

Angular and radial stabilities of spontaneously scalarized black holes in the presence of scalar-Gauss-Bonnet couplings

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We study the linear stability of spontaneously scalarized black holes (BHs) induced by a scalar field ϕ coupled to a Gauss-Bonnet (GB) invariant R_{GB}^2 . For the scalar-GB coupling $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$, where η and α are constants, we first show that there are no angular Laplacian instabilities of even-parity perturbations far away from the horizon for large multipoles $l \gg 1$. The deviation of angular propagation speeds from the speed of light is largest on the horizon, whose property can be used to put constraints on the model parameters. For $\alpha \gtrsim -1$, the region in which the scalarized BH is subject to angular Laplacian instabilities can emerge. Provided that $\alpha \lesssim -1$ and $-1/2 < \alpha\phi_0^2 < -0.1155$, where ϕ_0 is the field value on the horizon with a unit of the reduced Planck mass $M_{\text{Pl}} = 1$, there are scalarized BH solutions satisfying all the linear stability conditions throughout the horizon exterior. We also study the stability of spontaneously scalarized BHs in scalar-GB theories with a nonminimal coupling $-\beta\phi^2 R/16$, where β is a positive constant and R is a Ricci scalar. As the amplitude of the field on the horizon approaches an upper limit $|\phi_0| = 4/\sqrt{\beta}$, one of the squared angular propagation speeds $c_{\Omega-}^2$ enters the instability region $c_{\Omega-}^2 < 0$. So long as $|\phi_0|$ is smaller than a maximum value determined for each β in the range $\beta > 5$, however, the scalarized BHs are linearly stable in both angular and radial directions.

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I. INTRODUCTION

In general relativity (GR), the Schwarzschild black hole (BH) arises as a unique vacuum solution of the Einstein equation on a static and spherically symmetric background. The geometry of the Schwarzschild BH is solely characterized by a BH mass. In order to generate an additional “hair” to the BH mass on the same background, we need to take into account new degrees of freedom such as scalar and vector fields. In the presence of a $U(1)$ gauge-invariant vector field, the resulting solution is the Reissner-Nordström BH with an electric and/or magnetic charge besides the mass.

If we consider a canonical scalar field ϕ with the potential $V(\phi)$ satisfying certain conditions, it is known that static, spherically symmetric, and asymptotically flat BHs do not acquire new scalar hairs [1–3]. This no-hair

property also persists for k-essence [4] and for a non-minimally coupled scalar field with the Ricci scalar R of the form $\kappa(\phi)R$ [5–9]. A simple way to evade the no-hair theorems is to introduce time dependence to the scalar profile while keeping the time-independent metric.¹ In this paper, however, we shall not pursue this possibility and we assume that scalar-field profiles for the background solutions are time independent.

In the presence of a coupling between ϕ and the Gauss-Bonnet (GB) invariant R_{GB}^2 of the form $\xi(\phi)R_{\text{GB}}^2$, there are

¹This allows for, for example, stealth BH solutions, in which the metric is the same as in GR but the scalar profile is nontrivial (see Ref. [10] for the first stealth Schwarzschild solution in k-essence). For each stealth BH, if and only if the scalar profile is timelike, one can easily implement the so-called scordatura mechanism [11–13] to find the corresponding approximately stealth solution (see Ref. [10] for the first example), which is stealth for any practical purposes at astrophysical scales [10,14,15] and around which perturbations are weakly coupled all the way up to the scale associated with the background value of the derivative of the scalar field [11].

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hairy BH solutions endowed with scalar hairs [16–29]. Such Einstein-scalar-Gauss-Bonnet (EsGB) theories belong to a subclass of Horndeski theories with second-order field equations of motion [30–33]. Indeed, even in the framework of full Horndeski and degenerate higher-order scalar-tensor theories, it was shown that the existence of the scalar-GB coupling is necessary for realizing nonpathological (e.g., linearly stable) and asymptotically flat BH solutions with a regular scalar field on the horizon [34,35] (see also Ref. [36]).

In EsGB theories given by the coupling $\xi(\phi)R_{\text{GB}}^2$, it is known that a phenomenon called spontaneous scalarization of BHs and neutron stars (NSs) can occur for $\xi(\phi)$ containing even-power-law functions of ϕ [37–50]. This is analogous to spontaneous scalarization of NSs induced by the nonminimal coupling $\kappa(\phi)R$ [51,52]. In the latter case, the trace T of matter inside the star is coupled to the scalar field ϕ through a relation $M_{\text{Pl}}^2 R = -T$. For BHs, the Ricci scalar is vanishing due to the absence of matter ($T = 0$). However, the GB invariant does not vanish for a vacuum Schwarzschild or a Kerr background. Hence a BH can acquire a nontrivial scalar hair through tachyonic instability caused by the coupling between ϕ and R_{GB}^2 .

For the scalar-Gauss-Bonnet (sGB) coupling given by even-power-law functions like $\xi(\phi) = \eta\phi^2/8$ with η being constant, there is a nonvanishing scalar-field branch $\phi \neq 0$ besides the GR branch $\phi = 0$. On weak gravitational backgrounds like the Solar System, the scalar field stays in the latter trivial branch, whose property does not conflict with the experimental tests on gravity [53]. In the vicinity of BHs, the effective mass squared of ϕ , which is given by $m_{\text{eff}}^2 = -\xi_{,\phi\phi}R_{\text{GB}}^2$, can be negative at $\phi = 0$ under the condition $\xi_{,\phi\phi} \equiv d^2\xi/d\phi^2 > 0$. For the coupling $\xi(\phi) = \eta\phi^2/8$ this condition translates to $\eta > 0$, under which the GR branch can trigger tachyonic instability toward the other nontrivial branch to realize scalarized BHs (see also Refs. [54–63] for spontaneous scalarization caused by other couplings).

In EsGB theories given by the coupling $\xi(\phi) = \eta\phi^2/8$, however, the scalarized branch ($\phi \neq 0$) is unstable against radial perturbations everywhere [64]. Taking into account higher-order terms to the sGB coupling, e.g., $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$ with α being dimensionless coupling constant, there are parameter spaces in which the scalarized BH solutions are not subject to radial tachyonic instabilities [40,43,65]. In this paper we will study all the linear stability conditions for scalarized BHs with the sGB coupling $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$ in both angular and radial directions by exploiting linear stability conditions derived for Horndeski theories [66] (see also Refs. [67–69]). Indeed, there is a wide range of parameter spaces in which the scalarized BHs are subject to neither ghost nor Laplacian instabilities for large radial and angular momentum modes.

The BH scalarization model in EsGB theories was extended to include the nonminimal coupling $\kappa(\phi)R$, where

$\kappa(\phi)$ is an even-power-law function of ϕ [70–72]. Such Einstein-scalar-Gauss-Bonnet-Ricci (EsGBR) theories were motivated for realizing the asymptotic GR solution $\phi = 0$ as a cosmological attractor [70]. For the couplings $\kappa(\phi) = -\beta\phi^2/16$, with β being the dimensionless constant and $\xi(\phi) = \eta\phi^2/8$, the analysis of Ref. [72] showed that there are radially stable scalarized BH solutions if $\beta > 5$.

Recently, it was found that scalarized BHs in EsGBR theories can be subject to angular instabilities for the quadrupole perturbation ($l = 2$) [73]. This stability analysis is based on the computation of the BH entropy [74] integrated over the spatial cross section of the horizon. However, it is not yet clear what happens for the angular stability of BHs in the large multipole limit ($l \gg 1$). In particular, since l appears in the quadratic action for perturbations only through the expression $l(l+1)$ (which is an increasing function of l), if increasing l from 0 to 2 changes stable perturbations to unstable ones as claimed in Ref. [73], then the instability is naturally expected to be more prominent for larger l . Since EsGBR theories belong to a subclass of Horndeski theories, one can exploit the angular stability conditions derived in Ref. [66] for $l \gg 1$. In this paper, we will address both the angular and radial stabilities of scalarized BHs in EsGBR theories and find that the angular Laplacian instabilities can arise for large field values ϕ_0 on the horizon. However, we will show the existence of parameter spaces in which neither ghosts, angular/radial Laplacian instabilities, nor radial tachyonic instabilities are present.

This paper is organized as follows. In Sec. II, we briefly review conditions for tachyonic instabilities of the GR branch in both EsGB and EsGBR theories and also present the background equations of motion on the static and spherically symmetric background. In Sec. III, we revisit the linear stability conditions derived in [66] and apply them to EsGB and EsGBR theories in the small field regime ($|\phi| \ll 1$). In Sec. IV, we study the radial stability of the dynamical scalar-field perturbation $\delta\phi$ by considering a monopole mode ($l = 0$). After deriving a closed-form second-order differential equation for $\delta\phi$ with the backreaction of metric perturbations taken into account, we obtain the regime of model parameters consistent with the radial stability analysis performed in the literature [43,70]. In Secs. V and VI, we elucidate the parameter spaces of EsGB and EsGBR theories in which no linear instabilities of scalarized BHs are present. Section VII is devoted to conclusions.

II. ESGB AND ESGBR THEORIES

We begin with a general action of EsGBR theories given by

$$S = \frac{1}{2} \int d^4x \sqrt{-g} [R + X + \xi(\phi)R_{\text{GB}}^2 + \kappa(\phi)R], \quad (2.1)$$

where g is the determinant of metric tensor $g_{\mu\nu}$, R is the Ricci scalar, $X = -(1/2)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic term of a scalar field ϕ , R_{GB}^2 is the GB invariant, and $\xi(\phi)$ and $\kappa(\phi)$ are functions of ϕ . We will choose the unit of $M_{\text{Pl}} = 1$, where M_{Pl} represents the reduced Planck mass. Note that EsGB theories correspond to the special case where $\kappa(\phi) = 0$.

On the static and spherically symmetric background, spontaneous scalarization of BHs can be realized for the sGB coupling $\xi(\phi)$ with a Z_2 symmetry $\xi(-\phi) = \xi(\phi)$ [37,38,40–43]. If we apply these EsGB theories to cosmology, the asymptotic GR solution $\phi = 0$ is not generally realized without tuning the initial conditions [70,75,76]. If we allow the existence of a nonminimal coupling of the form $-\beta\phi^2 R/16$, the scalar field can approach a cosmological attractor being compatible with Solar System constraints on today's field value of ϕ [70–72]. In such EsGBR theories, it was recently recognized that scalarized BHs can be subject to a quadrupole angular instability below a critical value of the BH mass [73]. If this is the case, then the similar angular instability may also manifest itself for larger multipoles $l \gg 1$. In this paper, we will address this latter issue along with the radial stability by using general conditions derived in Ref. [66]. Moreover, we will investigate the angular and radial stabilities of BHs in EsGB theories.

The action (2.1) in EsGBR theories belongs to a subclass of Horndeski theories given by the action

$$\begin{aligned} \mathcal{S} = \int d^4x \sqrt{-g} \Big\{ & G_2 - G_3 \square\phi + G_4 R \\ & + G_{4,X} [(\square\phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi)] + G_5 G_{\mu\nu} \nabla^\mu \nabla^\nu \phi \\ & - \frac{1}{6} G_{5,X} [(\square\phi)^3 - 3(\square\phi)(\nabla_\mu \nabla_\nu \phi)(\nabla^\mu \nabla^\nu \phi) \\ & + 2(\nabla^\mu \nabla_\alpha \phi)(\nabla^\alpha \nabla_\beta \phi)(\nabla^\beta \nabla_\mu \phi)] \Big\}, \end{aligned} \quad (2.2)$$

where ∇_μ is a covariant derivative operator, $\square = \nabla^\mu \nabla_\mu$, $G_{\mu\nu}$ is the Einstein tensor, and

$$G_2 = \frac{1}{2} [X + 8\xi_{,\phi\phi\phi}(\phi)X^2(3 - \ln|X|)], \quad (2.3)$$

$$G_3 = 2\xi_{,\phi\phi\phi}(\phi)X[7 - 3\ln|X|], \quad (2.4)$$

$$G_4 = \frac{1}{2} [1 + \kappa(\phi)] + 2\xi_{,\phi\phi}(\phi)X(2 - \ln|X|), \quad (2.5)$$

$$G_5 = -2\xi_{,\phi}(\phi) \ln|X|, \quad (2.6)$$

with the notations $\xi_{,\phi}(\phi) = d\xi(\phi)/d\phi$, $\xi_{,\phi\phi}(\phi) = d^2\xi(\phi)/d\phi^2$, etc.

Varying the action (2.1) with respect to ϕ , we obtain the scalar-field equation of motion

$$\square\phi + R_{\text{GB}}^2 \xi_{,\phi} + R\kappa_{,\phi} = 0. \quad (2.7)$$

For the functions $\xi(\phi)$ and $\kappa(\phi)$ respecting the Z_2 symmetry, there are in general the GR branch $\phi = 0$ and the nonvanishing scalar-field branch $\phi \neq 0$.

