q,b,c)$ is a linear combination of characters and therefore a solution to the set of equations. Reversing the logic, the fact that $\mathcal{I}^{(0,2),2}_{0,4}(q,b,c)$ solves the set of modular differential equations predicts that it must be some linear combination of VOA characters.

2. SU(2) linear quivers

Consider the SU(2) linear quiver theory with $(n-1)$ SU(2) gauge nodes as in Fig. 4. The theory has manifest flavor symmetry $\mathrm{SU}(2)^2 \times \mathrm{U}(1)^n$. Its central charge is given by

$$c_L = 2(n+3), \qquad c_R = 3(n+3). \tag{2.36}$$

As explained in Appendix D of [27], (2.33) can be interpreted as the elliptic inversion formula:

$$\frac{\eta(q)^5 \vartheta_1(c_1^2 c_2^2)}{\vartheta_1(c_1^2)\vartheta_1(c_2^2)\vartheta_1(c_1 c_2 b_1^\pm b_2^\pm)} = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \frac{\eta(q)^8 \vartheta_1(a^{\pm 2})}{2\vartheta_4(c_1 b_1^\pm a^\pm)\vartheta_4(c_2 b_2^\pm a^\pm)}. \tag{2.37}$$

Therefore, starting from a collection of U_2 theories we can repeatedly gauge the SU(2) flavor symmetries to construct a linear quiver theory, and the elliptic genus takes the following simple form:

$$\begin{aligned}\mathcal{I}^{(0,2),2}_{0,n+2} &= \int_{JK} \frac{d\boldsymbol{a}}{2\pi i \boldsymbol{a}} \mathcal{I}^{(0,2)}_{U_2}(b_1,a_1;c_1)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_1)\mathcal{I}^{(0,2)}_{U_2}(b_2,a_{n-1};c_n)\prod_{i=1}^{n-2}\mathcal{I}^{(0,2)}_{U_2}(a_i,a_{i+1};c_{i+1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_{i+1}) \\ &= \frac{\eta(q)^{n+3}\vartheta_1(\prod_{i=1}^n c_i^2)}{\vartheta_\alpha(b_1^\pm b_2^\pm \prod_{i=1}^n c_i)\prod_{i=1}^n \vartheta_1(c_i^2)}, \qquad \begin{cases}\alpha = 1 & n \text{ even}, \\ \alpha = 4 & n \text{ odd}.\end{cases}\end{aligned} \tag{2.38}$$

As also found earlier in Sec. 4.3 of [47], the above linear quiver theory has an LG description, which consists of n chiral multiplets $\Phi_{k=1,\ldots,n}$ and one chiral meson multiplet $\tilde{\Phi}_{i,j=1,2}$ with $\mathrm{U}(1)_R$ charge $r=0$, and one Fermi multiplet Ψ with $\mathrm{U}(1)_R$ charge $r=1$, forming a J-type superpotential

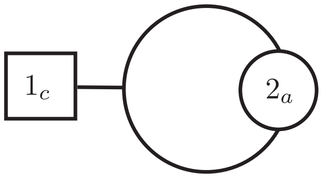

FIG. 5. Genus one with one puncture for SU(2).

$$W = \Psi\left(\prod_{i=1}^{n}\Phi_i + \det\tilde{\Phi}\right).$$

It is worth mentioning that with the shift of the $\mathrm{U}(1)_{c_i}$ flavor fugacities in (2.27), one must take this shift into account to correctly read off the $\mathrm{U}(1)_R$ charges of the superfields in the LG model from (2.38).

3. Genus one with one puncture and SU(2) with adjoint chiral

Let us consider the theory corresponding to the Riemann surface of genus one with one U(1) puncture. See Fig. 5 for an illustration of the quiver structure. Because of $\mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}$, the theory is a product of an SU(2) gauge theory with one adjoint chiral and a free chiral. The naive application of the c extremization leads to the “wrong” central charges,

$$c_L^{\text{naive}} = -9 - 1, \qquad c_R^{\text{naive}} = -9, \tag{2.39}$$

while the correct treatment provides

$$c_L = 3 - 1, \qquad c_R = 3, \tag{2.40}$$

where the SU(2) adjoint chiral contributes $\mathbf{3}$ and the free chiral contributes -1 to the left central charge. The elliptic genus can be evaluated as

$$\begin{aligned}\mathcal{I}_{1,1}^{(0,2),2} &= \int_{JK}\frac{da}{2\pi i a}\mathcal{I}_{U_2}^{(0,2)}(a,a^{-1};c_1)\mathcal{I}_{\text{vec}}^{(0,2)}(a)\\ &= \frac{\eta(q)}{\vartheta_1(c_1^2)} = \frac{\eta(q)}{\vartheta_4(c_1)}\cdot\frac{\vartheta_4(c_1)}{\vartheta_1(c_1^2)}.\end{aligned} \tag{2.41}$$

The first factor is the contribution of the free chiral, which is the vacuum character of the $\beta\gamma$ ghost, while the second factor is the elliptic genus of the SU(2) theory with one adjoint chiral. Following this computation, the SU(2) gauge theory with one adjoint chiral has an LG dual given by one chiral and one Fermi multiplet. We note that the elliptic genus enjoys the symmetry $c_1 \leftrightarrow c_1^{-1}$, which is the SU(2) Weyl group. Moreover, the elliptic genus admits the expansion with SU(2) characters

$$\begin{aligned}\mathcal{I}_{1,1}^{(0,2),2} = \frac{iq^{-\frac{1}{12}}}{c_1 - c_1^{-1}}\Big(&1 + \mathrm{ch}_{\frac12}^{\mathrm{SU}(2)}(c_1^2)q + \mathrm{ch}_{2}^{\mathrm{SU}(2)}(c_1^2)q^2\\ &+ \Big[1 + \mathrm{ch}_{\frac12}^{\mathrm{SU}(2)}(c_1^2) + \mathrm{ch}_{\frac32}^{\mathrm{SU}(2)}(c_1^2)\Big]q^3 + \cdots\Big),\end{aligned} \tag{2.42}$$

where $\mathrm{ch}_j^{\mathrm{SU}(2)}$ is the spin-j character of SU(2). Hence, one can observe that the $\mathrm{U}(1)_{c_1}$ flavor symmetry gets enhanced to $\mathrm{SU}(2)_{c_1}$ in the IR.

The 4D $\mathcal{N}=4$ theory with the SU(2) gauge group has the small $\mathcal{N}=4$ superconformal algebra as its associated VOA [7]. First of all, the central charge of the VOA is $c_{2D} = -9$, which agrees with the naive left-moving central charge of the SU(2) adjoint chiral (2.39). Moreover, the elliptic genus of the SU(2) adjoint chiral can be viewed as a special $bc\beta\gamma$ system [19],[3] and is also a linear combination of the characters of the associated VOA,

$$\frac{i\vartheta_4(c_1)}{\vartheta_1(c_1^2)} = \mathrm{ch}_0^{\mathcal{N}=4}(q,c_1) + \mathrm{ch}_1^{\mathcal{N}=4}(q,c_1). \tag{2.43}$$

Here $\mathrm{ch}_0^{\mathcal{N}=4}$ is the vacuum character and $\mathrm{ch}_1^{\mathcal{N}=4}$ is the character of the nonvacuum irreducible module of the VOA [52], and both characters are shown to satisfy three common flavored modular differential equations from null states in the VOA [31]. Note that the flavor symmetry enhancement to SU(2) is supported from the viewpoint of the VOA since both the small $\mathcal{N}=4$ superconformal algebra and the $bc\beta\gamma$-ghost VOA are endowed with an SU(2) flavor symmetry.[4]

An $\mathcal{N}=(0,2)$ vector multiplet and an $\mathcal{N}=(0,2)$ adjoint chiral multiplet form an $\mathcal{N}=(2,2)$ vector multiplet. Consequently, there is no distinction between the left- and right-moving sectors in the SU(2) adjoint chiral. This is supported by the equality of central charges for both sectors, which are both $c_L = c_R = 3$. Moreover, the elliptic genus of this theory can be expressed using characters of the small $\mathcal{N}=4$ superconformal algebra, as in (2.43). This suggests that the IR limit of the $\mathcal{N}=(2,2)$ SU(2) vector multiplet is equipped with the small $\mathcal{N}=4$ superconformal algebra in both the left- and right-moving sectors, suggesting the *supersymmetry enhancement*. It deserves further investigation to determine the exact IR Hilbert space on a torus by modular invariance, as demonstrated in [53–55].

4. Genus one with two punctures

Now consider the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,2}]$, which is an $\mathrm{SU}(2)^2$ gauge theory coupled to two bifundamentals. The central charges are given by

[3]The conformal weights of b, c, β, γ are $3/2, -1/2, 1, 0$.

[4]The SU(2) current of this particular $bc\beta\gamma$ system is given by $J^+ = \beta$, $J^0 = bc + 2\beta\gamma$, $J^- = \beta\gamma\gamma + \gamma bc - 3/2\partial\gamma$ [19].

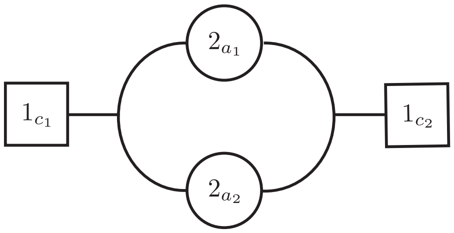

FIG. 6. (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,2}]$.

$$c_L = 4, \qquad c_R = 6. \tag{2.44}$$

The quiver diagram is given in Fig. 6.

The elliptic genus of this theory is computed by the JK residue computation of the integral

$$\begin{aligned}\mathcal{I}^{(0,2),2}_{1,2}(c_1,c_2) &= \int_{\mathrm{JK}} \frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}} \mathcal{I}^{(0,2)}_{U_2}(a_1,a_2;c_1)\\ &\quad\times \mathcal{I}^{(0,2)}_{U_2}(a_1^{-1},a_2^{-1};c_2)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_1)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_2)\\ &= -\frac{\eta(q)^2}{\prod_{i=1}^2 \vartheta_1(c_i^2)}.\end{aligned} \tag{2.45}$$

The elliptic genus implies the presence of an LG dual for the theory. This LG dual might comprise two free chiral multiplets. In additional to the two free chirals, there could be equal numbers of chiral and Fermi multiplets with identical charges which is not directly visible from the elliptic genus. As seen in (2.42), both of the $\mathrm{U}(1)_{c_i}$ flavor symmetries get enhanced to $\mathrm{SU}(2)_{c_i}$ in the IR.

Although the module structure of the corresponding VOA $\chi(\mathcal{T}_2[C_{1,2}])$ is not explicitly known, we argue that the elliptic genus is a linear combination of the module characters. One crucial piece of evidence is that the elliptic genus is a solution to the same flavored modular differential equations [23] that the Schur index of $\mathcal{T}_2[C_{1,2}]$ satisfies. For instance, the elliptic genus satisfies a weight-two differential equation,

$$\begin{aligned}0 = \Bigg[&D_q^{(1)} - \frac{1}{4}\sum_{i=1,2} D_{c_i}^2 - \frac{1}{4}\sum_{\alpha_i=\pm} E_1\left[\begin{matrix}1\\ c_1^{\alpha_1}c_2^{\alpha_2}\end{matrix}\right]\sum_{i=1,2}\alpha_i D_{c_i}\\ &- \sum_{i=1,2} E_1\left[\begin{matrix}1\\ c_i^2\end{matrix}\right] D_{c_i} + 2\Bigg(E_2 + \frac{1}{2}\sum_{\alpha_i=\pm} E_2\left[\begin{matrix}1\\ c_1^{\alpha_1}c_2^{\alpha_2}\end{matrix}\right]\\ &+ \sum_{i=1,2} E_2\left[\begin{matrix}1\\ c_i^2\end{matrix}\right]\Bigg)\Bigg]\mathcal{I}^{(0,2),2}_{1,2}.\end{aligned} \tag{2.46}$$

The definition of the twisted Eisenstein series is given in Appendix A 2.

Furthermore, by gauging the antidiagonal of the $\mathrm{U}(1)$ flavor symmetry of the two U_2 theories, one can construct two other quiver theories as $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theories, corresponding to the genus-one Riemann surface with two punctures, as in Fig. 7. Notably, these theories do not originate from class $\mathcal{S}$. Nonetheless, the central charges of these theories are computed as

$$c_L = 10, \qquad c_R = 12. \tag{2.47}$$

Let us first consider the left theory in Fig. 7. The elliptic genus of this theory is

$$\begin{aligned}\mathcal{I}'_{1,2}(b_1,b_2;d_1) = \eta(q)^2 &\int_{\mathrm{JK}} \frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}} \mathcal{I}^{(0,2)}_{U_2}(a_1,b_1;d_1a_2)\\ &\times \mathcal{I}^{(0,2)}_{U_2}(a_1^{-1},b_2;d_1a_2^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_1)\frac{\vartheta_4(a_2^{\pm 2})}{\eta(q)^2}.\end{aligned}$$

Here we introduce the $\mathrm{U}(1)_d$ flavor symmetry, which rotates the two chiral multiplets $U_2^{(0,2)}$ with the same phase and has a fugacity d_1. The detailed calculations of the JK residues can be found in Appendix C 1 b, and the final result is neatly summarized in the more simplified form

$$\begin{aligned}\mathcal{I}'_{1,2}(b_1,b_2;d_1) &= -2\frac{\eta(q)^2\vartheta_4(d_1^2)^2}{\vartheta_1(d_1^2 b_1^{\pm} b_2^{\pm})}\\ &= 2\frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)}\cdot\frac{\eta(q)^3\vartheta_1(d_1^{-4})}{\vartheta_1(d_1^2 b_1^{\pm} b_2^{\pm})}.\end{aligned} \tag{2.48}$$

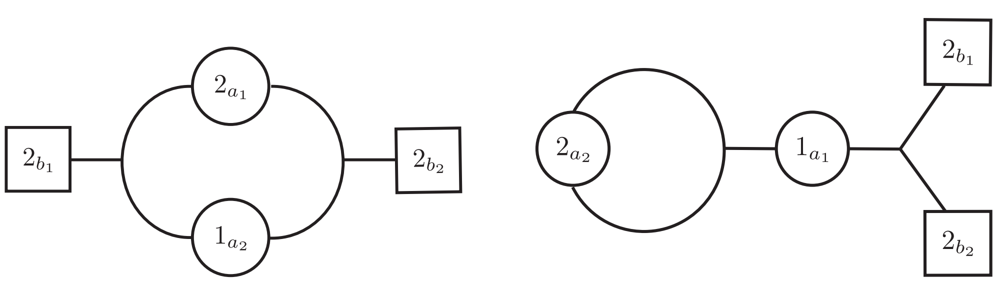

FIG. 7. $\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theories corresponding to the genus-one Riemann surface with two punctures.

The first factor always appears when the quiver gauge theory has a U(1) gauge node. This can be understood as the contribution of two Fermi fields $\Gamma_{1,2}$ with $\mathrm{U}(1)_{d_1}$ flavor charge 2 and one chiral field Φ with $\mathrm{U}(1)_{d_1}$ flavor charge 4. The second factor of the elliptic genus can be interpreted as the contribution from one Fermi field Ψ and one chiral meson field $\tilde{\Phi}$ with $\mathrm{SU}(2)_{b_1}\times\mathrm{SU}(2)_{b_2}$ flavor symmetry. The field content and their charges are summarized as follows:

	$\Gamma_{1,2}$	Ψ	Φ	$\tilde{\Phi}$
$\mathrm{U}(1)_{d_1}$	2	−4	4	2
$\mathrm{SU}(2)_{b_i}$	$\emptyset$	$\emptyset$	$\emptyset$	**2**

(2.49)

Therefore, a generic J-type superpotential is

$$W=\Psi(\Phi+\det\tilde{\Phi}).$$

The coefficient 2 in (2.48) means that the theory is dual to two (decoupled) copies of this LG model.

The elliptic genus of the right theory in Fig. 7 can be computed as the JK residue of the integrand

$$\mathcal{I}''_{1,2}(b_1,b_2;d_1)=\eta(q)^2\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\mathcal{I}^{(0,2)}_{U_2}(a_2,a_2^{-1};da_1)\times\mathcal{I}^{(0,2)}_{U_2}(b_1,b_2;da_1^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_2)\frac{\vartheta_4(a_1^{\pm2})}{\eta(q)^2}.$$

The detailed computations of the JK residues are provided in Appendix C 1 c, while the total residue is presented in the compact form

$$\mathcal{I}''_{1,2}(b_1,b_2;d_1)=\frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)}\cdot\frac{\eta(q)^3\vartheta_1(d_1^8)}{\vartheta_1(d_1^{-4}b_1^{\pm2}b_2^{\pm2})}\cdot\prod_{i=1}^{2}\frac{\vartheta_1(b_i^4)}{\vartheta_1(b_i^{-2})}.\tag{2.50}$$

The elliptic genus is different from (2.48) so that the two theories in Fig. 7 are not dual to each other. When the number of U(1) gauge nodes is equal to the genus, the theory is contingent on the quiver diagram, unlike class $\mathcal{S}$ theories.

Since (2.50) is expressed as a product of theta functions, the theory is also dual to the following LG model. The Fermi multiplets $\Gamma_{1,2}$ and Ψ and the chiral multiplets Φ and $\tilde{\Phi}_{\pm\pm}$ are similar to the aforementioned LG theory. A key distinction arises from the inclusion of two additional Fermi multiplets, $\Xi_{1,2}$, and two chiral multiplets, $\Sigma_{1,2}$. Because of these additions, the manifest flavor symmetry is $\mathrm{U}(1)_{b_1}\times\mathrm{U}(1)_{b_2}$ at UV. The charges of these fields are summarized as follows:

	$\Gamma_{1,2}$	Ψ	Ξ_j	Φ	$\tilde{\Phi}_{\epsilon_1\epsilon_2}$	Σ_j
$\mathrm{U}(1)_{d_1}$	2	8	0	4	−4	0
$\mathrm{U}(1)_{b_i}$	0	0	$4\delta_{ij}$	0	$\epsilon_i 2$	$-2\delta_{ij}$

where $\epsilon_i=\pm$. Then, a generic J-type superpotential is

$$W=\Psi\det\tilde{\Phi}+\Xi_1\Phi\tilde{\Phi}_{-+}\Sigma_1\Sigma_2+\Xi_2\Phi\tilde{\Phi}_{+-}\Sigma_1\Sigma_2.$$

The expansion of the elliptic genus (2.50) in terms of q reveals that the fugacities $b_{1,2}$ arrange themselves as characters of $\mathrm{SU}(2)_{b_1}\times\mathrm{SU}(2)_{b_2}$, implying the symmetry enhancement from $\mathrm{U}(1)_{b_1}\times\mathrm{U}(1)_{b_2}\to\mathrm{SU}(2)_{b_1}\times\mathrm{SU}(2)_{b_2}$ at IR.

For the quiver theories in Fig. 7, the corresponding VOA for the IR CFT is yet to be identified. Given their duality to the LG models, the technique in [56] offers a potential method for identifying the VOA. This remains an area for further study.

5. Genus one with three punctures

Now let us consider quiver theories of genus one with three punctures. The (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,3}]$ is the $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge theory in Fig. 8. The central charges of the theory are given by

$$c_L=6,\qquad c_R=9.\tag{2.51}$$

The elliptic genus is the JK residue of the integrand

$$\mathcal{I}^{(0,2),2}_{1,3}=\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\mathcal{I}^{(0,2)}_{U_2}(a_1^{-1},a_2;c_2)\mathcal{I}^{(0,2)}_{U_2}(a_2^{-1},a_3;c_1)\times\mathcal{I}^{(0,2)}_{U_2}(a_3^{-1},a_1;c_3)\prod_{i=1}^{3}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_i)=\frac{\eta(q)^3}{\prod_{i=1}^{3}\vartheta_1(c_i^2)},\tag{2.52}$$

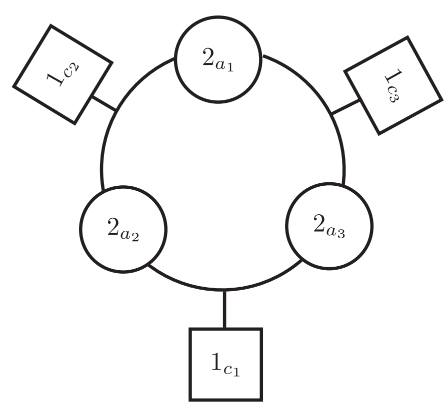

FIG. 8. (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{1,3}]$.

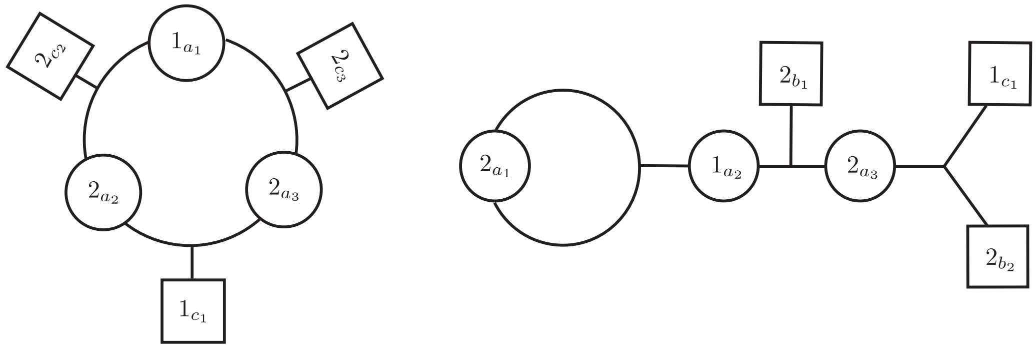

FIG. 9. $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theories corresponding to the genus-one Riemann surface with three punctures.

which signals the existence of an LG dual which includes three free chiral multiplets, associated with the three punctures. In addition to them, there could be equal numbers of chiral and Fermi multiplets with the same charges. All of the U(1) flavor symmetries get enhanced to SU(2) in the IR, as illustrated in (2.42).

Let us now consider $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theories for genus one with three punctures, as illustrated in Fig. 9. The central charges of the theory are given by

$$c_L = 12, \qquad c_R = 15. \tag{2.53}$$

The elliptic genus of the left quiver theory of Fig. 9 is

$$\begin{aligned}\mathcal{I}'_{1,3} &= \eta(q)^2\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\mathcal{I}^{(0,2)}_{U_2}(a_2,c_2^{-1};d_1a_1)\mathcal{I}^{(0,2)}_{U_2}(a_2^{-1},a_3;c_1)\\ &\quad\times\mathcal{I}^{(0,2)}_{U_2}(a_3^{-1},c_3;d_1a_1^{-1})\frac{\vartheta_4(a_1^{\pm2})}{\eta(q)^2}\prod_{i=2}^{3}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_i)\\ &= 2\frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)}\cdot\frac{\eta(q)^4\vartheta_1(c_1^2d_1^4)}{\vartheta_1(c_1^2)\vartheta_4(c_1d_1^2b_1^{\pm}b_2^{\pm})}.\end{aligned} \tag{2.54}$$

The computational details are given in Appendix C 2 b. The form of the elliptic genus signals an LG description. The LG model is similar to (2.49), but there is an additional chiral multiplet Φ_2 and $\mathrm{U}(1)_{c_1}$ flavor symmetry:

	$\Gamma_{1,2}$	Ψ	Φ_1	Φ_2	$\tilde{\Phi}$
$\mathrm{U}(1)_{d_1}$	2	4	−4	0	−2
$\mathrm{U}(1)_{c_1}$	0	2	0	−2	−1
$\mathrm{SU}(2)_{b_i}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	**2**

Therefore, a generic J-type superpotential is

$$W = \Psi(\Phi_1\Phi_2 + \det\tilde{\Phi}).$$

The coefficient 2 in (2.54) means that the theory is dual to two (decoupled) copies of this LG model.

