Equilibrium of slowly rotating polytropes in modified Einstein gravity

Shaswata Chowdhury,[1,†] Pritam Banerjee,[2,‡] and Aneta Wojnar[3,*]
[1]*Department of Physics, Indian Institute of Technology, Kanpur 208016, India*
[2]*Department of Physics, Indian Institute of Technology, Kharagpur 721302, India*
[3]*Departamento de Física Teórica, Universidad Complutense de Madrid, E-28040, Madrid, Spain*

(Received 30 December 2022; revised 4 December 2023; accepted 15 April 2024; published 17 May 2024)

A general formalism to find the density profile of a slowly rotating stellar object in modified Einstein gravity is presented. We derive a general form of the modified Lane-Emden equation in the presence of rotation and a general form of its possible solutions under the slow rotation approximation for a wide class of modified Einstein gravity theories.

DOI: 10.1103/PhysRevD.109.104049

I. INTRODUCTION

A polytropic equation of state (EOS) turns out to be a useful approximation to describe matter properties in substellar and stellar objects, as well as neutron stars. It has a simple form

$$P = K\rho^{1+\frac{1}{n}}, \tag{1.1}$$

relating stellar density ρ to the pressure P, where K and n are polytropic parameters or functions, taking different expressions and values being dependent upon the class of stellar objects we are considering. Because of that, it allows one to analyze, often analytically, a given astrophysical object in modified Einstein gravity theories [1–4] before applying a more complex approach, with more realistic microphysics. However, even with such a simple form, many sophisticated processes can be hidden in this EOS. The most important one is the electron degeneracy, crucial in modeling some layers of the Sun [5,6] and other main sequence stars [7,8], low-mass stars [9–11], brown dwarfs, and giant exoplanets [12] as well as white dwarfs [13–18]. Another improvement, which can also be incorporated into microphysics modeling and then rewritten in the polytropic form, is strongly coupled plasma [19], finite gas temperatures with phase transition points between metallic hydrogen and the molecular state [20]. Moreover, a merger of the third-order finite strain Birch-Murgnagham equation of state [21] with the Thomas-Fermi-Dirac one [22–26] turns out to be also approximated by the polytropic EOS [27], which is suitable to describe matter behavior in cold low-mass spheres such as terrestrial planets.

The set of following equations: polytropic EOS (1.1), Poisson equation (G is the Newton's gravitational constant, with U being the gravitational potential)

$$\nabla^2 U = -4\pi G\rho, \tag{1.2}$$

together with the equation of hydrostatic equilibrium in Newtonian gravity, both considered in the spherical-symmetric spacetime

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = -4\pi G\rho \tag{1.3}$$

$$\frac{dP}{dr} = \rho\frac{dU}{dr} \tag{1.4}$$

can be rewritten into the Lane-Emden equation (LEE)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{1.5}$$

where θ is a function of ξ, satisfying the boundary conditions $\theta(0) = 1$, $\theta'(0) = 0$, with $'$ denoting derivative with respect to ξ. To do so, one needs to introduce the dimensionless variables θ and ξ, such that

$$\rho = \rho_c\theta^n, \qquad r = r_c\xi \quad \text{with } r_c^2 = \frac{K(n+1)\rho_c^{(\frac{1}{n}-1)}}{4\pi G}, \tag{1.6}$$

where ρ_c denotes the central density. The solution of the LEE with a particular value of the polytropic index n and polytropic constant K provides the total stellar mass M, stellar radius R, and the density profile (1.6), pressure (1.1), temperature T as well as the core quantities ρ_c and T_c:

$$M = 4\pi r_c^3\rho_c\omega_n, \qquad R = \gamma_n\left(\frac{K}{G}\right)^{\frac{n}{3-n}} M^{\frac{1-n}{n-3}}, \tag{1.7}$$

*Corresponding author: awojnar@ucm.es
†shaswata@iitk.ac.in
‡pritam@phy.iitkgp.ac.in

$$\rho_c = \delta_n \left(\frac{3M}{4\pi R^3}\right), \qquad T = T_c\theta = K\frac{m_H\mu}{k_B}\rho_c^{\frac{1}{n}}\theta, \tag{1.8}$$

where k_B is Boltzmann's constant, m_H is the mass of Hydrogen atom, and μ being the mean molecular weight while

$$\omega_n = -\xi_1^2 \frac{d\theta}{d\xi}\bigg|_{\xi=\xi_1}, \tag{1.9}$$

$$\gamma_n = (4\pi)^{\frac{1}{n-3}}(n+1)^{\frac{n}{3-n}}\omega_n^{\frac{n-1}{3-n}}\xi_1, \tag{1.10}$$

$$\delta_n = -\frac{\xi_1}{3\frac{d\theta}{d\xi}|_{\xi=\xi_1}}. \tag{1.11}$$

For more properties, see [28].