We consider the static and spherically symmetric background given by the line element

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.8)$$

where t , r , and (θ, φ) represent the time, radial, and angular coordinates (in the ranges $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$, respectively), and f , h are functions of r . For the scalar field, we consider a radial-dependent profile $\phi(r)$. The gravitational and scalar-field equations of motion are given by

$$h' = \frac{4(h-1)[1 + \kappa - 4h(\phi''\xi_{,\phi} + \phi'^2\xi_{,\phi\phi})] + hr^2\phi'^2 + hr[4(r\phi'' + 2\phi')\kappa_{,\phi} + 4r\phi'^2\kappa_{,\phi\phi}]}{2[4\phi'(3h-1)\xi_{,\phi} - r(r\phi'\kappa_{,\phi} + 2\kappa + 2)]}, \quad (2.9)$$

$$f' = \frac{f[4(h-1)(1 + \kappa) - hr^2\phi'^2 + 8hr\phi'\kappa_{,\phi}]}{2h[4\phi'(3h-1)\xi_{,\phi} - r(r\phi'\kappa_{,\phi} + 2\kappa + 2)]}, \quad (2.10)$$

$$\phi'' + \left(\frac{2}{r} + \frac{f'}{2f} + \frac{h'}{2h}\right)\phi' + \frac{R_{\text{GB0}}^2}{h}\xi_{,\phi} + \frac{R_0}{h}\kappa_{,\phi} = 0, \quad (2.11)$$

where a prime represents the derivative with respect to r , and

$$R_{\text{GB0}}^2 = \frac{2[(2ff'' - f'^2)h^2 + \{(3f'h' - 2f'')f + f'^2\}h - ff'h']}{r^2f^2}, \quad (2.12)$$

$$R_0 = -\frac{(2ff'' - f'^2)hr^2 + rff'(rh' + 4h) + 4rf^2h' + 4f^2(h-1)}{2r^2f^2}. \quad (2.13)$$

Note that R_{GB0}^2 and R_0 are the background values of the GB invariant and the Ricci scalar, respectively. On the Schwarzschild background given by the metric components $f = h = 1 - r_h/r$, where r_h is the horizon radius, we have $R_{\text{GB0}}^2 = 12r_h^2/r^6$ and $R_0 = 0$, respectively. If we consider a scalar-field perturbation $\delta\phi$ about the GR branch $\phi = 0$, then Eq. (2.7) gives the linearized equation

$$\square\delta\phi + \frac{12r_h^2}{r^6}\xi_{,\phi\phi}\delta\phi = 0. \quad (2.14)$$

The GR branch can be subject to tachyonic instability for

$$\xi_{,\phi\phi} > 0. \quad (2.15)$$

The simplest choice of the sGB coupling allowing the existence of a nonvanishing ϕ branch besides $\phi = 0$ is $\xi(\phi) = \eta\phi^2/8$. So long as the condition $\eta > 0$ is satisfied, the GR branch can trigger tachyonic instability toward the other branch $\phi \neq 0$. This is the phenomenon of spontaneous scalarization of BHs induced by the sGB coupling.

In Ref. [64], it was shown that the scalarized branch ($\phi \neq 0$) for the coupling $\xi(\phi) = \eta\phi^2/8$ is unstable against radial perturbations in the even-parity sector, as the tachyonic instability is never quenched by nonlinearities in the field equations. This can be understood as the appearance of a negative effective potential $V_{\text{eff}}(r)$ for the scalar-field perturbation $\delta\phi$ in the vicinity of the horizon. This radial tachyonic instability can be avoided by taking into account a coupling proportional to ϕ^4 in $\xi(\phi)$ [40,43,65], i.e.,

$$\xi(\phi) = \frac{\eta}{8}(\phi^2 + \alpha\phi^4), \quad \kappa(\phi) = 0, \quad (2.16)$$

which generates nonlinearities in ϕ in the scalar-field equation. A negative constant α can lift up the effective potential toward the region $V_{\text{eff}}(r) > 0$. Without taking into account the backreaction of metric perturbations to the scalar-field perturbation $\delta\phi$, it was shown in Ref. [43] that the negative region of $V_{\text{eff}}(r)$ disappears for $\alpha\phi_0^2 < -0.1155$, where ϕ_0 is the field value on the horizon. We note that the sGB coupling of the form $\xi(\phi) = (\eta/8\beta)(1 - e^{-\beta\phi^2})$ can also address the radial tachyonic instability problem [37]. For both the quartic and exponential couplings, since the leading behavior for the small field amplitude is given by $\xi(\phi) \rightarrow \eta\phi^2/8$, bifurcation of a scalarized branch from the GR Schwarzschild branch occurs at the same point in the phase diagram.

There is yet another model of BH spontaneous scalarization based on EsGBR theories given by the coupling functions [70–72]

$$\xi(\phi) = \frac{\eta}{8}\phi^2, \quad \kappa(\phi) = -\frac{\beta}{16}\phi^2. \quad (2.17)$$

In this case, there are also the nontrivial solution $\phi \neq 0$ as well as the GR branch $\phi = 0$. For the latter branch, we have

$R_0 = 0$ and hence the perturbation $\delta\phi$ about $\phi = 0$ obeys Eq. (2.14). Then, under the condition $\eta > 0$, the GR solution can trigger tachyonic instability toward the other nontrivial branch. Since the Ricci scalar R acquires the scalar-field contribution for $\phi \neq 0$, the third term on the left-hand side of Eq. (2.7) affects the stability of the branch $\phi \neq 0$. Taking into account the backreaction of metric perturbations to the scalar-field perturbation $\delta\phi$, it was shown in Ref. [72] that² the radially stable BH solutions can be present for $\beta > 5$. Kleihaus *et al.* [73] found that a new branch of scalarized BHs can emerge from a spherical scalarized branch for the couplings (2.17) in EsGBR theories by taking the quadrupolar deformation of the horizon configuration (multipole $l = 2$) into consideration. It is not yet clear whether there are parameter spaces consistent with linear stability conditions along both angular and radial directions, which we will address in the following.

III. LINEAR STABILITY CONDITIONS FOR HIGH RADIAL AND ANGULAR MOMENTUM MODES

To study the linear stability of BHs for the modes with large wave numbers k and multipoles l , we consider metric perturbations $h_{\mu\nu}$ on the background line element (2.8) given by the metric tensor $\bar{g}_{\mu\nu}$, such that $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. We express all the perturbations in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$, with the parity $(-1)^{l+1}$ for odd modes and the parity $(-1)^l$ for even modes under a rotation in the (θ, φ) plane [77–79]. Without loss of generality, we set $m = 0$ and consider the case in which Y_{l0} depends on θ alone. We choose the gauge in which the metric components h_{ab} , where the subscripts a and b are either θ or φ , are vanishing. We also take $h_0 = 0$, where h_0 is the perturbation appearing in the metric components $h_{ta}(=h_{at})$ in the form $\sum_l h_0(t, r)\nabla_a Y_{l0}(\theta)$. These choices completely fix the residual gauge degrees of freedom. Then, the non-vanishing metric components are

$$\begin{aligned} g_{tt} &= -f(r) + f(r)\sum_l H_0(t, r)Y_{l0}(\theta), \\ g_{tr} &= g_{rt} = \sum_l H_1(t, r)Y_{l0}(\theta), \\ g_{t\varphi} &= g_{\varphi t} = -\sum_l Q(t, r)(\sin\theta)Y_{l0,\theta}(\theta), \\ g_{rr} &= h^{-1}(r) + h^{-1}(r)\sum_l H_2(t, r)Y_{l0}(\theta), \\ g_{r\theta} &= g_{\theta r} = \sum_l h_1(t, r)Y_{l0,\theta}(\theta), \\ g_{r\varphi} &= g_{\varphi r} = -\sum_l W(t, r)(\sin\theta)Y_{l0,\theta}(\theta), \\ g_{\theta\theta} &= r^2, \quad g_{\varphi\varphi} = r^2\sin^2\theta, \end{aligned} \quad (3.1)$$

²Our notation of β is 4 times as large as that used in Ref. [72].

where $Q(t, r)$ and $W(t, r)$ correspond to odd-parity perturbations and $H_0(t, r)$, $H_1(t, r)$, $H_2(t, r)$, and $h_1(t, r)$ to even-parity perturbations. The scalar field contains the even-parity perturbation $\delta\phi$, as

$$\phi = \bar{\phi}(r) + \sum_l \delta\phi(t, r) Y_{l0}(\theta), \quad (3.2)$$

where we will omit a bar from the background quantities.

In Horndeski theories given by Eq. (2.2), the second-order action containing seven perturbed variables mentioned above was derived in Refs. [66–68] (see also Refs. [69,80] for related works). After integrating out some of the nondynamical variables, the resulting quadratic-order action contains one dynamical gravitational perturbation in the odd-parity sector and two dynamical perturbations in the even-parity sector. This final action, which is expressed in terms of three dynamical perturbations, determines conditions for the absence of ghosts and Laplacian instabilities in the limits of high radial and angular momentum modes. In the following, we will briefly revisit such conditions and apply them to EsGBR theories.

In the odd-parity sector with the multipoles $l \geq 2$, there is one propagating degree of freedom given by

$$\chi = \dot{W} - Q' + \frac{2Q}{r}, \quad (3.3)$$

where a dot represents the derivative with respect to t . The dynamical field χ does not behave as a ghost for

$$\begin{aligned} \mathcal{G} &\equiv 2G_4 + 2h\phi'^2 G_{4,X} - h\phi'^2 \left(G_{5,\phi} + \frac{f'h\phi'G_{5,X}}{2f} \right) \\ &= 1 + \kappa - \frac{2f'h\phi'\xi_{,\phi}}{f} > 0. \end{aligned} \quad (3.4)$$

In the short-wavelength limit, the squared propagation speed of χ along the radial direction is

$$c_{r,\text{odd}}^2 = \frac{\mathcal{G}}{\mathcal{F}}, \quad (3.5)$$

where the absence of radial Laplacian instability requires that

$$\begin{aligned} \mathcal{F} &\equiv 2G_4 + h\phi'^2 G_{5,\phi} - h\phi'^2 \left(\frac{1}{2} h' \phi' + h\phi'' \right) G_{5,X} \\ &= 1 + \kappa - 4\xi_{,\phi\phi} h\phi'^2 - 2(2h\phi'' + h'\phi') \xi_{,\phi} > 0, \end{aligned} \quad (3.6)$$

together with the condition (3.4). In the limit of large multipoles ($l \gg 1$), the squared propagation speed of χ along the angular direction is

$$c_{\Omega,\text{odd}}^2 = \frac{\mathcal{G}}{\mathcal{H}}, \quad (3.7)$$

and hence the angular Laplacian instability is absent for

$$\begin{aligned} \mathcal{H} &\equiv 2G_4 + 2h\phi'^2 G_{4,X} - h\phi'^2 G_{5,\phi} - \frac{h^2 \phi'^3 G_{5,X}}{r} \\ &= 1 + \kappa - \frac{4h\phi'\xi_{,\phi}}{r} > 0, \end{aligned} \quad (3.8)$$

besides the condition (3.4).

For the monopole ($l = 0$), the total second-order action of odd-parity perturbations vanishes identically. For the dipole ($l = 1$), there is no propagating degree of freedom in the odd-parity sector [67]. Hence, in both cases, we do not have additional stability conditions besides those derived for $l \geq 2$.

In the even-parity sector with the multipoles $l \geq 2$, there are two dynamical perturbations arising from the matter and gravity sectors. One is the scalar-field perturbation $\delta\phi$, while the other is the gravitational perturbation given by

$$\psi = H_2 + \frac{a_4}{a_3} l(l+1) h_1 + \frac{a_1}{a_3} \delta\phi', \quad (3.9)$$

where

$$\begin{aligned} a_1 &= \frac{\sqrt{f\bar{h}}}{2} [r^2 \kappa_{,\phi} + 4(1-h)\xi_{,\phi}], & a_3 &= -\frac{1}{2} \phi' a_1 - r a_4, \\ a_4 &= \frac{\sqrt{f\bar{h}}}{2} \mathcal{H}. \end{aligned} \quad (3.10)$$

After integrating out the nondynamical fields H_0 and H_1 from the second-order action of even-parity perturbations, the resulting quadratic-order action can be expressed in the form

$$\mathcal{S}_{\text{even}} = \sum_l \int dt dr \left(\dot{\vec{\chi}}' \mathbf{K} \dot{\vec{\chi}} + \vec{\chi}' \mathbf{G} \vec{\chi} + \vec{\chi}' \mathbf{S} \vec{\chi}' + \vec{\chi}' \mathbf{M} \vec{\chi} \right), \quad (3.11)$$

where $\vec{\chi}' = (\psi, \delta\phi)$, and \mathbf{K} , \mathbf{G} , \mathbf{M} are the 2×2 symmetric matrices, while \mathbf{S} is the 2×2 antisymmetric matrix.