There is yet another quiver gauge theory with gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, as on the right side of Fig. 9, whose elliptic genus is

$$\begin{aligned}\mathcal{I}''_{1,3} &= \eta(q)^2\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\mathcal{I}^{(0,2)}_{U_2}(a_1,a_1^{-1};d_1a_2)\mathcal{I}^{(0,2)}_{U_2}(b_1,a_3;d_1a_1^{-1})\\ &\quad\times\mathcal{I}^{(0,2)}_{U_2}(a_3^{-1},b_2;c_2)\frac{\vartheta_4(a_1^{\pm2})}{\eta(q)^2}\prod_{i=2}^{3}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a_i)\\ &= \frac{\vartheta_4(d_1^2)^2}{\eta(q)\vartheta_1(d_1^4)}\cdot\frac{\eta(q)^4\vartheta_1(c_1^4d_1^8)}{\vartheta_1(c_1^2)\vartheta_1(c_1^{-2}d_1^{-4}b_1^{\pm2}b_2^{\pm2})}\cdot\prod_{i=1}^{2}\frac{\vartheta_1(b_i^4)}{\vartheta_1(b_i^{-2})},\end{aligned} \tag{2.55}$$

suggesting an LG description of the theory.

Since (2.55) is expressed as a product of theta functions, the theory is also dual to the following LG model. The Fermi multiplets $\Gamma_{1,2}$ and Ψ and the chiral multiplets $\Phi_{1,2}$ and $\tilde{\Phi}_{\pm\pm}$ are similar to the aforementioned LG theory. A key distinction arises from the inclusion of two additional Fermi multiplets, $\Xi_{1,2}$, and two chiral multiplets, $\Sigma_{1,2}$. Because of these additions, the manifest flavor symmetry is $\mathrm{U}(1)_{b_1}\times\mathrm{U}(1)_{b_2}$ at UV. The charges of these fields are summarized as follows:

	$\Gamma_{1,2}$	Ψ	Ξ_j	Φ_1	Φ_2	$\tilde{\Phi}_{\epsilon_1\epsilon_2}$	Σ_j
$\mathrm{U}(1)_{d_1}$	2	8	0	4	0	−4	0
$\mathrm{U}(1)_{c_1}$	0	4	0	0	2	−2	0
$\mathrm{U}(1)_{b_i}$	0	0	$4\delta_{ij}$	0	0	$\epsilon_i 2$	$-2\delta_{ij}$

where $\epsilon_i = \pm$. Then, a generic J-type superpotential is

$$W = \Psi\det\tilde{\Phi} + \Xi_1\Phi_1\Phi_2\tilde{\Phi}_{-+}\Sigma_1\Sigma_2 + \Xi_2\Phi_1\Phi_2\tilde{\Phi}_{+-}\Sigma_1\Sigma_2.$$

Like (2.50), the expansion of the elliptic genus (2.55) shows that a flavor symmetry is enhanced to $\mathrm{SU}(2)_{b_1}\times\mathrm{SU}(2)_{b_2}$ at IR.

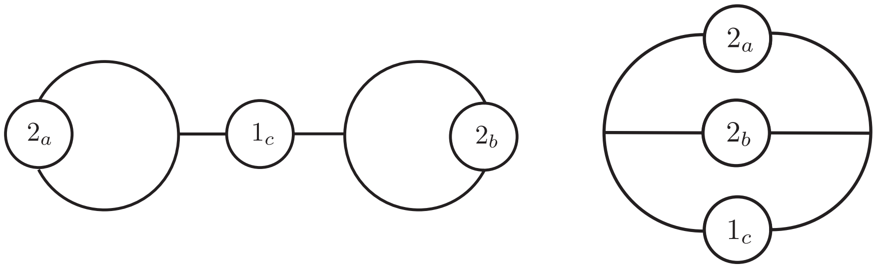

FIG. 10. Genus two for SU(2).

6. Genus two

For the genus-two case (with no puncture), we have two different quiver descriptions, as depicted in Fig. 10. The central charges of this theory are

$$c_L = 4, \qquad c_R = 3. \tag{2.56}$$

It is straightforward to show that the two different descriptions are dual to each other. Certainly, the Lagrangian descriptions provide the different expressions of the elliptic genera

$$\begin{aligned} \mathcal{I}^{(0,2),2}_{\bigcirc\!-\!\bigcirc} &= \int_{\mathrm{JK}} \frac{da}{2\pi i a}\frac{db}{2\pi i b}\frac{dc}{2\pi i c}\mathcal{I}^{(0,2)}_{U_2}(a,a^{-1};dc)\mathcal{I}^{(0,2)}_{U_2}(b,b^{-1};dc^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(b)\vartheta_4(c^{\pm 2}), \\ \mathcal{I}^{(0,2),2}_{\ominus} &= \int_{\mathrm{JK}} \frac{da}{2\pi i a}\frac{db}{2\pi i b}\frac{dc}{2\pi i c}\mathcal{I}^{(0,2)}_{U_2}(a,b;dc)\mathcal{I}^{(0,2)}_{U_2}(a^{-1},b^{-1};dc^{-1})\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a)\mathcal{I}^{(0,2)}_{\mathrm{vec}}(b)\vartheta_4(c^{\pm 2}), \end{aligned} \tag{2.57}$$

where the last factors $\vartheta_4(c^{\pm 2})$ correspond to the contributions from the U(1) vector multiplet and the two Fermi multiplets. Both JK residues can be straightforwardly evaluated, and the results agree, suggesting that the two theories are dual to each other.

Alternatively, the two elliptic genera can be computed from other building blocks. The former theory can be obtained by gluing two copies of the theory of genus one with one puncture whose elliptic genus is given in (2.41). Similarly, the latter can be obtained by gluing the two punctures in the theory of genus one with two punctures whose elliptic genus is given in (2.45). In this approach, it becomes more evident that they are identical and, moreover, the JK residue provides the remarkably simple result

$$\mathcal{I}^{(0,2),2}_{2,0} = \eta(q)^2 \int_{JK} \frac{dc}{2\pi i c}\frac{\vartheta_4(c^{\pm 2})}{\vartheta_1(d^2 c^{\pm 2})} = \frac{2\vartheta_4(d^2)^2}{\eta(q)\vartheta_1(d^4)}. \tag{2.58}$$

This can be compared to the S duality in class $\mathcal{S}$ theories. While the process of U(1) gauging may not naturally fit in the context of the (0,2) reduction of the class $\mathcal{S}$ theory, as explained below (2.30), the gluing procedure explained above is analogous to the class $\mathcal{S}$ construction. In fact, since the flavor symmetries get enhanced to SU(2) for genus one theories, we can gauge the antidiagonal SU(2) to obtain the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{2,0}]$ of genus two, whose elliptic genus is

$$\begin{aligned} &-\int_{JK} \frac{dc}{2\pi i c}\mathcal{I}^{(0,2),2}_{1,1}(cd)\frac{\vartheta_1(c^{\pm 2})}{2}\mathcal{I}^{(0,2),2}_{1,1}(c^{-1}d) \\ &= -\int_{JK} \frac{dc}{2\pi i c}\mathcal{I}^{(0,2),2}_{1,2}(cd, c^{-1}d)\frac{\vartheta_1(c^{\pm 2})}{2} \\ &= -\frac{\eta(q)^2}{2}\int_{JK} \frac{dc}{2\pi i c}\frac{\vartheta_1(c^{\pm 2})}{\vartheta_1(d^2 c^{\pm 2})} = \frac{\vartheta_1(d^2)^2}{\eta(q)\vartheta_1(d^4)}. \end{aligned} \tag{2.59}$$

Recalling that the Jacobi theta functions ϑ_1 and ϑ_4 are related by (A8), the result differs from (2.58) merely by the shift $d \to q^{1/4}d$ of the $\mathrm{U}(1)_d$ flavor fugacity, up to a factor. Indeed, comparing the SU(2) vector multiplet contribution (2.29) and U(1) gauging (2.30) at the level of elliptic genera, the difference appears only in $\vartheta_1(a^{\pm 2})$ and $\vartheta_4(a^{\pm 2})$, up to a factor of 2, which is the order of the Weyl group of SU(2). Hence, the duality between the two (0,2) theories in Fig. 10 is analogous to the S duality in the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{2,0}]$.

As a result, one can expect the relation between the elliptic genera (2.58) and (2.59) and characters of the VOA $\chi(\mathcal{T}_2[C_{2,0}])$. In 4D, the associated VOA of the genus-two A_1 theory $\mathcal{T}_2[C_{2,0}]$ of class $\mathcal{S}$ was studied in [57,58]. In particular, null states are present at levels four and six, which are expected to give rise to a weight-four and a weight-six flavor differential equation [23]. The above elliptic genus (2.59) is indeed a solution to these two equations once the U(1) fugacity d is rescaled $d \to d^{1/2}$. Therefore, it is natural to argue that the elliptic genus is

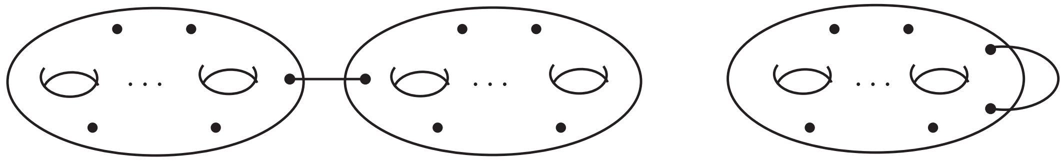

FIG. 11. Gluing minimal punctures leads to a new Riemann surface, and (0,2) elliptic genera are consistent with the cut-and-join procedure on Riemann surfaces $C_{g,n}$.

some linear combination of module characters of $\chi(\mathcal{T}_2[C_{2,0}])$,

$$\frac{\vartheta_1(d)^2}{\eta(q)\vartheta_1(d^2)} = \sum_i a_i \, \mathrm{ch}_{\lambda_i}^{\chi(\mathcal{T}_2[C_{2,0}])}(d), \qquad a_i \in \mathbb{Q}. \tag{2.60}$$

Likewise, we can argue that, upon rescaling $d \to d^{1/2}$, the elliptic genus (2.58) of the genus-two theory constructed from the U_2 theories can be written in a similar way as

$$\frac{2\vartheta_4(d)^2}{\eta(q)\vartheta_1(d^2)} = 2q^{1/4} d \sum_i a_i \, \mathrm{ch}_{\lambda_i}^{\chi(\mathcal{T}_2[C_{2,0}])}(q^{1/2} d). \tag{2.61}$$

7. General Riemann surfaces and TQFT structure

For a (0,2) quiver theory of genus $g > 0$ constructed from the U_2 theory, the minimal number of U(1) gauge groups is $g - 1$. Hence, based on the previous results, we can consider a (0,2) quiver theory analogous to the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g>0,n}]$, where the numbers of SU(2) and U(1) gauge groups are $2(g-1)+n$ and $g-1$, respectively. The central charges of the theory are

$$c_L = 2(2(g-1)+n), \qquad c_R = 3(g-1+n). \tag{2.62}$$

Regardless of their quiver descriptions (or frames), these theories all flow to the same IR theory. By introducing U(1) flavor fugacities c_i for the external punctures and d_i for the U(1) gauging in the quiver gauge theory, the elliptic genus of the theory can be expressed in the simple form

$$\mathcal{I}^{(0,2),2}_{g>0,n}(c_1, \ldots, c_n) = \prod_{j=1}^{g-1} \frac{2\vartheta_4(d_j^2)^2}{\eta(q)\vartheta_1(d_j^4)} \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_1(c_i^2)}. \tag{2.63}$$

Therefore, the flavor symmetry associated with each puncture gets enhanced to SU(2) at low energy, as seen in (2.42). Remarkably, the integral formula (2.58) guarantees that the above form of the (0,2) elliptic genera is consistent with the TQFT structure, as in Fig. 11,

$$\begin{aligned} \mathcal{I}^{(0,2),2}_{g=g_1+g_2,n_1+n_2-2} &= \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2),2}_{g_1,n_1}(\ldots, d_{g-1}a) \mathcal{I}^{(0,2),2}_{g_2,n_2}(d_{g-1}a^{-1}, \ldots) \vartheta_1(a^{\pm 2}), \\ \mathcal{I}^{(0,2),2}_{g+1,n-2} &= \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,2),2}_{g,n}(\ldots, d_g a, d_g a^{-1}) \vartheta_1(a^{\pm 2}). \end{aligned} \tag{2.64}$$

As we recall, the elliptic genus of the (0,2) reduction of a class $\mathcal{S}$ theory of genus one with n punctures is

$$\mathcal{I}^{(0,2),2}_{1,n}(c_1, \ldots, c_n) = \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_1(c_i^2)}, \tag{2.65}$$

which exhibits the enhancement to SU(2) for each puncture in the IR. To construct the (0,2) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_2[C_{g>0,n}]$ of type A_1, we can repeatedly gauge antidiagonal SU(2) of genus one theories with multiple punctures. Upon the rescaling $d \to q^{1/4} d$, the resulting elliptic genus differs from (2.63) by merely a factor of $(2q^{1/2}d^2)^{g-1}$. In this sense, the TQFT structure of the elliptic genus can be attributed to the class $\mathcal{S}$ construction.

From the perspective of the chiral algebra, we observe a connection to class $\mathcal{S}$ theories. The VOA corresponding to any Lagrangian $\mathcal{N} = 2$ SCFT can be constructed using the gauging method described in [7]. However, putting this method into practice is intricate. Detailing the VOA structure and its representation theory is notably challenging, even for class $\mathcal{S}$ theories of type A_1. Nevertheless, if we change $\vartheta_4(a^{\pm 2})$ to $\vartheta_1(a^{\pm 2})$ for the U(1) gauging (2.30), the JK integrand of the elliptic genus, when derived from the UV Lagrangian, coincides with that of the Schur index of $\mathcal{T}_2[C_{g>0,n}]$ up to a factor, and the JK residue gives

$$\prod_{j=1}^{g-1} \frac{2\vartheta_1(d_j^2)^2}{\eta(q)\vartheta_1(d_j^4)} \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_1(c_i^2)}. \tag{2.66}$$

Building upon the results in [23,31,32], certain residues of the integrand correspond to the Schur index with surface

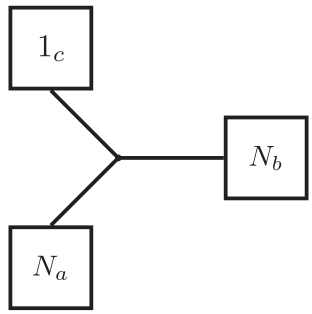

FIG. 12. A building block U_N, representing N^2 (0,2) chiral multiplets with $\mathrm{SU}(N)_a \times \mathrm{SU}(N)_b \times \mathrm{U}(1)_c$ flavor symmetry.

defects of Gukov-Witten type.[5] Given its role as a surface defect index, one expects (2.66) to satisfy all of the flavored modular differential equations associated with the VOA $\chi(\mathcal{T}_2[C_{g,n}])$. In light of these observations, we propose that the elliptic genus (2.63) can be written in terms of a linear combination of characters of the VOA $\chi(\mathcal{T}_2[C_{g>0,n}])$ upon an appropriate redefinition of the U(1) fugacities and a certain overall factor.

When the number of U(1) gauge groups is g for a theory of genus g, it admits an LG dual, but it depends on the quiver description, as we saw in the examples of Secs. II B 4 and II B 5. It would be interesting to study the VOA structures for theories of this type.

C. SU(N) × U(1) gauge theories, LG duals, and VOAs

Now let us move on to cases of higher rank. For class $\mathcal{S}$ of type A_{N-1}, punctures are classified by a partition of N and theories do not admit a Lagrangian description in general. As seen in Sec. II A 3, we perform the (0,2) reduction for 4D $\mathcal{N}=2$ Lagrangian theories of class $\mathcal{S}$. Consequently, the basic building block is a sphere with two maximal punctures and one minimal puncture, corresponding to N^2 hypermultiplets. (See the right side of Fig. 16.) Its (0,2) reduction yields (0,2) N^2 chiral multiplets, with the flavor symmetry represented as $\mathrm{U}(N^2)$ and includes the subgroup $\mathrm{SU}(N)_a \times \mathrm{SU}(N)_b \times \mathrm{U}(1)_x$. This particular (0,2) theory, labeled as U_N, serves as our fundamental building block, with its quiver illustrated in Fig. 12. As highlighted in (2.23), the c extremization is invalid for theories of this class, and the $\mathrm{U}(1)_R$ charge of the (0,2) chiral multiplet is $r=0$. The NS elliptic genus of U_N is given by

$$\mathcal{I}^{(0,2)}_{U_N}(a,b,c)=\prod_{i,j=1}^{N}\frac{\eta(q)}{\vartheta_4(q^{-\frac{1}{2}}\tilde{c}a_ib_j)}=\prod_{i,j=1}^{N}\frac{\eta(q)}{\vartheta_4(ca_ib_j)}, \quad (2.67)$$

where we redefine the $\mathrm{U}(1)_c$ flavor fugacity by $c=q^{-\frac{1}{2}}\tilde{c}$. Note that we impose the condition $\prod_{i=1}^{N}a_i=1=\prod_{j=1}^{N}b_j$ on the SU(N) fugacities. On a related note, as seen in (2.5), the Schur limit of the superconformal index for a sphere with two maximal punctures and one minimal puncture is given by

$$\mathcal{I}^{\mathrm{4D}}=\prod_{i,j=1}^{N}\Gamma(\sqrt{t}(ca_ib_j)^{\pm1})\xrightarrow{t\to q}\mathcal{I}^{\mathrm{Schur}}=\prod_{i,j=1}^{N}\frac{\eta(q)}{\vartheta_4(ca_ib_j)}. \quad (2.68)$$

Consequently, by redefining the $\mathrm{U}(1)_c$ flavor fugacity as in (2.67), the elliptic genus of the U_N theory agrees with the Schur index above.

The gauging procedure of the U_N theories is as usual. The contribution of a vector multiplet is the same in both the Schur index (2.6) and the elliptic genus (2.12). The SU(N) vector multiplet contribution is

$$\mathcal{I}^{(0,2)}_{\mathrm{vec}}(a)=\frac{\eta(q)^{2N}}{N!}\prod_{A\neq B}i\frac{\vartheta_1(a_A/a_B)}{\eta(q)}, \quad (2.69)$$

and the SU(N) gauging leads to no gauge anomaly. In this way, the integrand of the superconformal index and the (0,2) elliptic genus agree for a class $\mathcal{S}$ Lagrangian theory.

However, in 2D (0,2) theories, one can also gauge the U(1) symmetry of the U_N theory. This U(1) gauging is similar to the U_2 case, but the U(1) gauge charges of the two Fermi multiplets must be $\pm N$ to avoid gauge anomalies. Following the c extremization, the $\mathrm{U}(1)_R$ charge for these Fermi multiplets is assigned to be $r=0$. Consequently, the gauging procedure is then applied to the elliptic genus as described:

$$\frac{\eta(q)}{\vartheta_4(q,c_1\cdots)}\frac{\eta(q)}{\vartheta_4(q,c_2\cdots)} \to \eta(q)^2\int_{\mathrm{JK}}\frac{da}{2\pi ia}\frac{\eta(q)}{\vartheta_4(da\cdots)}\frac{\eta(q)}{\vartheta_4(da^{-1}\cdots)}\frac{\vartheta_4(a^{\pm N})}{\eta(q)^2}. \quad (2.70)$$

Once a (0,2) quiver theory involves a U(1) gauging of the U_N theories, the theory is no longer the (0,2) reduction of a class $\mathcal{S}$ theory. Nonetheless, one can consider the elliptic genus of the theory in the IR.

1. Gauge/LG duality for linear quivers

The (0,2) reduction of a class $\mathcal{S}$ theory of higher rank is quite restricted because the class $\mathcal{S}$ theory should have a Lagrangian description. Consequently, the theories that we focus on are constructed by gauging the SU(N) flavor symmetry of the U_N theories. One such example is a linear quiver, as in Fig. 13. The central charges are given by

$$c_L=2(N^2+n-1), \qquad c_R=3(N^2+n-1). \quad (2.71)$$

[5]Up to some prefactors of q to account for the different stress-energy tensors involved [59].

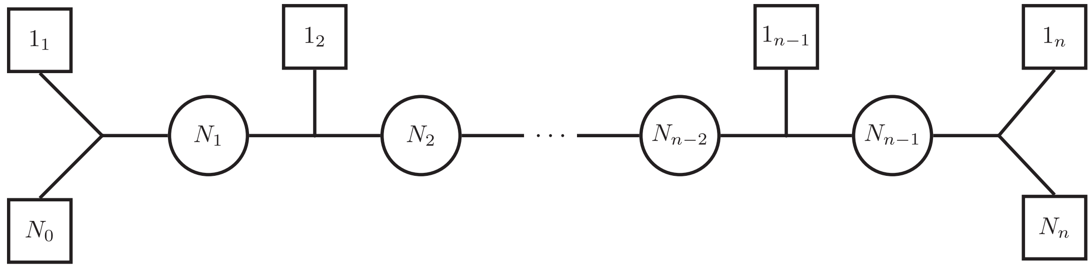

FIG. 13. An SU(N) linear quiver where the subscripts are added solely for node numbering purposes.

As the simplest example, we can consider the (0,2) SU(N) SQCD with N fundamentals and N antifundamentals. As found in Eq. (4.7) of [27], the computation of its elliptic genus is equivalent to the higher-rank rendition of the elliptic inversion formula (2.37),

$$\frac{\eta(q)^{N^2+1}\vartheta_1(c_1^N c_2^N)}{\vartheta_\alpha(c_1^N)\vartheta_\alpha(c_2^N)\prod_{A,B=1}^N \vartheta_1(c_1c_2b_{0,A}b_{2,B}^{-1})} = \frac{\eta(q)^{2N^2}}{N!}\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\prod_{A,B,i=1}^{N}\frac{\prod_{j\neq i}\vartheta_1(a_i/a_j)}{\vartheta_4(c_1b_{0,A}a_i^{-1})\vartheta_4(c_2b_{2,B}^{-1}a_i)}, \tag{2.72}$$

where $\alpha=1$ for even N and $\alpha=4$ for odd N. To evaluate the elliptic genus of a linear quiver, we repeatedly apply the elliptic inversion formula (2.72), and it therefore takes the simple form

$$\mathcal{I}_{0,n,2}^{(0,2),N} = \frac{\eta(q)^{N^2+n-1}\vartheta_\alpha(\prod_{i=1}^n c_i^N)}{\prod_{i=1}^n \vartheta_\beta(c_i^N)\cdot\prod_{A,B=1}^N\vartheta_\gamma(b_{0,A}b_{n,B}^{-1}\prod_{i=1}^n c_i)}, \tag{2.73}$$

where

$$\alpha=\begin{cases}1 & n\cdot N\text{ even},\\ 4 & n\cdot N\text{ odd},\end{cases}\qquad \beta=\begin{cases}1 & N\text{ even},\\ 4 & N\text{ odd},\end{cases}\qquad \gamma=\begin{cases}1 & n\text{ even},\\ 4 & n\text{ odd}.\end{cases} \tag{2.74}$$

Taking into account the shift of the $\mathrm{U}(1)_{c_i}$ fugacities in (2.27), the form of the elliptic genus tells us that the theory is dual to an LG model with one Fermi multiplet Ψ, n chiral multiplets Φ, and one chiral meson multiplet $\tilde{\Phi}_{i,j=1,\ldots,n}$, forming a J-type superpotential

$$W=\Psi\left(\prod_{i=1}^n\Phi_i+\det\tilde{\Phi}\right).$$

2. Gauge/LG duality for circular quivers

The other class of the (0,2) reduction of class $\mathcal{S}$ theories is a circular quiver, as in Fig. 14. The central charges for genus 1 with n punctures are given by

$$c_L=2n,\qquad c_R=3n. \tag{2.75}$$

Some of the explicit JK residue computations of elliptic genera can be found in Appendices C 5 and C 6.

