Island of Reissner-Nordström anti–de Sitter black holes in the large D limit

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We study the information paradox of Reissner-Nordström anti–de Sitter black holes in the large dimension limit by using the island formula. The entanglement entropy of Hawking radiation is calculated both for the nonextremal and the extremal cases, in which the boundary of the radiation region is close to the outer horizon. For the nonextremal case, the entanglement entropy of Hawking radiation obeys the Page curve, i.e., the entanglement entropy of Hawking radiation increases with time and reaches saturation about twice Bekenstein-Hawking entropy at the Page time. For the extremal case, the entanglement entropy of Hawking radiation becomes ill defined in the absence of the island due to the appearance of the singularity at the origin of the radial coordinate, while when the island exists, the entanglement entropy is found to be equal to the Bekenstein-Hawking entropy. In addition, for the case where the boundary of the radiation region is close to the horizon, there are some obvious constraints required by the existence of the island solution for both nonextremal and extremal cases, which can be utilized to put constraints on the size of the black hole. These results reveal new features of the semiclassical large D black holes from the island perspective.

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I. INTRODUCTION

In 1974, Hawking discovered that the black hole can emit thermal radiation, which is called Hawking radiation [1]. In this way, the black hole has a temperature and can evaporate. However, a consequent problem called the black hole information paradox [2] is noticed during the black hole evaporation in the semiclassical gravity: essentially, a black hole formed by the collapse of the pure state becomes the mixed state after thermal Hawking radiation. Obviously, this process is not unitary evolution, which is in conflict with the standard rules of quantum mechanics. To avoid the information loss problem in the semiclassical gravity, Page proposed that the evolution of the entanglement entropy produced in the radiation process of the black hole should satisfy the Page curve [3,4]. Therefore, the key point to solve the black hole information loss problem is to reproduce the Page curve in the semiclassical description.

Recently, significant progress has been made in the study of the black hole information paradox in the light of the AdS/CFT correspondence [5]. In Refs. [6–9], the island formula was proposed to calculate the entanglement entropy of Hawking radiation, which helped to reproduce the Page curve in semiclassical gravity. This formula comes from the holographic entanglement entropy (HEE) formula [10,11] and its quantum version, i.e., the quantum extremal surface [12,13]. It is worth noting that the island formula can be derived through the gravitational path integral formalism [14,15]. The island formula for the entanglement entropy of Hawking radiation is given by

$$\begin{aligned} S(R) &= \min\{\mathrm{ext}\{S_{\mathrm{gen}}\}\} \\ &= \min\left\{\mathrm{ext}\left\{\frac{\mathrm{Area}(\partial I)}{4G_N} + S^{\mathrm{finite}}_{\mathrm{matter}}(R\cup I)\right\}\right\}, \end{aligned} \tag{1.1}$$

where I is the island and R is the radiation region outside the black hole. Note that due to the short distance cutoff [16,17], the entanglement entropy of matter has UV divergence, which can be absorbed by renormalizing the Newton constant G_N [18]. S_{gen} is the generalized entropy composed of two parts: the first area term is the contribution of the island, and the second term is the finite part of the entanglement entropy of the matter on the union of the radiation region and the island. The entanglement island, or simply called island, is a region in the black hole interior, which is found to be a part of the entanglement wedge of the Hawking radiation outside the black hole. The island formula is expected to be applicable to different types of black holes. So far, the island formula has been applied into the $(1+1)$-dimensional gravitational models [19–34] and also some higher-dimensional models [34–53]. In addition, there were also studies on equivalent descriptions of the

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entanglement islands based on the anti–de Sitter/boundary conformal field theory correspondence [54,55], see [56–65] for examples.

In the present paper, we mainly consider the entanglement island of $D = (d+1)$-dimensional Reissner-Nordström anti–de Sitter (RN-AdS$_{d+1}$) black holes coupled to an auxiliary thermal bath in the large D limit for both nonextremal and extremal cases, in which the gravitational theory shows interesting decoupling properties. For a large D black hole, its near-horizon geometry will be driven into a fixed point and its dynamics will decouple with that in the far region, and the effective theory reduces to a two-dimensional dilaton gravity. In addition, the gravitational perturbations will go into the hydrodynamic limit, which makes analytical calculations for the equations of motion of the gravitational perturbations available [66–73]. Thus, large D black holes provide a good platform for studying the entanglement island analytically. On the other hand, the properties of the entanglement island for extremal black holes have not been completely understood yet. Previous works attempted to use the island formula to analyze the extremal black hole, but their calculations were mainly based on taking the extremal limit from the nonextremal case [40,42]. Subsequent studies suggested that the nonextremal and extremal cases should be analyzed separately based on their different Penrose diagrams [41,43,74]. Moreover, although the Hawking temperature of a extremal black hole is zero, its Bekenstein-Hawking area entropy is nonzero, which also has been verified from counting microstates of solitons for extremal black holes in string theory [75] and calculating the microscopic entropy of the conformal field theory (CFT) holographically dual to the extremal black hole [76]. Now, for the large D RN-AdS$_{d+1}$ black hole, it has been shown that it contains new dual CFT description in the (near) extremal limit. Therefore, it would be interesting to further study the entanglement island of the large D RN-AdS$_{d+1}$ black hole, which will give deeper understanding of the microscopic entropy and entanglement property of the extremal black holes.

This paper is organized as follows. In Sec. II, we will briefly review RN-AdS_{d+1} black holes in the large D limit, both for the nonextremal and extremal cases. In Sec. III, we review the formulas to calculate the entanglement entropy of matter. In Secs. IV and V, we mainly study the entanglement entropy of Hawking radiation for the nonextremal and extremal cases by using the island formula. In Sec. VI, we discuss the constraints in the presence of the island in more detail. In Sec. VII, the Page curve and Page time are discussed. Finally, conclusions and discussion are given in Sec. VIII.

II. REVIEW OF THE RN-AdS$_{d+1}$ BLACK HOLE

In this section, we review the RN-AdS$_{d+1}$ black hole and its metric in the large D limit [68]. Based on the previous results, we rewrite the metric for the nonextremal and extremal cases in the Kruskal coordinates.

The action of $(d+1)$-dimensional Einstein-Maxwell theory has the form

$$I = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{L^2} - \frac{L^2}{g_s^2}F_{\mu\nu}F^{\mu\nu}\right), \tag{2.1}$$

where G_N is the Newton constant, R is the Ricci scalar, L is the curvature radius of the asymptotically AdS$_{d+1}$ spacetime, and g_s is the dimensionless coupling constant of the U(1) gauge field. The equations of motion can be found as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{d(d-1)}{2L^2}g_{\mu\nu} = \frac{L^2}{2g_s^2}(4F_{\mu\lambda}F_{\nu}{}^{\lambda} - g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$$
$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0, \tag{2.2}$$

which admits the following RN-AdS$_{d+1}$ black hole solution:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-1}^2,$$
$$A = \mu\left(1 - \frac{r_+^{d-2}}{r^{d-2}}\right)dt, \tag{2.3}$$

with

$$f(r) = 1 - \frac{M}{r^{d-2}} + \frac{Q^2}{r^{2d-4}} + \frac{r^2}{L^2},$$
$$\mu = \sqrt{\frac{d-1}{2(d-2)}}\frac{g_s Q}{L r_+^{d-2}}, \tag{2.4}$$

where r_+ is the radius of the outer horizon, μ is the chemical potential, and M and Q are the mass and charge of the RN-AdS$_{d+1}$ black hole.