The determinants of principal submatrices of \mathbf{K} are positive for [68]

$$\mathcal{K} \equiv 2\mathcal{P}_1 - \mathcal{F} > 0, \quad (3.12)$$

where the quantity \mathcal{P}_1 is defined by

$$\mathcal{P}_1 \equiv \frac{h\mu_s}{2fr^2\mathcal{H}^2} \left(\frac{fr^4\mathcal{H}^4}{\mu_s^2 h} \right)', \quad \mu_s \equiv -\frac{4a_3}{\sqrt{f\bar{h}}}. \quad (3.13)$$

Under the inequality (3.12), the ghosts arise neither for ψ nor $\delta\phi$.

In the short-wavelength limit, the radial propagation speeds of ψ and $\delta\phi$ can be obtained by assuming the solutions to the perturbation equations in the form $\vec{\mathcal{X}} \propto \vec{\mathcal{X}}_0 e^{i(\omega t - kr)}$. For large values of ω and k , terms containing the matrix components of \mathbf{K} and \mathbf{G} are the dominant contributions to the radial dispersion relation. The squared radial propagation speeds of ψ and $\delta\phi$, which are measured in terms of the rescaled radial coordinate $r_s = \int dr/\sqrt{h}$ and the proper time $\tau = \int dt\sqrt{f}$, are given, respectively, by

$$c_{r1,\text{even}}^2 = \frac{\mathcal{G}}{\mathcal{F}}, \quad (3.14)$$

$$c_{r2,\text{even}}^2 = \frac{4\phi'}{(fh)^{3/2}(2\mathcal{P}_1 - \mathcal{F})\mu_s^2} \left[8r^2 h a_4 c_4 (\phi' a_1 + r a_4) - \sqrt{fh} \phi' a_1^2 \mathcal{G} + 2r^2 a_4^2 \left(\frac{f'}{f} a_1 + 2c_2 \right) \right], \quad (3.15)$$

where

$$B_1 = \frac{a_4 r^3 [4h(\phi' a_1 + 2r a_4) \beta_1 + \beta_2 - 4\phi' a_1 \beta_3] - 2fh\mathcal{G}[2r a_4(2\mathcal{P}_1 - \mathcal{F})(\phi' a_1 + r a_4) + \phi'^2 a_1^2 \mathcal{P}_1]}{4\sqrt{fh} a_4 (\phi' a_1 + 2r a_4)^2 (2\mathcal{P}_1 - \mathcal{F})}, \quad (3.19)$$

$$B_2 = -r^2 \frac{r^2 h \beta_1 [2fh\mathcal{F}\mathcal{G}(\phi' a_1 + 2r a_4) + r^2 \beta_2] - r^4 \beta_2 \beta_3 - fh\mathcal{F}\mathcal{G}(\phi' fh\mathcal{F}\mathcal{G} a_1 + 4r^3 a_4 \beta_3)}{fh\mathcal{F}\phi' a_1 (\phi' a_1 + 2r a_4)^2 (2\mathcal{P}_1 - \mathcal{F})}, \quad (3.20)$$

where $\beta_1, \beta_2, \beta_3$ are given in the Appendix. To ensure the angular Laplacian stability, we require that $c_{\Omega+}^2 > 0$ and $c_{\Omega-}^2 > 0$. These conditions are satisfied if

$$B_1^2 \geq B_2 > 0 \quad \text{and} \quad B_1 < 0. \quad (3.21)$$

The violation of the condition $B_1^2 - B_2 \geq 0$ gives rise to the imaginary values of $c_{\Omega\pm}^2$, one of which leads to the Laplacian instability. Even for $B_1^2 - B_2 \geq 0$, the inequalities $B_1 < 0$ and $B_2 > 0$ need to be satisfied further to ensure that $c_{\Omega\pm}^2 > 0$.

Before ending this section, we consider theories given by the coupling functions

$$\xi(\phi) = \frac{\eta}{8}(\phi^2 + \alpha\phi^4), \quad \kappa(\phi) = -\frac{\beta}{16}\phi^2, \quad (3.22)$$

which accommodate both the models (2.16) and (2.17). For the scalarized BH solution where $|\phi(r)|$ is much smaller than 1, we expand the quantities \mathcal{G} , \mathcal{F} , \mathcal{H} , \mathcal{K} , and $c_{r2,\text{even}}^2$ around $\phi = 0$. On using the background

$$c_2 = \frac{\sqrt{h}}{4\sqrt{f}} [r^2 \phi' f - r(rf' + 4f)\kappa_{,\phi} + 4(3h-1)f'\xi_{,\phi}],$$

$$c_4 = \frac{1}{2r\sqrt{fh}} (rf\kappa_{,\phi} - 2f'h\xi_{,\phi}). \quad (3.16)$$

Since $c_{r1,\text{even}}^2$ is the same as $c_{r,\text{odd}}^2$, the Laplacian instability of ψ is absent under the two conditions (3.4) and (3.6). The radial Laplacian stability of $\delta\phi$ requires that

$$c_{r2,\text{even}}^2 > 0. \quad (3.17)$$

In the limit of large multipoles l and high frequencies ω , the dominant contributions to the angular dispersion relation arise from the matrix components of \mathbf{K} and \mathbf{M} . The squared angular propagation speeds of even-parity dynamical perturbations for $l \gg 1$, which are measured in terms of the infinitesimal angular distance $r d\theta$ and the proper time $d\tau$, are expressed, respectively, by

$$c_{\Omega\pm}^2 = -B_1 \pm \sqrt{B_1^2 - B_2}, \quad (3.18)$$

where

equations (2.9)–(2.11) to eliminate the derivative terms f' , h' , and ϕ'' , it follows that

$$\mathcal{G} = 1 - \frac{\eta(4 - 4h + hr^2\phi'^2)\phi'}{8r}\phi + \mathcal{O}(\phi^2), \quad (3.23)$$

$$\mathcal{F} = 1 - \eta h \phi'^2 + \frac{\eta(4 + 12h + hr^2\phi'^2)\phi'}{8r}\phi + \mathcal{O}(\phi^2), \quad (3.24)$$

$$\mathcal{H} = 1 - \frac{\eta h \phi'}{r}\phi + \mathcal{O}(\phi^2), \quad (3.25)$$

$$\mathcal{K} = \frac{r^2\phi'^2}{4} \left[1 + \frac{\{8\eta(h-1) + \beta r^2\}\phi'}{8r}\phi + \mathcal{O}(\phi^2) \right], \quad (3.26)$$

$$c_{r2,\text{even}}^2 = 1 - \frac{\eta h \phi'^2}{64} \left[\beta + \frac{8\eta(h-1)}{r^2} \right] \left[3\beta - 8 + \frac{24\eta(h-1)}{r^2} \right] \phi^2 + \mathcal{O}(\phi^3). \quad (3.27)$$

We study the case in which $\phi(r)$ is a positive decreasing function of r outside the horizon ($r \geq r_h$), with $|r\phi'|$ being at most of order ϕ . Then, the scalarized solution has the

largest field value ϕ_0 on the horizon ($r = r_h$). So long as the condition

$$\bar{\eta}\phi_0^2 \ll 1 \quad (3.28)$$

is satisfied, where

$$\bar{\eta} \equiv \frac{\eta}{r_h^2} \quad (3.29)$$

is a dimensionless coupling constant, all of the dominant contributions to \mathcal{G} , \mathcal{F} , and \mathcal{H} are 1 outside the horizon. If the inequality

$$\beta\phi_0^2 \ll 1 \quad (3.30)$$

holds in addition to the condition (3.28), the dominant contribution to \mathcal{K} is $r^2\phi^2/4$ and hence it is positive. If the two inequalities (3.28) and (3.30) are satisfied, then $c_{r2,\text{even}}^2$ is close to 1 for

$$\bar{\eta} \lesssim \mathcal{O}(1). \quad (3.31)$$

Thus, the stability conditions $\mathcal{G} > 0$, $\mathcal{F} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$, and $c_{r2,\text{even}}^2 > 0$ hold for the field value ϕ_0 and the couplings $\bar{\eta}$, β in the ranges (3.28), (3.30), and (3.31). Since the angular stability conditions associated with $c_{\Omega\pm}^2$ are more involved, we will study them in detail in Secs. V and VI.

IV. RADIAL TACHYONIC STABILITY OF SCALARIZED BHS

In this section, we will revisit the radial tachyonic stability of scalarized BHs in both EsGB and EsGBR theories. This amounts to studying a monopole perturbation without restricting the analysis to the short-wavelength limit $kr_h \gg 1$. In other words, we discuss the radial stability of the dynamical perturbation $\delta\phi$ for $l = 0$ by paying particular attention to an effective potential induced by its effective mass term. For $l = 0$, it was shown in Refs. [66,68,69] that the gravitational perturbation ψ does not propagate and hence the scalar-field perturbation $\delta\phi$ is the only propagating degree of freedom in the even-parity sector.³ As we will see below, the radial propagation speed squared c_r^2 for $l = 0$ is equivalent to Eq. (3.15) obtained for $l \geq 2$.

In Horndeski theories, the second-order action of even-parity perturbations was derived in Refs. [66,68,69] for the

perturbed metric components (3.1). Focusing on the monopole mode ($l = 0$), the action is expressed in the form $\mathcal{S}_{\text{even}} = \int dt dr \mathcal{L}$, where

$$\begin{aligned} \mathcal{L} = & H_0[a_1\delta\phi'' + a_2\delta\phi' + (a_2' - a_1'')\delta\phi + a_3H_2' + a_3'H_2] \\ & - \frac{2}{f}H_1\left[a_1\dot{\delta\phi}' + (a_2 - a_1')\dot{\delta\phi} + a_3\dot{H}_2\right] \\ & + c_1\dot{\delta\phi}H_2 + (c_2\delta\phi' + c_3\delta\phi)H_2 + c_6H_2^2 + e_1\dot{\delta\phi}^2 \\ & + e_2\delta\phi'^2 + e_3\delta\phi^2, \end{aligned} \quad (4.1)$$

where the explicit forms of the r -dependent coefficients are given in Eqs. (3.10), (3.16), (A3), and (A4) in the Appendix. For $l = 0$, the contributions to \mathcal{L} arising from the metric perturbation h_1 identically vanish. Hence we have the three metric perturbations H_0 , H_1 , H_2 and the scalar-field perturbation $\delta\phi$ in the second-order action of even-parity perturbations. We note that the radial stability of scalarized BHs was also discussed in Refs. [40,64,71,72] by considering the perturbations H_0 , H_2 , and $\delta\phi$. While we need to include the field H_1 for the consistent analysis, it does not eventually contribute to the dynamics of perturbations as we will see below.