As the simplest example, let us consider the theory at genus one with one puncture. In other words, it is an SU(N) gauge theory with one adjoint chiral, which is the (0,2) reduction of the 4D $\mathcal{N}=4$ SU(N) theory, and an additional free chiral multiplet. The evaluation of its elliptic genus can be understood as an elliptic inversion formula of another kind,

$$\frac{\eta(q)}{\vartheta_\alpha(c^N)}=\frac{\eta(q)^N}{N!\vartheta_4(c)^N}\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\prod_{j\neq i}\frac{\vartheta_1(a_i/a_j)}{\vartheta_4(ca_i/a_j)}, \tag{2.76}$$

where $\alpha=1$ for even N and $\alpha=4$ for odd N. We can remove a free hypermultiplet factor $\eta(\tau)/\vartheta_4(c)$ from the above expression and obtain

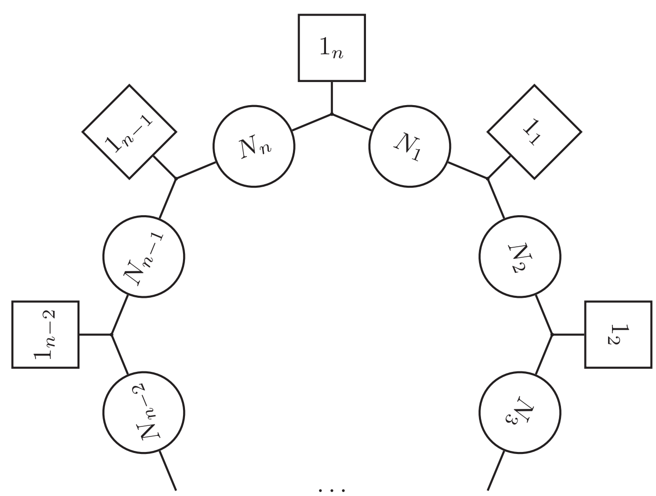

FIG. 14. An SU(N) circular quiver where the subscripts are added solely for node numbering purposes.

$$\mathcal{I}^{(0,2),N}_{\mathcal{N}=4} = \frac{\vartheta_4(c)}{\vartheta_\alpha(c^N)}. \tag{2.77}$$

We note that this expression is precisely the vacuum character of $N-1$ copies of $bc\beta\gamma$ systems labeled by $i = 1, \ldots, N-1$, with the following conformal weights h and U(1) charges m:

	h	m
b_i	$\frac{1}{2}(d_i+1)$	$\frac{1}{2}(d_i-1)$
c_i	$-\frac{1}{2}(d_i-1)$	$-\frac{1}{2}(d_i-1)$
β_i	$\frac{1}{2}d_i$	$\frac{1}{2}d_i$
γ_i	$1-\frac{1}{2}d_i$	$-\frac{1}{2}d_i$

Here $d_i = i+1$ denotes the degree of the ith invariant of SU(N). The vacuum character reads (up to a factor of i)

$$q^{\frac{1}{8}(N^2-1)} \prod_{i=1}^{N-1} \frac{(c^{d_i-1} q^{\frac{d_i+1}{2}};q)(c^{-d_i+1} q^{\frac{1-d_i}{2}};q)}{(c^{d_i} q^{\frac{d_i}{2}};q)(c^{-d_i} q^{1-\frac{d_i}{2}};q)} = \frac{\vartheta_4(c)}{\vartheta_\alpha(c^N)}. \tag{2.78}$$

The $bc\beta\gamma$ system serves as a free-field realization of the chiral algebra $\chi^{\mathcal{N}=4,N}$ of the 4D $\mathcal{N}=4$ SU(N) supersymmetric Yang-Mills theory [19], and therefore is a reducible module of $\chi^{\mathcal{N}=4,N}$. Hence, the above vacuum character is naturally a reducible module character of $\chi^{\mathcal{N}=4,N}$.

As discussed at the end of Sec. II B 3, the combination of a (0,2) vector multiplet and an adjoint chiral multiplet forms a (2,2) vector multiplet. Consequently, there is no distinction between the left- and right-moving sectors. The analysis above suggests that the IR theory is described by the chiral algebra $\chi^{\mathcal{N}=4,N}$.

As a generalization of this case, the circular quiver in Fig. 14 can be obtained by SU(N) gauging of the ends of the linear quiver in Fig. 13. The elliptic genus is given by

$$\mathcal{I}^{(0,2),N}_{1,n} = \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_\alpha(c_i^N)}, \tag{2.79}$$

where again $\alpha = 1$ for even N and $\alpha = 4$ for odd N. Extrapolating from the $\mathcal{N}=4$ discussion, we conjecture that the elliptic genus continues to be a module character of the chiral algebra of the 4D $\mathcal{N}=2$ circular quiver theory.

3. General Riemann surfaces

Other quiver theories cannot be obtained from the (0,2) reduction of class $\mathcal{S}$ theories because they involve a U(1) gauge group. Nonetheless, one can consider an SU(N) × U(1) quiver gauge theory of genus $g > 0$ with n punctures, where the numbers of SU(N) and U(1) gauge groups are $2(g-1)+n$ and $g-1$, respectively, whose central charges are give by

$$c_L = 2(2(g-1)+n), \qquad c_R = 3(g-1+n). \tag{2.80}$$

Compared with (2.62), the central charge depends on the genus and puncture, but not on the rank of gauge groups for this specific class of theory.

By introducing U(1) flavor fugacities c_i for the external punctures and d_i for the U(1) gauging, the elliptic genus of the theory can be expressed in the simple form (up to a sign)

$$\mathcal{I}^{(0,2),N}_{g>0,n} = \prod_{j=1}^{g-1} \frac{(-1)^\beta N \vartheta_\beta(d_j^N)^2}{\eta(q)\vartheta_1(d_j^{2N})} \prod_{i=1}^{n} \frac{\eta(q)}{\vartheta_\alpha(c_i^N)}, \tag{2.81}$$

where $\alpha = 1, \beta = 4$ for even N and $\alpha = 4, \beta = 1$ for odd N. This form is independent of quiver descriptions. Therefore, regardless of the quiver descriptions, we claim that these theories all flow to the same IR theory. Applying the formula

$$\eta(q)^2 \int_{JK} \frac{dc}{2\pi i c} \frac{\vartheta_4(c^{\pm N})}{\vartheta_\alpha(d^N c^{\pm N})} = \frac{(-1)^\beta N \vartheta_\beta(d^N)^2}{\eta(q)\vartheta_1(d^{2N})}, \tag{2.82}$$

one can convince oneself that (2.81) is consistent with the TQFT structure, as in Fig. 11.

The distinctions between SU(2) and SU(N) become evident in theories of genus g that have g U(1) gauge nodes. As demonstrated in Appendix C 6 b, the elliptic genus evaluation for the left quiver theory depicted in Fig. 15 reveals that it is dual to an LG model. Contrarily, the explicit evaluation shows that the elliptic genus of the right quiver theory in Fig. 15 does not factorize into theta functions. This observation implies that the right quiver theory does not possess an LG dual description.

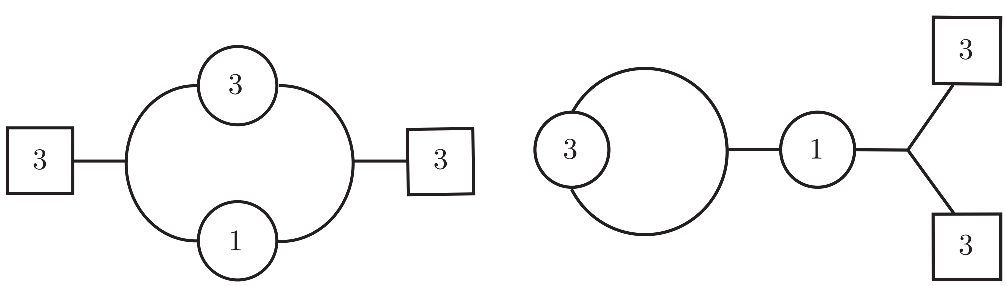

FIG. 15. SU(3) × U(1) gauge theories corresponding to the genus-one Riemann surface with two punctures.

D. Comments on non-Lagrangian cases

The (0,2) reduction described in Sec. II A 3 can potentially be applied to non-Lagrangian class $\mathcal{S}$ theories, including the trinion theories T_N and the Argyres-Douglas theories. However, to perform a consistent reduction, the (0,2) U(1) $\mathfrak{R}$ charges, represented as $\mathfrak{R} = R + \frac{r-f}{2}$, must take integer values. This imposes stringent conditions on which class $\mathcal{S}$ theories can undergo consistent reduction. Upon examining the Higgs and Coulomb branch operators, the reduction seems possible for simple theories such as the T_3 theory and the (A_1, D_4) Argyres-Douglas theory. On the other hand, the (A_1, D_{2n+1}) theories do not admit consistent reduction since the dimensions of their Coulomb branch operators are fractional, resulting in a nonintegral value for $R + \frac{r-f}{2}$. In the following, we discuss potential candidates for the (0,2) elliptic genus in these non-Lagrangian cases.

Recall that from Sec. II A 2, the (0,2) elliptic genus in the Ramond sector[6] is expected to be a Jacobi form of weight 0 with a nonzero index which captures the 't Hooft anomaly of the flavor symmetry. Additionally, we further conjecture in (2.26) that the (0,2) elliptic genus for a class $\mathcal{S}$ theory $\mathcal{T}[C_{g,n}]$ should be some linear combination of the module characters of the associated VOA $\chi(\mathcal{T}[C_{g,n}])$. These two conditions are expected to place strong constraints on candidates of a (0,2) elliptic genus. For example, under an S transformation, the elliptic genus should transform back to itself up to a phase.[7]

The T_3 trinion theory of type A_2 is endowed with E_6 flavor symmetry [60,61]. As uncovered in Fig. 19 of [1], the SU(2) gauging of the T_3 trinion theory leads to the infinite coupling limit of SU(3) $N_f = 6$ superconformal theory. The corresponding VOA is the affine Lie algebra $(\hat{\mathfrak{e}}_6)_{-3}$ with level -3 [7]. The algebra $(\hat{\mathfrak{e}}_6)_{-3}$ has irreducible representations with the following highest weights [62]:

$$\begin{aligned} &0, \qquad -3\omega_1, \qquad -3\omega_6, \qquad \omega_1 - 2\omega_3, \\ &\omega_6 - 2\omega_5, \qquad -2\omega_2, \qquad -\omega_4. \end{aligned} \tag{2.83}$$

Using the pure spinor formalism [48,63], the following combination of the $(\hat{\mathfrak{e}}_6)_{-3}$ characters is considered:

$$\mathcal{I}^{\mathfrak{e}_6}(\boldsymbol{m}, q) = \mathrm{ch}_0^{(\hat{\mathfrak{e}}_6)_{-3}}(\boldsymbol{m}, q) - \mathrm{ch}_{-3\omega_1}^{(\hat{\mathfrak{e}}_6)_{-3}}(\boldsymbol{m}, q). \tag{2.84}$$

We would like to analyze if this partition function is a candidate of the (0,2) elliptic genus coming from the (0,2) reduction. Concretely, $\mathcal{I}^{\mathfrak{e}_6}$ can be expressed as a combination of theta functions as follows:

$$\mathcal{I}^{\mathfrak{e}_6}(\boldsymbol{m}, q) = \frac{\eta(q)^{10}(\Theta^{\mathfrak{e}_6}_{\omega_1}(\tilde{\boldsymbol{m}}, q) - \Theta^{\mathfrak{e}_6}_{\omega_6}(\tilde{\boldsymbol{m}}, q))}{\prod_{w\in\mathbf{S}}\vartheta_1(\boldsymbol{m}^w_{\mathfrak{d}_5})}, \tag{2.85}$$

where the two theta functions $\Theta^{\mathfrak{e}_6}_{\omega_1,\omega_6}$ are defined [64,65] as

$$\begin{aligned} \Theta^{\mathfrak{e}_6}_{\omega_1}(\boldsymbol{m}, q) &= \frac{q^{1/6}}{2}\sum_{k=1}^{4}\sigma_k m_0 \vartheta_k(m_0^3 q)\prod_{j=1}^{5}\vartheta_k(m_j), \\ \Theta^{\mathfrak{e}_6}_{\omega_6}(\boldsymbol{m}, q) &= \frac{q^{1/6}}{2}\sum_{k=1}^{4}\sigma_k m_0^{-1} \vartheta_k(m_0^3 q^{-1})\prod_{j=1}^{5}\vartheta_k(m_j), \end{aligned}$$

with $-\sigma_1 = \sigma_2 = \sigma_3 = -\sigma_4 = 1$. Here, $\boldsymbol{m} = (m_0, \boldsymbol{m}_{\mathfrak{d}_5}) = (m_0, m_1, \ldots, m_5)$ are the fugacities for $\mathfrak{e}_6$ ($m_{i>0}$ are also fugacities for the subalgebra $\mathfrak{d}_5$) in the orthogonal basis.[8] In the numerator of (2.85) we use $\tilde{\boldsymbol{m}} = (m_0^2, m_1, \ldots, m_5)$, and in the denominator $\mathbf{S} = [0,0,0,0,1]$ is the spin representation of $\mathfrak{d}_5$.

Using the branching rules, one can establish the relationships between the $\mathfrak{e}_6$ fugacities $\boldsymbol{m}$ and the $\mathfrak{a}_3 \oplus \mathfrak{a}_3 \oplus \mathfrak{a}_3$ fugacities x_i, y_j, z_k (with the usual relation $\prod_{i=1}^3 x_i = \prod_{i=1}^3 y_i = \prod_{i=1}^3 z_i = 1$) in the fundamental weight basis (or the omega basis in [66,67]):

$$\begin{aligned} m_0 &= x_1^{\frac{1}{2}} z_2^{-\frac{1}{2}}, \qquad & m_3 &= x_1^{\frac{1}{2}} y_1^{-1} z_2^{\frac{1}{2}}, \\ m_1 &= x_1^{-\frac{1}{2}} x_2^{-1} z_1 z_2^{\frac{1}{2}}, \qquad & m_4 &= x_1^{\frac{1}{2}} y_1 y_2^{-1}, \\ m_2 &= x_1^{\frac{1}{2}} x_2 z_1 z_2^{\frac{1}{2}}, \qquad & m_5 &= x_1^{\frac{1}{2}} y_2 z_2^{\frac{1}{2}}. \end{aligned} \tag{2.86}$$

Recall from (2.19) that the 't Hooft anomaly can be read off from the shift property of (2.85). In particular, we focus on the shift behavior with respect to the SU(3) flavor fugacities x_i, y_j, z_k:

$$\begin{aligned} x_1 &\to x_1 q, \qquad x_2 \to x_2/q, \qquad \mathcal{I}^{\mathfrak{e}_6} \to \frac{q^2 x_1 z_2^2}{x_2^3}\mathcal{I}^{\mathfrak{e}_6}, \\ y_1 &\to y_1 q, \qquad y_2 \to y_2/q, \qquad \mathcal{I}^{\mathfrak{e}_6} \to (q y_1/y_2)^9 \mathcal{I}^{\mathfrak{e}_6}, \\ z_1 &\to z_1 q, \qquad z_2 \to z_2/q, \qquad \mathcal{I}^{\mathfrak{e}_6} \to \frac{q^2 z_1^3}{x_1 z_2}\mathcal{I}^{\mathfrak{e}_6}. \end{aligned} \tag{2.87}$$

In four dimensions, the E_6 theory is related to the SU(3) SQCD through the Argyres-Seiberg duality, where two of the SU(3) flavor symmetries of the former theory are identified with the two SU(3) flavor symmetries of the latter theory. If the above $\mathcal{I}^{\mathfrak{e}_6}$ truly represents the elliptic genus of the reduced E_6 theory, then its SU(3)^2 't Hooft anomaly should match with that of the reduced SU(3)

[6] In the previous subsections, we considered the (0,2) elliptic genus in the NS sector. Nonetheless, it is straightforward to transform it to the Ramond sector simply by replacing ϑ_4 with ϑ_1.

[7] We will encounter Kac-Moody algebras as the associated VOAs, which are all bosonic: under the S transformation, the transformed characters remain within the standard (untwisted) sector.

[8] Contrarily, the fugacities in [48] were expressed in the alpha basis [66,67].

SQCD. The elliptic genus $\mathcal{I}^{(0,2),3}_{0,2,2}$ of 2D (0,2) SU(3) SQCD with three fundamentals and three antifundamentals is given by the $N = 3$ specialization of (2.72), which has the simple shift property

$$c_i \to c_i q, \quad \mathcal{I}^{(0,2),3}_{0,2,2} \to q^{\frac{9}{2}} c_i^9 \mathcal{I}^{(0,2),3}_{0,2,2},$$
$$b_{i,1} \to b_{i,1} q, \quad b_{i,2} \to b_{i,2}/q, \quad \mathcal{I}^{(0,2),3}_{0,2,2} \to (q b_{i,1}/b_{i,2})^9 \mathcal{I}^{(0,2),3}_{0,2,2}.$$

Here c_i are the $\mathrm{U}(1)^2$ fugacities and $b_{i,j}$ are the two SU(3) fugacities. However, by comparison, the two shift behaviors do not match if we identify the SU(3) fugacities as $y_1 = b_{0,1}, y_2 = b_{0,2}, z_1 = b_{2,1}, z_2 = b_{2,2}$. Hence, when we perform the SU(2) gauging on the expression given in (2.85), it appears that we do not arrive at the (0,2) elliptic genus for SU(3) SQCD with $N_f = 6$, and $\mathcal{I}^{\mathfrak{e}_6}$ fails to be a candidate for the desired elliptic genus. Note, however, that $\mathcal{I}^{\mathfrak{e}_6}$ is not the only candidate. It may be worth exploring alternative combinations of $(\hat{\mathfrak{e}}_6)_{-3}$ characters, distinct from (2.84), in order to compare with the SU(3) SQCD with $N_f = 6$. Indeed, by the logic of [48], a linear combination of the vacuum character and the character with the highest weight $-\omega_4$ is the most promising starting point. We leave this to future study.

Argyres-Douglas theories [68,69] constitute another interesting class of non-Lagrangian theories, whose construction involves a higher-order pole of the Higgs field in the Hitchin system [70].

Let us first consider the (A_1, D_4) theory. The rank-one theory contains a Coulomb branch operator with conformal dimension $\Delta = -3/2$, and therefore an integral r charge $r = 2\Delta = 3$, suggesting a possible S^2 reduction and a corresponding (0,2) elliptic genus. The associated VOA is given by the Kac-Moody algebra $\widehat{\mathfrak{su}}(3)_{-3/2}$ [6,71]. The level $k = -3/2$ with respect to the SU(3) flavor symmetry is called *boundary admissible* in the mathematics literature [72]. There are four irreducible admissible highest-weight modules with affine weights:

$$-\frac{3}{2}\hat{\omega}_0, \qquad -\frac{3}{2}\hat{\omega}_1, \qquad -\frac{3}{2}\hat{\omega}_2, \qquad \hat{\rho} = -\frac{1}{2}\sum_{i=0}^{2}\hat{\omega}_i, \tag{2.88}$$

where $\hat{\rho}$ is the affine Weyl vector. The characters are given by

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} = \frac{\eta(\tau)\vartheta_1(\mathsf{b}_1 - 2\mathsf{b}_2|2\tau)\vartheta_1(-\mathsf{b}_1 - \mathsf{b}_2|2\tau)\vartheta_1(-2\mathsf{b}_1 + \mathsf{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathsf{b}_1 - 2\mathsf{b}_2|\tau)\vartheta_1(-\mathsf{b}_1 - \mathsf{b}_2|\tau)\vartheta_1(-2\mathsf{b}_1 + \mathsf{b}_2|\tau)},$$
$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_1} = -\frac{\eta(\tau)\vartheta_4(\mathsf{b}_1 - 2\mathsf{b}_2|2\tau)\vartheta_4(-\mathsf{b}_1 - \mathsf{b}_2|2\tau)\vartheta_1(-2\mathsf{b}_1 + \mathsf{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathsf{b}_1 - 2\mathsf{b}_2|\tau)\vartheta_1(-\mathsf{b}_1 - \mathsf{b}_2|\tau)\vartheta_1(-2\mathsf{b}_1 + \mathsf{b}_2|\tau)},$$
$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_2} = -\frac{\eta(\tau)\vartheta_1(\mathsf{b}_1 - 2\mathsf{b}_2|2\tau)\vartheta_4(-\mathsf{b}_1 - \mathsf{b}_2|2\tau)\vartheta_4(-2\mathsf{b}_1 + \mathsf{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathsf{b}_1 - 2\mathsf{b}_2|\tau)\vartheta_1(-\mathsf{b}_1 - \mathsf{b}_2|\tau)\vartheta_1(-2\mathsf{b}_1 + \mathsf{b}_2|\tau)},$$
$$\mathrm{ch}_{-\frac{1}{2}\hat{\rho}} = -\frac{\eta(\tau)\vartheta_4(\mathsf{b}_1 - 2\mathsf{b}_2|2\tau)\vartheta_1(-\mathsf{b}_1 - \mathsf{b}_2|2\tau)\vartheta_4(-2\mathsf{b}_1 + \mathsf{b}_2|2\tau)}{\eta(2\tau)\vartheta_1(\mathsf{b}_1 - 2\mathsf{b}_2|\tau)\vartheta_1(-\mathsf{b}_1 - \mathsf{b}_2|\tau)\vartheta_1(-2\mathsf{b}_1 + \mathsf{b}_2|\tau)},$$

where the first is the vacuum character of $\widehat{\mathfrak{su}}(3)_{-3/2}$ as well as the Schur index of the (A_1, D_4) theory. The modular S matrix is given by

$$S = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{2.89}$$

An elliptic genus should be a weight-0 Jacobi form and therefore should transform back to itself under S, up to a prefactor. We look for linear combinations of the characters that have such simple behavior. There are four eigenvectors of S, with eigenvalues $(-1,-1,-1,1)$, respectively,

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} - \mathrm{ch}_{-\frac{3}{2}\hat{\rho}}, \qquad \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} + \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_1}, \qquad \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} + \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_2}, \tag{2.90}$$

and finally,

$$\mathrm{ch}_{-\frac{3}{2}\hat{\omega}_0} - \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_1} - \mathrm{ch}_{-\frac{3}{2}\hat{\omega}_2} + \mathrm{ch}_{-\frac{3}{2}\hat{\rho}}. \tag{2.91}$$

Unfortunately, neither the $+1$ eigenvector nor any linear combination of the -1 eigenvectors behaves consistently under the shift of both SU(3) flavor fugacities b_1 and b_2 that can reflect the 't Hooft anomaly. Consequently, no linear combination of the $\widehat{\mathfrak{su}}(3)_{-3/2}$ characters appears as a valid candidate of the desired elliptic genus.