A. The nonextremal large D RN-AdS$_{d+1}$ black hole

The dual CFT description of the large D RN-AdS$_{d+1}$ black hole has been studied in [68]. To analyze the large D RN-AdS$_{d+1}$ black hole, defining $\rho \equiv \frac{r^{d-2}}{M}$ and $M \equiv r_o^{d-2}$, and taking the large dimension limit $d \to \infty$ together with the near-horizon limit $r - r_+ \ll r_+$, the metric (2.3) becomes

$$ds^2 = -f(\rho)dt^2 + \frac{r^2}{(d-2)^2\rho^2 f(\rho)}d\rho^2 + r_o^2\rho^{\frac{2}{d-2}}d\Omega_{d-1}^2,$$
$$A = \mu\left(1 - \frac{\rho_+}{\rho}\right)dt, \tag{2.5}$$

where

$$f(\rho) = 1 - \frac{1}{\rho} + \frac{Q^2}{M^2}\frac{1}{\rho^2} + \frac{r_o^2 \rho^{\frac{2}{d-2}}}{L^2}$$
$$\simeq 1 - \frac{1}{\rho} + \frac{Q^2}{M^2}\frac{1}{\rho^2} + \frac{r_o^2}{L^2}. \tag{2.6}$$

Note that in order to obtain Eq. (2.5) we have used $\rho^{\frac{2}{d-2}} \simeq 1$ (for finite ρ) once for $d \to \infty$, but we still keep the exact form of $\rho^{\frac{2}{d-2}}$ that appears in the spherical term of the metric.[1] The inner and outer horizon radius of large D RN-AdS$_{d+1}$ black holes are

$$\rho_\pm = \frac{1 \pm \sqrt{1 - \frac{4kQ^2}{M^2}}}{2k}, \qquad k = 1 + \frac{r_o^2}{L^2}. \tag{2.7}$$

Thus, the metric (2.5) can be rewritten as

$$ds^2 = -\frac{k(\rho-\rho_+)(\rho-\rho_-)}{\rho^2}dt^2 + \left(k(\rho-\rho_+)(\rho-\rho_-)\left(\frac{d-2}{r_o}\right)^2\right)^{-1}d\rho^2 + r_o^2\rho^{\frac{2}{d-2}}d\Omega_{d-1}^2. \tag{2.8}$$

The temperature and entropy associated with the outer horizon are, respectively,

$$T = \left(\frac{d-2}{r_o}\right)\frac{k(\rho_+ - \rho_-)}{4\pi\rho_+}, \qquad S_{\rm BH} = \frac{\Omega_{d-1} r_o^{d-1}\rho_+}{4G_N}, \tag{2.9}$$

where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the unit sphere S^{d-1}. In addition, the tortoise coordinate is

$$\rho_* = \int^\rho \frac{r_o \rho d\rho}{(d-2)k(\rho-\rho_+)(\rho-\rho_-)} = \frac{1}{2\kappa_+}\log(\rho-\rho_+) + \frac{1}{2\kappa_-}\log(\rho-\rho_-), \tag{2.10}$$

where $\kappa_\pm = (\frac{d-2}{r_o})(\frac{k(\rho_+-\rho_-)}{2\rho_\pm})$ are the surface gravity on the inner and outer horizons, respectively. Then the Kruskal coordinates are

$$U = -e^{-\kappa_+(t-\rho_*)}, \qquad V = e^{\kappa_+(t+\rho_*)}. \tag{2.11}$$

Finally, we can rewrite the metric (2.8) in terms of the Kruskal coordinates as

$$ds^2 = -g^2(\rho)dUdV + r_o^2\rho^{\frac{2}{d-2}}d\Omega_{d-1}^2, \tag{2.12}$$

where

$$g^2(\rho) = \frac{k(\rho-\rho_+)(\rho-\rho_-)}{\rho^2}\frac{1}{\kappa_+^2}e^{-2\kappa_+\rho_*}. \tag{2.13}$$

B. The extremal large D RN-AdS$_{d+1}$ black hole

The extremal condition is $M^2 = 4kQ^2 = r_o^{2d-4}$, in which the outer horizon coincides with the inner horizon (i.e., $\rho_+ = \rho_- = \frac{1}{2k} \equiv \rho_h$). Then the metric (2.8) reduces to

$$ds^2 = -\frac{k(\rho-\rho_h)^2}{\rho^2}dt^2 + \left(k(\rho-\rho_h)^2\left(\frac{d-2}{r_o}\right)^2\right)^{-1}d\rho^2 + r_o^2\rho^{\frac{2}{d-2}}d\Omega_{d-1}^2, \tag{2.14}$$

the tortoise coordinate now is

$$\rho_* = \int^\rho \frac{r_o\rho d\rho}{(d-2)k(\rho-\rho_h)^2} = \frac{r_o}{(d-2)k}\left[-\frac{\rho_h}{\rho-\rho_h} + \log(\rho-\rho_h)\right], \tag{2.15}$$

and the corresponding Kruskal coordinates are

$$U = -e^{-\kappa(t-\rho_*)}, \qquad V = e^{\kappa(t+\rho_*)}, \tag{2.16}$$

where we have still adopted the form of the Kruskal coordinates defined in Eq. (2.11). Note that, in the extremal case, the surface gravity $\kappa_+ = \kappa_- \equiv \kappa$ becomes zero, so we assume that the limit $\kappa \to 0$ to approach the final result of the extremal case. Now, the metric (2.14) in terms of the Kruskal coordinates becomes

$$ds^2 = -w^2(\rho)dUdV + r_o^2 d\Omega_{d-1}^2, \tag{2.17}$$

where

$$w^2(\rho) = \frac{k(\rho-\rho_h)^2}{\rho^2}\frac{1}{\kappa^2}e^{-2\kappa\rho_*}, \tag{2.18}$$

and the area entropy of the large D extremal RN-AdS$_{d+1}$ black hole is

$$S_{\rm BH} = \frac{\Omega_{d-1}r_o^{d-1}\rho_h}{4G_N}. \tag{2.19}$$

III. FORMULAS OF THE ENTANGLEMENT ENTROPY OF MATTER FIELDS

In this section, in order to calculate the entanglement entropy of the radiation in any $(d+1) \geq 4$-dimensional

[1]Note that in the sphere radius $r_o\rho^{\frac{1}{d-2}}$ we are retaining an apparent term $\rho^{\frac{1}{d-2}}$ since it provides a contribution whenever the area element $r_o^{d-1}\rho^{\frac{d-1}{d-2}} \simeq r_o^{d-1}\rho$ is involved [71]. Essentially, this detail is crucial when we use Eq. (3.5) to calculate the entanglement entropy. As you can see from Eq. (4.8), if we omit this term $\rho^{\frac{1}{d-2}}$, the location of the island cannot be determined.

curved spacetime through the island formula (1.1), we will take some assumptions and adopt some limits to calculate the entanglement entropy of free massless matter fields in higher-dimensional spacetime [34,51]:

(i) If the distance between region A and region B is larger than the correlation length of the massive modes in the Kaluza-Klein tower of the spherical part, the Hawking radiation is assumed to be described by the two-dimensional s-wave modes (with the zero angular momentum) and influence from the higher angular momentum modes can be ignored. Then the finite part of the entanglement entropy of massless matter fields is approximated by the mutual information of the two-dimensional massless fields as [34]

$$S^{\text{finite}}_{\text{matter}} = -I(A:B) = \frac{c}{3}\log d(A,B), \tag{3.1}$$

where c is the central charge, and $d(A,B)$ is the distance between the boundaries of region A and region B in flat spacetime. More specifically, in two-dimensional conformally flat spacetime, the metric can be written in terms of the Kruskal coordinates as

$$ds^2 = -\Omega^2 dUdV, \tag{3.2}$$

where Ω is the conformal factor. Under a Weyl transformation, the distance $d(A,B)$ between two points in conformally flat spacetime can be written as [25,34,47]

$$d(A,B) = \sqrt{\Omega(A)\Omega(B)[U(B)-U(A)][V(A)-V(B)]}. \tag{3.3}$$

Therefore, the finite part of the entanglement entropy of matter fields is given by

$$S^{\text{finite}}_{\text{matter}} = \frac{c}{6}\log \Omega(A)\Omega(B)[U(B) - U(A)][V(A) - V(B)]. \tag{3.4}$$

As the U, V parts of metrics (2.12) and (2.17) are conformally flat, so the entropy formula in Eq. (3.4) is applicable.