Let us introduce the following perturbed quantity:

$$\Phi = a_1\delta\phi' + (a_2 - a_1')\delta\phi + a_3H_2. \quad (4.2)$$

Then, the Lagrangian density (4.1) can be written as

$$\begin{aligned} \mathcal{L} = & H_0\Phi' - \frac{2}{f}H_1\dot{\Phi} + c_1\dot{\delta\phi}H_2 + (c_2\delta\phi' + c_3\delta\phi)H_2 \\ & + c_6H_2^2 + e_1\dot{\delta\phi}^2 + e_2\delta\phi'^2 + e_3\delta\phi^2. \end{aligned} \quad (4.3)$$

Varying \mathcal{L} with respect to H_0 and H_1 , respectively, we obtain

$$\Phi' = 0, \quad \dot{\Phi} = 0, \quad (4.4)$$

so that $\Phi = \mathcal{C} = \text{constant}$. Since the integration constant \mathcal{C} is irrelevant to the perturbation dynamics, we set $\mathcal{C} = 0$ in the following. Then, we have

$$a_1\delta\phi' + (a_2 - a_1')\delta\phi + a_3H_2 = 0, \quad (4.5)$$

with the Lagrangian density

$$\begin{aligned} \mathcal{L} = & c_1\dot{\delta\phi}H_2 + (c_2\delta\phi' + c_3\delta\phi)H_2 + c_6H_2^2 + e_1\dot{\delta\phi}^2 \\ & + e_2\delta\phi'^2 + e_3\delta\phi^2. \end{aligned} \quad (4.6)$$

Solving Eq. (4.5) for H_2 and taking its time derivative, we can eliminate the H_2 -dependent terms in Eq. (4.6) to give

$$\mathcal{L} = K_0\dot{\delta\phi}^2 - G_0\delta\phi'^2 - M_0\delta\phi^2 + S_0\delta\phi'\delta\phi, \quad (4.7)$$

³For the dipole mode $l = 1$, we can choose the gauge $\delta\phi = 0$ to fix the residual gauge degree of freedom. In this case, ψ is the only propagating degree of freedom in the even-parity sector. Again, the no-ghost condition and the radial propagation speed of even-parity perturbations are the same as Eqs. (3.12) and (3.15), respectively [66,68,69].

where

$$K_0 = e_1 + \frac{[a_1 c'_1 + (3a'_1 - 2a_2)c_1]a_3 - a_1 c_1 a'_3}{2a_3^2}, \quad (4.8)$$

$$G_0 = -e_2 + \frac{a_1(a_3 c_2 - a_1 c_6)}{a_3^2}, \quad (4.9)$$

$$M_0 = -e_3 + \frac{(a_2 - a'_1)[c_3 a_3 - c_6(a_2 - a'_1)]}{a_3^2}, \quad (4.10)$$

$$S_0 = \frac{(a_2 - a'_1)(2a_1 c_6 - a_3 c_2) - a_1 a_3 c_3}{a_3^2}. \quad (4.11)$$

Thus, for $l = 0$, the dynamics of even-parity perturbations is governed by the single propagating degree of freedom $\delta\phi$. The ghost is absent under the condition $K_0 > 0$. Since K_0 can be expressed as $K_0 = (2\mathcal{P}_1 - \mathcal{F})/(\sqrt{f}h\phi'^2)$ [69], the no-ghost condition translates to $2\mathcal{P}_1 - \mathcal{F} > 0$. This is equivalent to the condition (3.12) derived for $l \geq 2$. The radial propagation speed squared of $\delta\phi$ measured in terms of the rescaled radial coordinate $r_s = \int dr/\sqrt{h}$ and the proper time $\tau = \int dt\sqrt{f}$ is given by

$$c_r^2 = \frac{G_0}{fhK_0}, \quad (4.12)$$

which is equivalent to $c_{r2,\text{even}}^2$ obtained for $l \geq 2$. Thus the radial Laplacian instability is absent for $c_r^2 > 0$, which translates to $G_0 > 0$ under the no-ghost condition $K_0 > 0$.

Varying (4.7) with respect to $\delta\phi$, it follows that

$$g_s^2(r)\delta\ddot{\phi} - \delta\phi'' + C_1(r)\delta\phi' + U(r)\delta\phi = 0, \quad (4.13)$$

where

$$g_s^2(r) \equiv \frac{K_0}{G_0} \frac{1}{fhc_r^2}, \quad C_1(r) \equiv -\frac{G'_0}{G_0}, \quad U(r) \equiv \frac{2M_0 + S'_0}{2G_0}. \quad (4.14)$$

To study the radial stability of $\delta\phi$ associated with the potential $U(r)$, we consider the solution to Eq. (4.13) in the form

$$\delta\phi(t, r) = C_0(r)\delta\varphi(r)e^{-i\omega t}, \quad (4.15)$$

where ω is an angular frequency, and $C_0(r)$ and $\delta\varphi(r)$ are r -dependent functions. We introduce the tortoise coordinate $r_* = \int g_s dr$ and choose $C_0(r)$ to eliminate the first derivative $d\delta\varphi/dr_*$ in the equation of motion for $\delta\varphi$ following from Eq. (4.13). Then, we obtain

$$\frac{C'_0}{C_0} = \frac{1}{2} \left(C_1 - \frac{g'_s}{g_s} \right), \quad (4.16)$$

and

$$\left(-\frac{d^2}{dr_*^2} + V_{\text{eff}} \right) \delta\varphi = \omega^2 \delta\varphi, \quad (4.17)$$

where

$$V_{\text{eff}} = \frac{1}{g_s^2} \left(U + \frac{C_1^2}{4} - \frac{C'_1}{2} - \frac{3g_s^2}{4g_s^2} + \frac{g_s''}{2g_s} \right). \quad (4.18)$$

If the effective potential $V_{\text{eff}}(r_*)$ is negative at some distance r_* , it signals the presence of radial tachyonic instability. In the following, we will study the shapes of $V_{\text{eff}}(r)$ in both EsGB theories and EsGBR theories.

A. EsGB theories

In EsGB theories characterized by the coupling functions $\xi(\phi) \neq 0$ and $\kappa(\phi) = 0$, the quantity e_3 appearing in the mass term M_0 in Eq. (4.10) is given by

$$e_3 = \frac{r^2 \sqrt{f}}{4\sqrt{h}} \xi_{,\phi\phi} R_{\text{GB}0}^2. \quad (4.19)$$

The dominant contributions to f and h are the Schwarzschild metric components $f = h = 1 - r_h/r$. Then, the leading-order contribution to the background GB invariant (2.12) is given by $R_{\text{GB}0}^2 = 12r_h^2/r^6$. The metric perturbation H_2 affects the quantities K_0 , G_0 , and M_0 through the terms containing c_1 , c_2 , c_3 , and c_6 in Eqs. (4.8)–(4.11). If we neglect the backreaction of metric perturbations to $\delta\phi$, the potential U in Eq. (4.14) is approximately given by

$$U \simeq \frac{M_0}{G_0} \simeq \frac{e_3}{e_2} = -\frac{1}{h} \xi_{,\phi\phi} R_{\text{GB}0}^2, \quad (4.20)$$

where we used $e_2 = -r^2 \sqrt{f}h/4$.

Let us consider the coupling $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$ and focus on the regime in which the two conditions (3.28) and (3.31) hold, i.e.,

$$\bar{\eta}\phi_0^2 \ll 1, \quad \bar{\eta} \lesssim \mathcal{O}(1), \quad (4.21)$$

where we recall that ϕ_0 is the field value on the horizon. Then, as we showed in Eq. (3.27), c_r^2 is close to 1. In this regime, we can exploit the approximation that the backreaction of the scalar field on the metric is negligible. This means that, for the estimations of g_s and C_1 , the metric components f and h are approximated as $f \simeq h \simeq 1 - r_h/r$. Since we have $g_s \simeq (1 - r_h/r)^{-1}$ and $G_0 \simeq (r^2/4)(1 - r_h/r)$, the effective potential (4.18) reduces to

$$V_{\text{eff}} \simeq \frac{r_h(r - r_h)}{r^7} (r^3 - 12r_h \xi_{,\phi\phi}), \quad (4.22)$$

where $\xi_{,\phi\phi} = \eta(1 + 6\alpha\phi^2)/4$. This approximate effective potential coincides with the one derived in Ref. [43]. For $\alpha = 0$ and $\bar{\eta} = \mathcal{O}(1)$, we have $V_{\text{eff}}(r) < 0$ in the vicinity of the horizon and hence the radial instability is present for the scalarized branch. For $\alpha < 0$, it is possible to realize $V_{\text{eff}}(r) > 0$ throughout the horizon exterior. The analysis of Ref. [43] based on the approximate potential (4.22) shows that the negative region of $V_{\text{eff}}(r)$ disappears for

$$\alpha\phi_0^2 < -0.1155. \quad (4.23)$$

This condition was derived by neglecting the backreaction of metrics to the scalar-field perturbation $\delta\phi$.

In order to confirm whether the condition (4.23) gives a good criterion for the radial tachyonic stability of $\delta\phi$, we numerically compute $V_{\text{eff}}(r)$ in Eq. (4.18) without using the approximations for U , \mathcal{C}_1 , and g_s explained above. In Fig. 1, we plot V_{eff} versus r/r_h for $\phi_0 = 0.01$ by choosing three different combinations of $(\bar{\eta}, \alpha)$. In case (a), where the condition (4.23) is violated, we have $V_{\text{eff}}(r) < 0$ in the vicinity of $r = r_h$. Case (b) corresponds to the border value $\alpha\phi_0^2 = -0.1155$, below which the region with $V_{\text{eff}}(r) < 0$ starts to disappear. In case (c), where the inequality (4.23) is satisfied, $V_{\text{eff}}(r)$ is positive for $r \geq r_h$. Thus, the full analysis of the effective potential shows that the condition (4.23) is a trustable criterion for the radial stability of

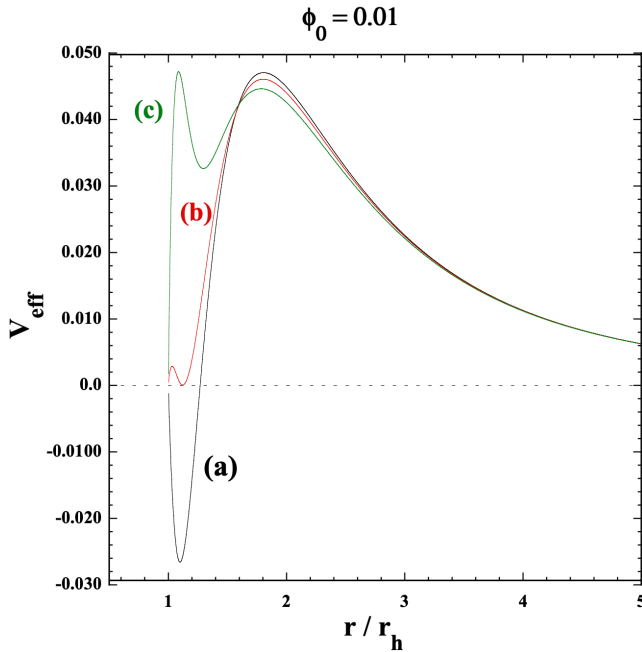


FIG. 1. The effective potential V_{eff} versus r/r_h in EsGB theories with the coupling $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$ and the field value $\phi_0 = 0.01$ on the horizon. Each case corresponds to the model parameters (a) $\bar{\eta} = 0.7742$, $\alpha = -500$, (b) $\bar{\eta} = 0.8500$, $\alpha = -1155$, and (c) $\bar{\eta} = 0.9807$, $\alpha = -2000$, respectively. For $\alpha\phi_0^2 < -0.1155$, the effective potential is positive throughout the horizon exterior.

scalarized BHs. We confirm that this is also the case for other field values ϕ_0 in the range (4.21).

B. EsGB theories

In EsGB theories given by the coupling functions $\xi(\phi) = \eta\phi^2/8$ and $\kappa(\phi) = -\beta\phi^2/16$, the quantity e_3 in M_0 is given by

$$e_3 = \frac{r^2\sqrt{f}}{32\sqrt{h}}(2\eta R_{\text{GB}0}^2 - \beta R_0). \quad (4.24)$$

On the GR branch ($\phi = 0$) with the Schwarzschild background ($f = h = 1 - r_h/r$), we have $R_{\text{GB}0}^2 = 12r_h^2/r^6$ and $R_0 = 0$. Then, for $\eta > 0$, this GR solution can be subject to tachyonic instability toward the scalarized branch ($\phi \neq 0$) due to the dominance of a negative term in M_0 induced by the sGB coupling. After the solution reaches the scalarized branch, the Ricci scalar R_0 acquires the contribution of ϕ through the coupling between gravity and the scalar field. For $\beta > 0$, the $-\beta R_0$ term in e_3 gives rise to a positive contribution to M_0 .

Moreover, since we are considering the scalar-field contribution to the gravity sector at the background level, we cannot generally neglect the backreaction of the metric perturbation H_2 on the dynamics of $\delta\phi$. In other words, we should exploit the full expressions of K_0 , G_0 , M_0 , and S_0 in Eqs. (4.8)–(4.11) to accurately compute the quantities U , \mathcal{C}_1 , and g_s^2 appearing in the effective potential $V_{\text{eff}}(r)$. Indeed, unlike EsGB theories discussed in Sec. IV A, the analysis based on ignoring the backreaction of H_2 on $\delta\phi$ leads to different forms of $V_{\text{eff}}(r)$ especially in the vicinity of the horizon. In other words, even if the full analysis without using any approximation gives $V_{\text{eff}}(r) > 0$ throughout the horizon exterior, it can happen that the approximation neglecting the metric backreaction generates the region of negative values of $V_{\text{eff}}(r)$.