Let us also consider the (A_1, D_{2n+1}) Argyres-Douglas theories, which enjoy SU(2) flavor symmetry. Since these theories contain Coulomb branch operators with fractional

r charge, a valid (0,2) reduction is *not* anticipated. Below, we contend that from the VOA perspective a (0,2) elliptic genus does not exist. The associated VOAs are given by $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$ [6,71]. In this case, the level $k=-\frac{4n}{2n+1}=-2+\frac{2}{2n+1}$ is also boundary admissible, and the VOA has admissible affine weights given by (where $u\coloneqq 2n+1$) [72,73]

$$\hat{\lambda}_{k,j}=\left(k+\frac{2j}{u}\right)\hat{\omega}_0-\frac{2j}{u}\hat{\omega}_1,\quad j=0,1,2,\ldots,u-1=2n. \tag{2.92}$$

They are the highest weights of irreducible highest-weight modules $L(\hat{\lambda}_{k,j})$ of $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$, whose characters are given by the simple formula

$$\operatorname{ch}L(\hat{\lambda}_{k,j})=z^{-\frac{2j}{u}}q^{\frac{j^2}{2u}}\frac{\vartheta_1(2\mathsf{z}-j\tau|u\tau)}{\vartheta_1(2\mathsf{z}|\tau)}. \tag{2.93}$$

Here z is the flavor SU(2) fugacity. The modular S matrix is given by

$$S_{jj'}=\sqrt{\frac{2}{u^2(k+2)}}e^{\pi i(j+j')}e^{\pi i(jj'(k+2))}\sin\left(\frac{\pi}{k+2}\right). \tag{2.94}$$

However, none of the eigenvectors of the S matrix transform themselves (up to a factor) under the shift $\mathsf{z}\to\mathsf{z}+\tau$ since each character transforms in the following manner (the subscript follows a cyclic rule such that $j\sim j+2n+1$):

$$\operatorname{ch}(\hat{\lambda}_{k,j})\xrightarrow{\mathsf{z}\to\mathsf{z}+\tau}(b^2q)^{-k}\operatorname{ch}(\hat{\lambda}_{k,j-2}). \tag{2.95}$$

This implies that a Jacobi form from $\operatorname{ch}(\hat{\lambda}_{k,j})$ must take the form

$$\text{const}\cdot\sum_{j=0}^{2n}\operatorname{ch}(\hat{\lambda}_{k,j}), \tag{2.96}$$

which is never an eigenvector of the S matrix (2.94). Therefore, we conclude that no linear combination of the $\widehat{\mathfrak{su}}(2)_{k=-\frac{4n}{2n+1}}$ characters satisfies the expected properties of a (0,2) elliptic genus. To summarize, we have been unable to identify a (0,2) elliptic genus for non-Lagrangian theories.

III. $\mathcal{N}=(0,4)$ ELLIPTIC GENERA FOR CLASS $\mathcal{S}$ THEORIES ON S^2

In this section, we study $\mathcal{N}=(0,4)$ theories obtained by a distinct twisted compactification of class $\mathcal{S}$ theories of type A on S^2. In these theories, we perform a topological twist on $\mathrm{U}(1)_{S^2}$ with 4D $\mathcal{N}=2$ superconformal R symmetry $\mathrm{U}(1)_r\subset\mathrm{SU}(2)_R\times\mathrm{U}(1)_r$, as discussed in [27,74]. Referencing Table I, the four supercharges $Q^I_-,\tilde{Q}^I_-$ $(I=1,\,2)$ survive under this twist and they possess the same $\mathrm{U}(1)_{T^2}$ charge. Therefore, this twist preserves 2D $\mathcal{N}=(0,4)$ supersymmetry and thus we refer to this twisted compactification as *the (0,4) reduction* of class $\mathcal{S}$ theories. For 4D $\mathcal{N}=2$ SCFT, $\mathrm{U}(1)_r$ charges of operators are integral, eliminating the need for an additional twist by a flavor symmetry. The 2D $\mathcal{N}=(0,4)$ supersymmetry has $SO(4)_R\cong\mathrm{SU}(2)^-_R\times\mathrm{SU}(2)^+_R$ as the UV R symmetry, where 4D $\mathrm{SU}(2)_R$ is identified with 2D $\mathrm{SU}(2)^-_R\subset\mathrm{SO}(4)_R$. This subgroup subsequently evolves into the affine $\widehat{\mathfrak{su}}(2)$ Lie algebra within the small $\mathcal{N}=4$ superconformal algebra in the IR. Given the (0,4) reduction of a class $\mathcal{S}$ theory, we consider its IR SCFT on the Higgs branch where $\mathrm{SU}(2)^+_R$ becomes the small $\mathcal{N}=4$ superconformal R symmetry in the right-moving sector. For a detailed analysis of the symmetries within this context, readers are directed to [27].

In the (0,4) reduction, a 4D $\mathcal{N}=2$ hypermultiplet reduces to a 2D $\mathcal{N}=(0,4)$ hypermultiplet [two (0,2) chirals with opposite charges]. Likewise, a 4D $\mathcal{N}=2$ vector multiplet reduces to a 2D $\mathcal{N}=(0,4)$ vector multiplet [(0,2) vector + (0,2) adjoint Fermi]. Consequently, for a Lagrangian theory, the basic building blocks in 2D are as follows.

For type A_1, it corresponds to a sphere with three punctures. For type A_{N-1}, a sphere with one minimal puncture and two maximal punctures gives rise to this building block. For simplicity in notation (without distinguishing types of punctures), we denote its contribution to a (0,4) elliptic genus as $\mathcal{I}^{(0,4)}_{0,3}$. The explicit contributions of this and the vector multiplet are

$$\begin{aligned}\mathcal{I}^{(0,4)}_{0,3}(a,b,c)&=\prod_{i,j=1}^{N}\frac{\eta(q)^2}{\vartheta_1(v(ca_ib_j)^{\pm})},\\ \mathcal{I}^{(0,4)}_{\mathrm{vec}}(a)&=\frac{(\vartheta_1(v^2)\eta(q))^{N-1}}{N!}\\ &\quad\times\prod_{\substack{A,B=1\\A\neq B}}^{N}\frac{\vartheta_1(v^2a_A/a_B)\vartheta_1(a_A/a_B)}{\eta(q)^2},\end{aligned} \tag{3.1}$$

where the $\mathrm{SU}(N)$ fugacities condition is implicitly imposed,

$$\prod_{i=1}^{N}a_i=1=\prod_{i=1}^{N}b_i. \tag{3.2}$$

The fugacity v is for the Cartan subgroup of the antidiagonal of $\mathrm{SU}(2)^-_R\times\mathrm{SU}(2)^+_R$ R symmetry that commutes with the supercharges. In this physical setup, it was argued in [27,75] that the (0,4) elliptic genus is expected to be the Vafa-Witten partition function [76] on $C_{g,n}\times S^2$. In [27], the S duality of (0,4) theories with genus zero was confirmed by evaluating the elliptic genera. Notably, using the elliptic inversion formula (2.37), the elliptic genus of the (0,4) reduction of the non-Lagrangian T_3 trinion theory was obtained there. The primary focus of this paper is to explore the (0,4) theories with a genus greater than zero, using elliptic genera.

In theories with a genus of zero, one can determine the right-moving central charge using the $\mathrm{SU}(2)_R^+$ anomaly,

$$c_R = 6(2k_R) = 6(n_h - n_v). \tag{3.3}$$

Here, k_R denotes the $\mathrm{SU}(2)_R^+$ anomaly coefficient, which can be computed as in (2.14) in the ultra-violet theory, and $2k_R$ represents the level of the affine $\mathrm{SU}(2)_R^+$ symmetry. For a class $\mathcal{S}$ theory $\mathcal{T}_N[C_{g=0,n}]$ lacking a Lagrangian description, n_h and n_v can be evaluated from partitions of N assigned to punctures. The explicit treatment can be found in [27,77], and we omit the details here. The left-moving central charge c_L can be derived from the gravitational anomaly (2.18). Note that $2k_R$ represents the quaternionic dimension of the Higgs branch. Moreover, the $q \to 0$ limit of the elliptic genus agrees with the Hilbert series of the Higgs branch [78]. (The computational techniques were developed in [79–81].)

On the other hand, the situation drastically changes for theories with a genus greater than zero. In a theory with genus $g > 0$, the $\mathrm{U}(1)^g$ gauge symmetry is unbroken at a generic point of the moduli space of the hypermultiplets, and therefore it was called the *Kibble branch* in [78]. It was conjectured in [27] that the computation of the right-moving central change is modified from (3.3) as

$$c_R = 6(2k_R + g) = 6(n_h - n_v + g). \tag{3.4}$$

As we will see below, the $q \to 0$ limit of the elliptic genus is no longer equal to the Hilbert series of the Kibble branch. This is very similar to the relation between the Hall-Littlewood index and the Higgs branch Hilbert series [5] in which the agreement can be seen for theories with genus zero but not higher.

In the following, we present closed-form expressions for the (0,4) elliptic genera of theories where the genus $g > 0$. If a theory has a Lagrangian description with a gauge group of adequately low total rank, one can straightforwardly compute the elliptic genus through the JK-residue method. To determine the elliptic genus of non-Lagrangian theories at higher genus, we exploit the inversion formula in [27,82,83], namely, by performing additional gauging in Lagrangian theories. For detailed calculations, readers can refer to Appendix D, which provides explicit JK-residue computations of (0,4) elliptic genera. The resulting closed-form expressions are remarkably simple, aligning well with the TQFT structure on punctured Riemann surfaces $C_{g,n}$.

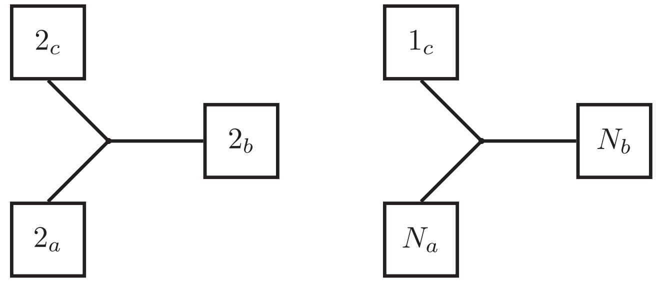

FIG. 16. Left: a building block for type A_1, representing eight (0,2) chiral multiplets with $\mathrm{SU}(2)^3$ flavor symmetry. Right: a building block for Lagrangian theories of type A_{N-1}, representing a (0,4) free hypermultiplet with $\mathrm{SU}(N)_a \times \mathrm{SU}(N)_b \times \mathrm{U}(1)_c$ flavor symmetry.

A. Type A_1

Class $\mathcal{S}$ theories of type A_1 all have Lagrangian descriptions and are completely specified by the genus g and the number of (regular) punctures n. We focus on theories at genus $g \geq 1$ with an arbitrary number of punctures. To compute elliptic genera, we can gauge the basic building block illustrated on the left side of Fig. 16 by using (3.1). For $g = 1$, $n = 1$, the elliptic genus is computed from the JK residue where one only encounters nondegenerate poles,

$$\mathcal{I}_{1,1}^{(0,4),2}(c) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{0,3}^{(0,4)}(a, a^{-1}, c) \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) = \frac{\eta(q)^2 \vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2 c^{\pm 2})}. \tag{3.5}$$

This expression is a simple ratio of the theta functions, and the LG dual theory was described in Sec. 2.2.3 of [27]. A similar computation can be performed for $g = 1$, $n \geq 1$, which yields

$$\mathcal{I}_{1,n}^{(0,4),2}(c_1, \ldots, c_n) = \prod_{i=1}^{n} \frac{\eta(\tau)^2 \vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2 c_i^{\pm 2})}. \tag{3.6}$$

While the Kibble branch Hilbert series was computed in Sec. 4.2.2 of [78] for $n = 2$, the relation between the elliptic genus and the Hilbert series is unclear. Consequently, although the form of the elliptic genus suggests the existence of an LG dual theory, its precise description remains unknown to us.

It is straightforward to obtain the elliptic genus for higher genera. For example, the theory of genus two is

$$\begin{aligned} \mathcal{I}_{2,0}^{(0,4),2} &= \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}_{1,2}^{(0,4),2}(a, a^{-1}) \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) \\ &= \int_{\mathrm{JK}} \frac{da}{2\pi i a} (\mathcal{I}_{1,1}^{(0,4),2}(a))^2 \mathcal{I}_{\mathrm{vec}}^{(0,4)}(a) \\ &= \frac{\vartheta_1(v^2)\vartheta_1(v^4)}{\eta(q)^2}. \end{aligned} \tag{3.7}$$

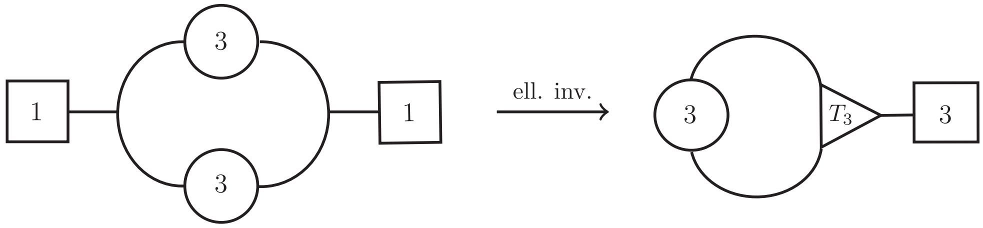

FIG. 17. Application of the elliptic inversion formula leads to the elliptic genus for a maximal puncture.

Moreover, we can increase the genus g by gluing together any number of pairs of punctures, and the final result takes the simple form

$$\mathcal{I}^{(0,4),2}_{g,n}(c_1,\ldots,c_n)=\left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)}{\eta(q)^2}\right)^{g-1}\prod_{i=1}^{n}\frac{\eta(q)^2\vartheta_1(v^4)}{\vartheta_1(v^2)\vartheta_1(v^2c_i^{\pm 2})}. \tag{3.8}$$

This result is consistent with the cut-and-join TQFT structure on $C_{g,n}$, so that

$$\mathcal{I}^{(0,4),2}_{g+1,n}(c_1,\ldots,c_n)=\int_{\mathrm{JK}}\frac{da}{2\pi i a}\mathcal{I}^{(0,4),2}_{g,n+2}(c_1,\ldots,c_n,a,a^{-1})\mathcal{I}^{(0,4)}_{\mathrm{vec}}(a),$$
$$\mathcal{I}^{(0,4),2}_{g_1+g_2,n_1+n_2}(c_1,\ldots,c_{n_1+n_2})=\int_{\mathrm{JK}}\frac{da}{2\pi i a}\mathcal{I}^{(0,4),2}_{g_1,n_1+1}(c_1,\ldots,a)\mathcal{I}^{(0,4)}_{\mathrm{vec}}(a)\mathcal{I}^{(0,4),2}_{g_2,n_2+1}(c_{n_1+1},\ldots,a^{-1}).$$

B. Type A_2

For class $\mathcal{S}$ theories of type A_2, there are two types of regular punctures: the minimal punctures with flavor symmetry $\mathrm{U}(1)$ and the maximal punctures with flavor symmetry SU(3). We denote the number of these punctures as n_1 and n_3, respectively. First, we can gauge the basic building block illustrated on the right side of Fig. 16 by using (3.1) to compute the elliptic genus for $g=1$ with several minimal punctures. This can be swiftly computed using the JK residue, from which we can postulate the general formula

$$\mathcal{I}^{(0,4),3}_{g=1;n_1,0}(c_1,\ldots,c_{n_1})=\prod_{i=1}^{n_1}\frac{\eta(q)^2\vartheta_1(v^6)}{\vartheta_1(v^2)\vartheta_1(v^3c_i^{\pm 3})}. \tag{3.9}$$

To access the elliptic genus in the presence of maximal punctures, we apply the elliptic inversion formula in [27] that computes the (0,4) elliptic genus of the T_3 theory. Specifically, starting from $g=1, n_1=2$, we use the inversion formula (2.37) to obtain $g=1, n_3=1$ (Fig. 17),

$$\begin{aligned}\mathcal{I}^{(0,4),3}_{g=1;0,n_3=1}(b)&=\frac{\eta(q)^5}{2\vartheta_1(v^2z^{\pm 2})}\int_{\mathrm{JK}}\frac{ds}{2\pi i s}\frac{\vartheta_1(s^{\pm 2})\vartheta_1(v^{-2})}{\vartheta_1(v^{-1}s^{\pm 1}z^{\pm 1})}\\&\quad\times\mathcal{I}^{(0,4),3}_{g=1;n_1,0}(s^{\frac{1}{3}}/r,s^{-\frac{1}{3}}/r)\\&=\frac{\eta(q)^6\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)}{\prod_{A,B=1}^{3}\vartheta_1(v^2b_A/b_B)},\end{aligned} \tag{3.10}$$

where the SU(3) fugacities for the maximal puncture are identified by $(b_1,b_2,b_3)=(rz,r/z,r^{-2})$. The detailed computations of the elliptic genus for type A_2 theories are collected in Appendix D 2. In summary, the elliptic genus for the (0,4) reduction of the class $\mathcal{S}$ theory $\mathcal{T}_3[C_{g;n_1,n_3}]$ is given by the following simple form:

$$\mathcal{I}^{(0,4),3}_{g;n_1,n_3}=\left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)^2\vartheta_1(v^6)}{\eta(q)^4}\right)^{g-1}\mathcal{I}^{(0,4),3}_{1;n_1,0}\mathcal{I}^{(0,4),3}_{1;0,n_3}, \tag{3.11}$$

where

$$\mathcal{I}^{(0,4),3}_{g=1;0,n_3}=\prod_{i=1}^{n_3}\frac{\eta(q)^6\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)}{\prod_{A,B=1}^{3}\vartheta_1(v^2b_{iA}/b_{iB})}. \tag{3.12}$$

C. Type A_3

For type A_3 theories, there are four types of regular punctures whose partitions and flavor symmetries are given as follows:

(1) [3, 1] U(1).
(2) [2, 1, 1] $\mathrm{SU}(2)\times\mathrm{U}(1)$.
(3) [2, 2] SU(2).
(4) [1, 1, 1, 1] SU(4).

We use the notations n_1, n_2, n_3, n_4 to denote the numbers of these punctures, respectively. For $g=1$ only with minimal

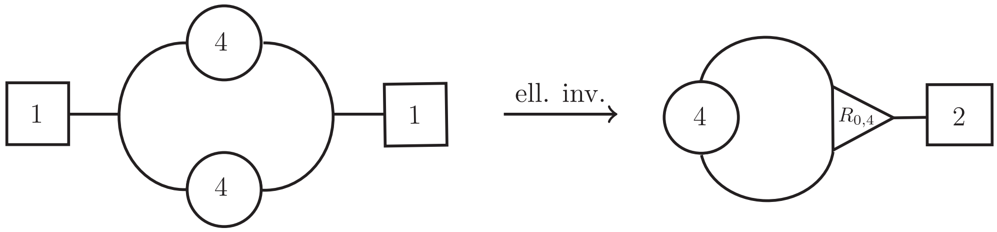

FIG. 18. Application of the elliptic inversion formula leads to the elliptic genus for a [2,1,1] puncture.

punctures ($n_2 = n_3 = n_4 = 0$), it is straightforward to compute the elliptic genus,

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1} \frac{\eta(q)^2 \vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4 c_i^{\pm 4})}. \tag{3.13}$$

Applying the elliptic inversion formula in [83], we can calculate the elliptic genus of genus-one theories that have [2,1,1], and [2,2] punctures. As a specific case, when there is only one [2,1,1] puncture, the elliptic genus is as follows:

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{g=1;0,1,0,0} &= \frac{\eta(q)^5}{2\vartheta_1(v^2 z^{\pm 2})} \int_{\mathrm{JK}} \frac{ds}{2\pi i s} \frac{\vartheta_1(s^{\pm 2})\vartheta_1(v^{-2})}{\vartheta_1(v^{-1} s^{\pm 1} z^{\pm 1})} \\ &\quad \times \mathcal{I}^{(0,4),3}_{g=1;2,0,0,0}(s^{\frac{1}{4}}/r, s^{-\frac{1}{4}}/r) \\ &= \frac{\eta(q)^6 \vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)^2 \vartheta_1(v^2 z^{\pm 2})\vartheta_1(v^3 z^{\pm} r^{\pm 4})},\end{aligned} \tag{3.14}$$

where z, r denote the SU(2) and U(1) fugacities, respectively. (See Fig. 18.) With only one [2, 2] puncture, the elliptic genus is

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{g=1;0,0,1,0} &= \frac{\eta(q)^5 \vartheta_1(v^2)\vartheta_1(v^{-2})}{2\vartheta_1(v^4)} \\ &\quad \times \int_{\mathrm{JK}} \frac{ds}{2\pi i s} \frac{\vartheta_1(s^{\pm 2})}{\vartheta_1(s^{\pm 1})\vartheta_1(v^{-2} s^{\pm 1})} \\ &\quad \times \mathcal{I}^{(0,4),3}_{g=1;2,0,0,0}(s^{\frac{1}{4}}/w, s^{-\frac{1}{4}}/w) \\ &= \frac{\eta(q)^4 \vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^4 w^{\pm 4})\vartheta_1(v^2 w^{\pm 4})}.\end{aligned} \tag{3.15}$$

While the derivation of the elliptic genus for the theory of genus one with a maximal puncture remains unknown, an extrapolation from the results in (3.5) and (3.10) allows us to propose the following expression:

$$\mathcal{I}^{(0,4),4}_{g=1;0,0,0,1} = \frac{\eta(q)^{12} \vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)\vartheta_1(v^8)}{\prod_{A,B=1}^4 \vartheta_1(v^2 b_A/b_B)}. \tag{3.16}$$

The validity of our proposed formula can be tested by examining the S duality in Fig. 19. Gauging this theory as in the right side of the figure leads to the theory of genus one with three minimal punctures, and thus we can compare the result with (3.13). The detailed computations of the elliptic genera are collected in Appendix D 3.

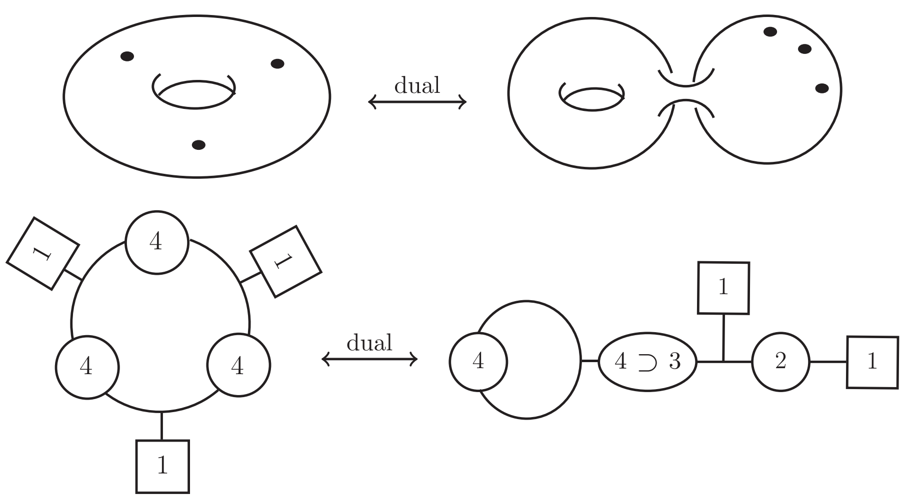

FIG. 19. S duality in the class $\mathcal{S}$ theory of type A_3. The right theory involves gauging the A_3 trinion theory.