(ii) If the distance L between region A and region B is sufficiently small, then the finite part of the entanglement entropy of massless matter fields can be evaluated by [34,77,78]

$$S^{\text{finite}}_{\text{matter}} = -I(A:B) = -\kappa_{d+1} c \frac{\text{Area}}{L^{d-1}}, \tag{3.5}$$

where c is the central charge and κ_{d+1} is a dimensionally dependent constant. Note that this formula is only valid for the flat spacetime. We expect the above formula can also be applied in curved spacetime as long as we require that the length scale of the curvature is much larger than the distance L.

IV. THE ENTANGLEMENT ENTROPY IN NONEXTREMAL LARGE D RN-AdS$_{d+1}$ BLACK HOLE

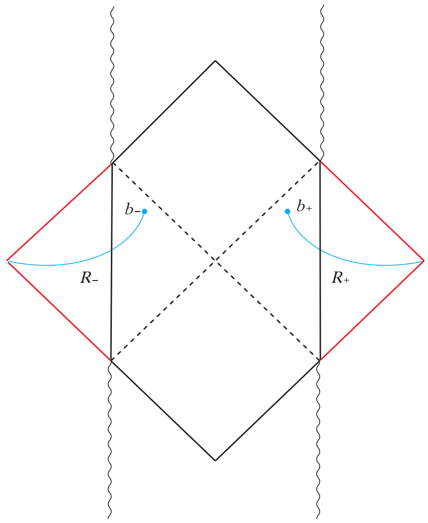
In this section, we will calculate the entanglement entropy of radiation in the nonextremal large D RN-AdS$_{d+1}$ black hole. Before discussing the island rule, we first give a specific description to the model we studied. In order to investigate the evaporation process of black holes in AdS spacetime, we expect to couple a bath at the asymptotically AdS boundary of the black hole. Here, we couple two flat thermal bath systems that have no gravitational effect at the boundary of the RN-AdS$_{d+1}$ black hole and make it transparent [46,79]. For the thermal bath, suppose that the bath is in thermal equilibrium with the black hole. The Penrose diagram of whole spacetime (RN-AdS$_{d+1}$ + bath) is shown in Fig. 1.

Furthermore, it is worth noting that we choose the boundary of radiation region $b_\pm$ near horizon, which is crucial in our later calculation of the entanglement entropy of Hawking radiation. As we can see, due to the special features of large D geometry near the horizon, the metric of the large D $\mathrm{RN-AdS}_{d+1}$ black hole behaves like in asymptotically flat spacetime [66,71]. Then a natural setup is to choose the boundary of the radiation region $b_\pm$ that is to be near the horizon, which is similar to the previous models [34,51]. To explain this, we can focus on the black hole solution $f(r)$; the horizon radius r_+ is satisfied with $f(r_+) = 0$ and $r_+ < r_o$. At large D,

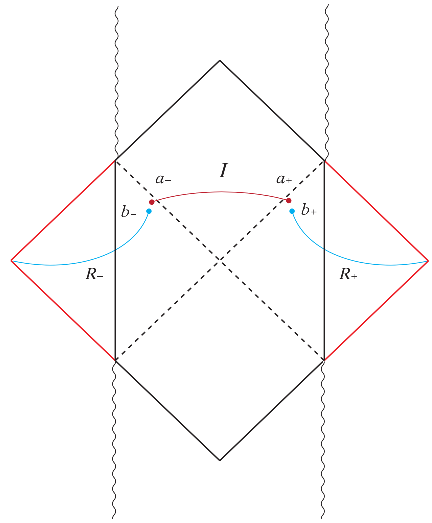
$$r_+ \simeq r_o \left(\frac{1+\sqrt{1-\frac{4kQ^2}{M^2}}}{2k}\right)^{\frac{1}{d-2}} \simeq r_0\left(1 - \frac{1}{d}\ln\left(1+\frac{r_0^2}{L^2}\right) + O(d^{-2})\right), \tag{4.1}$$

when taking $d \to \infty$, $r_+ \to r_o$. It implies that the gravitational effect of the black hole quickly disappears outside horizon $r > r_o$ in the large D limit. In fact, there is a small region around the horizon on the r/d scale where the gravitational effect of the black hole is still appreciable, more precisely, within the region

$$r - r_o \lesssim \frac{r_o}{d} + O(d^{-2}), \tag{4.2}$$



(a) Without island



(b) With island

FIG. 1. The Penrose diagram of the nonextremal RN-AdS$_{d+1}$ black hole coupled to two auxiliary thermal baths. The black part and red triangle represent the black hole and the auxiliary at spacetime, respectively. (a) $R_\pm$ are the radiation regions on the right and left wedges, and $b_\pm$ are the boundaries of the radiation region $R_\pm$. (b) The boundaries of the island are supposed at $a_\pm$, and the inner boundaries of the radiation regions correspond to $b_\pm$.

i.e., $\rho = (\frac{r_o}{r})^{d-2} = O(d^0)$. Thus, the gravitational influence of the black hole is mainly concentrated in the near horizon and vanishes in the far zone[2] in the large D limit. Therefore, it allows us to choose the boundary of the radiation region $b_\pm$ to be near the horizon as in asymptotically flat spacetime. So, we are able to calculate the entanglement entropy in large D $\mathrm{RN}-\mathrm{AdS}_{d+1}$ black hole spacetime by using the formulas (3.4) and (3.5) as in the following sections.

A. Without island

There are two points $b_\pm$ corresponding to the boundaries of radiation regions on left wedge R_- and right wedge R_+ [see Fig. 1(a)]. Here $b_+ = (t_b, \rho_b)$ and $b_- = (-t_b + i\beta/2, \rho_b)$, respectively. Note that, for the case without island, it can be inferred from the island formula that the entanglement entropy of Hawking radiation only receives the contribution from the matter fields. If the distance between the boundaries of R_- and R_+ is large, the entanglement entropy of Hawking radiation can be approximated by formula (3.4)

$$S_R = S_{\mathrm{matter}}^{\mathrm{finite}}(R) = -I(R_+ :\ R_-) = \frac{c}{3}\log d(b_+, d_-), \qquad (4.3)$$

then calculating in the Kruskal coordinates (2.12), we have

$$\begin{aligned} S_R &= \frac{c}{6}\log[g^2(b)((U(b_-) - U(b_+))(V(b_+) - V(b_-)))] \\ &= \frac{c}{6}\log\left[4\frac{k(\rho-\rho_+)(\rho-\rho_-)}{\rho^2}\frac{1}{\kappa_+^2}e^{-4\kappa_+\rho_*(b)}\cosh^2(\kappa_+ t_b)\right]. \end{aligned} \qquad (4.4)$$

At late time, we assume that $t_b \gg \rho_b > \rho_+$, thus

$$S_R \simeq \frac{c}{3}\kappa_+ t_b = \frac{d-2}{r_o}\frac{k(\rho_+ - \rho_-)}{6\rho_+} t_b. \qquad (4.5)$$

We can see that the entanglement entropy of the radiation increases linearly with time at late time and becomes larger than the Bekenstein-Hawking entropy. This clearly does not satisfy the unitary, which requires the entanglement entropy to follow the Page curve. However, this problem will be solved once we introduce the contribution of the island after the Page time.