In Fig. 2, we plot V_{eff} versus r/r_h for $\phi_0 = 0.01$ with four different combinations of $(\bar{\eta}, \beta)$. Note that this effective potential is numerically computed without neglecting the backreaction of H_2 on $\delta\phi$. For $\beta = 1$, there is the region with $V_{\text{eff}}(r) < 0$ around the horizon. In the other three cases, which correspond to $\beta = 8, 20, 50$, we have $V_{\text{eff}}(r) > 0$ throughout the horizon exterior. This means that, with the field value $\phi_0 = 0.01$, the radial tachyonic instability of $\delta\phi$ is absent for $\beta > \mathcal{O}(1)$. The study of Ref. [72] shows that the radial stability is ensured for

$$\beta > 5, \quad (4.25)$$

irrespective of the field value ϕ_0 on the horizon. Our analysis is consistent with this condition. Even with small values of $|\phi_0|$ of order 10^{-3} , we confirm that $V_{\text{eff}}(r) > 0$ outside the horizon for β in the range (4.25). We stress that this positivity of $V_{\text{eff}}(r)$ in the range $|\phi_0| \ll 1$ arises by consistently incorporating the backreaction of metric

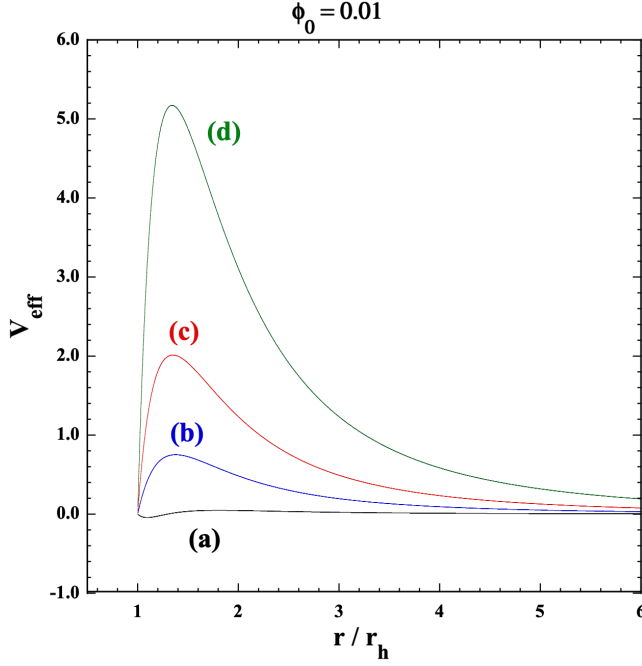


FIG. 2. The effective potential V_{eff} versus r/r_h in EsGB theories with the couplings $\xi(\phi) = \eta\phi^2/8$ and $\kappa(\phi) = -\beta\phi^2/16$ for the field value $\phi_0 = 0.01$ on the horizon. Each case corresponds to the model parameters (a) $\bar{\eta} = 0.72565$, $\beta = 1$, (b) $\bar{\eta} = 0.72568$, $\beta = 8$, (c) $\bar{\eta} = 0.72612$, $\beta = 20$, and (d) $\bar{\eta} = 0.72701$, $\beta = 50$, respectively.

perturbations on the dynamics of $\delta\phi$. Interestingly, the absence of radial tachyonic instability of $\delta\phi$ does not particularly impose the minimal values of $|\phi_0|$. This property is different from EsGB theories discussed in Sec. IV A, where the amplitude of ϕ_0 is constrained to be $|\phi_0| > \sqrt{0.1155/|\alpha|}$.

V. BH LINEAR STABILITY IN ESGB THEORIES

In this section, we investigate the angular Laplacian stability of scalarized BHs in EsGB theories given by the couplings (2.16). We first analytically study the Laplacian stability conditions far away from the horizon ($r \gg r_h$). Then, we consider the regime in the vicinity of the horizon and find the parameter region in which both $c_{\Omega+}^2$ and $c_{\Omega-}^2$ are positive. We also numerically compute $c_{\Omega\pm}^2$ outside the horizon and study their behavior in an intermediate regime. Then, we elucidate the parameter space in which the scalarized BHs are linearly stable in both angular and radial directions.

A. Region far away from the horizon

At spatial infinity, we impose a boundary condition of the vanishing scalar field, i.e., $\phi(\infty) = 0$. At large distances

the scalar field acquires a charge q_s , so that the solution can be expressed in the form $\phi = q_s/r + \mathcal{O}(r^{-2})$. The Schwarzschild metric components $f = h = 1 - m/r$, where m corresponds to twice the Arnowitt-Deser-Misner (ADM) mass, are modified by the scalar charge (note that $m = r_h$ for the Schwarzschild metric). The background solutions to ϕ , f , and h , which are expanded up to the fifth order in $1/r$, are given by

$$\begin{aligned} \phi = & \frac{q_s}{r} + \frac{q_s m}{2r^2} - \frac{q_s(q_s^2 - 8m^2)}{24r^3} - \frac{q_s m(q_s^2 - 3m^2)}{12r^4} \\ & + \frac{9q_s^5 - 8q_s(29q_s^2 + 36\eta)m^2 + 384q_s m^4}{1920r^5} \\ & + \mathcal{O}(r^{-6}), \end{aligned} \quad (5.1)$$

$$\begin{aligned} f = & 1 - \frac{m}{r} + \frac{q_s^2 m}{24r^3} + \frac{q_s^2 m^2}{24r^4} - \frac{q_s^2 m(3q_s^2 - 24m^2 - 256\eta)}{640r^5} \\ & + \mathcal{O}(r^{-6}), \end{aligned} \quad (5.2)$$

$$\begin{aligned} h = & 1 - \frac{m}{r} + \frac{q_s^2}{4r^2} + \frac{q_s^2 m}{8r^3} + \frac{q_s^2 m^2}{12r^4} - \frac{q_s^2 m(q_s^2 - 24m^2 - 384\eta)}{384r^5} \\ & + \mathcal{O}(r^{-6}). \end{aligned} \quad (5.3)$$

While the coupling constant η appears at the fifth-order expansions of ϕ , f , h in $1/r$, the other coupling constant α arises at their seventh-order expansions. In order to correctly estimate the quantities associated with stability of scalarized BHs, we need to expand ϕ , f , h at least up to ninth order, which will be used in the following.

The quantities associated with the stability of odd-parity perturbations are given by

$$\begin{aligned} \mathcal{G} = & 1 + \frac{\eta q_s^2 m}{2r^5} + \mathcal{O}(r^{-6}), \quad \mathcal{F} = 1 - \frac{3\eta q_s^2}{r^4} + \mathcal{O}(r^{-5}), \\ \mathcal{H} = & 1 + \frac{\eta q_s^2}{r^4} + \mathcal{O}(r^{-5}), \end{aligned} \quad (5.4)$$

whose leading-order terms are all positive.

In the even-parity sector, the no-ghost quantity (3.12) yields

$$\mathcal{K} = \frac{q_s^2}{4r^2} + \frac{q_s^2 m}{2r^3} + \mathcal{O}(r^{-4}), \quad (5.5)$$

so that $\mathcal{K} > 0$ at leading order. The radial propagation speed squared (3.15) is estimated as $c_{r2,\text{even}}^2 = 1 + \mathcal{O}(r^{-8})$ and hence it is very close to 1. The quantities B_1 and B_2 defined in Eqs. (3.19) and (3.20) are given, respectively, by

$$B_1 = -1 + \frac{\eta q_s^2}{2r^4} - \frac{\eta[5q_s^2 m^2 - (1 + 24\alpha)q_s^4]}{24r^6} + \frac{\eta q_s^2 m[(1 + 12\alpha)q_s^2 - 4m^2]}{12r^7} - \frac{\eta q_s^2[q_s^4 + 3q_s^2\{40\eta - (11 + 40\alpha)m^2\} + 101m^4 + 1188\eta m^2]}{240r^8} + \mathcal{O}(r^{-9}), \quad (5.6)$$

$$B_2 = 1 - \frac{\eta q_s^2}{r^4} + \frac{\eta[5q_s^2 m^2 - (1 + 24\alpha)q_s^4]}{12r^6} - \frac{\eta q_s^2 m[(1 + 12\alpha)q_s^2 - 4m^2]}{6r^7} + \frac{\eta q_s^2[q_s^4 + 3q_s^2\{40\eta - (11 + 40\alpha)m^2\} + 101m^4 + 108\eta m^2]}{120r^8} + \mathcal{O}(r^{-9}). \quad (5.7)$$

Then, it follows that

$$B_1^2 - B_2 = \frac{\eta^2 q_s^2 (q_s^2 + 36m^2)}{4r^8} + \mathcal{O}(r^{-9}), \quad (5.8)$$

whose leading-order contribution to $B_1^2 - B_2$ is positive.⁴ Our new result (5.8) means that both $c_{\Omega+}^2$ and $c_{\Omega-}^2$ are real. Since B_1 and B_2 approach -1 and 1 as $r \rightarrow \infty$, respectively, the conditions (3.21) are all satisfied at large distances and hence there are no angular Laplacian instabilities of even-parity perturbations.

B. Region in the vicinity of the horizon

Around the horizon characterized by the distance r_h , we expand the metric components and the scalar field, as

$$f = \sum_{i=1} f_i (r - r_h)^i, \quad h = \sum_{i=1} h_i (r - r_h)^i, \quad (5.9)$$

$$\phi = \sum_{i=0} \phi_i (r - r_h)^i,$$

where f_i , h_i , and ϕ_i are constants. The first component f_1 is not particularly constrained from the background equations, while h_1 and ϕ_1 are given by

$$h_1 = -\frac{\phi_1}{3\bar{\eta}\phi_0(2\alpha\phi_0^2 + 1)},$$

$$\phi_1 = -\frac{1 - \sqrt{1 - 6\bar{\eta}^2\phi_0^2(2\alpha\phi_0^2 + 1)^2}}{\bar{\eta}r_h\phi_0(2\alpha\phi_0^2 + 1)}, \quad (5.10)$$

where $\bar{\eta}$ is defined in Eq. (3.29). To realize a scalar-field profile where $|\phi(r)|$ is a decreasing function of r around the horizon, we require that $\phi_0\phi_1 < 0$. For $\bar{\eta} > 0$, this

⁴There was an error in the calculation of Ref. [81], which led to the incorrect result that the leading-order term of $B_1^2 - B_2$ is negative. In deriving the results in Ref. [81], there was a typo in the input of the coefficient d_3 defined in Eq. (A2) in the *Mathematica* code. This led to a wrong result in the $\mathcal{O}(r^{-7})$ terms of B_1 and B_2 . After correcting this, the $\mathcal{O}(r^{-7})$ terms in $B_1^2 - B_2$ cancel out, which gives rise to no instabilities in the angular directions in the large-distance limit.

demands the inequality $2\alpha\phi_0^2 + 1 > 0$. Combining this with Eq. (4.23), we obtain

$$-\frac{1}{2} < \alpha\phi_0^2 < -0.1155, \quad (5.11)$$

whose conditions are required for the existence of hairy BH solutions without the radial tachyonic instability. To ensure that ϕ_1 is real, we also need the inequality $6\bar{\eta}^2\phi_0^2(2\alpha\phi_0^2 + 1)^2 < 1$. For $\bar{\eta}\phi_0$ smaller than the order 1, this condition is satisfied under the bound (5.11).

On using the expansion (5.9), we can estimate \mathcal{G} , \mathcal{F} , and \mathcal{H} at the horizon $r = r_h$ to be

$$\mathcal{G}(r_h) = \mathcal{F}(r_h)$$

$$= \frac{2}{\sqrt{1 - 6\bar{\eta}^2\phi_0^2(2\alpha\phi_0^2 + 1)^2} + 1}, \quad \mathcal{H}(r_h) = 1, \quad (5.12)$$

which are positive. In the regime where the inequality $\bar{\eta}\phi_0^2 \ll 1$ is satisfied, the leading-order contribution to Eq. (3.26) on the horizon is given by

$$\mathcal{K}(r_h) = \frac{r^2\phi(r)^2}{4} \Big|_{r=r_h} = \frac{9\bar{\eta}^2\phi_0^2}{4}, \quad (5.13)$$

which is positive. We also find that $c_{r2,\text{even}}^2$ is equivalent to 1 at $r = r_h$. As for the squared angular propagation speeds in the even-parity sector on the horizon, the expansions of B_1 and B_2 for small values of ϕ_0 close to 0 give

$$B_1(r_h) = -1 - \frac{21}{4}\bar{\eta}^2\phi_0^2 + \mathcal{O}(\phi_0^4),$$

$$B_2(r_h) = 1 + \frac{3}{2}\bar{\eta}^2\phi_0^2 + \mathcal{O}(\phi_0^4), \quad (5.14)$$

and hence $B_1(r_h)^2 - B_2(r_h) = 9\bar{\eta}^2\phi_0^2 + \mathcal{O}(\phi_0^4)$. Then, the angular Laplacian stability conditions (3.21) are satisfied in this regime. For ϕ_0 close to the order 1, the approximation (5.14) loses its validity. In Sec. V C, we will numerically find the parameter space in which all the linear stability conditions are satisfied.