This allows us to further increase the genus g and, finally, the elliptic genus of a general A_3 theory is given by

$$\mathcal{I}^{(0,4),4}_{g;n_1,n_2,n_3,n_4} = \left(\frac{\vartheta_1(v^2)\vartheta_1(v^4)^2\vartheta_1(v^6)^2\vartheta_1(v^8)}{\eta(q)^6}\right)^{g-1} \times \mathcal{I}^{(0,4),4}_{1;n_1,0,0,0}\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0}\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0}\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4},$$

where

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1}\frac{\eta(q)^2\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4c_i^{\pm 4})}, \tag{3.17}$$

$$\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0} = \prod_{i=1}^{n_3}\frac{\eta(q)^6\vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)^2\vartheta_1(v^2z_i^{\pm 2})\vartheta_1(v^3z_i^{\pm}r_i^{\pm 4})}, \tag{3.18}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0} = \prod_{i=1}^{n_2}\frac{\eta(q)^4\vartheta_1(v^6)\vartheta_1(v^8)}{\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^2w_i^{\pm 4})\vartheta_1(v^4w_i^{\pm 4})}, \tag{3.19}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} = \prod_{i=1}^{n_4}\frac{\eta(q)^{12}\vartheta_1(v^2)\vartheta_1(v^4)\vartheta_1(v^6)\vartheta_1(v^8)}{\prod_{A,B=1}^{4}\vartheta_1(v^2b_{iA}/b_{iB})}. \tag{3.20}$$

D. Type A_{N-1} and TQFT structure

From the above results, we can observe a simple TQFT structure in the $\mathcal{N}=(0,4)$ elliptic genus of the class $\mathcal{S}$ theory at genus $g \geq 1$. This structure suggests that the elliptic genus can be expressed as a straightforward product of contributions from individual punctures and handles. Therefore, the $\mathcal{N}=(0,4)$ elliptic genus corresponding to type A_{N-1} is expected to have the following form:

$$\mathcal{I}^{(0,4),N}_{g,n} = (\mathcal{H}_N)^{g-1}\prod_{i=1}^{n}\mathcal{I}^{(0,4),N}_{\lambda_i}(b_i), \tag{3.21}$$

where g is the genus of the associated Riemann surface and n collectively denotes the number of punctures, with their internal data represented by partitions (Young diagrams). The function $\mathcal{I}_{\lambda_i}$ captures the contribution from the ith puncture labeled by a partition λ_i, and $\mathcal{H}_N$ encapsulates the contribution originating from a handle. Loosely speaking, this expression resembles the TQFT expression of the 4D $\mathcal{N}=2$ superconformal index [4,5], which involves an infinite sum over representations of SU(N) schematically as

$$\mathcal{I}^{4\mathrm{D}} = \sum_{\mu} H_\mu^{2g-2+n}\prod_i \psi_\mu^{(\lambda_i)}(b_i). \tag{3.22}$$

We expect the elliptic genus $\mathcal{I}_{g,n}$ to obey a TQFT structure under cutting and gluing. Let us consider the maximal puncture corresponding to the integer partition $[1^N]$, which contributes

$$\mathcal{I}^{(0,4),N}_{[1^N]}(b) = \frac{\eta(q)^{N^2-N}\prod_{M=1}^{N}\vartheta_1(v^{2M})}{\prod_{A,B}^{N}\vartheta_1(v^2b_A/b_B)}, \tag{3.23}$$

where b_A denotes the SU(N) flavor fugacities with the constraint $b_1\cdots b_N = 1$.

Consider two Riemann surfaces, labeled as C_{g_1,n_1} and C_{g_2,n_2}, each with a maximal puncture. By SU(N) gauging, these two maximal punctures can be joined together, which results in a new Riemann surface $C_{g_1+g_2,n_1+n_2-2}$. Similarly, if a Riemann surface $C_{g,n}$ possesses more than two maximal punctures, by gauging the diagonal of the SU$(N)^2$ flavor symmetry originating from these two maximal punctures, we can transform this surface into a new Riemann surface $C_{g+1,n-2}$. These processes is visualized in Fig. 20. For the form of the (0,4) elliptic genus (3.21) to be compatible with these procedures, the handle contribution must be

$$\begin{aligned}\mathcal{H}_N &= \int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\mathcal{I}^{(0,4),N}_{[1^N]}(\boldsymbol{a})\mathcal{I}^{(0,4),N}_{[1^N]}(\boldsymbol{a}^{-1})\mathcal{I}^{(0,4)}_{\mathrm{vec}}(a)\\ &= \frac{\prod_{M=1}^{N}\vartheta_1(v^{2M})^2}{N!\eta(q)^{N-1}\vartheta_1(v^2)^{N+1}}\int_{\mathrm{JK}}\frac{d\boldsymbol{a}}{2\pi i\boldsymbol{a}}\prod_{A\neq B}\frac{\vartheta_1(a_A/a_B)}{\vartheta_1(v^2a_A/a_B)},\\ &= \frac{\prod_{M=1}^{N}\vartheta_1(v^{2M})^2}{\eta(q)^{2(N-1)}\vartheta_1(v^2)\vartheta_1(v^{2N})}.\end{aligned} \tag{3.24}$$

The JK integral is analogous to that in (2.76). The results (3.23) and (3.24) reduce to those in the previous examples

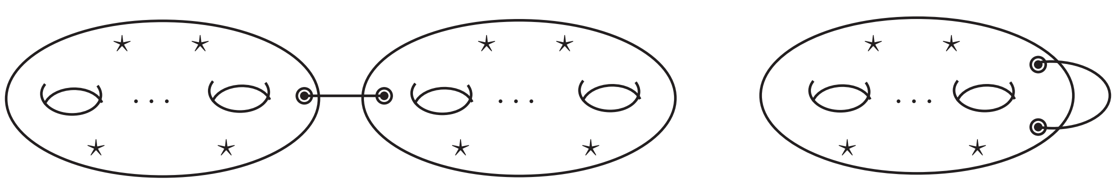

FIG. 20. Gluing maximal punctures leads to a new Riemann surface, and (0,4) elliptic genera are consistent with the cut-and-join procedure on Riemann surfaces $C_{g,n}$.

when $N = 2, 3, 4$. Furthermore, given that the (0,4) elliptic genus form in (3.21) receives only local contributions, verifying the following properties is straightforward:

$$\mathcal{I}^{(0,4),N}_{g_1+g_2,n_1+n_2-2}(b,c) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,4),N}_{g_1,n_1}(b,a) \times \mathcal{I}^{(0,4),N}_{g_2,n_2}(c,a^{-1}) \mathcal{I}^{(0,4)}_{\mathrm{vec}}(a),$$
$$\mathcal{I}^{(0,4),N}_{g+1,n-2}(b) = \int_{\mathrm{JK}} \frac{da}{2\pi i a} \mathcal{I}^{(0,4),N}_{g,n}(b,a,a^{-1}) \mathcal{I}^{(0,4)}_{\mathrm{vec}}(a). \tag{3.25}$$

The contribution from other punctures can be derived from the maximal one using nilpotent Higgsing. When transitioning from the maximal one to another type, an operator $\mathcal{O}$, which is charged under the $\mathrm{SU}(N)$ flavor symmetry, acquires a nilpotent vacuum expectation value $\langle\mathcal{O}\rangle$ with Jordan blocks of sizes 1 and $N-1$, which specifies an embedding $\mathrm{SU}(2) \hookrightarrow \mathrm{SU}(N)$ [77]. Following [83], we propose that this nilpotent Higgsing procedure can be implemented at the level of the elliptic genus as follows. The contribution from a puncture defined by an integer partition λ of N is given by

$$\mathcal{I}_\lambda(c) = \lim_{b\to c} \left[\frac{K_\lambda(c)}{K_{[1^N]}(b)}\right]_{\Gamma(t^\alpha z)\to\frac{\eta(q)}{\vartheta_1(v^{2\alpha}z)}} \mathcal{I}_{[1^N]}(b). \tag{3.26}$$

Here b denotes the flavor fugacities associated with the puncture and the function K is defined using the plethystic exponential (A5) as

$$K_\lambda(c) \coloneqq \mathrm{PE}\left[\sum_j \frac{t^{j+1} - pqt^j}{(1-p)(1-q)} \mathrm{ch}^f_{\mu_j}(c)\right]. \tag{3.27}$$

The ratio $K_\lambda/K_{[1^N]}$ in (3.26) can always be expressed by elliptic gamma functions, which will be shown at the end of this section.

The replacement $b \to c$ and the K_λ should be understood in the following way [84]. Recall that the integer partition λ captures an embedding of SU(2) in $\mathrm{SU}(N)$. The adjoint representation of $\mathrm{SU}(N)$ decomposes with respect to this embedding,

$$\mathbf{adj} = \oplus_j \mu_j \otimes \sigma_j, \tag{3.28}$$

where σ_j denotes the spin-j representation of the embedded SU(2) and μ_j denotes a representation of the commutant, namely, the flavor symmetry f of the puncture. Under the decomposition, the character of the adjoint representation can also be decomposed as[9]

$$\mathrm{ch}_{\mathbf{adj}}(b) = \sum_j \mathrm{ch}^f_{\mu_j}(c)\,\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}), \tag{3.30}$$

where $\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}) = \sum_{m=-j}^{j} t^m$.

For example, the partition $\lambda = [1^N]$ corresponding to the maximal puncture simply means a trivial embedding of SU(2), and therefore $f = \mathrm{SU}(N)$, $j = 0$, and $\mu_0 = \mathbf{adj}$. The K function then reads

$$K_{[1^N]}(b) = \mathrm{PE}\left[\frac{t - pq}{(1-p)(1-q)} \mathrm{ch}_{\mathbf{adj}}(b)\right], \tag{3.31}$$

where $b = (b_1, b_2, \ldots, b_{N-1}, b_N)$ denotes the fugacities of the flavor $\mathrm{SU}(N)$. As another example, when $N = 4$, $\lambda = [2,2]$, the flavor symmetry is $f = \mathrm{SU}(2)$. The adjoint of SU(4) decomposes as $\mathbf{adj} = ([j=1] \oplus [j=0]) \otimes \sigma_{j=1} \oplus [j=1] \otimes \sigma_{j=0}$, where $[j]$ simply denotes the spin-j representation with respect to f. The replacement $b \to c$ reads $(b_1, b_2, b_3, b_4) \to (ct^{1/2}, ct^{-1/2}, c^{-1}t^{1/2}, c^{-1}t^{-1/2})$, and the K function is given by

$$K_{[2^2]}(c) = \mathrm{PE}\left[\frac{(t-pq)}{(1-q)(1-p)}\left(c^2 + \frac{1}{c^2} + 1\right) + \frac{(t^2 - pqt)}{(1-q)(1-p)}\left(c^2 + \frac{1}{c^2} + 2\right)\right]. \tag{3.32}$$

Another case of interest is the principal embedding $\lambda = [N]$ corresponding to trivial flavor symmetry. The fugacity takes the principal specialization

$$(b_1, \ldots, b_N) \to (t^{-\frac{N-1}{2}}, t^{-\frac{N-3}{2}}, \ldots, t^{\frac{N-3}{2}}, t^{-\frac{N-1}{2}}). \tag{3.33}$$

The adjoint of $\mathrm{SU}(N)$ is decomposed into $\mathbf{adj} = \oplus_{i=1}^{N-1} \sigma_i$ and the K function is

$$K_{[N]}(c) = \mathrm{PE}\left[\sum_{i=1}^{N-1} \frac{t^{i+1} - pqt^i}{(1-p)(1-q)}\right]. \tag{3.34}$$

Note that there is no real c dependence since the corresponding flavor symmetry is trivial. We should regard the

[9]To actually obtain the decomposition (3.30) of character, one may start by finding the replacement $b \to c$ associated with the embedding. The process to find $b \to c$ effectively starts by addressing a simpler equation,

$$\mathrm{ch}_{\mathbf{fund}}(b) = \sum_j \mathrm{ch}^f_{\mu'_j}(c)\,\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}), \tag{3.29}$$

with respect to the decomposition of the $\mathrm{SU}(N)$ fundamental $\mathbf{fund} = \oplus_j \mu'_j \otimes \sigma_j$. Then, the replacement can then be reintroduced into (3.30) and help determine pairs μ, σ of representations.

principal embedding as removing the puncture entirely. Hence,

$$\lim_{b\to c}\left[\frac{K_{[N]}(c)}{K_{[1^N]}(b)}\right]_{\Gamma(t^\alpha z)\to\frac{\eta(\tau)}{\vartheta_1(2\alpha\mathsf{v}+\mathsf{z})}}\mathcal{I}_{[1^N]}(b)=1. \tag{3.35}$$

This illustrates the closure of a puncture.

In this way, the contribution from a puncture with a hook-type Young diagram can be read as

$$\mathcal{I}_{[N-K,1^K]}=\frac{\prod_{M=N-K+1}^{N}\vartheta_1(v^{2M})}{\eta(q)^K}\frac{\eta(q)^{2K}}{\prod_{A=1}^{K}\vartheta_1(v^{N-K+1}(r^Nc_A)^{\pm})}\times\frac{\eta(q)^{K^2}}{\prod_{A,B=1}^{K}\vartheta_1(v^2c_A/c_B)}, \tag{3.36}$$

where r and c_A represent the flavor fugacities.

In all of the above ratios of K, we make the replacement $\Gamma(t^\alpha z)\to\frac{\eta(\tau)}{\vartheta_1(2\alpha\mathsf{v}+\mathsf{z})}$ at the end of the computation. This is possible thanks to the fact that $\lim_{b\to c}K_\lambda/K_{[1^N]}$ is always a product of elliptic gamma functions. This can be seen by explicitly writing out the plethystic exponential

$$\lim_{b\to c}\frac{K_\lambda(c)}{K_{[1^N]}(b)}=\mathrm{PE}\left[\frac{t-pq}{(1-p)(1-q)}\times\sum_j\mathrm{ch}^f_{\mu_j}(c)(t^j-\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}))\right]. \tag{3.37}$$

For each j in the above sum, we write explicitly

$$(t-pq)(t^j-\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}))=-(t-pq)(t^{-j}+t^{-j+1}+\cdots+t^{j-2}+t^{j-1}). \tag{3.38}$$

Note that the adjoint representation $\mathbf{ad\,j}$ is real, and hence we have

$$\mathrm{ch}^f_{\mu_j}(c)=\mathrm{ch}^f_{\mu_j}(c^{-1}). \tag{3.39}$$

Therefore,

$$\begin{aligned}&(t-pq)(t^j-\mathrm{ch}^{\mathrm{SU}(2)}_{\sigma_j}(t^{1/2}))\mathrm{ch}^f_{\mu_j}(c)\\&=-(t^{-j+1}+t^{-j+2}+\cdots+t^{j-1}+t^j)\mathrm{ch}^f_{\mu_j}(c)\\&\quad+pq(t^{-j+1}+t^{-j+2}+\cdots+t^{j-1}+t^j)\mathrm{ch}^f_{\mu_j}(c)|_{\substack{t\to t^{-1}\\c\to c^{-1}}}.\end{aligned}$$

Here we see that the structure of $x-\frac{pq}{x}$ emerges, and the ratio of K precisely forms a product of elliptic gamma functions. At the end of the computation, the replacement $\Gamma(t^\alpha z)\to\frac{\eta(\tau)}{\vartheta_1(2\alpha\mathsf{v}+\mathsf{z})}$ should be performed to derive the contribution from a puncture to the (0,4) elliptic genus.

Our findings give rise to various intriguing questions and potential research directions. Thus, we conclude this section by highlighting a few prospective avenues for future exploration.

Simplicity of forms: The (0,4) elliptic genus manifests in surprisingly simple forms, primarily as products of theta functions. However, such simplicity is observed only in theories where the genus is greater than zero. The underlying reasons for this remain mysterious.

Nonlinear sigma model: Our investigations point toward an $\mathcal{N}=(0,4)$ nonlinear sigma model as an IR theory. The target space of this model is the moduli space of the hypermultiplet with a nontrivial left-moving bundle. Notably, the form of the (0,4) elliptic genus strongly indicates the existence of an LG dual theory for this class of theories. An immediate challenge is to identify the superpotential of the LG model, which realizes the target space of the nonlinear sigma model.

Relation with Schur indices: Prior works, such as [85–88], brought up the relationship between (0,4) elliptic genera and Schur indices. However, the findings in these works remain observational and lack a foundational understanding. Therefore, a deeper analysis of the (0,4) elliptic genus presented in this paper, in light of these observations, is a promising avenue.

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APPENDIX A: NOTATIONS AND CONVENTIONS

In this paper, the symbol q is defined as $q\coloneqq e^{2\pi i\tau}$, where τ is a complex structure of a two-torus. Throughout the paper, single symbols written in sans-serif type are used to represent chemical potentials. The fugacity z and the chemical potential z for either gauge or flavor symmetry are related by the equation $z=e^{2\pi i\mathsf{z}}$. Abusing notation, functions with fugacities and chemical potentials will be used interchangeably. For example, the following two notations represent the same theta function:

$$\vartheta_1(\mathsf{z})=\vartheta_1(z). \tag{A1}$$

The notation $f(a^\pm b^\pm)$ is a shorthand notation used to denote the multiplication of all possible combinations of signs in the arguments. It is defined as follows:

$$f(a^{\pm}b^{\pm}) \coloneqq f(ab)f(a^{-1}b)f(\mathrm{ab}^{-1})f(a^{-1}b^{-1}),$$
$$g(\pm \mathsf{a} \pm \mathsf{b}) \coloneqq g(\mathsf{a}+\mathsf{b})g(-\mathsf{a}+\mathsf{b})g(\mathsf{a}-\mathsf{b})g(-\mathsf{a}-\mathsf{b}). \quad \text{(A2)}$$

Throughout this paper, we use the following notation for q-Pochhammer symbols:

$$(z;q) \coloneqq \prod_{k=0}^{\infty}(1-zq^k). \quad \text{(A3)}$$

The elliptic gamma function is defined by

$$\Gamma(z;p,q) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z} = \mathrm{PE}\left[\frac{z-\frac{pq}{z}}{(1-q)(1-p)}\right], \quad \text{(A4)}$$

where PE represents the plethystic exponential

$$\mathrm{PE}[f(x,y,\cdots)] \equiv \exp\left[\sum_{d=1}^{\infty}\frac{1}{d}f(x^d,y^d,\cdots)\right], \quad \text{(A5)}$$

which brings the single-particle index f to the multiparticle index. We often use the shorthand notation $\Gamma(z)$ for the elliptic gamma function.

The Dedekind eta function is

$$\eta(\tau) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n), \quad \text{(A6)}$$

where $q = e^{2\pi i \tau}$ and $\mathrm{Im}\tau > 0$. Often, we also use the notation $\eta(q)$. Its modular properties are

$$\eta(\tau+1) = e^{\frac{i\pi}{12}}\eta(\tau), \qquad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau). \quad \text{(A7)}$$

1. Jacobi theta functions

The Jacobi theta functions are defined as a Fourier series,

$$\vartheta_1(\mathsf{z}|\tau) \coloneqq -i\sum_{r\in\mathbb{Z}+\frac{1}{2}}(-1)^{r-\frac{1}{2}}e^{2\pi i r\mathsf{z}}q^{\frac{r^2}{2}},$$
$$\vartheta_2(\mathsf{z}|\tau) \coloneqq \sum_{r\in\mathbb{Z}+\frac{1}{2}}e^{2\pi i r\mathsf{z}}q^{\frac{r^2}{2}},$$
$$\vartheta_3(\mathsf{z}|\tau) \coloneqq \sum_{n\in\mathbb{Z}}e^{2\pi i n\mathsf{z}}q^{\frac{n^2}{2}},$$
$$\vartheta_4(\mathsf{z}|\tau) \coloneqq \sum_{n\in\mathbb{Z}}(-1)^n e^{2\pi i n\mathsf{z}}q^{\frac{n^2}{2}},$$

where $q = e^{2\pi i\tau}$ and $z = e^{2\pi i \mathsf{z}}$. The Jacobi theta functions can be rewritten in the triple-product form

$$\vartheta_1(\mathsf{z}|\tau) = iq^{\frac{1}{8}}z^{-\frac{1}{2}}(q;q)(z;q)(z^{-1}q;q),$$
$$\vartheta_2(\mathsf{z}|\tau) = q^{\frac{1}{8}}z^{-\frac{1}{2}}(q;q)(-z;q)(-z^{-1}q;q),$$
$$\vartheta_3(\mathsf{z}|\tau) = (q;q)(-zq^{1/2};q)(-z^{-1}q^{1/2};q),$$
$$\vartheta_4(\mathsf{z}|\tau) = (q;q)(zq^{1/2};q)(z^{-1}q^{1/2};q).$$

From the Jacobi triple products, we can easily find the relation between ϑ_1 and ϑ_4 as

$$\vartheta_4(\mathsf{z}|\tau) = -iq^{\frac{1}{8}}z^{\frac{1}{2}}\vartheta_1\left(\mathsf{z}+\frac{\tau}{2}|\tau\right). \quad \text{(A8)}$$

We also use the notation $\vartheta_i(z,q)$. In either notation, the q and τ are often omitted, and we simply write $\vartheta_i(z)$ or $\vartheta_i(\mathsf{z})$.

Let us spell out some properties of the function $\vartheta_1(\mathsf{z}|\tau)$ we use in the main text. Under shifts of z, we have

$$\vartheta_1(\mathsf{z}+a+b\tau|\tau) = (-1)^{a+b}e^{-2\pi i b\mathsf{z}-i\pi b^2\tau}\vartheta_1(\mathsf{z}|\tau) \quad \text{(A9)}$$

for $a,b\in\mathbb{Z}$. Furthermore, ϑ_1 is odd with respect to z, while the others are even,

$$\vartheta_1(-\mathsf{z}|\tau) = -\vartheta_1(\mathsf{z}|\tau), \qquad \vartheta_{i=2,3,4}(-\mathsf{z}|\tau) = \vartheta_i(\mathsf{z}|\tau).$$

The function $\vartheta_1(\mathsf{z}|\tau)$ has simple zeros in z at $\mathsf{z} = \mathbb{Z}+\tau\mathbb{Z}$, and no poles. When computing JK residues, it is notable that the derivative of z at 0 relates to $\eta(\tau)$ as follows:

$$\vartheta_1'(0|\tau) = 2\pi\eta(q)^3.$$

From this relationship, we deduce a pole at $z=0$ as

$$\frac{1}{\vartheta_1(\mathsf{z})} = \frac{1}{2\pi\eta(\tau)^3}\frac{1}{z} + \mathcal{O}(\mathsf{z}), \quad \text{(A10)}$$

from which one easily extracts residues of ratios of Jacobi theta functions.

Under the modular transformation $\tau \xrightarrow{T} \tau+1$, $(\mathsf{z},\tau) \xrightarrow{S} (\frac{\mathsf{z}}{\tau}, -\frac{1}{\tau})$, the Jacobi theta function ϑ_1 transforms as

$$\vartheta_1(\mathsf{z}|\tau+1) = e^{\frac{\pi i}{4}}\vartheta_1(\mathsf{z}|\tau),$$
$$\vartheta_1\left(\frac{\mathsf{z}}{\tau}|-\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\,e^{\pi i\mathsf{z}^2/\tau}\vartheta_1(\mathsf{z}|\tau).$$

2. Eisenstein series

The twisted Eisenstein series, denoted by $E_k\left[\begin{smallmatrix}\phi\\ \theta\end{smallmatrix}\right]$ with characteristics $\left[\begin{smallmatrix}\phi\\ \theta\end{smallmatrix}\right]$, are defined as a series in q,

$$E_{k\geq 1}\begin{bmatrix}\phi\\ \theta\end{bmatrix} \coloneqq -\frac{B_k(\lambda)}{k!}+\frac{1}{(k-1)!}\sum_{r\geq 0}{}' \frac{(r+\lambda)^{k-1}\theta^{-1}q^{r+\lambda}}{1-\theta^{-1}q^{r+\lambda}} + \frac{(-1)^k}{(k-1)!}\sum_{r\geq 1}\frac{(r-\lambda)^{k-1}\theta q^{r-\lambda}}{1-\theta q^{r-\lambda}}.$$

Here $\phi \equiv e^{2\pi i\lambda}$ determines $0\leq\lambda<1$. $B_k(x)$ represents the kth Bernoulli polynomial. When $\phi=\theta=1$, the prime in the sum indicates that the $r=0$ term is omitted.