B. With island

Now we consider the contribution of the island to the entanglement entropy of Hawking radiation. We will focus on the situation where the inner boundary of the radiation region is near the outer horizon, characterized by $\rho_b - \rho_+ \ll \rho_+$. We set the boundaries of the island as $a_+ = (t_a, \rho_a)$ and $a_- = (-t_a + i\beta/2, \rho_a)$, respectively [see Fig. 1(b)]. Here we will apply the formula (3.5) to calculate the entanglement entropy of matter fields, as we have set the boundary of the island to be outside and near the outer horizon, namely, $\rho_a - \rho_+ < \rho_b - \rho_+ \ll \rho_+$. By formula (3.5), we have

[2]The definition of two distinct regions in the geometry: near-horizon region: $r - r_o \ll r_o$, i.e., $\ln\rho \ll d$, and far region: $r - r_o \gg \frac{r_o}{d}$, i.e., $\ln\rho \gg 1$.

$$S_{\rm matter}^{\rm finite}(R\cup I) = -2I(R_+ : I) = -2\kappa_{d+1}c\frac{\rm Area}{L^{d-1}}, \quad (4.6)$$

and the geodesic distance between the boundary of region I and that of region R is [34]

$$L = \int_{\rho_a}^{\rho_b}\frac{r_o}{d-2}\frac{d\rho}{\sqrt{k(\rho-\rho_+)(\rho-\rho_-)}}$$
$$\simeq 2\frac{r_o}{d-2}\frac{1}{\sqrt{k(\rho_+-\rho_-)}}(\sqrt{\rho_b-\rho_+}-\sqrt{\rho_a-\rho_+}). \quad (4.7)$$

Thus, the generalized entropy is

$$S_{\rm gen} = 2\frac{{\rm Area}(\partial I)}{4G_N} - 2\kappa_{d+1}c\frac{{\rm Area}(\partial R)}{L^{d-1}}$$
$$= \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{2G_N} - 2\kappa_{d+1}c\frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{(\sqrt{\rho_b-\rho_+}-\sqrt{\rho_a-\rho_+})^{d-1}}$$
$$\times\frac{(k(\rho_+-\rho_-))^{\frac{d-1}{2}}}{2^{d-1}}\left(\frac{d-2}{r_o}\right)^{d-1}. \quad (4.8)$$

The factor 2 is due to the double contributions from the left and right wedges. For convenience, we define new variables $x \equiv \sqrt{\frac{\rho_a-\rho_+}{\rho_+}}$ and $y \equiv \sqrt{\frac{\rho_b-\rho_+}{\rho_+}}$, thus $x < y \ll 1$ because of $\rho_a-\rho_+ < \rho_b-\rho_+ \ll \rho_+$. Then Eq. (4.8) becomes

$$S_{\rm gen} = \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{2G_N} - 2\kappa_{d+1}c\frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{\left(1-\frac{x}{y}\right)^{d-1}}$$
$$\times\frac{(k(\rho_+-\rho_-))^{\frac{d-1}{2}}}{\rho_+^{\frac{d-1}{2}}y^{d-1}2^{d-1}}\left(\frac{d-2}{r_o}\right)^{d-1}. \quad (4.9)$$

According to the island formula, the entanglement entropy is given by the minimal value among all extremal solutions of the generalized entropy. Note that the expression of the generalized entropy by using formula (3.5) does not explicitly include time. We just take the derivative with respect to position ρ_a and solve the following equation:

$$\frac{\partial S_{\rm gen}}{\partial\rho_a} = \frac{d-1}{d-2}\frac{\Omega_{d-1}M^{\frac{d-1}{d-2}}\rho_a^{\frac{1}{d-2}}}{2G_N} - (d-1)\kappa_{d+1}c\frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{\left(1-\frac{x}{y}\right)^{d}xy\rho_+}$$
$$\times\frac{(k(\rho_+-\rho_-))^{\frac{d-1}{2}}}{\rho_+^{\frac{d-1}{2}}y^{d-1}2^{d-1}}\left(\frac{d-2}{r_o}\right)^{d-1} = 0, \quad (4.10)$$

which gives

$$\frac{x}{y}\left(1-\frac{x}{y}\right)^d = G_N\kappa_{d+1}c\frac{(k(\rho_+-\rho_-))^{\frac{d-1}{2}}(d-2)^d}{\rho_+^{\frac{d-1}{2}}y^{d+1}2^{d-2}r_o^{d-1}}. \quad (4.11)$$

1. The existence of general island solution

Now let us study the general island solution of Eq. (4.11); we will show that the existence of the island solution requires some constraints on the large D RN-AdS$_{d+1}$ black hole. Similar observations in Schwarzschild black hole spacetime have been noticed in Refs. [38,44,45,51]. By defining new variables as

$$u \equiv \frac{x}{y} \in (0,1)$$

and $$\lambda \equiv \frac{\rho_+^{\frac{d-1}{2}}y^{d+1}2^{d-2}r_o^{d-1}}{c\kappa_{d+1}G_N(k(\rho_+-\rho_-))^{\frac{d-1}{2}}(d-2)^d(d+1)}, \quad (4.12)$$

where $x < y \ll 1$, Eq. (4.11) becomes

$$u(1-u)^d = \frac{1}{(d+1)\lambda}. \quad (4.13)$$

Note there exists a local maximum value of function $F(u)$, that is,

$$F(u_m) = \frac{d^d}{(d+1)^{d+1}} \quad \text{at } u_m = \frac{1}{d+1}. \quad (4.14)$$

Function $F(u) = u(1-u)^d$ monotonically increases with u in the interval $(0, u_m)$ and monotonically decreases in the interval $[u_m, 1)$ (see Fig. 2). Obviously, there exists an island solution to Eq. (4.13) only if $\frac{1}{(d+1)\lambda} < F(u_m)$. Therefore, in the large D limit, we obtain

$$\lambda > \left(1+\frac{1}{d}\right)^d \to e. \quad (4.15)$$

As shown in Fig. 3, if the constraint $\lambda > e$ is satisfied, there would exist two solutions u_1 and u_2 ($0 < u_1 < u_m < u_2 < 1$). The local minimum and maximum values of the generalized entropy are located at $u = u_1$ and $u = u_2$,

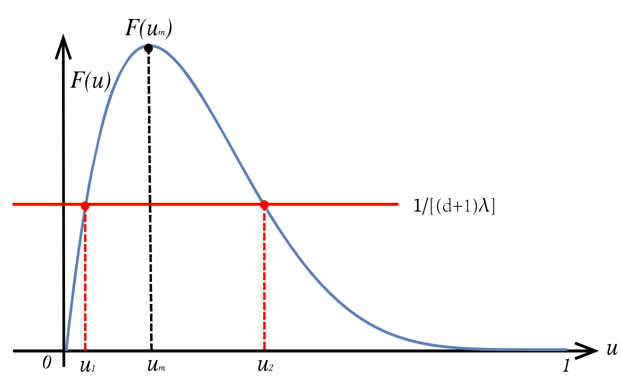


FIG. 2. The schematic diagram of function $F(u)$ in the interval $u \in (0, 1)$. A local maximum value of function $F(u)$ is located at u_m. There are two solutions u_1 and u_2 when $F(u_m) > 1/[(d+1)\lambda]$. The island solution is just u_1.

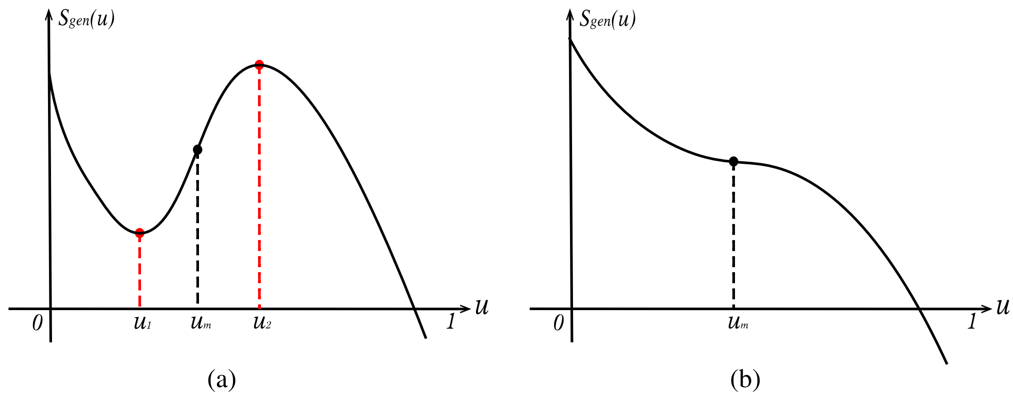


FIG. 3. The entropy curve of $S_{\rm gen}$ with u. (a) When $\lambda > e$, the generalized entropy $S_{\rm gen}$ reaches its local minimum and local maximum at u_1 and u_2, respectively. (b) If $\lambda \leq e$, there would be no solutions.

respectively. Thus, the island solution is exactly given by $u = u_1$. When $\lambda = e$, points u_1 coincide with u_2 at u_m, the generalized entropy $S_{\rm gen}$ will monotonically decrease with u in the interval (0, 1). Thus, there would be no local minimum value of $S_{\rm gen}$ and no nontrivial island solution can be found, similar to the case $\lambda < e$. Physically speaking, the existence of the island puts a constraint on the large dimensional RN-AdS$_{d+1}$ black hole. As according to the island formula, we need to make sure of the existence of the island in order to save the unitarity in our case, and this makes the constraint (4.15) meaningful.