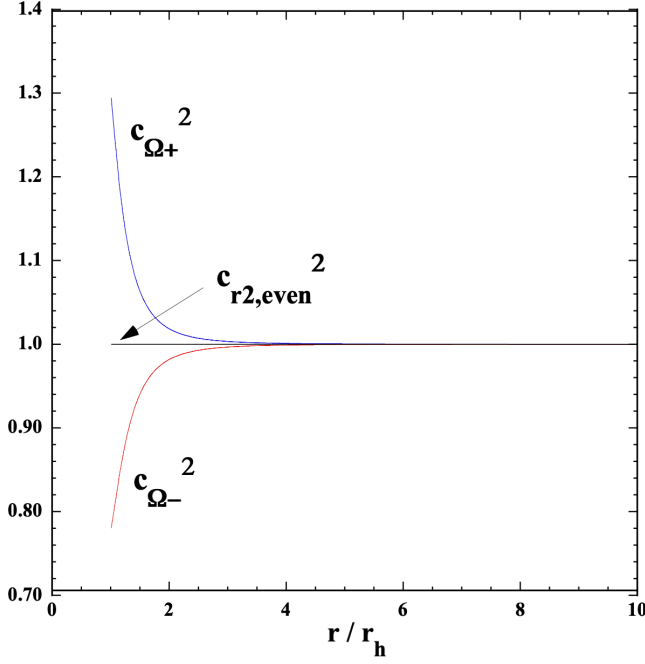


FIG. 3. We plot $c_{\Omega+}^2$, $c_{\Omega-}^2$, and $c_{r2,even}^2$ versus r/r_h for a scalarized BH with $\alpha = -10$ and $\bar{\eta} = 1.0257$ in EsGB theories given by the coupling (2.16). In this case, the field value on the horizon realizing the asymptotic behavior $\phi(\infty) = 0$ is $\phi_0 = 0.15$.

C. Parameter space consistent with the linear stability

In Secs. VA and VB, we showed that there are model parameter spaces consistent with all the linear stability conditions both at large distances and the horizon. In order to confirm the stability in an intermediate regime between the horizon and spatial infinity, we numerically compute quantities associated with the angular Laplacian stability by

using the background scalarized BH solutions. For given α and $\bar{\eta}(= \eta/r_h^2)$, we obtain the field value ϕ_0 on the horizon leading to the boundary condition $\phi(\infty) = 0$ at spatial infinity. Without loss of generality, we will focus on the case $\phi_0 > 0$.

In Fig. 3, we show an example for the radial dependence of $c_{\Omega+}^2$, $c_{\Omega-}^2$, and $c_{r2,even}^2$ outside the horizon. For these values of α and $\bar{\eta}$, the squared angular propagation speeds computed from Eq. (3.18) are $c_{\Omega+}^2 = 1.30$ and $c_{\Omega-}^2 = 0.78$ on the horizon, which are in good agreement with their numerical values in Fig. 3. As r increases from the horizon, both $c_{\Omega+}^2$ and $c_{\Omega-}^2$ continuously approach the asymptotic value 1. Since $c_{\Omega+}^2 > 0$ and $c_{\Omega-}^2 > 0$ throughout the horizon exterior, there are no angular Laplacian instabilities for even-parity perturbations. As we also observe in Fig. 3, $c_{r2,even}^2$ is close to 1 outside the horizon. Numerically, we also confirmed that all the other linear stability conditions $\mathcal{G} > 0$, $\mathcal{F} > 0$, $\mathcal{H} > 0$, and $\mathcal{K} > 0$ are satisfied.

We also carry out numerical simulations for the values of α and $\bar{\eta}$ different from those used in Fig. 3. The general property is that $c_{\Omega+}^2 > 1$ and $c_{\Omega-}^2 < 1$ outside the horizon. Then, for the angular Laplacian stability of even-parity perturbations, we only need to study whether $c_{\Omega-}^2$ is positive or not. Moreover, the deviation of $c_{\Omega-}^2$ from 1 is most significant at $r = r_h$ (see Fig. 3). Then, what determines the angular BH stability in the even-parity sector is the value $c_{\Omega-}^2$ on the horizon. In Fig. 4, we plot $c_{\Omega-}^2(r_h)$ together with $c_{\Omega+}^2(r_h)$ and $\bar{\eta}$ as functions of ϕ_0 by choosing three different values of α . For given α and ϕ_0 , the other parameter $\bar{\eta}$ is uniquely determined for the realization of scalarized BH solutions with the boundary condition $\phi(\infty) = 0$. The dashed lines in Fig. 4, which are outside the range (5.11), are excluded by the radial tachyonic instability.

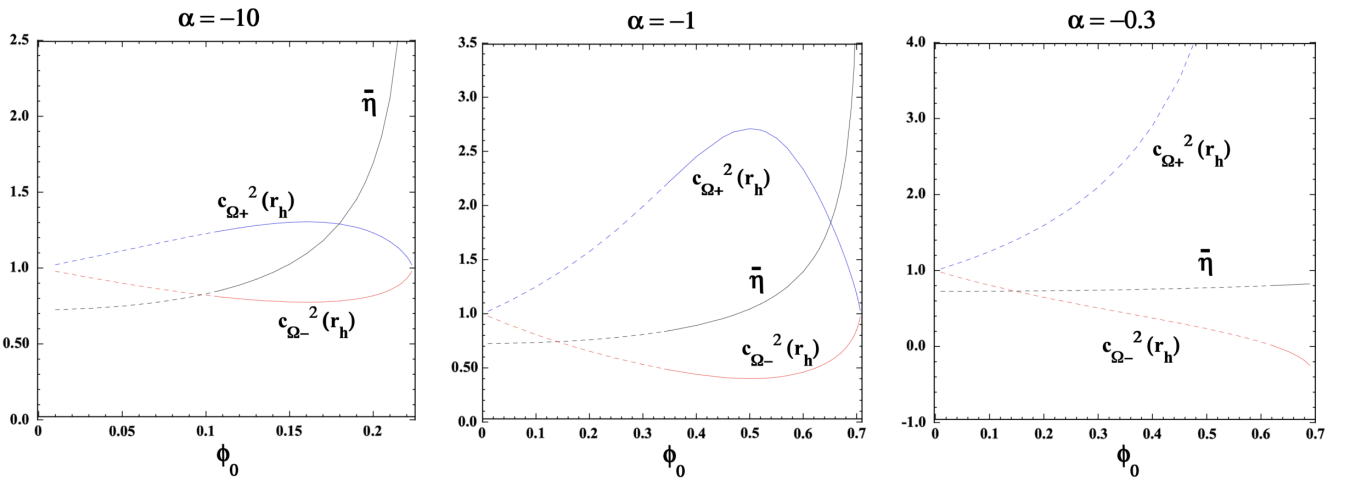


FIG. 4. The squared angular propagation speeds $c_{\Omega+}^2$ and $c_{\Omega-}^2$ on the horizon and $\bar{\eta}$ as functions of $\phi_0 (> 0)$ in EsGB theories with the coupling (2.16). The left, middle, and right panels correspond to $\alpha = -10, -1, -0.3$, respectively. The dashed lines represent the regions excluded by the radial tachyonic instability, whereas the solid lines belong to the parameter range $-1/2 < \alpha\phi_0^2 < -0.1155$.

For $\alpha = -10$, $c_{\Omega-}^2(r_h)$ has a minimum value 0.77 around $\phi_0 = 0.16$ and increases as a function of ϕ_0 in the range $\phi_0 > 0.16$. For $\alpha = -1$, $c_{\Omega-}^2(r_h)$ exhibits a similar behavior, with the minimum value 0.40 around $\phi_0 = 0.50$. The solid lines in Fig. 4 correspond to the range $-1/2 < \alpha\phi_0^2 < -0.1155$ [Eq. (5.11)], in which the radial tachyonic instability is absent. As we increase ϕ_0 , the product $\alpha\phi_0^2$ approaches the lower limit $-1/2$. On using the expansion $\alpha\phi_0^2 = -(1/2)(1 - \epsilon)$, where ϵ is a small positive parameter close to 0, the squared angular propagation speeds can be estimated as

$$\begin{aligned} c_{\Omega+}^2(r_h) &= 1 + 3\bar{\eta}\phi_0\epsilon + \mathcal{O}(\epsilon^2), \\ c_{\Omega-}^2(r_h) &= 1 - 3\bar{\eta}\phi_0\epsilon + \mathcal{O}(\epsilon^2). \end{aligned} \quad (5.15)$$

In the limit $\alpha\phi_0^2 \rightarrow -1/2$, both $c_{\Omega+}^2(r_h)$ and $c_{\Omega-}^2(r_h)$ approach 1. Indeed, this behavior can be seen in the left and middle panels of Fig. 4. For $\alpha \lesssim -1$, we have $c_{\Omega-}^2(r_h) > 0$ for the field value ϕ_0 in the range $-1/2 < \alpha\phi_0^2 < -0.1155$. In such regions of the parameter space, we also confirmed that all the other linear stability conditions $\mathcal{G} > 0$, $\mathcal{F} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$, and $c_{r2,\text{even}}^2 > 0$ are satisfied throughout the horizon exterior.

If $\alpha \gtrsim -1$, i.e., $|\alpha|$ less than the order 1, then we require larger values of ϕ_0 to satisfy the condition $\alpha\phi_0^2 < -0.1155$ as compared to the case $\alpha \lesssim -1$. As we observe in the right panel of Fig. 4, which corresponds to $\alpha = -0.3$, the field value in the range $\phi_0 < 0.62$ is excluded by the radial tachyonic instability. For ϕ_0 exceeding 0.62, $c_{\Omega-}^2(r_h)$ starts to enter the region $c_{\Omega-}^2(r_h) < 0$. When ϕ_0 approaches the limit $\alpha\phi_0^2 = -1/2$, we numerically find that it is difficult to realize scalarized hairy BH solutions with $\alpha \gtrsim -1$ due to the largeness of ϕ_0 and the small variation of $\phi(r)$ around $r = r_h$. For $\alpha = -0.3$, $c_{\Omega-}^2(r_h)$ does not start to grow from negative values toward 1 with the increase of ϕ_0 . In the right panel of Fig. 4, we see that there are almost no parameter spaces consistent with both angular Laplacian stability and radial tachyonic stability.

From the above discussion, so long as the coupling α and the horizon field value ϕ_0 are in the ranges

$$\alpha \lesssim -1 \quad \text{and} \quad -\frac{1}{2} < \alpha\phi_0^2 < -0.1155, \quad (5.16)$$

there are scalarized BH solutions compatible with all the linear stability conditions outside the horizon.

VI. BH LINEAR STABILITY IN ESGR THEORIES

Finally, we study the angular Laplacian stability of scalarized BHs in ESGR theories given by the couplings

(2.17). We first consider the two asymptotic regimes far away from the horizon and around the horizon and then find the parameter space consistent with all the linear stability conditions.

A. Region far away from the horizon

Imposing the boundary conditions $f \rightarrow 1$, $h \rightarrow 1$, and $\phi \rightarrow 0$ at spatial infinity, the background solutions to f , h , and ϕ , which are expanded far away from the horizon ($r \gg r_h$), are given by

$$\begin{aligned} f &= 1 - \frac{m}{r} + \frac{\beta q_s^2}{16r^2} - \frac{(3\beta - 4)m q_s^2}{96r^3} \\ &\quad - \frac{q_s^2(3\beta^3 q_s^2 - 20\beta^2 q_s^2 + 16\beta q_s^2 + 112\beta m^2 - 128m^2)}{3072r^4} \\ &\quad + \mathcal{O}(r^{-5}), \end{aligned} \quad (6.1)$$

$$\begin{aligned} h &= 1 - \frac{m}{r} - \frac{(\beta - 2)q_s^2}{8r^2} - \frac{(3\beta - 4)m q_s^2}{32r^3} \\ &\quad + \frac{q_s^2(3\beta^3 q_s^2 - 5\beta^2 q_s^2 + 4\beta q_s^2 - 56\beta m^2 + 64m^2)}{768r^4} \\ &\quad + \mathcal{O}(r^{-5}), \end{aligned} \quad (6.2)$$

$$\begin{aligned} \phi &= \frac{q_s}{r} + \frac{m q_s}{2r^2} - \frac{q_s(3\beta^2 q_s^2 - 8\beta q_s^2 + 16q_s^2 - 128m^2)}{384r^3} \\ &\quad - \frac{q_s m(9\beta^2 q_s^2 - 36\beta q_s^2 + 64q_s^2 - 192m^2)}{768r^4} \\ &\quad + \mathcal{O}(r^{-5}), \end{aligned} \quad (6.3)$$

where m corresponds to twice the ADM mass, and q_s is the scalar charge. Unlike the coupling α in EsGB theories, the nonminimal coupling constant β appears at the order of r^{-2} in f and h and of r^{-3} in ϕ . To obtain all the linear stability conditions correctly, we need to perform the expansions of f , h , and ϕ up to the order r^{-9} .