Additionally, we define

$$E_0\begin{bmatrix}\phi\\ \theta\end{bmatrix} = -1.$$

The standard, or untwisted, Eisenstein series E_{2n} is obtained from the $\theta,\phi\to 1$ limit of $E_{2n}[{}^{\phi}_{\theta}]$,

$$E_{2n}(\tau) = E_{2n}\begin{bmatrix}+1\\ +1\end{bmatrix}.$$

Contrarily, taking the limit $\theta,\phi\to 1$ for odd k results in 0, with the exception of $E_1[{}^{\phi}_{\theta}]$, which is singular.

The Eisenstein series with $\phi=\pm 1$ enjoy the useful symmetry property

$$E_k\begin{bmatrix}\pm 1\\ z^{-1}\end{bmatrix} = (-1)^k E_k\begin{bmatrix}\pm 1\\ z\end{bmatrix}.$$

For instance, under transformations $z\to qz$ or $z\to q^{\frac{1}{2}}z$, the twisted Eisenstein series intermix with those of lower weight:

$$E_n\begin{bmatrix}\pm 1\\ zq^{\frac{k}{2}}\end{bmatrix} = \sum_{\ell=0}^{n}\left(\frac{k}{2}\right)^{\ell}\frac{1}{\ell!}E_{n-\ell}\begin{bmatrix}(-1)^k(\pm 1)\\ z\end{bmatrix}.$$

Similarly, for the modular S transformation, an inhomogeneous behavior is observed. For instance,

$$E_n\begin{bmatrix}+1\\ +z\end{bmatrix} \xrightarrow{S} \left(\frac{1}{2\pi i}\right)^n\left[\left(\sum_{k\geq 0}\frac{1}{k!}(-\log z)^k y^k\right) \times\left(\sum_{\ell\geq 0}(\log q)^{\ell}y^{\ell}E_{\ell}\begin{bmatrix}+1\\ z\end{bmatrix}\right)\right]_n, \tag{A11}$$

where $[\cdots]_n$ implies taking the coefficient of y^n.

APPENDIX B: JK RESIDUE INTEGRALS

In this appendix, we provide a detailed overview of the JK-residue computation [38–41] related to the elliptic genus addressed in the main text. Since the elliptic genera receive contributions from both nondegenerate and degenerate poles in general, a thorough review of the JK-residue integral definition is beneficial for the paper to be self-contained.

In the context of a rank-r gauge theory, the elliptic genus computed through the JK-residue technique integrates an r-form over specific cycles, and is conventionally represented as

$$\mathcal{I}^{\mathrm{2D}} = \oint_{\mathrm{JK}}\prod_{i=1}^{r}\frac{da_i}{2\pi i a_i}\mathcal{Z}(\boldsymbol{a}) = \oint_{\text{special cycles}}\mathcal{Z}(\boldsymbol{a})d\mathsf{a}_1\wedge\ldots\wedge d\mathsf{a}_r. \tag{B1}$$

As stated in the main text, $a_i = e^{2\pi i\mathsf{a}_i}$, and similarly for other variables except for $q=e^{2\pi i\tau}$. For our purpose, the integrand $\mathcal{Z}$, as a function of a_i, is separately elliptic in each a_i, namely,

$$\mathcal{Z}(\ldots,\mathsf{a}_i+\tau,\ldots) = \mathcal{Z}(\ldots,\mathsf{a}_i,\ldots), \mathcal{Z}(\ldots,\mathsf{a}_i+1,\ldots) = \mathcal{Z}(\ldots,\mathsf{a}_i,\ldots). \tag{B2}$$

More concretely, $\mathcal{Z}$ takes the form of certain ratios of the Jacobi theta functions ϑ_1, and poles come from the zeros of ϑ_1 in the denominator. Each pole is given as a solution to a set of pole equations,

$$\sum_{i=1}^{r}Q_a^i\mathsf{a}_i+\mathsf{b}_a = m_a+n_a\tau,\qquad Q_a^i\in\mathbb{Z},\qquad m_a, n_a\in\mathbb{N},\quad a=1,2,\ldots,r, \tag{B3}$$

coming from some factors $\vartheta_1(Q_a^i\mathsf{a}_i+\mathsf{b}_a)^{N_a}$ in the denominator of $\mathcal{Z}$. Note that m_a, n_a only take values in a finite range in $\mathbb{N}$ that will be determined by the charge vectors $Q_a=(Q_a^1,Q_a^2,\ldots,Q_r^r)$. A few remarks follow.

1. (1) Zeros from numerators may arise in certain solutions of the pole equations, reducing the pole's order. If the total order of the pole is below r, it is not included in the JK residue.
2. (2) At some poles a_*, there may be $n>r$ factors of ϑ_1^N's simultaneously made zero by a_*, associated with n different charge vectors $Q_1,\ldots,Q_n$. This is referred to as a degenerate pole.
3. (3) A pole associated with precisely r different ϑ_1^N factors and therefore r different charge vectors $Q_1,\ldots,Q_r$ is referred to as a nondegenerate pole.
4. (4) The range of m_a, n_a is not unique. We start by rearranging the Q_a terms such that $Q_i^i\neq 0$. Then, m_a, n_a are defined by methods like the Hermite or Smith normal form decomposition of the (reordered) integral square matrix $(Q)_{ai}\coloneqq Q_a^i$. In the Smith decomposition,

$$UQV = D,$$
$$U, V \text{ are integral and } |\det U| = |\det V| = 1,$$
$$D \text{ is diagonal.}$$

Then, we fix the range $m_a, n_a, = 0, 1, \ldots, D_{aa} - 1$. Alternatively, in Hermite decomposition, $UQ = T$ with a unimodular integral U, and T is an upper triangular integral matrix. In this case, $m_a, n_a = 0, 1, \ldots, T_{aa} - 1$. Note that although $T_{aa} \neq D_{aa}$ in general, the final result of the JK-residue computation will be the same.

For any pole a_* satisfying (B3), we need to compute the corresponding JK residue. We follow the constructive definition of the JK residue [38]. To begin, one picks a generic reference vector $\eta \in \mathfrak{h}^*$ of the gauge group. If a_* is a nondegenerate pole with charge vectors $Q_1, \ldots, Q_r$, its contribution is given by

$$\underset{\mathsf{a}_*}{\text{JK-Res}}(\eta)\mathcal{Z} = \delta(Q, \eta)\frac{1}{|\det Q|}\text{Res}_{\epsilon_r=0}\cdots \text{Res}_{\epsilon_1=0}\mathcal{Z}\Big|_{Q_a\mathsf{a}+\mathsf{b}_a=m_a+n_a\tau+\epsilon_a}, \tag{B4}$$

where $\delta(Q, \eta)$ equals one when η is inside the cone spanned by $Q_1, \ldots, Q_r$, and zero otherwise. The residues are calculated in sequence.

For a degenerate pole, we identify an associated set of charge vectors, $Q_* = \{Q_1, \ldots, Q_n\}$, with $n > r$. From the set Q_*, a collection of geometric objects can be defined.

(1) Given any r sequence of linearly independent charge vectors $(Q_{a_1}, \ldots, Q_{a_r})$ from Q_*, we can construct a flag F. This flag is essentially a series of nested subspaces of $\mathbb{R}^r$:

$$\{0\} \subset F_1 \subset \ldots \subset F_r = \mathbb{R}^r,$$
$$F_\ell = \text{span}\{Q_{a_1}, \ldots, Q_{a_\ell}\}. \tag{B5}$$

Note that different sequences may give rise to the same flag. When this happens, we only consider one of them. The sequence $(Q_{a_1}, \ldots, Q_{a_r})$ is often called a basis $\mathcal{B}(F, Q_*)$ of F in Q_*. Given an F, the basis in Q_* is generally not unique, but we pick an arbitrary one.

(2) From each flag F and its basis $\mathcal{B}(F, Q_*)$, one constructs a sequence of vectors

$$\kappa(F, Q_*) \coloneqq (\kappa_1, \ldots, \kappa_r), \qquad \kappa_a = \sum_{\substack{Q\in Q_*\\ Q\in F_a}} Q. \tag{B6}$$

One further defines $\text{sign}F \coloneqq \text{sign}\det \kappa(F, Q_*)$.

(3) For each $\kappa(F)$, one constructs a closed-cone $\mathsf{c}(F, Q_*)$ spanned by $\kappa(F, Q_*)$.

With these objects defined, the JK residue of the given degenerate pole a_* is given by

$$\underset{\mathsf{a}_*}{\text{JK-Res}}(\eta)\mathcal{Z} = \sum_F \delta(F, \eta)\frac{\text{sign}F}{\det \mathcal{B}(F, Q_*)}\text{Res}_{\epsilon_r=0}\cdots \times \text{Res}_{\epsilon_1=0}\mathcal{Z}\Big|_{\substack{Q_{a_1}\mathsf{a}+\mathsf{b}_{a_1}=m_{a_1}+n_{a_1}\tau+\epsilon_1\\ \cdots\\ Q_{a_r}\mathsf{a}+\mathsf{b}_{a_r}=m_{a_r}+n_{a_r}\tau+\epsilon_r}}, \tag{B7}$$

where the sum is over all flags constructed out of Q_* associated with a_*. Again, $\delta(F, \eta)$ equals one if the closed-cone $\mathsf{c}(F, Q_*)$ contains η, and zero otherwise. This definition of JK-Res naturally extends to nondegenerate poles, where there are precisely r vectors in Q_*, and

$$\kappa(F, Q_*) = \mathcal{B}(F, Q_*), \qquad \frac{\text{sign}F}{\det \mathcal{B}(F, Q_*)} = \frac{1}{|\det \mathcal{B}(F, Q_*)|}. \tag{B8}$$

The result clearly reduces to the previous definition of the JK residue for the nondegenerate case. Finally, given a generic η,

$$\int_{\text{JK}} \prod_{i=1}^r \frac{da_i}{2\pi i a_i}\mathcal{Z}(\boldsymbol{a}) = \sum_{\mathsf{a}_*} \underset{\mathsf{a}_*}{\text{JK-Res}}(\eta)\mathcal{Z}(\boldsymbol{a}). \tag{B9}$$

Although the structure of poles and the results of individual JK residues often differ drastically when η varies across chambers, the overall result is independent of the choice of η.

In the following, we apply the JK-residue prescription to a number of quiver gauge theories discussed in the main text, presenting details of the computations. We first focus on cases with SU(2) and U(1) gauge groups, followed by those with SU(N) and U(1) gauge groups.

APPENDIX C: JK RESIDUES OF $\mathcal{N} = (0,2)$ ELLIPTIC GENERA

In this appendix, we provide detailed computations of JK-residue integrals for elliptic genera of 2D $\mathcal{N} = (0, 2)$ quiver gauge theories, complementing the main text. To elucidate the JK-residue computations, here we use notations based on chemical potentials instead of fugacities.

1. $g = 1$, $n = 2$

Given genus $g = 1$ and the number of punctures $n = 2$, one can write down different quiver gauge theories with different gauge groups.

a. SU(2)^2 gauge theory

The first theory is an SU(2)^2 gauge theory coupled to two bifundamentals,

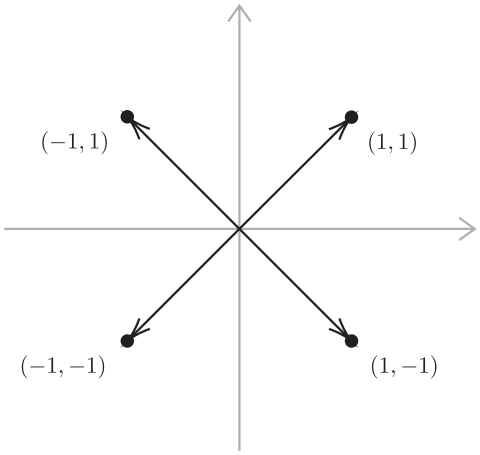

FIG. 21. Charge vectors of the genus-one theory as an $SU(2)^2$ gauge theory.

	$SU(2)_1 \times SU(2)_2$
ϕ_1	$(\mathbf{2},\mathbf{2})$
ϕ_2	$(\mathbf{2},\mathbf{2})$

The quiver is shown in Fig. 6. The elliptic genus of this theory is computed by the JK-residue computation of the integral

$$\mathcal{I}_{1,2} = \int_{\mathrm{JK}} \frac{da_1}{2\pi i a_1}\frac{da_2}{2\pi i a_2}\frac{\eta(\tau)^8}{4}\frac{\prod_{i=1}^2 \vartheta_1(\pm 2\mathsf{a}_i)}{\prod_{i=1}^2 \vartheta_4(\pm\mathsf{a}_1 \pm \mathsf{a}_2 + \mathsf{c}_i)}. \tag{C1}$$

Here $\mathsf{c}_{1,2}$ represent the flavor $U(1)\times U(1)$ fugacities. The charge covectors are shown in Fig. 21. Various reference vectors η can be chosen, all of which yield the same result. For example, picking $\eta = (1,0)$ picks out one cone in $\mathbb{R}^2$ spanned by the charge vectors (1,1) and $(1,-1)$. The corresponding poles are given by the set of equations

$$\begin{aligned} \mathsf{a}_1 - \mathsf{a}_2 + \mathsf{c}_1 + \frac{\tau}{2} &= m_1 + n_1\tau, \\ \mathsf{a}_1 + \mathsf{a}_2 + \mathsf{c}_2 + \frac{\tau}{2} &= m_1' + n_1'\tau \end{aligned} \tag{C2}$$

and

$$\begin{aligned} \mathsf{a}_1 - \mathsf{a}_2 + \mathsf{c}_2 + \frac{\tau}{2} &= m_2 + n_2\tau, \\ \mathsf{a}_1 + \mathsf{a}_2 + \mathsf{c}_1 + \frac{\tau}{2} &= m_2' + n_2'\tau, \end{aligned} \tag{C3}$$

where $m_i, n_i = 0$, and $m_i, n_i = 0, 1$. There are in total eight poles, all of which contribute $-\frac{1}{8}\frac{\eta(\tau)^2}{\vartheta_1(\mathsf{c}_1)\vartheta_1(\mathsf{c}_2)}$, and therefore

$$\mathcal{I}_{1,2} = -\frac{\eta(\tau)^2}{\prod_{i=1}^2 \vartheta_1(2\mathsf{c}_i)}. \tag{C4}$$

b. First $SU(2)\times U(1)$ gauge theory

Additionally, there are two other quiver theories as $SU(2)\times U(1)$ gauge theories that correspond to the genus-one Riemann surface with two punctures, with the quiver diagrams on the left side of Fig. 7. The first such gauge theory has an elliptic genus described by

$$\mathcal{I}'_{1,2} = \int_{\mathrm{JK}} \frac{da_1}{2\pi i a_1}\frac{da_2}{2\pi i a_2}\frac{\eta(\tau)^8}{2} \times \frac{\vartheta_1(\pm 2\mathsf{a}_1)\vartheta_4(\pm 2\mathsf{a}_2)}{\vartheta_4(\mathsf{a}_1 \pm \mathsf{a}_2 \pm \mathsf{b}_1 + \mathsf{d}_1)\vartheta_4(-\mathsf{a}_1 \pm \mathsf{a}_2 \pm \mathsf{b}_2 + \mathsf{d}_1)}.$$

Here we turn on the U(1) flavor symmetry that rotates the chiral multiplets from the two $U_2^{(0,2)}$ with the same phase with fugacity d_1. One can also check that the integrand remains separately elliptic with respect to the variables $\mathsf{a}_{1,2}$. The charge vectors are still

$$(1,1), \qquad (-1,1), \qquad (1,-1), \qquad (-1,-1). \tag{C5}$$

One can pick any η inside the four quadrants, and the JK-residue computation yields the same result,

$$\mathcal{I}'_{1,2} = \frac{2\eta(\tau)^2\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(2\mathsf{d}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2)}. \tag{C6}$$

c. Second $SU(2)\times U(1)$ gauge theory

A distinct $SU(2)\times U(1)$ gauge theory is depicted by the quiver diagram on the right side of Fig. 7. The elliptic genus can be computed as the JK residue of the integrand

$$\mathcal{Z}''_{1,2} = \frac{\eta(\tau)^{10}\vartheta_4(\pm 2\mathsf{a}_1)\vartheta_1(\pm 2\mathsf{a}_2)}{4\vartheta_4(\mathsf{a}_1+\mathsf{d}_1)^2\prod_{\pm}\vartheta_4(\mathsf{a}_1 \pm 2\mathsf{a}_2 + \mathsf{d}_1)\vartheta_4(-\mathsf{a}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2 + \mathsf{d}_1)}.$$

From the denominator, we can deduce the charge vectors,

$$(-1,0), \qquad (1,-2), \qquad (1,0), \qquad (1,2). \tag{C7}$$

See Fig. 22. Clearly, there are several choices for η. Let us start with $\eta = (-1,-1)$. In this case, only the cones spanned by $(-1,0)$ and $(1,-2)$ contribute, corresponding to the poles from the equation (with four different choices of the signs $\pm$)

$$\begin{aligned} -\mathsf{a}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_1 + n_1\tau \\ \mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_2 + n_2\tau, \end{aligned} \tag{C8}$$

where m_1, $n_1 = 0$, $m_2, n_2 = 0$, 1. In total, there are 4×4 different poles. For example, poles having $-\mathsf{a}_1 - \mathsf{b}_1 - \mathsf{b}_2 + \mathsf{d}_1 + \frac{\tau}{2} = 0$ contribute

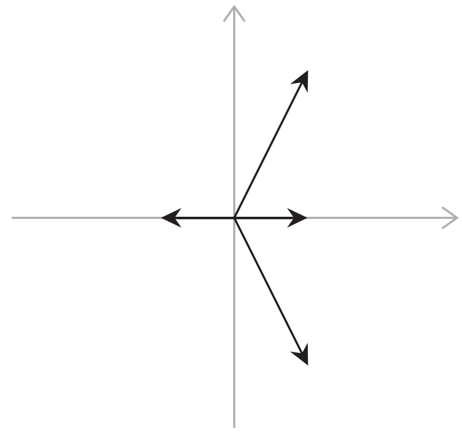

FIG. 22. Charge vectors of the JK residues for the second $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge theory.

$$\frac{\eta(\tau)^4 \vartheta_4(-2\mathsf{b}_1 - 2\mathsf{b}_2 + 2\mathsf{d}_1)^2}{\vartheta_1(2\mathsf{b}_1 + 2\mathsf{b}_2)\vartheta_1(2\mathsf{b}_1 + 2\mathsf{b}_2)\vartheta_1(-2\mathsf{b}_1 - 2\mathsf{b}_2 + 4\mathsf{d}_1)\prod_{i=1}^2 \vartheta_1(2\mathsf{b}_i)}.$$

To summarize, the elliptic genus reads

$$\mathcal{I}''_{1,2} = \frac{\eta(\tau)^2}{\prod_{i=1}^2 \vartheta_1(2\mathsf{b}_i)} \sum_{\alpha,\beta=\pm} \frac{\alpha\beta\vartheta_4(2\alpha\mathsf{b}_1 + 2\beta\mathsf{b}_2 + 2\mathsf{d}_1)}{\vartheta_1(2\alpha\mathsf{b}_1 + 2\beta\mathsf{b}_2)\vartheta_1(2\alpha\mathsf{b}_1 + 2\beta\mathsf{b}_2 + 4\mathsf{d}_1)}. \tag{C9}$$

Alternatively, one can also choose $\eta = (1,1)$. In this case, the relevant cones are spanned by the charge vectors

$$(1,2) \text{ and } (1,0), \qquad (1,2) \text{ and } (1,-2). \tag{C10}$$

The corresponding poles are from the equations

$$\begin{aligned} \mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_1 + n_1\tau, \\ \mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_2 + n_2\tau, \end{aligned} \tag{C11}$$

with m_1, $n_1 = 0$, $m_2, n_2 = 0$, 1, 2, 3, and the equations

$$\begin{aligned} \mathsf{a}_1 + \mathsf{d}_1 + \frac{\tau}{2} &= m_1 + n_1\tau, \\ \mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} &= m_2 + n_2\tau, \end{aligned} \tag{C12}$$

with m_1, $n_1 = 0$, $m_2, n_2 = 0$, 1. The JK-residue computation is, however, more subtle in this setup, due to the presence of degenerate poles

$$(\mathsf{a}_1, \mathsf{a}_2) = \left(-\mathsf{d}_1 - \frac{\tau}{2}, \frac{m + n\tau}{2}\right), \qquad m, n = 0, 1. \tag{C13}$$

Note also that these degenerate poles are precisely the common solutions to (C11) and (C12) (up to a shift of a_1 by full periods $1, 1+\tau, \tau$). At these poles, the factors

$$\vartheta_4(\mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1), \quad \vartheta_4(\mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1), \quad \vartheta_4(\mathsf{a}_1 + \mathsf{d}_1)^2 \tag{C14}$$

simultaneously vanish. Therefore, there are 12 nondegenerate poles and four degenerate poles that contribute to the elliptic genus. The former is straightforward to compute. For example,

$$\mathrm{JK}_{\substack{\mathsf{a}_1 - 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} = 0 \\ \mathsf{a}_1 + 2\mathsf{a}_2 + \mathsf{d}_1 + \frac{\tau}{2} = \tau}} \mathcal{Z} = -\frac{\vartheta_4(2\mathsf{d}_1)^2}{16\vartheta_4(\pm\mathsf{b}_1 \pm \mathsf{b}_2 + 2\mathsf{d}_1)}. \tag{C15}$$

For the degenerate poles, we follow [38]. The relevant charge vectors can be grouped into

$$Q_* = \{(1,-2), (1,2), (1,0)\} \tag{C16}$$

and that gives rise to three flags $F_{1,-2}, F_{1,2}, F_{1,0}$ led by the three vectors in Q_*. The corresponding $\kappa(F)$ and other relevant information is collected in the following table.