2. Two specific analytical island solutions

Note that we cannot give an analytical expression of the island solution for the general case $x < y \ll 1$. In this section, we will give two analytical island solutions for the special cases $x \ll y/d$ and $x \sim y/d$ in the large D limit.

First, we consider a more special case satisfying $x \ll y/d$ with $d \to \infty$. Starting with Eq. (4.10), $(1 - \frac{x}{y})^{-d}$ can be expanded as

$$\left(1 - \frac{x}{y}\right)^{-d} \simeq 1 + d\frac{x}{y} + \mathcal{O}\left(\left(d\frac{x}{y}\right)^2\right). \tag{4.16}$$

This allows us to ignore the higher-order terms of Eq. (4.16) in the large D limit for the special case $x \ll y/d$. Note that, in the finite dimension case, the condition $x \ll y$ would be enough to drop the higher-order terms of Eq. (4.16). However, in the large dimension case, the condition $x \ll y$ is not enough and we need to further assume $x \ll y/d$. Substituting Eq. (4.16) into Eq. (4.10) and omitting the higher-order terms, we find an island solution as

$$x = \frac{y}{\frac{y^{d+1} r_o^{d-1} 2^{d-2}}{(d-2)^d \kappa_{d+1} c G_N} \left(\frac{\rho_+}{k(\rho_+ - \rho_-)}\right)^{\frac{d-1}{2}} - d} = \frac{y}{(d+1)\lambda - d}, \tag{4.17}$$

which is a special island solution that is valid for $x \ll y/d$ with $d \to \infty$.

Now we try to give another special analytical solution in the large D limit. We assume $d\frac{x}{y} = \eta \sim O(1)$, so that $\lim_{d\to\infty} (1 - \frac{\eta}{d})^d = e^{-\eta}$. Then from Eq. (4.13), we get

$$-\eta e^{-\eta} = -\frac{1}{\lambda}. \tag{4.18}$$

The solution of the above equation can be expressed as the Lambert W function (or the product logarithmic function). It can be seen in Fig. 4 that there are two real number solutions $\eta_1 = -W(-\frac{1}{\lambda})$ and $\eta_2 = -W_{-1}(-\frac{1}{\lambda})$, where $-\frac{1}{\lambda} \in (-\frac{1}{e}, 0)$. The island solution is just given by $\eta_1 = -W(-\frac{1}{\lambda})$, that is,

$$x = \frac{y}{d}\eta_1 = -\frac{y}{d} W\left(-\frac{1}{\lambda}\right), \tag{4.19}$$

which is a special island solution that is valid for $x \sim y/d$ with $d \to \infty$.

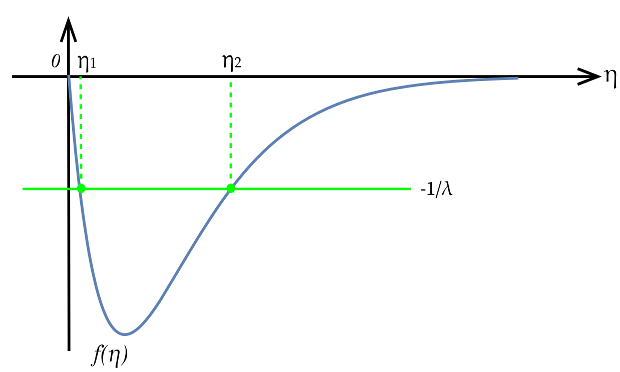


FIG. 4. The schematic diagram of the function $f(\eta) = -\eta e^{-\eta}$. There are two solutions η_1 and η_2 of Eq. (4.18). The island solution is just η_1.

3. The entanglement entropy with island

In previous sections, we have confirmed the existence of the island solution for the general case $x < y \ll 1$. Now let us evaluate the entanglement entropy of Hawking radiation when the island exists. For the special case with $x \ll y/d$ and $d \to \infty$, we note that the solution (4.17) should satisfy the constraint $\lambda \gg 1$. Thus,

$$x = \frac{y}{(d+1)\lambda - d} \simeq \frac{y}{(d+1)\lambda}. \tag{4.20}$$

Then by plugging the solution (4.20) back into the formula (4.9), the entanglement entropy in this special case is given by

$$\begin{aligned} S_R &\simeq \frac{\Omega_{d-1}(M\rho_+)^{\frac{d-1}{d-2}}}{2G_N} - \kappa_{d+1} c \frac{\Omega_{d-1}(M\rho_+)^{\frac{d-1}{d-2}}}{y^{d-1}} \\ &\quad \times \frac{(k(\rho_+ - \rho_-))^{\frac{d-1}{2}}}{\rho_+^{\frac{d-1}{2}} 2^{d-2}} \left(\frac{d-2}{r_o}\right)^{d-1} \left(1 + (d-1)\frac{x}{y}\right) \\ &\simeq 2S_{\rm BH}\left(1 - \frac{2y^2}{(d+1)(d-2)\lambda}\right) \simeq 2S_{\rm BH}, \end{aligned} \tag{4.21}$$

where we used $\lambda \gg 1$ and $y/d \ll y \ll 1$. Therefore, for this special case, the entanglement entropy of Hawking radiation S_R is approximately equal to $2S_{\rm BH}$.

For another special case with $d\frac{x}{y} = \eta \sim O(1)$, from Eqs. (4.18) and (4.19) we have the following expression:

$$\left(1 - \frac{x}{y}\right)^{d-1} \simeq e^{-\eta_1} = \frac{1}{\lambda \eta_1}. \tag{4.22}$$

Substituting Eq. (4.9) into Eq. (4.22), we obtain

$$\begin{aligned} S_R &\simeq 2S_{\rm BH}\left(1 - \frac{2y^2}{d^2\lambda}\frac{1}{(1-\frac{x}{y})^{d-1}}\right) \\ &\simeq 2S_{\rm BH}\left(1 - \frac{2y^2}{d^2}\eta_1\right) \simeq 2S_{\rm BH}, \end{aligned} \tag{4.23}$$

where we have used the solution $\eta_1 \sim O(1)$ and $y/d \ll y \ll 1$, Therefore, we obtained the entanglement entropy of Hawking radiation which is approximately equal to $2S_{\rm BH}$ for two special cases.

Moreover, it can be shown that, for the general case $x < y \ll 1$, one still has $S_R \simeq 2S_{\rm BH}$, since from Eq. (4.9), we have

$$S_R \simeq 2S_{\rm BH}\left(1 - \frac{2y^2}{(d+1)(d-2)\lambda}\left(1 - \frac{x}{y}\right)^{1-d}\right) < 2S_{\rm BH}. \tag{4.24}$$

For the general case $x < y \ll 1$, the island solution satisfies $x/y = u_1 < u_m = 1/(d+1)$ under the constraint $\lambda > e$. Then we have

$$\begin{aligned} S_R &> 2S_{\rm BH}\left(1 - \frac{2y^2}{(d+1)(d-2)\lambda}\left(1 - \frac{1}{d+1}\right)^{1-d}\right) \\ &= 2S_{\rm BH}\left(1 - \frac{2y^2 e}{(d+1)(d-2)\lambda}\right) \simeq 2S_{\rm BH}. \end{aligned} \tag{4.25}$$

Thus, we obtain $S_R \simeq 2S_{\rm BH}$ for the general case $x < y \ll 1$. The leading term is given by double Bekenstein-Hawking entropy, which comes from the boundary area term of the island. The subleading term has been ignored, which reflects the contribution from the quantum effects of matter fields.