The quantities relevant to the stability of odd-parity perturbations are given by

$$\begin{aligned} \mathcal{G} &= 1 - \frac{\beta q_s^2}{16r^2} + \mathcal{O}(r^{-3}), \quad \mathcal{F} = 1 - \frac{\beta q_s^2}{16r^2} + \mathcal{O}(r^{-3}), \\ \mathcal{H} &= 1 - \frac{\beta q_s^2}{16r^2} + \mathcal{O}(r^{-3}), \end{aligned} \quad (6.4)$$

whose dominant terms are all positive. The leading-order contribution to \mathcal{K} is the same as the first term in Eq. (5.5), so that the no-ghost condition is satisfied at large distances. We also have $c_{r2,\text{even}}^2 = 1 + \mathcal{O}(r^{-8})$ and hence the radial Laplacian stability of even-parity perturbations is ensured.

The quantities B_1 and B_2 have the following asymptotic behaviors:

$$B_1 = -1 + \frac{\eta q_s^2}{2r^4} - \frac{\eta q_s^2[(15\beta^2 - 16\beta - 16)q_s^2 + 80m^2]}{384r^6} - \frac{\eta q_s^2 m[(15\beta^2 - 16\beta - 32)q_s^2 + 128m^2]}{384r^7} \\ + \frac{\eta q_s^2[(\beta\{320 + \beta[592 + 9\beta(41\beta - 108)]\} - 512)q_s^4 + 96\{176 + 5\beta(2 - 5\beta)\}q_s^2 m^2 - 51712m^4 - 6144\eta(10q_s^2 + 99m^2)]}{122880r^8} \\ + \mathcal{O}(r^{-9}), \quad (6.5)$$

$$B_2 = 1 - \frac{\eta q_s^2}{r^4} + \frac{\eta q_s^2[(15\beta^2 - 16\beta - 16)q_s^2 + 80m^2]}{192r^6} + \frac{\eta q_s^2 m[(15\beta^2 - 16\beta - 32)q_s^2 + 128m^2]}{192r^7} \\ - \frac{\eta q_s^2[(\beta\{320 + \beta[592 + 9\beta(41\beta - 108)]\} - 512)q_s^4 + 96\{176 + 5\beta(2 - 5\beta)\}q_s^2 m^2 - 51712m^4 - 6144\eta(10q_s^2 + 99m^2)]}{61440r^8} \\ + \mathcal{O}(r^{-9}). \quad (6.6)$$

This gives

$$\beta\phi_0^2 < 16. \quad (6.11)$$

$$B_1^2 - B_2 = \frac{\eta^2 q_s^2 (q_s^2 + 36m^2)}{4r^8} + \mathcal{O}(r^{-9}). \quad (6.7)$$

The leading-order contribution to $B_1^2 - B_2$, which is the same as Eq. (5.8), is always positive. Since we have $B_1 \rightarrow -1$ and $B_2 \rightarrow 1$ as $r \rightarrow \infty$, the angular Laplacian stability conditions (3.21) are consistently satisfied.

B. Region in the vicinity of the horizon

Around the horizon radius r_h , we expand f , h , ϕ in the same form as Eq. (5.9) and derive the coefficients f_i , h_i , ϕ_i by using the background equations of motion. While the coefficient f_1 is undetermined, h_1 and ϕ_1 are given by

$$h_1 = \frac{2}{r_h} \left[1 + \sqrt{\frac{64 + [\beta(3\beta - 4 + 24\bar{\eta}) - 384\bar{\eta}^2]\phi_0^2}{64 + \beta(3\beta - 4 - 24\bar{\eta})\phi_0^2}} \right]^{-1}, \\ \phi_1 = \frac{12\bar{\eta}\phi_0(\beta\phi_0^2 - 16)}{64 + \beta(3\beta - 4 - 24\bar{\eta})\phi_0^2} h_1. \quad (6.8)$$

To have a positive real value of h_1 , we need the following condition:

$$[64 + \beta(3\beta - 4 - 24\bar{\eta})\phi_0^2] \\ \times \{64 + [\beta(3\beta - 4 + 24\bar{\eta}) - 384\bar{\eta}^2]\phi_0^2\} > 0, \quad (6.9)$$

which holds for small ϕ_0 close to 0. Let us consider the regime in which the inequality

$$64 + \beta(3\beta - 4 - 24\bar{\eta})\phi_0^2 > 0 \quad (6.10)$$

is satisfied. To realize a decreasing function of $|\phi(r)|$ around $r = r_h$ for $\bar{\eta} > 0$, we require that

As $|\phi_0|$ approaches the upper limit $4/\sqrt{\beta}$, it tends to be difficult to obtain the scalarized BH solutions with the asymptotic behavior $\phi(\infty) = 0$.

Let us consider the regime where $|\phi_0| \ll 1$, with β and $\bar{\eta}$ at most of order 10. On using Eq. (6.8) together with the expansion around $\phi_0 = 0$, the quantities \mathcal{G} , \mathcal{F} , and \mathcal{H} at $r = r_h$ can be estimated as

$$\mathcal{G}(r_h) = 1 + \left(\frac{3\bar{\eta}^2}{2} - \frac{\beta}{16} \right) \phi_0^2 + \mathcal{O}(\phi_0^4), \\ \mathcal{F}(r_h) = 1 + \left(\frac{3\bar{\eta}^2}{2} - \frac{\beta}{16} \right) \phi_0^2 + \mathcal{O}(\phi_0^4), \\ \mathcal{H}(r_h) = 1 - \frac{\beta}{16} \phi_0^2, \quad (6.12)$$

and hence the odd-parity linear stability is ensured for small $|\phi_0|$. In the same regime, the leading-order contribution to $\mathcal{K}(r_h)$ is $9\bar{\eta}^2\phi_0^2/4$, which is positive. The squared radial propagation speed $c_{r2,\text{even}}^2$ is also equivalent to 1 on the horizon. Up to the order of ϕ_0^2 , the quantities $B_1(r_h)$ and $B_2(r_h)$ are of the same forms as those in Eq. (5.14). Hence, in the $\phi_0 \rightarrow 0$ limit, the angular Laplacian stability of even-parity perturbations is also ensured, with $c_{\Omega\pm}^2(r_h) \rightarrow 1$.

C. Parameter space consistent with the linear stability

Since the approximation of small $|\phi_0|$ used in Sec. VI B starts to lose its validity for $|\phi_0|$ exceeding the order 0.1, we need to compute the values of $B_1(r_h)$, $B_2(r_h)$, etc. without resorting to such an approximation. For given β and $\bar{\eta}$, we search for field values ϕ_0 that lead to the asymptotic behavior $\phi(\infty) = 0$. We will consider the case $\phi_0 > 0$ without loss of generality. We numerically compute the quantities \mathcal{G} , \mathcal{F} , \mathcal{H} , \mathcal{K} , $c_{r2,\text{even}}^2$, $c_{\Omega+}^2$, and $c_{\Omega-}^2$ from the

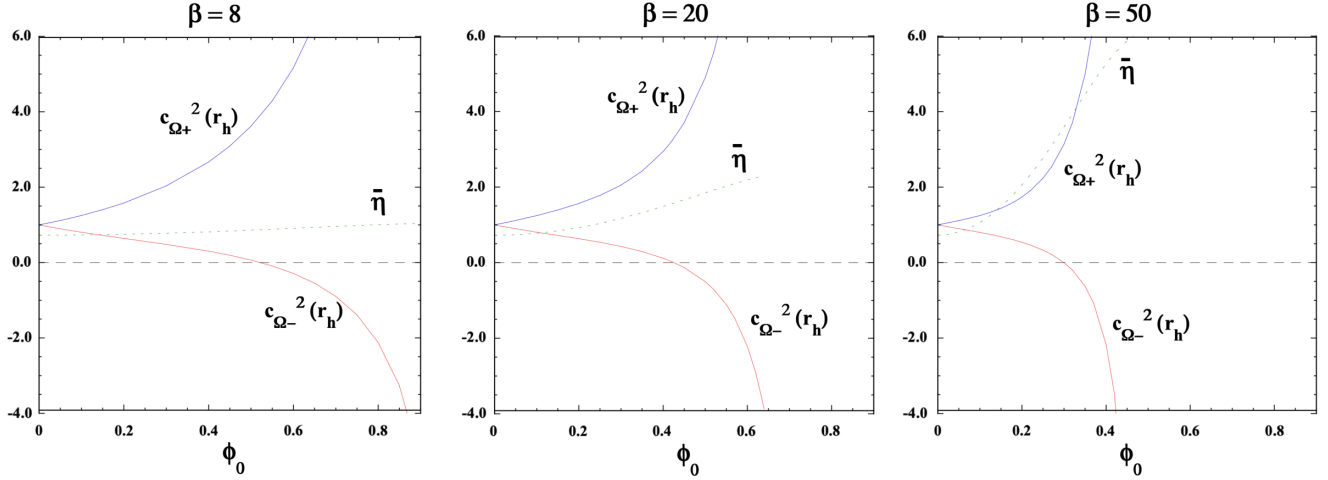


FIG. 5. The squared angular propagation speeds $c_{\Omega+}^2$ and $c_{\Omega-}^2$ at $r = r_h$ and $\bar{\eta}$ versus ϕ_0 in EsGBR theories with the coupling (2.17). The left, middle, and right panels correspond to $\beta = 8, 20, 50$, respectively. The field values ϕ_0 that give $c_{\Omega-}^2(r_h) < 0$ are excluded by the angular Laplacian instability of even-parity perturbations.

horizon to a sufficiently large distance. As in EsGB theories given by the couplings (2.16), we find that the second squared angular propagation speed $c_{\Omega-}^2$ on the horizon determines the linear stability of scalarized BHs. In other words, if the condition $c_{\Omega-}^2(r_h) > 0$ is satisfied, all the other linear stability conditions also hold outside the horizon. Hence we will focus on the behavior of $c_{\Omega-}^2(r_h)$ in the following discussion.

In Fig. 5, we plot $c_{\Omega+}^2(r_h)$, $c_{\Omega-}^2(r_h)$, and $\bar{\eta}$ as functions of ϕ_0 for three different values of β . While $c_{\Omega+}^2(r_h)$ is larger than 1 in all these cases, $c_{\Omega-}^2(r_h)$ decreases from the value close to 1 as ϕ_0 increases. For $\beta = 8$, $c_{\Omega-}^2(r_h)$ becomes negative in the region $\phi_0 > 0.52$. Even if ϕ_0 increases toward the upper limit $\phi_0 = 4/\sqrt{\beta}$, neither $c_{\Omega+}^2(r_h)$ nor $c_{\Omega-}^2(r_h)$ approaches 1. This property is different from that in EsGB theories given by the coupling (2.16). Instead, we find that $c_{\Omega-}^2(r_h)$ in EsGBR theories with the coupling (2.17) monotonically decreases with the increase of ϕ_0 . Hence, for given values of β , there are upper bounds on ϕ_0 , below which the angular Laplacian stability condition $c_{\Omega-}^2(r_h) > 0$ is satisfied.

As we observe in Fig. 5, the maximum values of ϕ_0 constrained from the condition $c_{\Omega-}^2(r_h) > 0$ tend to be smaller for larger β . From the requirement $c_{\Omega-}^2(r_h) > 0$, we obtain the following bounds:

$$\phi_0 < 0.52, \quad \text{for } \beta = 8, \quad (6.13)$$

$$\phi_0 < 0.42, \quad \text{for } \beta = 20, \quad (6.14)$$

$$\phi_0 < 0.30, \quad \text{for } \beta = 50, \quad (6.15)$$

respectively. As we discussed in Sec. IV B, the radial tachyonic instability of even-parity perturbations is absent

for $\beta > 5$ irrespective of the values of ϕ_0 . We also confirmed that, so long as the horizon field values are in the ranges (6.13)–(6.15), all the other required conditions $\mathcal{G} > 0$, $\mathcal{F} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$, and $c_{r2,\text{even}}^2 > 0$ hold outside the horizon. Thus, for $\beta > 5$, there are viable parameter spaces of ϕ_0 such as (6.13)–(6.15) consistent with all the linear stability conditions.