F	$F_{1,-2}$	$F_{1,2}$	$F_{1,0}$
$\kappa(F)$	$((1,-2),(3,0))$	$((1,2),(3,0))$	$((1,0),(3,0))$
$\operatorname{sign}\det(\kappa(F))$	1	-1	0
$\eta \in \mathsf{c}(F, Q_*)$	False	True	False

From the table, we can compute the contribution from the degenerate poles,

$$\frac{\eta(\tau)^2 \vartheta_4(2\mathsf{d}_1)^2}{4\vartheta_1(\pm\mathsf{b}_1 \pm \mathsf{b}_2 + 2\mathsf{d}_1)}. \tag{C17}$$

In the end, the elliptic genus computed using $\eta = (1,1)$ is

$$\mathcal{I}''_{1,2} = \frac{\eta(\tau)^4}{2}\vartheta_4(2\mathsf{d}_1)^2 \sum_{i=1}^{4}(-1)^i \frac{1}{\vartheta_i(2\mathsf{d}_1 \pm \mathsf{b}_1 \pm \mathsf{b}_2)}. \tag{C18}$$

Although they look different, the elliptic genera (C9) and (C18) are actually identical and are equal to

$$\mathcal{I}''_{1,2} = -\eta(\tau)^2\vartheta_4(2\mathsf{d}_1)^2 \frac{\vartheta_1(8\mathsf{d}_1)}{\vartheta_1(4\mathsf{d}_1)} \prod_{i=1}^{2}\frac{\vartheta_1(4\mathsf{b}_i)}{\vartheta_1(2\mathsf{b}_i)} \times \frac{1}{\vartheta_1(4\mathsf{d}_1 \pm 2\mathsf{b}_1 \pm 2\mathsf{b}_2)}. \tag{C19}$$

The equality of the expressions can be checked by the power expansion in q explicitly; for instance, the leading q term of reads

$$\mathcal{I}''_{1,2} = \frac{b_1^3(b_1^2+1)b_2^3(b_2^2+1)d_1^4(d_1^2-1)^2(d_1^4+1)}{(b_1^2b_2^2-d_1^4)(b_1^2d_1^4-b_2^2)(b_1^2-b_2^2d_1^4)(b_1^2b_2^2d_1^4-1)} q^{-\frac{1}{6}} + \cdots. \tag{C20}$$

2. $g=1$, $n=3$

There are several different 2D $\mathcal{N}=(0,2)$ theories corresponding to the genus-one Riemann surfaces $C_{1,3}$ with three punctures.

a. SU(2)^3 gauge theory

Let us first consider the $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge theory shown in Fig. 8. The elliptic genus is the JK residue of the integrand

$$\mathcal{Z}_{1,3} = -\frac{\eta(\tau)^{12}\vartheta_1(\pm 2\mathsf{a}_1)\vartheta_1(\pm 2\mathsf{a}_2)\vartheta_1(\pm 2\mathsf{a}_3)}{\prod_{A<B}\vartheta_4(\pm\mathsf{a}_A \pm \mathsf{a}_B + \mathsf{c}_1)}. \tag{C21}$$

We can choose $\eta = (1, 1+\frac{1}{1000}, 1+\frac{1}{2000})$. There is no degenerate pole, and the elliptic genus is given by

$$\mathcal{I}_{1,3} = \frac{\eta(\tau)^3}{\prod_{i=1}^{3}\vartheta_1(2\mathsf{c}_i)}. \tag{C22}$$

b. First SU(2)^2 × U(1) theory

Let us now consider an $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ gauge theory illustrated on the left side of Fig. 9, whose elliptic genus is the JK residue of the integrand

$$\mathcal{Z}'_{1,3} = 2\frac{\eta(\tau)^{12}\vartheta_4(\pm 2\mathsf{a}_1)\vartheta_1(\pm 2\mathsf{a}_2)\vartheta_1(\pm 2\mathsf{a}_3)}{4\vartheta_4(\pm\mathsf{a}_2 \pm \mathsf{a}_3 + \mathsf{c}_1)\vartheta_4(\mathsf{a}_1 \pm \mathsf{a}_2 \pm \mathsf{b}_2 + \mathsf{d}_1)\vartheta_4(-\mathsf{a}_1 \pm \mathsf{a}_3 \pm \mathsf{b}_3 + \mathsf{d}_1)}. \tag{C23}$$

An arbitrary reference vector can be chosen, such as $\eta = (\frac{999}{1000}, \frac{1999}{2000}, \frac{2999}{3000})$ which leads to 40 nondegenerate poles. The JK residue is straightforward, yielding

$$\frac{\mathcal{I}'_{1,3}}{\eta(\tau)^3} = -\frac{1}{2\vartheta_1(2\mathsf{c}_1)\prod_{i=2}^{3}\vartheta_1(2\mathsf{b}_i)}\sum_{\alpha,\beta=\pm}\frac{\alpha\beta\vartheta_1(\alpha\mathsf{b}_2+\beta\mathsf{b}_3+\mathsf{c}_1)^2}{\vartheta_4(\alpha\mathsf{b}_2+\beta\mathsf{b}_3+\mathsf{c}_1\pm 2\mathsf{d}_1)} + \frac{\vartheta_4(2\mathsf{d}_1)^2}{2\vartheta_1(2\mathsf{b}_2)\vartheta_1(2\mathsf{c}_1)}\sum_{\alpha=\pm}\frac{\alpha\vartheta_1(2\alpha\mathsf{b}_2+2\mathsf{c}_1)}{\vartheta_4(\alpha\mathsf{b}_2+\mathsf{c}_1\pm\mathsf{b}_3\pm 2\mathsf{d}_1)} + \frac{\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(2\mathsf{b}_3)\vartheta_1(4\mathsf{d}_1)}\sum_{\alpha=\pm}\frac{\alpha\vartheta_1(2\alpha\mathsf{b}_3+4\mathsf{d}_1)}{\vartheta_4(\alpha\mathsf{b}_3+2\mathsf{d}_1\pm\mathsf{b}_2\pm\mathsf{c}_1)}. \tag{C24}$$

It is straightforward to check that this complicated expression is actually equal to

$$\frac{\mathcal{I}'_{1,3}}{\eta(\tau)^3} = 2\frac{\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(4\mathsf{d}_1)}\frac{\vartheta_1(2\mathsf{c}_1+4\mathsf{d}_1)}{\vartheta_1(2\mathsf{c}_1)}\frac{1}{\vartheta_4(\mathsf{c}_1+2\mathsf{d}_1\pm\mathsf{b}_2\pm\mathsf{b}_3)}, \tag{C25}$$

which signals an LG description of the quiver gauge theory.

c. Second $\mathrm{SU}(2)^2\times\mathrm{U}(1)$ theory

Another quiver gauge theory with the gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$ is depicted on the right side of Fig. 9. Its elliptic genus is determined by the JK residue of the integrand

$$\mathcal{Z} = \mathcal{I}^{(0,2)}_{U_2}(\mathsf{c}_1,\mathsf{b}_1,\mathsf{a}_3)\mathcal{I}^{(0,2)}_{U_2}(\mathsf{a}_2+\mathsf{d}_1,\mathsf{b}_2,-\mathsf{a}_3) \times\mathcal{I}^{(0,2)}_{U_2}(-\mathsf{a}_2+\mathsf{d}_1,\mathsf{a}_1,-\mathsf{a}_1)\vartheta_4(\pm\mathsf{a}_2)\prod_{i=\{1,3\}}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_i).$$

The six charge vectors are given by

$$(\pm1,0,0),\qquad(\pm1,0,1),\qquad(0,\pm2,-1). \tag{C26}$$

There are various choices for η. For example, let us begin with

$$\eta=\left(\frac{999}{1000},\frac{1999}{2000},\frac{2999}{3000}\right). \tag{C27}$$

With this choice, there are 64 nondegenerate poles, giving the elliptic genus

$$\frac{\mathcal{I}''_{0,3}}{\eta(\tau)^3} = \frac{1}{\vartheta_1(2\mathsf{c}_1)\prod_{i=1}^2\vartheta_1(2\mathsf{b}_i)}\sum_{\alpha\beta=\pm}\frac{\alpha\beta\vartheta_4(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1+2\mathsf{d}_1)^2}{\vartheta_1(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1)\vartheta_1(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1+4\mathsf{d}_1)} + \frac{2\vartheta_4(2\mathsf{d}_1)^2}{\vartheta_1(4\mathsf{d}_1)}\sum_{i=1}^4\frac{(-1)^i}{2\vartheta_i(\pm\mathsf{b}_1\pm\mathsf{b}_2+\mathsf{c}_1)}. \tag{C28}$$

For instance, another intriguing choice for η is

$$\eta=\left(-\frac{999}{1000},\frac{1}{1000},1\right). \tag{C29}$$

With this choice of η, there are 56 nondegenerate poles and eight degenerate poles. The latter poles are given by

$$\begin{aligned} &\mathsf{a}_1=-\frac{m+n\tau}{2},\qquad \mathsf{a}_2=\mathsf{d}_1+\frac{\tau}{2},\qquad \mathsf{a}_3=+\mathsf{b}_2-2\mathsf{d}_1-\tau,\qquad m,n=0,1\\ \text{or,}\quad&\mathsf{a}_1=-\frac{m+n\tau}{2},\qquad \mathsf{a}_2=\mathsf{d}_1+\frac{\tau}{2},\qquad \mathsf{a}_3=-\mathsf{b}_2-2\mathsf{d}_1-\tau,\qquad m,n=0,1. \end{aligned} \tag{C30}$$

The elliptic genus reads

$$4\frac{\vartheta_1(2\mathsf{b}_2)\vartheta_1(4\mathsf{d}_1)}{\eta(\tau)^3\vartheta_4(2\mathsf{d}_1)^2}\mathcal{I}''_{0,3} = \sum_{i=1}^4\sum_{\alpha=\pm}\frac{(-1)^i\alpha\vartheta_1(2\alpha\mathsf{b}_2+4\mathsf{d}_1)}{\vartheta_i(\pm\mathsf{b}_1+\alpha\mathsf{b}_2\pm\mathsf{c}_1+2\mathsf{d}_1)} + \sum_{\alpha,\beta,\gamma=\pm}\frac{2\alpha\beta\vartheta_1(4\mathsf{d}_1)}{\vartheta_1(2\mathsf{b}_1)\vartheta_1(2\mathsf{c}_1)\vartheta_4(2\mathsf{d}_1)^2}\frac{\vartheta_4(2\alpha\mathsf{b}_1+2\beta\mathsf{b}_2+2\mathsf{c}_1+2\gamma\mathsf{d}_1)^2}{\vartheta_1(2(\alpha\mathsf{b}_1+\beta\mathsf{b}_2+\mathsf{c}_1+2\gamma\mathsf{d}_1))\vartheta_1(2(\alpha\mathsf{b}_1+\beta\mathsf{b}_2+\mathsf{c}_1))}. \tag{C31}$$

Although the two expressions for $\mathcal{I}''_{1,3}$ look different, they are both equal to the simple ratio

$$\frac{\mathcal{I}''_{1,3}}{\eta(\tau)^3} = \frac{\vartheta_4(2\mathsf{d}_1)^2\vartheta_1(4\mathsf{c}_1+8\mathsf{d}_1)}{\vartheta_1(4\mathsf{d}_1)\vartheta_1(2\mathsf{c}_1)}\frac{1}{\vartheta_1(\pm2\mathsf{b}_1\pm2\mathsf{b}_2+2\mathsf{c}_1+4\mathsf{d}_1)}\prod_{i=1}^2\frac{\vartheta_1(4\mathsf{b}_i)}{\vartheta_1(2\mathsf{b}_i)}, \tag{C32}$$

suggesting an LG description of the theory.

3. $g=2$, $n=0$

There are two possible quiver gauge theories with $\mathrm{SU}(2)^2\times \mathrm{U}(1)$ gauge groups associated with the genus-two Riemann surface with no puncture (see Fig. 10).

a. First $\mathrm{SU}(2)^2\times \mathrm{U}(1)$ theory

The theory on the left side of Fig. 10 has an elliptic genus given by the JK residue of the integrand

$$\begin{aligned}\mathcal{Z}_{2,0} &= 2\mathcal{I}^{(0,2)}_{U_2}(\mathsf{a}_3+\mathsf{d}_1,\mathsf{a}_1,\mathsf{a}_2)\\ &\quad\times \mathcal{I}^{(0,2)}_{U_2}(-\mathsf{a}_3+\mathsf{d}_1,-\mathsf{a}_1,-\mathsf{a}_2)\\ &\quad\times \vartheta_4(\pm 2\mathsf{a}_3)\prod_{i=1}^{2}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_i). \end{aligned}\tag{C33}$$

The eight charge vectors are given by $(\pm1,\pm1,\pm1)$. There are many essentially equivalent choices of η. For example, we pick

$$\eta=\left(1,\frac{999}{1000},\frac{4999}{5000}\right).\tag{C34}$$

With this choice of η, there are 32 nondegenerate poles. All of them contribute identically to the total JK residue. Finally, the elliptic genus reads

$$\mathcal{I}_{2,0}=-\frac{2\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)}.\tag{C35}$$

b. Second $\mathrm{SU}(2)^2\times \mathrm{U}(1)$ theory

The theory on the right side of Fig. 10 has an elliptic genus integrand

$$\begin{aligned}\mathcal{Z}'_{2,0} &= 2\mathcal{I}^{(0,2)}_{U_2}(\mathsf{a}_3+\mathsf{d}_1,\mathsf{a}_1,-\mathsf{a}_1)\\ &\quad\times \mathcal{I}^{(0,2)}_{U_2}(-\mathsf{a}_3+\mathsf{d}_1,-\mathsf{a}_2,-\mathsf{a}_2)\\ &\quad\times \vartheta_4(\pm 2\mathsf{a}_3)\prod_{i=1}^{2}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_i). \end{aligned}\tag{C36}$$

The charge vectors are given by

$$(\pm2,0,1),\qquad (0,\pm2,-1),\qquad (0,0,\pm1).\tag{C37}$$

Let us consider the choice of η

$$\eta=\left(\frac{1001}{1000},\frac{501}{500},\frac{1003}{1000}\right).\tag{C38}$$

With this choice, there are 48 nondegenerate poles and 16 degenerate poles. The latter take the form

$$\mathsf{a}_1=\frac{m_1+n_1\tau}{2},\qquad \mathsf{a}_2=-\mathsf{d}_1-\frac{\tau}{2}+\frac{m_2+n_2\tau}{2},\qquad \mathsf{a}_3=-\mathsf{d}_1-\frac{\tau}{2}.\tag{C39}$$

It turns out that all 64 poles share identical contributions to the elliptic genus. In the end, we have

$$\mathcal{I}'_{2,0}=\frac{2\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)}.\tag{C40}$$

Apparently, up to a sign,

$$\mathcal{I}_{2,0}=\mathcal{I}'_{2,0}.\tag{C41}$$

4. Genus two with n punctures

Let us briefly summarize the computation for genus-two theories with n punctures. Given n, there are essentially only two $\mathrm{SU}(2)^{n+2}\times \mathrm{U}(1)$ quiver gauge theories one can consider: if the U(1) node in the quiver diagram is removed/ungauged, one frame continues to have a connected quiver diagram, while the other frame is cut into two disconnected pieces.

The first frame is simpler. The integrand reads

$$\begin{aligned}\mathcal{Z}_{2,n} &= \vartheta_4(\pm 2\mathsf{a}_{n+3})\prod_{j=1}^{n+2}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_j)\\ &\quad\cdot\prod_{i=1}^{n+2}\mathcal{I}^{(0,2)}_{U_2}(\mathsf{c}_i,-\mathsf{a}_{i-1},\mathsf{a}_i)\Big|_{\substack{\mathsf{c}_{n+1}=\mathsf{a}_{n+3}+\mathsf{d}_1\\ \mathsf{c}_{n+2}=-\mathsf{a}_{n+3}+\mathsf{d}_1\\ \mathsf{a}_0=\mathsf{a}_{n+2}}}. \end{aligned}\tag{C42}$$

One can pick a simple and generic $\eta=(\eta_1,\ldots,\eta_{n+3})$, such as

$$\eta_i=1-\frac{1}{1000000i}.\tag{C43}$$

With this choice, there are only 2^{2g+n+1} nondegenerate poles, which lead to the simple elliptic genus

$$\mathcal{I}_{2,n}=2(-1)^{n+1}\frac{\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)}\prod_{i=1}^{n}\frac{\eta(\tau)}{\vartheta_1(2\mathsf{c}_i)}.\tag{C44}$$

The second frame has an integrand

$$\begin{aligned}\mathcal{Z}'_{2,n} &= \vartheta_4(\pm 2\mathsf{a}_{n+3})\prod_{j=1}^{n+2}\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_j)\\ &\quad\cdot\mathcal{I}^{(0,2)}_{U_2}(-\mathsf{a}_{n+3}+\mathsf{d},\mathsf{a}_{n+2},-\mathsf{a}_{n+2})\\ &\quad\times\prod_{i=1}^{n+1}\mathcal{I}^{(0,2)}_{U_2}(\mathsf{c}_i,-\mathsf{a}_{i-1},\mathsf{a}_i)\Big|_{\substack{\mathsf{c}_{n+1}=\mathsf{a}_{n+3}+\mathsf{d}_1\\ \mathsf{a}_0=\mathsf{a}_{n+1}}}. \end{aligned}\tag{C45}$$

There are different η to choose from. For example,

$$\eta = (\eta_1, \ldots), \qquad \eta_i = 1 - \frac{1}{1000 \times i}. \tag{C46}$$

With this choice, only nondegenerate poles contribute, yielding

$$\mathcal{I}'_{2,n} = 2(-1)^n \frac{\vartheta_4(2\mathsf{d}_1)^2}{\eta(\tau)\vartheta_1(4\mathsf{d}_1)} \prod_{i=1}^{n} \frac{\eta(\tau)}{\vartheta_1(2\mathsf{c}_i)}. \tag{C47}$$

While other choices of η may lead to degenerate poles, the final result of the elliptic genus remains independent of η. Up to a sign, both $\mathrm{SU}(2)^{n+3} \times \mathrm{U}(1)$ quiver gauge theories share the same elliptic genus,

$$\mathcal{I}_{2,n} = \mathcal{I}'_{2,n}. \tag{C48}$$

5. SU(3) theory for $g = 1$, $n = 1$

The elliptic genus for the $\mathcal{N} = (0,2)$ SU(3) theory coupled to an adjoint chiral multiplet is given by the integral

$$\mathcal{I}^{(0,2),3}_{1,1} = \int_{\mathrm{JK}} \frac{da_1}{2\pi i a_1} \frac{da_2}{2\pi i a_2} \mathcal{I}^{(0,2)}_{U_3}(\mathsf{c}, \mathsf{a}, -\mathsf{a}) \mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}), \tag{C49}$$

where

$$\mathcal{I}^{(0,2)}_{U_3}(\mathsf{c}, \mathsf{a}, \mathsf{b}) = \prod_{A,B=1}^{3} \frac{\eta(\tau)}{\vartheta_4(\mathsf{c} + \mathsf{a}_A + \mathsf{b}_B)}\Bigg|_{\substack{\mathsf{a}_3 = -\mathsf{a}_1 - \mathsf{a}_2 \\ \mathsf{b}_3 = -\mathsf{b}_1 - \mathsf{b}_2}} \tag{C50}$$

$$\mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}) = \frac{\eta(\tau)^3}{3!} \prod_{\substack{A,B=1 \\ A \neq B}}^{3} \frac{\vartheta_1(\mathsf{a}_A - \mathsf{a}_B)}{\eta(\tau)}. \tag{C51}$$

Subsequent discussions involving SU(N) gauge group will continue to use these basic building blocks. One can pick some generic reference vector η, such as

$$\eta = (1, 0), \quad \text{or} \quad (0, 1). \tag{C52}$$

There are always 18 nondegenerate poles, yielding

$$\mathcal{I}^{(0,2),3}_{1,1} = \frac{\eta(\tau)}{\vartheta_4(3\mathsf{c})}. \tag{C53}$$

6. $g = 1$, $n = 2$ with SU(3) and/or U(1) gauge group

Several A_2 type $\mathcal{N} = (0,2)$ theories can be defined for the genus-one Riemann surface with two punctures.

a. $\mathrm{SU}(3)^2$ gauge theory

The simplest theory is an $\mathrm{SU}(3)^3$ theory, with the integrand

$$\mathcal{Z} = \mathcal{I}^{(0,2)}_{U_3}(\mathsf{c}_1, \mathsf{a}_1, \mathsf{a}_2) \mathcal{I}^{(0,2)}_{U_3}(\mathsf{c}_2, -\mathsf{a}_1, -\mathsf{a}_2) \prod_{i=1}^{2} \mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_i). \tag{C54}$$

One can pick, for example,

$$\eta = \left(1, \frac{1}{1000}, -\frac{1}{19}, \frac{16}{17}\right). \tag{C55}$$

There are 108 nondegenerate poles, giving

$$\mathcal{I}^{(0,2),3}_{1,2} = \frac{\eta(\tau)^2}{\vartheta_4(3\mathsf{c}_1)\vartheta_4(3\mathsf{c}_2)}. \tag{C56}$$

b. $\mathrm{SU}(3) \times \mathrm{U}(1)$ gauge theory

The integrand of the elliptic genus for the $\mathrm{SU}(3) \times \mathrm{U}(1)$ theory, depicted on the left side of Fig. 15, is

$$\mathcal{Z} = \mathcal{I}^{(0,2)}_{U_3}(\mathsf{a}_2 + \mathsf{d}_1, \mathsf{a}_1, \mathsf{b}) \mathcal{I}^{(0,2)}_{U_3}(-\mathsf{a}_2 + \mathsf{d}_1, -\mathsf{a}_1, \mathsf{b}') \times \mathcal{I}^{(0,2)}_{\mathrm{vec}}(\mathsf{a}_1) \vartheta_4(\pm 3\mathsf{a}_2), \tag{C57}$$

where $\vartheta_4(\pm 3\mathsf{a}_2)$ accounts for the contribution from two Fermi multiplets with ± 3 U(1) gauge charges. With the simple choice of

$$\eta = \left(\frac{999}{1000}, \frac{1999}{2000}, \frac{2999}{3000}\right), \tag{C58}$$

there are 162 nondegenerate poles, giving a fairly complicated expression for the elliptic genus,

$$\mathcal{I}'_{1,2} = -\eta(\tau)^7 \sum_{A,A'=1}^{3} \frac{\vartheta_1(-\mathsf{b}_A - \mathsf{b}'_{A'} + \mathsf{d}_1)^2}{\prod_{B \neq A} \vartheta_1(\mathsf{b}_B - \mathsf{b}_A) \prod_{B' \neq A'} \vartheta_1(\mathsf{b}'_{B'} - \mathsf{b}'_{A'})} \times \frac{1}{\vartheta_1(\mathsf{b}_A + \mathsf{b}'_{A'} + 2\mathsf{d}_1) \prod_{B \neq A, B' \neq A'} \vartheta_1(\mathsf{b}_B + \mathsf{b}'_{B'} + 2\mathsf{d}_1)}. \tag{C59}$$

Although the expression is complicated, it can be reformulated as the simpler ratio

$$\mathcal{I}'_{1,2} = \frac{3\eta(\tau)^7 \vartheta_1(3\mathsf{d}_1)^2}{\prod_{A,B=1}^{3} \vartheta_1(\mathsf{b}_A + \mathsf{b}'_B + 2\mathsf{d}_1)}. \tag{C60}$$

APPENDIX D: JK RESIDUES OF $\mathcal{N} = (0,4)$ ELLIPTIC GENERA

In this appendix, we present the detailed computation of the $\mathcal{N} = (0,4)$ elliptic genus. Recall that the basic building blocks of the elliptic genus for the Lagrangian theories of type A_{N-1}, as illustrated in Fig. 16, are given by [27]

$$\mathcal{I}_{0,3}^{(0,4),N}(\mathsf{c},\mathsf{a},\mathsf{b}) = \prod_{A,B=1}^{N} \frac{\eta(\tau)}{\vartheta_1(\mathsf{v}\pm(\mathsf{c}+\mathsf{a}_A+\mathsf{b}_B))}, \quad \text{(D1)}$$

$$\mathcal{I}_{\mathrm{vec}}^{(0,4),N} = \frac{1}{N!}(\vartheta_1(2\mathsf{v})\eta(\tau))^{N-1} \times \prod_{\substack{A,B=1\\ A\neq B}}^{N} \frac{\vartheta_1(2\mathsf{v}+\mathsf{a}_A-\mathsf{a}_B)\vartheta_1(\mathsf{a}_A-\mathsf{a}_B)}{\eta(\tau)^2}, \quad \text{(D2)}$$

where all of the SU(N) chemical potentials satisfy the traceless condition, such as $\sum_{A=1}^{N}\mathsf{a}_A=0$. The chemical potentials $\mathsf{a},\mathsf{b},\mathsf{c}$ make manifest the flavor symmetry $\mathrm{SU}(N)\times\mathrm{SU}(N)\times\mathrm{U}(1)\subset\mathrm{SU}(2N)\times\mathrm{U}(1)$. Note that our convention is slightly different from that in [27] by some constant factor.