V. THE ENTANGLEMENT ENTROPY IN EXTREMAL LARGE D RN-AdS$_{d+1}$ BLACK HOLE

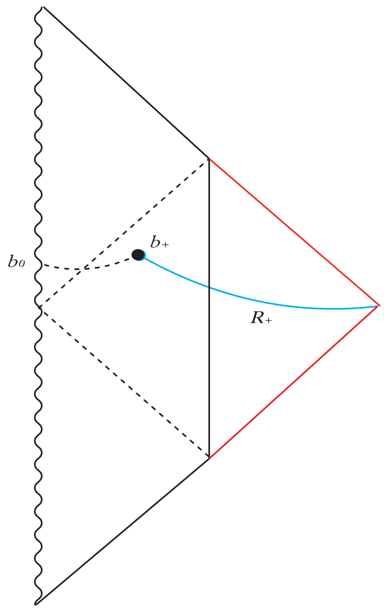
In this section, we consider the extremal large D RN-AdS$_{d+1}$ black hole. In the same way, we attach an auxiliary bath to the AdS boundary of the extremal large D RN-AdS$_{d+1}$ black hole. In Refs. [41,43], the authors argued that one cannot calculate the entanglement entropy of the extremal black hole by taking the extremal limit from the entanglement entropy of the nonextremal black hole, because the Penrose diagram of the extremal black hole is not a continuous limit of the nonextremal case, one should start from the extremal setup. We will start from the Penrose diagram of the extremal black hole to calculate the entanglement entropy in the extremal large D RN-AdS$_{d+1}$ black hole.

A. Without island

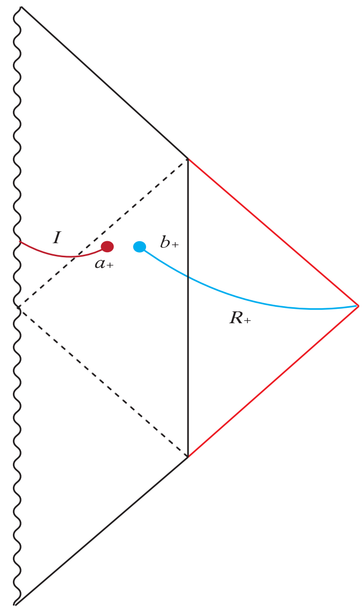
As showed in Fig. 5(a), the Cauchy surface including $b_+ = (t_b, \rho_b)$ touches the singularity at $b_0 = (t_b, 0)$. By using Eq. (3.4), the entanglement entropy of Hawking radiation is given by

$$\begin{aligned} S_R &= \lim_{\kappa\to 0}\frac{c}{3}\log d(b_+, b_0) \\ &= \lim_{\kappa\to 0}\frac{c}{6}\log[w(\rho_b)w(0)(U(b_0) \\ &\quad - U(b+))(V(b_+) - V(b_0)))] \\ &= \frac{c}{12}\log\left[f(0)f(\rho_b)(\rho_*(\rho_b) - \rho_*(0))^2\right], \end{aligned} \tag{5.1}$$

where ρ_* is defined in Sec. II B, and $f(\rho) = k(\rho - \rho_h)^2/\rho^2$. We can find that $f(0)$ is singular for the extremal large dimensional RN-AdS$_{d+1}$ black hole and the entanglement entropy is divergent at $\rho = 0$. This means that we cannot give a well-behaved entanglement entropy of Hawking radiation for the extremal case. This problem also was noticed in Refs. [41,43,52]. However, we can still give the



(a) Without island



(b) With island

FIG. 5. The Penrose diagram of the extremal RN-AdS$_{d+1}$ black hole coupled to an auxiliary thermal bath. The black part and red triangle represent the black hole and the auxiliary at spacetime, respectively. (a) b_+ is the boundary surface of the radiation region R_+ and b_0 is the singularity with $\rho = 0$. (b) The island region extends from $\rho = 0$ to $\rho = \rho_a$.

entanglement entropy of Hawking radiation when the island exists in the extremal case.

B. With island

In the presence of the island, we set the boundary of the island as $a_+ = (t_a, \rho_a)$. Similarly, we consider the situation where $\rho_b - \rho_h \ll \rho_h$. We assume that the boundary of the island is outside and near the horizon, thus $\rho_a - \rho_h < \rho_b - \rho_h \ll \rho_h$ [see Fig. 5(b)]; we still use the formula (3.5) for analysis. Before calculating the entanglement entropy, let us first give the geodesic distance between a_+ and b_+,

$$L = \int_{\rho_a}^{\rho_b} \frac{r_o}{d-2} \frac{d\rho}{k^{\frac{1}{2}}(\rho-\rho_h)} = \frac{r_o}{k^{\frac{1}{2}}(d-2)} \log\left(\frac{\rho_b-\rho_h}{\rho_a-\rho_h}\right). \quad (5.2)$$

By using Eq. (3.5), the generalized entropy is

$$\begin{aligned} S_{\text{gen}} &= \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{4G_N} - \kappa_{d+1} c \frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{L^{d-1}} \\ &= \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{4G_N} - \kappa_{d+1} c \frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{\left(\log\left(\frac{\rho_b-\rho_h}{\rho_a-\rho_h}\right)\right)^{d-1}} \\ &\quad \times \frac{k^{\frac{d-1}{2}}(d-2)^{d-1}}{r_o^{d-1}}. \end{aligned} \quad (5.3)$$

We still adopt the definitions $x \equiv \sqrt{\frac{\rho_a-\rho_+}{\rho_+}}$ and $y \equiv \sqrt{\frac{\rho_b-\rho_+}{\rho_+}}$, thus $x < y \ll 1$. The generalized entropy becomes

$$S_{\text{gen}} = \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{4G_N} - \kappa_{d+1} c \frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{(\log\frac{y^2}{x^2})^{d-1}} \frac{k^{\frac{d-1}{2}}(d-2)^{d-1}}{r_o^{d-1}}. \quad (5.4)$$

Then from the equation $\frac{\partial S_{\text{gen}}}{\partial \rho_a} = 0$, we obtain

$$\frac{x^2}{y^2}\left(\log\frac{y^2}{x^2}\right)^d = 4\kappa_{d+1} c G_N \frac{(d-2)^d k^{\frac{d-1}{2}}}{r_o^{d-1} y^2}. \quad (5.5)$$

Defining $z \equiv x^2/y^2 \in (0, 1)$, the above equation becomes

$$z\left(\log\frac{1}{z}\right)^d = 4\kappa_{d+1} c G_N \frac{(d-2)^d k^{\frac{d-1}{2}}}{r_o^{d-1} y^2} \equiv F_0. \quad (5.6)$$

The function $F(z) = z(\log z^{-1})^d$ monotonically increases with z in the interval $(0, z_m)$ and monotonically decreases with z in the interval $(z_m, 1)$; the local maximum value of function $F(z)$ is located at z_m (see Fig. 6), that is,

$$F(z_m) = \frac{d^d}{e^d} \quad \text{at } z_m = e^{-d}. \quad (5.7)$$

Therefore, the existence of the island solution of Eq. (5.6) requires the constraint

$$\frac{d^d}{e^d} > F_0. \quad (5.8)$$

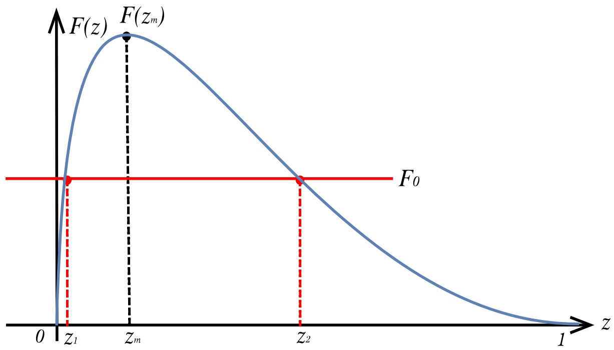


FIG. 6. The schematic diagram of function $F(z)$ in the interval $z \in (0,1)$. The maximal value $F(z_m)$ is located at z_m. There are two solutions z_1 and z_2 when $F(z_m) > F_0$, and the island solution is given by z_1.