VII. CONCLUSIONS

In this paper, we studied the linear stability of scalarized BHs in scalar-tensor theories given by the action (2.1). EsGB theories have the sGB coupling $\xi(\phi)R_{\text{GB}}^2$ with $\kappa(\phi) = 0$, whereas EsGBR theories possess the nonminimal coupling $\kappa(\phi)R$ besides the sGB coupling. Provided that $\xi(\phi)$ and $\kappa(\phi)$ are even-power-law functions of ϕ , there are in general a nonvanishing scalar-field branch ($\phi \neq 0$) and a GR branch ($\phi = 0$). On the strong gravitational background like the vicinity of BHs, the GR branch can trigger a tachyonic instability toward the other scalarized branch under the condition $\xi_{,\phi\phi} > 0$. A simple choice of the sGB coupling satisfying this inequality is $\xi(\phi) = \eta\phi^2/8$ with $\eta > 0$. However, for the same coupling, the scalarized branch is subject to tachyonic instability against radial perturbations. This problem can be circumvented by implementing higher-order terms in EsGB theories, e.g., $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$, or by taking into account the nonminimal coupling in EsGBR theories, e.g., $\xi(\phi) = \eta\phi^2/8$ and $\kappa(\phi) = -\beta\phi^2/16$.

In Sec. III, we revisited conditions for avoiding ghosts and Laplacian instabilities along the radial and angular directions on the background (2.8). For the multipoles $l \geq 2$, there are three dynamical degrees of freedom: χ in the odd-parity sector and ψ and $\delta\phi$ in the even-parity sector. For large radial and angular momentum modes, there are

neither ghosts nor Laplacian instabilities under the seven conditions $\mathcal{G} > 0$, $\mathcal{F} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$, $c_{r2,\text{even}}^2 > 0$, $c_{\Omega+}^2 > 0$, and $c_{\Omega-}^2 > 0$. In general, a scalarized BH solution has the largest scalar-field amplitude $|\phi_0|$ on the horizon. For the couplings $\xi(\phi) = (\eta/8)(\phi^2 + \alpha\phi^4)$ and $\kappa(\phi) = -\beta\phi^2/16$, which accommodate both the models in EsGB and EsGBR theories mentioned above, we showed that the five conditions $\mathcal{G} > 0$, $\mathcal{F} > 0$, $\mathcal{H} > 0$, $\mathcal{K} > 0$, and $c_{r2,\text{even}}^2 > 0$ hold for small $\phi_0 (\ll 1)$ and the couplings $\bar{\eta} = \eta/r_h^2$ and β in the ranges (3.28), (3.30), and (3.31). The angular stability relevant to the squared propagation speeds $c_{\Omega\pm}^2$ in the even-parity sector requires more detailed analyses, which we performed in later Secs. V and VI.

In Sec. IV, we addressed the radial tachyonic stability of scalarized BHs by considering the monopole perturbation $l = 0$. In this case, the second-order action of odd-parity perturbations vanishes identically. In the even-parity sector, we showed that the quadratic action reduces to that of a single propagating degree of freedom $\delta\phi$, see Eq. (4.7) for the corresponding Lagrangian. We derived an effective potential $V_{\text{eff}}(r)$ for $\delta\phi$ along the radial direction in the form (4.18), which captures the backreaction of metric perturbations on the scalar-field perturbation. In EsGB theories, the leading-order contribution to $V_{\text{eff}}(r)$ arises from the potential $U \simeq -\xi_{,\phi\phi} R_{\text{GB}0}^2/h$, where the background GB invariant $R_{\text{GB}0}^2$ does not vanish even in the vacuum. In this case, we showed that the analysis of neglecting the backreaction of metric perturbations on $\delta\phi$ is a good approximation for estimating $V_{\text{eff}}(r)$. For the sGB coupling (2.16), we confirmed that the full study of $V_{\text{eff}}(r)$ without using such an approximation gives the condition $\alpha\phi_0^2 < -0.1155$ [43] for the absence of radial tachyonic instability. In EsGBR theories, we found that it is important to implement the backreaction of metric perturbations on the effective potential of $\delta\phi$. The radial stability is ensured for $\beta > 5$ even for $|\phi_0|$ much smaller than 1, whose result is consist with the one obtained in Ref. [72].

In Sec. V, we studied the linear stability of scalarized BHs in EsGB theories given by the coupling (2.16), by paying particular attention to the angular stability conditions of even-parity perturbations in the limit $l \gg 1$. On using the background solutions expanded at large distances, we first showed that there are no angular Laplacian instabilities associated with $c_{\Omega\pm}^2 = -B_1 \pm \sqrt{B_1^2 - B_2}$ far away from the horizon, see Eqs. (5.6)–(5.8). To realize a radially stable scalarized BH with a decreasing function of $|\phi(r)|$ around the horizon, we require that the product $\alpha\phi_0^2$ needs to be in the range (5.11). We found that the tightest linear stability condition comes from the value $c_{\Omega-}^2$ on the horizon. For $\alpha \lesssim -1$, $c_{\Omega-}^2(r_h)$ can remain positive in the parameter range $-1/2 < \alpha\phi_0^2 < -0.1155$. On the other hand, for $\alpha \gtrsim -1$, the region with $c_{\Omega-}^2(r_h) < 0$ starts to appear and the parameter space compatible with the radial

stability tends to be smaller. Thus, so long as $\alpha \lesssim -1$ and $-1/2 < \alpha\phi_0^2 < -0.1155$, the scalarized BHs can be consistent with all the linear stability conditions.

In Sec. VI, we addressed the linear stability issue for scalarized BHs in EsGBR theories given by the coupling (2.17). As we observe in Eqs. (6.5)–(6.7), the angular Laplacian instability far away from the horizon is absent in this case as well. The horizon field value should be in the range $\beta\phi_0^2 < 16$ to realize the scalarized BH with a decreasing function of $|\phi(r)|$ around $r = r_h$. As seen in Fig. 5, the second angular propagation speed squared on the horizon tends to enter the region $c_{\Omega-}^2(r_h) < 0$ as ϕ_0 increases toward the upper limit $4/\sqrt{\beta}$. However, for each β larger than the order 1, there are the regions of ϕ_0 in which the condition $c_{\Omega-}^2(r_h) > 0$ is satisfied, see Eqs. (6.13)–(6.15). Provided that $\beta > 5$, these small values of ϕ_0 are also consistent with the radial tachyonic stability as well as other linear stability conditions.

We have thus shown that the scalarized BHs present in EsGB and EsGBR theories have the theoretically allowed parameter spaces in which all the linear stability conditions are satisfied throughout the horizon exterior. It will be of interest to explore how the observations of gravitational waveforms emitted from BH-BH binaries and of quasinormal modes during the ringdown phase put constraints on the model parameters further. One could also extend the present work to the case of spontaneously vectorized BHs [82–88]. It will also be intriguing to see how the linear scalar-GB coupling (without breaking the shift symmetry) will imprint nonstealth modifications to the known stealth and approximately stealth BH solutions (see footnote 1 for a brief description of such solutions). These issues are left for future works.

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APPENDIX: COEFFICIENTS IN THE SECOND-ORDER ACTION OF EVEN-PARITY PERTURBATIONS

The coefficients that appear in the quantities B_1 and B_2 in Eqs. (3.19) and (3.20) are given by

$$\begin{aligned}
\beta_1 &= \phi'^2 a_4 e_4 - 2\phi' c_4 a'_4 + \left[\left(\frac{f'}{f} + \frac{h'}{h} - \frac{2}{r} \right) a_4 + \frac{\sqrt{f h} \mathcal{F}}{r} \right] \phi' c_4 + \frac{f \mathcal{F} \mathcal{G}}{2r^2}, \\
\beta_2 &= \left[\frac{\sqrt{f h} \mathcal{F}}{r^2} \left(2hr\phi'^2 c_4 + \frac{rf'\phi' a_4}{f} - \sqrt{f h} \phi' \mathcal{G} \right) - \frac{2\sqrt{f h} \phi' a_4 \mathcal{G}}{r} \left(\frac{\mathcal{G}}{f} - \frac{a'_4}{a_4} + \frac{f'}{f} + \frac{1}{2} \frac{h'}{h} - \frac{1}{r} \right) \right] a_1 - \frac{4\mathcal{F} \mathcal{G} f h a_4}{r}, \\
\beta_3 &= \left(h c'_4 - \frac{d_3}{2} + \frac{1}{2} h' c_4 \right) \phi' a_4 + \left(\frac{h'}{2h} - \frac{1}{r} + \frac{f'}{2f} - \frac{a'_4}{a_4} \right) \left(\frac{a_4 f'}{2f} + 2h\phi' c_4 + \frac{\sqrt{f h} \mathcal{G}}{2r} \right) a_4 \\
&\quad + \frac{\sqrt{f h} \mathcal{F}}{4r} \left(\frac{f'}{f} a_4 + 2h\phi' c_4 + \frac{3\sqrt{f h} \mathcal{G}}{r} \right), \tag{A1}
\end{aligned}$$

where

$$\begin{aligned}
d_3 &= -\frac{1}{r^2} \left(\frac{2\phi''}{\phi'} + \frac{h'}{h} \right) a_1 + \frac{2f}{(f'r - 2f)\phi'} \left(\frac{2\phi''}{h\phi'r} + \frac{f'^2}{f^2} - \frac{f'h'}{fh} - \frac{2f'}{fr} + \frac{2h'}{hr} + \frac{h'}{h^2 r} \right) a_4 \\
&\quad + \frac{f'r - 2f}{fr} \frac{\partial a_4}{\partial \phi} + \frac{\sqrt{f}}{\phi' \sqrt{hr^2}} \mathcal{F} - \frac{f^{3/2}}{\sqrt{h}(f'r - 2f)\phi'} \left(\frac{f'}{fr} + \frac{2\phi''}{\phi'r} + \frac{h'}{hr} - \frac{2}{r^2} \right) \mathcal{G}, \\
e_4 &= \frac{1}{\phi'} c'_4 - \frac{1}{2} \frac{f'}{f\phi'^2 h} a'_4 - \frac{1}{2} \frac{\sqrt{f}}{\phi'^2 \sqrt{hr}} \mathcal{G}' + \frac{1}{h\phi' r^2} \left(\frac{\phi''}{\phi'} + \frac{1}{2} \frac{h'}{h} \right) a_1 \\
&\quad + \frac{1}{4h\phi'^2} \left[\frac{(f'r - 6f)f'}{f^2 r} + \frac{h'(f'r + 4f)}{hrf} - \frac{4f(2\phi''h + h'\phi')}{\phi' h^2 r(f'r - 2f)} \right] a_4 + \frac{1}{2} \frac{h'}{h\phi'} c_4 - \frac{f'r - 2f}{4r\phi' \sqrt{f h}} \mathcal{H}_{,\phi} \\
&\quad + \frac{1}{2} \frac{f'hr - f}{r^2 \sqrt{f} \phi'^2 h^{3/2}} \mathcal{F} + \frac{1}{2} \frac{\sqrt{f}}{r\phi'^2 h^{3/2}} \left[\frac{f(2\phi''h + h'\phi')}{h\phi'(f'r - 2f)} + \frac{1}{2} \frac{2f - f'hr}{fr} \right] \mathcal{G}. \tag{A2}
\end{aligned}$$

In the Lagrangian density (4.1), the coefficients which are not presented in the main text are

$$\begin{aligned}
a_2 &= \sqrt{f h} \left(\frac{a_1}{\sqrt{f h}} \right)' - \left(\frac{\phi''}{\phi'} - \frac{f'}{2f} \right) a_1 + \frac{r}{\phi'} \left(\frac{f'}{f} - \frac{h'}{h} \right) a_4, \quad c_1 = -\frac{1}{f h} a_1, \\
c_3 &= -\frac{1}{4\sqrt{f h}} [2(rf'h + fh - f)\kappa_{,\phi} + h\phi' \{ r(rf' + 4f)\kappa_{,\phi\phi} - 4f'(3h - 1)\xi_{,\phi\phi} \}], \\
c_6 &= \frac{f'\phi'}{8f} a_1 + \frac{f'r}{2f} a_4 - \frac{1}{4} \phi' c_2 + \frac{1}{2} h\phi' r c_4 + \frac{1}{4} \sqrt{f h} \mathcal{G}, \\
e_1 &= \frac{1}{\phi' f h} \left[\left(\frac{f'}{f} + \frac{h'}{2h} \right) a_1 - 2a'_1 + a_2 - 2rha_6 \right], \quad e_2 = -\frac{1}{4} r^2 \sqrt{f h}, \\
e_3 &= \frac{r^2 \sqrt{f}}{4\sqrt{h}} (\xi_{,\phi\phi} R_{\text{GB0}}^2 + \kappa_{,\phi\phi} R_0), \tag{A3}
\end{aligned}$$

where

$$a_6 = -\frac{\sqrt{f}}{\phi' \sqrt{h}} \left[\frac{a'_4}{\sqrt{f h}} - \frac{a_4(hf' + fh')}{2(fh)^{3/2}} + \frac{a_4}{r\sqrt{f h}} - \frac{\mathcal{F}}{2r} \right]. \tag{A4}$$

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