1. Type A_{N-1} theories with minimal punctures at genus one

We begin with the theory of type A_2 at genus one with one minimal puncture associated with an U(1) flavor symmetry. The elliptic genus can be computed by the JK residue

$$\mathcal{I}_{1,1,0}^{(0,4),3}(\mathsf{c}) = \int_{\mathrm{JK}} \prod_{A=1}^{2} \frac{da_A}{2\pi i a_A} \mathcal{I}_{0,3}^{(0,4),3}(\mathsf{c},\mathsf{a},-\mathsf{a})\mathcal{I}_{\mathrm{vec}}^{(0,4),3}(\mathsf{a}). \quad \text{(D3)}$$

The integration simplifies when choosing a suitable reference vector, such as $\eta=(1,1-\frac{1}{1000})$, resulting in only 54 nondegenerate poles. The result of the JK-residue computation is

$$\mathcal{I}_{1,1,0}^{(0,4),3}(\mathsf{c}) = \eta(\tau)^2\left[-\frac{\vartheta_1(4\mathsf{v}-2\mathsf{c})\vartheta_1(3\mathsf{v}-\mathsf{c})}{\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(3\mathsf{v}-3\mathsf{c})\vartheta_1(2\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})} + \frac{\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})}{\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})} + \frac{\vartheta_1(3\mathsf{v}+\mathsf{c})\vartheta_1(4\mathsf{v}+2\mathsf{c})}{\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(2\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})\vartheta_1(3\mathsf{v}+3\mathsf{c})}\right]. \quad \text{(D4)}$$

The sum of the three terms can be reorganized into the simple ratio

$$\begin{aligned}\mathcal{I}_{1,1,0}^{(0,4),3}(\mathsf{c}) &= \frac{\eta(\tau)^2\vartheta_1(6\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(3\mathsf{v}+3\mathsf{c})\vartheta_1(3\mathsf{v}-3\mathsf{c})} \\ &= q^{-1/6}(1+2v^2+(c^3+c^{-3})v^3+3v^4+(-v^{-6}-2v^{-4}-v^{-2}+6+2(c^3+c^{-3})v+15v^2+\ldots)q+\ldots). \end{aligned} \quad \text{(D5)}$$

The same computation can be done for SU(4), but it is more tedious. There are both nondegenerate and degenerate poles, and in the end the elliptic genus reads

$$\begin{aligned}\mathcal{I}_{1,1,0}^{(0,4),4}(\mathsf{c}) = \eta(\tau)^2\Bigg[&-\frac{\vartheta_1(5\mathsf{v}-3\mathsf{c})\vartheta_1(4\mathsf{v}-2\mathsf{c})\vartheta_1(3\mathsf{v}-\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(2\mathsf{v}-4\mathsf{c})\vartheta_1(4\mathsf{v}-4\mathsf{c})\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})} - \frac{\vartheta_1(4\mathsf{v}-2\mathsf{c})\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(4\mathsf{c})\vartheta_1(2\mathsf{v}-4\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})} \\ &+ \frac{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})}{(\vartheta_1(2\mathsf{c}))^2\vartheta_1(\mathsf{v}-3\mathsf{c})\vartheta_1(2\mathsf{v}-2\mathsf{c})\vartheta_1(2\mathsf{v}+2\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})} - \frac{\vartheta_1(3\mathsf{v}-\mathsf{c})\vartheta_1(3\mathsf{v}+\mathsf{c})\vartheta_1(4\mathsf{v}+2\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(4\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+\mathsf{c})\vartheta_1(2\mathsf{v}+4\mathsf{c})} \\ &+ \frac{\vartheta_1(3\mathsf{v}+\mathsf{c})\vartheta_1(4\mathsf{v}+2\mathsf{c})\vartheta_1(5\mathsf{v}+3\mathsf{c})}{\vartheta_1(2\mathsf{c})\vartheta_1(\mathsf{v}-\mathsf{c})\vartheta_1(\mathsf{v}+3\mathsf{c})\vartheta_1(2\mathsf{v}+4\mathsf{c})\vartheta_1(4\mathsf{v}+4\mathsf{c})}\Bigg]. \end{aligned} \quad \text{(D6)}$$

Similar to the SU(3) case, the expression can be recast into the simple ratio

$$\mathcal{I}_{1,1,0}^{(0,4),N}(\mathsf{c}) = \frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c})}, \qquad N=4. \quad \text{(D7)}$$

Here the expression on the right is expected to hold for all genus-one A_{N-1} theories with one minimal puncture. Furthermore, through an even more intricate computation, we derive the elliptic genus for a circular $\mathrm{SU}(3)^2$ quiver with $\mathrm{U}(1)^2_{x_1,x_2}$ flavor symmetry or, equivalently, a theory of type A_2 at genus one with two minimal punctures. The expression from the direct JK-residue computation is too complicated to detail here; however, it can be reorganized into the simple form

$$\mathcal{I}^{(0,4),2}_{g=1,n_1=2}(\mathsf{c}_1,\mathsf{c}_2) = \frac{\eta(\tau)^4\vartheta(6\mathsf{v})^2}{\vartheta_1(2\mathsf{v})^2\prod_{i=1}^2\vartheta(3\mathsf{v}\pm 3\mathsf{c}_i)}. \quad \text{(D8)}$$

Here n denotes the number of minimal punctures. Extrapolating from the above results, it is natural to conjecture that for all of the $\mathrm{SU}(N)$-type theories at genus one with n_1 minimal punctures, the elliptic genus can be written as

$$\mathcal{I}^{(0,4),N}_{g=1,n_1}(\mathsf{c}_1,\ldots,\mathsf{c}_{n_1}) = \prod_{i=1}^{n_{\mathrm{U}(1)}}\frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c}_i)}. \quad \text{(D9)}$$

We will apply this conjecture to later computations.

2. Type A_2 theories at genus $g\geq 1$

The elliptic genus of $\mathcal{N}=(0,4)T_3$ theory was computed in [27] using an elliptic inversion formula. In more detail, the elliptic genus of the $\mathcal{N}=(0,4)T_3$ theory and the SU(3) SQCD with six fundamental flavors is related by the Argyres-Seiberg duality,

$$\mathcal{I}^{(0,4),N=3}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}) = \int_{\mathrm{JK}}\frac{dz}{2\pi i z}\frac{\eta(\tau)}{\vartheta_1(\mathsf{v}\pm\mathsf{s}\pm\mathsf{z})} \times\mathcal{I}^{(0,4),2}_{\mathrm{vec}}(\mathsf{z})\mathcal{I}_{T^3}(\mathsf{a},\mathsf{b},\mathsf{c}). \quad \text{(D10)}$$

Within the integral, we gauge an $\mathrm{SU}(2)\subset\mathrm{SU}(3)_c$, leading to $\mathsf{c}_1=\mathsf{r}+\mathsf{z},\mathsf{c}_2=\mathsf{r}-\mathsf{z},\mathsf{c}_3=-2\mathsf{r}$. The integral can be inverted to compute the elliptic genus $\mathcal{I}_{T_3}$. Explicitly, it is given by the simple JK-residue computation

$$\mathcal{I}_{T_3}(\mathsf{a},\mathsf{b},\mathsf{c}) = \frac{\eta(\tau)^5}{2\vartheta_1(2\mathsf{v}\pm 2\mathsf{z})}\int_{\mathrm{JK}}\frac{ds}{2\pi i s}\frac{\vartheta_1(\pm 2\mathsf{s})\vartheta_1(\pm 2\mathsf{v})}{\vartheta_1(-\mathsf{v}\pm\mathsf{s}\pm\mathsf{z})} \times\mathcal{I}^{(0,4),N=3}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}).$$

Note that the coefficient in front of the integral has been adjusted according to our convention. On the right, $\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}$ represent the $\mathrm{SU}(3)_a\times\mathrm{SU}(3)_b\times\mathrm{U}(1)_x\times\mathrm{U}(1)_y$ flavor chemical potentials, where

$$\mathsf{x}=\frac{\mathsf{s}}{3}+\mathsf{r},\qquad \mathsf{y}=\frac{\mathsf{s}}{3}-\mathsf{r}. \quad \text{(D11)}$$

After the integral, z,r combine into SU(3) chemical potentials,

$$\mathsf{c}_1=\mathsf{r}+\mathsf{z},\qquad \mathsf{c}_2=\mathsf{r}-\mathsf{z},\qquad \mathsf{c}_3=-2\mathsf{r}. \quad \text{(D12)}$$

The $\mathsf{a},\mathsf{b},\mathsf{c}$ denote the $\mathrm{SU}(3)^3\subset E_6$ chemical potentials for the $\mathcal{N}=(0,4)E_6$ theory.

From the elliptic inversion formula for $\mathcal{I}_{T_3}$, it is evident that the diagonal of the two SU(3) flavor subgroups can be gauged, which yields the elliptic genus of the SU(3)-type genus-one theory with one maximal puncture associated with an $\mathrm{SU}(3)_c$ flavor symmetry. Now the situation is much better: the right-hand side involves $\mathcal{I}^{(0,4),3}_{g=1,n_1=2}$, which is shown to be a simple ratio of elliptic theta functions, where n_1 denotes the number of minimal punctures (see Fig. 17). When performing the JK-residue computation, we encounter only nondegenerate poles and obtain a simple result for the (0,4) elliptic genus with $n_3=1$ maximal puncture,

$$\mathcal{I}^{(0,4),3}_{g=1,n_1=0,n_3=1}(\mathsf{c}) = \frac{\eta(\tau)^6\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(6\mathsf{v})}{\prod_{A,B=1}^3\vartheta_1(2\mathsf{v}+\mathsf{b}_{iA}-\mathsf{b}_{iB})}. \quad \text{(D13)}$$

Instead of directly gauging the diagonal of the two existing SU(3) groups, one can alternatively gauge the diagonal of the $\mathrm{SU}(3)^2$ flavor subgroup of $\mathcal{I}_{T_3}$ and an SU(3) linear quiver. This approach can be used to generate a genus-one theory with additional minimal punctures. In other words,

$$\mathcal{I}^{(0,4),3}_{g=1,n_1,n_3=1} = \text{elliptic inversion of } \mathcal{I}^{(0,4),3}_{g=1,n_1+2,n_3=0}. \quad \text{(D14)}$$

Effectively, the elliptic inversion formula represents the fusion of any two minimal punctures into a single maximal one. Therefore, the inversion, or fusing two minimal punctures, can be performed successively, yielding more maximal punctures. Since the elliptic genera on the right are all simple ratios, the JK residue can be easily carried out, and at each step of the fusion the outcome continues to be a simple ratio. In conclusion, we obtain the following general result for the theory of type A_2 for any $g\geq 1$ that has n_1 minimal and n_3 maximal punctures:

$$\mathcal{I}^{(0,4),3}_{g,n_1,n_3} = \left(\frac{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})^2\vartheta_1(6\mathsf{v})}{\eta(\tau)^4}\right)^{g-1}\mathcal{I}^{(0,4),3}_{g=1,n_1,0}\mathcal{I}^{(0,4),3}_{g=1,0,n_3}, \quad \text{(D15)}$$

where with $N=3$,

$$\mathcal{I}^{(0,4),3}_{g=1,n_1,0} = \prod_{i=1}^{n_1}\frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c}_i)}, \quad \text{(D16)}$$

$$\mathcal{I}^{(0,4),3}_{g=1,0,n_3} = \prod_{i=1}^{n_3}\frac{\eta(\tau)^6\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(6\mathsf{v})}{\prod_{A,B=1}^3\vartheta_1(2\mathsf{v}+\mathsf{b}_{iA}-\mathsf{b}_{iB})}. \quad \text{(D17)}$$

3. Type A_3 theories at genus $g\geq 1$

Let us now consider theories of type A_3 of class $\mathcal{S}$. There are four punctures (and the corresponding flavor symmetry) to be considered: minimal [U(1)], $[2,1^2]$ $[\mathrm{SU}(2)\times\mathrm{U}(1)]$, $[2^2]$ [SU(2)], and maximal [SU(4)]. To begin, we have

$$\mathcal{I}^{(0,4),N=4}_{g=1;n_1,0,0,0} = \prod_{i=1}^{n_1}\frac{\eta(\tau)^2\vartheta_1(2N\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(N\mathsf{v}\pm N\mathsf{c}_i)},\quad N=4, \quad \text{(D18)}$$

where the subscripts n_1, n_2, n_3, and n_4 denote the number of punctures associated with the respective flavor symmetries U(1), SU(2) × U(1), SU(2), and SU(4).

In the context of 4D $\mathcal{N}=2$ SCFTs, $R_{0,N}$ are non-Lagrangian theories with $\mathrm{SU}(2N)\times\mathrm{SU}(2)$ flavor symmetry, and they arise from a strong-coupling limit of the $\mathcal{N}=2$ SU(N) theory with $2N$ fundamental hypermultiplets. Concretely, as a theory of A_3-type class $\mathcal{S}$, $R_{0,4}$ corresponds to the three-punctured sphere with two maximal punctures and a $[2,1^2]$ puncture, where the manifest flavor subgroup is $\mathrm{SU}(4)^2\times\mathrm{U}(1)\times\mathrm{SU}(2)$.

In 2D, the $\mathcal{N}=(0,4)$ elliptic genus of $R_{0,4}$ theory was computed in [83] using an elliptic inversion. The Argyres-Seiberg-like duality implies an integral equality between the elliptic genus of SU(4) SQCD with eight fundamental flavors and that of the $R_{0,4}$ theory,

$$\mathcal{I}^{(0,4),N=4}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s})=\int_{\mathrm{JK}}\frac{dz}{2\pi iz}\frac{\eta(\tau)}{\vartheta_1(\mathsf{v}\pm\mathsf{s}\pm\mathsf{z})}\times\mathcal{I}^{(0,4),2}_{\mathrm{vec}}(\mathsf{z})\mathcal{I}^{(0,4)}_{R_{0,4}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{z}),\quad\text{(D19)}$$

which can be inverted to give

$$\mathcal{I}^{(0,4)}_{R_{0,4}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{z})=\frac{\eta(\tau)^5}{2\vartheta_1(2\mathsf{v}\pm2\mathsf{z})}\times\int_{\mathrm{JK}}\frac{ds}{2\pi is}\frac{\vartheta_1(\pm2\mathsf{s})\vartheta_1(-2\mathsf{v})}{\vartheta_1(-\mathsf{v}\pm\mathsf{s}\pm\mathsf{z})}\times\mathcal{I}^{(0,4),N=4}_{\mathrm{SQCD}}(\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}).$$

Here $\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{z}$ denote the chemical potentials of $\mathrm{SU}(4)\times\mathrm{SU}(4)\times\mathrm{U}(1)\times\mathrm{SU}(2)\subset\mathrm{SU}(8)\times\mathrm{SU}(2)$ flavor symmetry of the $R_{0,4}$ theory. On the right, $\mathsf{a},\mathsf{b},\mathsf{r},\mathsf{s}$ are the chemical potentials of the $\mathrm{U}(8)=\mathrm{SU}(8)_{\mathsf{a},\mathsf{b},\mathsf{r}}\times\mathrm{U}(1)_{\mathsf{s}}$ flavor symmetry of the SU(4) gauge theory with eight fundamental hypermultiplets. The chemical potentials r,s are related to the standard x,y [associated with the two minimal punctures of the SU(4) SQCD] by

$$\mathsf{x}=\frac{\mathsf{s}}{4}+\mathsf{r},\qquad\mathsf{y}=\frac{\mathsf{s}}{4}-\mathsf{r}.\quad\text{(D20)}$$

One can start gauging in the theory of 4^2 free hypermultiplets or handles to the punctures associated with a,b on both sides of the above equation, so that the $\mathcal{I}^{(0,4),N=4}_{\mathrm{SQCD}}$ on the right becomes $\mathcal{I}^{(0,4),N=4}_{g,n_1+2,0,0,n_4}$, while on the left it becomes $\mathcal{I}^{(0,4),4}_{g;n_1,0,1,n_4}$. Effectively, the elliptic inversion merges two minimal punctures into a $[2,1^2]$ puncture (see Fig. 18). In the simplest case, when $g=1$, $n_1=0$, $n_4=0$,

$$\mathcal{I}^{(0,4),4}_{g=1;0,1,0,0}=\frac{\eta(\tau)^6\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})^2\vartheta_1(2\mathsf{v}\pm2\mathsf{z})\vartheta_1(3\mathsf{v}\pm\mathsf{z}\pm4\mathsf{r})}.\quad\text{(D21)}$$

The [2, 1, 1] puncture can be further Higgsed to a [2, 2] puncture with SU(2) flavor symmetry. The class $\mathcal{S}$ theory corresponding to a three-punctured sphere with two maximal and one [2, 2] puncture has an enhanced E_7 flavor symmetry. The elliptic genus of the corresponding 2D $\mathcal{N}=(0,4)$ theory can also be computed using an elliptic inversion formula [83]. Following the same approach as above, we perform additional gluing/gauging operations on the two maximal punctures and obtain the contribution from one [2, 2] puncture,

$$\begin{aligned}\mathcal{I}^{(0,4),4}_{g=1;0,0,1,0}(\mathsf{w})&=\int_{\mathrm{JK}}\frac{ds}{2\pi is}\frac{\eta(\tau)^5\vartheta_1(\pm2\mathsf{v})}{2\vartheta_1(4\mathsf{v})}\times\frac{\vartheta_1(\pm2\mathsf{s})}{\vartheta_1(\pm\mathsf{s})\vartheta_1(-2\mathsf{v}\pm\mathsf{s})}\mathcal{I}^{(0,4),4}_{1;2,0,0,0}(\mathsf{c}_1,\mathsf{c}_2)\\&=\frac{\eta(\tau)^6\vartheta_1(6\mathsf{v})\vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})\vartheta_1(4\mathsf{v}\pm4\mathsf{w})\vartheta_1(2\mathsf{v}\pm4\mathsf{w})}.\end{aligned}\quad\text{(D22)}$$

Again, w represents the SU(2) flavor chemical potential. As before, the two U(1) chemical potentials on the right are related to w by $\mathsf{c}_1=\frac{\mathsf{s}}{4}+\mathsf{w}$ and $\mathsf{c}_2=\frac{\mathsf{s}}{4}-\mathsf{w}$.

The contribution from the maximal puncture can be obtained by considering the generalized S duality, as shown in Fig. 19, which requires

$$\mathcal{I}^{(0,4),4}_{1;3,0,0,0}(\mathsf{x}_{1,2,3})=\int_{\mathrm{JK}}\frac{db_1}{2\pi ib_1}\prod_{A=1}^{2}\frac{da_A}{2\pi ia_A}\mathcal{I}^{(0,4),4}_{1;0,0,0,1}(\mathsf{c})|_{\mathsf{c}_{A=1,2,3}=\mathsf{a}_A+\mathsf{r}}\mathcal{I}^{(0,4),3}_{\mathrm{vec}}(\mathsf{a})\prod_{A=1}^{3}\prod_{i=1}^{2}\frac{\eta(\tau)^2}{\vartheta_1(\mathsf{v}\pm(-\mathsf{a}_A+\mathsf{b}_i+\mathsf{x}'_1))}\times\mathcal{I}^{(0,4),2}_{\mathrm{vec}}(\mathsf{b})\prod_{i=1}^{2}\frac{\eta(\tau)^2}{\vartheta_1(\mathsf{v}\pm(-\mathsf{b}_i+\mathsf{x}'_2))}.\quad\text{(D23)}$$

In the above, a_A and b_i are SU(3) and SU(2) fugacities with $a_3=1/(a_1a_2)$ and $b_1=1/b_2$. The U(1) fugacities on both sides are identified by

$$x_1' = \frac{x_1^{4/3} x_3^{2/3}}{x_2^{2/3}}, \qquad x_2' = x_2^2 x_3^2, \qquad r = \left(\frac{x_3}{x_1 x_2}\right)^{1/3}. \tag{D24}$$

Explicitly, the elliptic genus of the $\mathcal{N} = (0,4)$ genus-one theory with one maximal puncture reads

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,1} = \frac{\eta(\tau)^{12} \vartheta_1(2\mathsf{v}) \vartheta_1(4\mathsf{v}) \vartheta_1(6\mathsf{v}) \vartheta_1(8\mathsf{v})}{\prod_{A,B=1}^4 \vartheta_1(2\mathsf{v} + \mathsf{c}_A - \mathsf{c}_B)}. \tag{D25}$$

Finally, by gauging the maximal punctures, we determine the elliptic genus for genus $g \geq 1$ with arbitrary punctures,

$$\mathcal{I}^{(0,4),4}_{g;n_1,n_2,n_3,n_4} = \left(\frac{\vartheta_1(2\mathsf{v})\vartheta_1(4\mathsf{v})^2\vartheta_1(6\mathsf{v})^2\vartheta_1(8\mathsf{v})^2}{\eta(\tau)^6}\right)^{g-1} \times \mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} \mathcal{I}^{(0,4),4}_{1;0,n_2,0,0} \mathcal{I}^{(0,4),4}_{1;0,0,n_3,0} \mathcal{I}^{(0,4),4}_{1;0,0,0,n_4},$$

where

$$\mathcal{I}^{(0,4),4}_{1;n_1,0,0,0} = \prod_{i=1}^{n_1} \frac{\eta(\tau)^2 \vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v}) \vartheta_1(4\mathsf{v} \pm 4\mathsf{x}_i)}, \tag{D26}$$

$$\mathcal{I}^{(0,4),4}_{1;0,n_2,0,0} = \prod_{i=1}^{n_3} \frac{\eta(\tau)^6 \vartheta_1(6\mathsf{v}) \vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v})^2 \vartheta_1(2\mathsf{v} \pm 2\mathsf{z}_i) \vartheta_1(3\mathsf{v} \pm \mathsf{z}_i \pm 4\mathsf{r}_i)}, \tag{D27}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,n_3,0} = \prod_{i=1}^{n_2} \frac{\eta(\tau)^6 \vartheta_1(6\mathsf{v}) \vartheta_1(8\mathsf{v})}{\vartheta_1(2\mathsf{v}) \vartheta_1(4\mathsf{v}) \vartheta_1(2\mathsf{v} \pm 4\mathsf{w}_i) \vartheta_1(4\mathsf{v} \pm 4\mathsf{w}_i)}, \tag{D28}$$

$$\mathcal{I}^{(0,4),4}_{1;0,0,0,n_4} = \prod_{i=1}^{n_4} \frac{\eta(\tau)^{12} \vartheta_1(2\mathsf{v}) \vartheta_1(4\mathsf{v}) \vartheta_1(6\mathsf{v}) \vartheta_1(8\mathsf{v})}{\prod_{A,B=1}^4 \vartheta_1(2\mathsf{v} + \mathsf{b}_{iA} - \mathsf{b}_{iB})}. \tag{D29}$$

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