If the constraint is satisfied, there exist two solutions z_1 and z_2 ($z_1 < z_m < z_2$). The generalized entropy $S_{\rm gen}$ reaches its local minimum and local maximum at z_1 and z_2, respectively, thus the island solution is given by z_1. If the constraint (5.8) is violated, there would be no nontrivial island solution in this case. This constraint obtained in the extremal case is different from the constraint (4.15) in the nonextremal case.

We have shown the existence of the island solution for $x < y \ll 1$ in the extremal case, which requires the constraint (5.8), though we do not give the exact expression of the island solution in this case. Now we consider the corresponding entanglement entropy of the Hawking radiation. Note that the island solution satisfies $x^2/y^2 = z_1 < z_m = e^{-d}$; by plugging the solution back to Eq. (5.4), we can get the entanglement entropy

$$\begin{aligned} S_R &= \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{4G_N} - \kappa_{d+1} c \frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{(\log\frac{y^2}{x^2})^{d-1}} \frac{k^{\frac{d-1}{2}}(d-2)^{d-1}}{r_o^{d-1}} \\ &> \frac{\Omega_{d-1}(M\rho_a)^{\frac{d-1}{d-2}}}{4G_N} - \kappa_{d+1} c \frac{\Omega_{d-1}(M\rho_b)^{\frac{d-1}{d-2}}}{d^{d-1}} \frac{k^{\frac{d-1}{2}}(d-2)^{d-1}}{r_o^{d-1}} \\ &> S_{\rm BH}\left(1 - 4\kappa_{d+1} c G_N \frac{k^{\frac{d-1}{2}}(d-2)^{d-1}}{d^{d-1} r_o^{d-1}}\right) \\ &> S_{\rm BH}\left(1 - \frac{y^2}{e^d}\right) \simeq S_{\rm BH}, \end{aligned} \tag{5.9}$$

where $y \ll 1$. We have utilized the following relation from the constraint (5.8), i.e.,

$$4\kappa_{d+1} c G_N \frac{k^{\frac{d-1}{2}}(d-2)^{d-1}}{d^{d-1} r_o^{d-1}} < \frac{d\cdot y^2}{e^d(d-2)} < \frac{y^2}{e^d}. \tag{5.10}$$

At the same time, we have $S_R < S_{\rm BH}$, so the value of S_R is approximately equal to $S_{\rm BH}$.

In summary, we find that the entanglement entropy is equal to Bekenstein-Hawking entropy for the extremal case. This result is the same as in Refs. [41,43], in which the authors consider the situation where the boundary of the radiation region is far from the horizon. While we focus on the situation where the boundary of the radiation region is taken to be near the horizon in large dimensional RN-AdS$_{d+1}$ black holes, we mainly calculate the entanglement entropy with the island by using the formula (3.5).

VI. THE CONSTRAINTS IN THE PRESENCE OF ISLAND

In Secs. IV and V, we have studied the island in nonextremal and extremal large D RN-AdS$_{d+1}$ black holes. In order to ensure the existence of the island to save the unitarity in our cases, the constraints (4.15) and (5.8) should be satisfied. In this section, we would like to analyze these constraints on the large D RN-AdS$_{d+1}$ black hole in more detail.

In the nonextremal case, the constraint (4.15) should be satisfied, i.e.,

$$\lambda = \frac{\rho_+^{\frac{d-1}{2}} y^{d+1} 2^{d-2} r_o^{d-1}}{c\kappa_{d+1}\ell_p^{d-1}(k(\rho_+ - \rho_-))^{\frac{d-1}{2}}(d-2)^d(d+1)} > e, \tag{6.1}$$

where $G_N = \ell_p^{d-1}$. Taking the large D limit and utilizing the approximation $\kappa_{d+1} = \Gamma[\frac{d-1}{2}]/(2^{d+3}\pi^{(d-1)/2})$ [77] in large dimensions, we have

$$\frac{r_o}{\ell_p^2 T} > \frac{d^2}{8y^2} \gg \frac{d^2}{8}, \tag{6.2}$$

where T is the Hawking temperature. Equation (6.2) is a more general constraint that provides the limitation on the black hole size r_o and temperature T, but note that the constraint (6.2) holds when $y \ll 1$. Moreover, if one sets charge $Q = 0$ and $k = 1$, the temperature changes to

$$T = \frac{d-2}{4\pi r_o}. \tag{6.3}$$

We can find that the constraint (6.2) can be written as $r_o/\ell_p \gg d^{3/2}/\sqrt{32\pi e}$, which is the constraint of Schwarzschild black hole in large D limit [51].

In the extremal case, the constraint (5.8) should be satisfied, equivalently,

$$\frac{d^d}{e^d} > 4\kappa_{d+1} c G_N \frac{(d-2)^d k^{\frac{d-1}{2}}}{r_o^{d-1} y^2} \equiv \chi\kappa_{d+1}\ell_p^{d-1}\frac{(d-2)^d k^{\frac{d-1}{2}}}{r_o^{d-1}}, \tag{6.4}$$

where $\chi \equiv 4c/y^2$. In the large D limit, we have

$$\chi\kappa_{d+1}\ell_p^{d-1}\frac{(d-2)^d k^{\frac{d-1}{2}}}{r_o^{d-1}} \simeq \chi\frac{d^{\frac{3d}{2}}\ell_p^d k^{\frac{d}{2}}}{8^{\frac{d}{2}}\pi^{\frac{d}{2}}e^{\frac{d}{2}}r_o^d}. \tag{6.5}$$

Then by reorganizing Eq. (6.4), we obtain

$$\frac{r_o^2}{\ell_p^2} > \frac{ek}{8\pi}\chi^{\frac{2}{d}}d \simeq \frac{e}{8\pi}\left(1+\frac{r_o^2}{L^2}\right)d, \tag{6.6}$$

where we have used $\chi^{\frac{2}{d}} \to 1$ when $d \to \infty$ (as we take a finite y even it is small). To find a constraint on r_o, we write the above inequality as

$$\left(\frac{8\pi}{e\ell_p^2 d} - \frac{1}{L^2}\right)r_o^2 - 1 > 0, \tag{6.7}$$

which reproduces a quadratic inequality of variable r_o. Note that this inequality further requires

$$\frac{8\pi}{e\ell_p^2 d} > \frac{1}{L^2}, \tag{6.8}$$

which puts a constraint on the value of the AdS radius L, and r_o should satisfy the following relation:

$$\frac{r_o}{\ell_p} > \frac{1}{\sqrt{\frac{8\pi}{ed} - \frac{\ell_p^2}{L^2}}} > \sqrt{\frac{e}{8\pi}}d^{\frac{1}{2}}. \tag{6.9}$$

This indicates that there is a universal lower bound on the radius of the large D extremal RN-AdS$_{d+1}$ black hole.

In short, we find new constraints on the large D RN-AdS$_{d+1}$ black hole in the presence of the island. Similar results have been found in Schwarzschild black holes in Refs. [51,80]. In Ref. [80], the authors provided a constraint, i.e., $r_o/\ell_p \gtrsim d^{3/2}$, on the size of Schwarzschild black holes through the large D analysis. Instead, in Ref. [51] the authors also provided a constraint by the existence of the island, i.e., $r_o/\ell_p \gg d^{3/2}/\sqrt{32\pi e}$ for the large dimensional Schwarzschild black hole. In the present paper, we focused on finding new constraints in the large D limit RN-AdS$_{d+1}$ black hole. Indeed, we find constraints (4.15) for the nonextremal case and constraint (5.8) for the extremal case, which leads to the constraints on the size of the large D RN-AdS$_{d+1}$ black hole, i.e., Eqs. (6.2) and (6.9), respectively. It is interesting to note that the constraint on r_o for the extremal case is scaling as $d^{1/2}$, which is very different from that in Schwarzschild black holes [51,80].

VII. PAGE CURVE AND PAGE TIME

In this section, we would like to estimate the Page time. For the nonextremal case with $x < y \ll 1$, the entanglement entropy S_R without island is given by Eq. (4.5), which grows linearly with t at late time; whereas the entanglement entropy S_R with island is given by

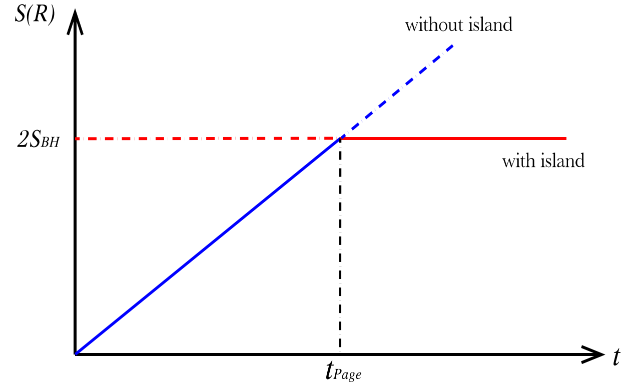


FIG. 7. The Page curve for the nonextremal large dimensional eternal RN-AdS$_{d+1}$ black hole. For the eternal black hole, the entanglement entropy of Hawking radiation remains unchanged and constrained by twice Bekenstein-Hawking entropy after the Page time.

Eq. (4.25), approximatively as $S_R \simeq 2S_{\rm BH}$. This helps us reproduce the Page curve of the nonextremal large dimensional eternal RN-AdS$_{d+1}$ black hole (see Fig. 7). So the Page time in this case is given by

$$t_{\rm Page} \simeq \frac{6S_{\rm BH}}{c\kappa_+} = \frac{3S_{\rm BH}}{\pi cT}, \tag{7.1}$$

where T is the Hawking temperature of the nonextremal large D RN-AdS$_{d+1}$ black hole.

For the extremal case with $x < y \ll 1$, although we have analyzed the Penrose diagram with island and give its entanglement entropy as $S_R \simeq S_{\rm BH}$, the entanglement entropy S_R without island is ill defined. This is mainly because $f(\rho)$ is ill defined at the singularity $b_0 = (t_b, 0)$. Therefore, we cannot provide the Page time for the extremal black hole.

In summary, for the nonextremal large D RN-AdS$_{d+1}$ black hole, by combining Eqs. (4.5) and (4.21), we get the Page curve of the entanglement entropy of Hawking radiation. Before the Page time, the entanglement entropy increases approximately linearly with time, and there is no island. After the Page time, the island appears and its boundary is near the horizon, thus the entanglement entropy becomes approximately twice the Bekenstein-Hawking entropy. The Page time is obtained in nonextremal case, which is the same as the result in the Reissner-Nordström black hole in four dimensions [39]. Whereas, for the

TABLE I. The summary of results for large dimensional RN-AdS$_{d+1}$ black hole.

Black holes	Without island	With island	Page time
Nonextremal case	$S_R \simeq \frac{c}{3}\kappa_+ t_b$	$S_R \simeq 2S_{\rm BH}$	$t_{\rm Page} \simeq \frac{3S_{\rm BH}}{\pi cT}$
Extremal case	Ill defined	$S_R \simeq S_{\rm BH}$	Ill defined

extremal case, we cannot give a well-defined Page time (see Table I).

VIII. CONCLUSION AND DISCUSSION

In this paper, we investigated the entanglement entropy of the Hawking radiation in the nonextremal and extremal cases of large D RN-AdS$_{d+1}$ black holes coupled to an auxiliary bath at the boundary of the black hole via the island formula. We mainly considered the situation in which the boundary of the radiation region is close to the outer horizon of the black hole.

For the nonextremal case, we showed the existence of the general island solution, and we obtained two analytical island solutions in special cases $x \ll y/d$ and $x \sim y/d$ with $d \to \infty$, i.e., Eqs. (4.17) and (4.19). Although we did not give the analytical expression of the island solution for general $x < y \ll 1$, we found a constraint (4.15) that is required by the existence of the island in this case. The entanglement entropy of Hawking radiation has been obtained for both cases with and without the island. Meanwhile, the Page curve and Page time are also obtained (see Fig. 7).

For the extremal case, we showed that the entanglement entropy without island is ill defined. As shown in the Penrose diagram [i.e., Fig. 5(a)], the region extends from the boundary of the radiation region to the singularity $\rho = 0$, while the conformal factor [see Eq. (5.1)] is divergent at $\rho = 0$. It has been pointed out in Ref. [74] that, when taking the extremal limit for nonextremal RN black holes, the black hole geometry will divide into an extremal black hole and a disconnected AdS$_2$ part. While the microscopic entropy of the extremal RN black hole as shown can be calculated from its near-horizon geometry either from the RN/CFT correspondence [81] or from the HEE perspective [82]. However, the island formula will involve the black hole singularity in the absence of island for the extremal case, which will cause the semiclassical calculation to be invalid when the left boundary reaches the singularity b_0 [see Fig. 5(a)]. In previous works, such as [41,43,52], the authors mainly studied the case where the boundary radiation region is far from the outer horizon and utilized the formula (3.4) for the island phase. This differs from our analysis, as we mainly used the formula (3.5) to calculate the entanglement entropy with the island phase. We showed that the entanglement entropy of Hawking radiation in the extremal case is approximately equal to the Bekenstein-Hawking entropy ($S_R \simeq S_{\rm BH}$), which is consistent with the results in previous studies [41,43,52]. However, since the entanglement entropy without island is not yet clear in the extremal case, therefore, the Page curve and Page time are also not well defined in this case.

Moreover, we showed that the existence of the island will put some constraints on the black holes (also see related papers [38,44,45,51]). For large D RN-AdS$_{d+1}$ black holes, we found the constraints both for the nonextremal and extremal cases, which are

$$\frac{r_o}{\ell_p^2 T} > \frac{d^2}{8y^2} \gg \frac{d^2}{8} \quad \text{(for nonextremal case)}, \tag{8.1}$$

$$\frac{r_o}{\ell_p} > \frac{1}{\sqrt{\frac{8\pi}{ed} - \frac{\ell_p^2}{L^2}}} > \sqrt{\frac{e}{8\pi}} d^{\frac{1}{2}}$$

$$\text{and} \quad \frac{L}{\ell_p} > \sqrt{\frac{e}{8\pi}} d^{\frac{1}{2}} \quad \text{(for extremal case)}, \tag{8.2}$$

as required by the existence of island in the case $y \ll 1$.

Now let us discuss some issues needed to be solved in the future. First, we focused on studying the case of one island in our paper. In general, the configuration of multiple islands is allowed, and it can soften the turning point of the Page curve at the Page time. Second, our calculation was mainly based on the two-dimensional approximation formula in Sec. III; a more general formula is needed to calculate the entanglement entropy of matter in the high-dimensional curved spacetime. Finally, the constraints we obtained are only valid in the case of $x < y \ll 1$. For more general case $y > 0$, we do not yet know whether there is a similar constraint, and it deserves to generalize the calculation of the entanglement entropy in high-dimensional black holes for general case $y > 0$.

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