On the other hand, modified Einstein gravity, which we will refer to as modified gravity (MG) later in the text, often introduces additional terms to the Poisson equation [11,29–32], which we can write in a generic form as

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = -4\pi G\rho + LV_{\mathrm{mod0}}(r), \tag{1.12}$$

where the modification term $LV_{\mathrm{mod0}}(r)$ is a general function that is different for different classes of modified gravity theories. As an example, we can have a look at three of the most popular and working models of modified gravity theories, in which the corresponding Poisson equation takes the following forms:

(i) In generalized beyond-Horndeski theories [33–35] the Poisson equation takes the form (because of the partial breaking of Vainshtein mechanism [36–38]) [11,16]:

$$\nabla^2 V \sim -\frac{\kappa}{2}\left(\rho + \frac{\Upsilon}{4}\nabla^2(r^2\rho)\right) \tag{1.13}$$

where Υ is the modified gravity parameter for the beyond-Horndeski class of theories.

(ii) In Palatini $f(\mathcal{R})$ gravity, the Poisson equation reads [31]

$$\nabla^2 V \sim -\frac{\kappa}{2}(\rho + 2\beta\nabla^2\rho) \tag{1.14}$$

where $\kappa = 8\pi G$ and β is a constant[1] associated to the $O(\mathcal{R}^2)$ term of the function $f(\mathcal{R})$, with $\mathcal{R}$ being the Palatini-Ricci scalar. The constant β thus parametrizes this particular class of modified gravity theories.

(iii) In Eddington-inspired Born-Infeld (EiBI) gravity, the Poisson equation reads [29,39–41]

$$\nabla^2 V \sim -\frac{\kappa}{2}\left(\rho + \frac{\epsilon}{2}\nabla^2\rho\right) \tag{1.15}$$

where $\epsilon = 1/M_{\mathrm{BI}}$, with M_{BI} being the Born-Infeld mass. ϵ is the modified gravity parameter for this class of theories.

We will discuss the significance of such forms of the Poisson equation in MG theories, in the further part of the article.

As in the preceding case of Newtonian gravity, introducing to the generic Poisson equation (1.12) in MG theories, the polytropic EOS (1.1), the pressure balance equation (1.4) along with the dimensionless quantities (1.6), we can write down the modified Lane-Emden equation (MLEE)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n + g_{\mathrm{mod0}}(\xi) \tag{1.16}$$

where the extra term appearing in the above

$$g_{\mathrm{mod0}} = \frac{LV_{\mathrm{mod0}}}{4\pi G\rho_c} \tag{1.17}$$

is a dimensionless term induced by a given MG. The boundary condition for θ remains the same as for Eq. (1.5).

So far, most of the stellar and substellar objects have been studied in MG with some form of polytropic EOS, however, in the spherical-symmetric spacetime (for review, see [30,42,43]). In order to obtain limiting masses, such as, for instance, the Chandrasekhar mass limit of white dwarfs [16,44–52], the minimum main sequence mass [53–56], minimum mass for deuterium burning [57], Jeans [58] and opacity mass [59], the authors were using the considered EOS. To constrain models of gravity with the use of seismic data from stars [5,6] and rocky planets [60–63], the polytropic EOS was also applied to describe at least some of the object's layers. In a similar manner, to obtain light elements' abundances [64], the polytrope was also adopted, the same as in the evolutionary phases of various astrophysical objects [65–75].

Although the astrophysical objects rotate, rotating polytropes [76–84] (for the second order approximation, see [85]) have not been studied widely in the nonrelativistic framework of MG. Several studies of stellar rotation in MG in the relativistic framework are found in the literature. Rotating neutron stars have been studied in the context of various MG theories using the polytropic equation of state such as scalar-tensor theories [86], dilatonic Einstein-Gauss-Bonnet theory [87], and Rastall's gravity [88]. In the relativistic context [89,90] one usually uses the numerical approach, which is even more inevitable in the case of

[1] β is of dimension $[\mathrm{L}]^2$, where [L] corresponds to length dimension.

modifications introduced to the Tolman-Oppenheimer-Volkoff equation by MG (for review, see [30] and references therein). Because of this poorly studied branch, we are going to look for an analytic density profile given for a wide class of MG in the nonrelativistic regime. Interestingly, MG's effects in the stellar interior are prominent even in the nonrelativistic limit. However, let us comment that since the slow-rotation approximation resembles anisotropic fluid in the nonrotating case [91–94], such a fluid in MG could possibly also mimic a slowly rotating object in the framework of a given theory of gravity [52,95–100].

The paper is organized as follows. In Sec. II, we provide a general form of the modified Lane-Emden equation for a rotating polytrope together with its solution. Those are given for a wide class of modified gravity theories, which satisfies three specific conditions as demonstrated in the further part of the section. Section III is devoted to specific examples of MG, which allows us to put forward a corollary on the conditions to be satisfied by the Poisson equation in such theories, such that the presented formalism is valid. We draw our conclusions in Sec. IV. Appendix A deals with an insightful derivation, explicitly highlighting the dependence of the internal potential on the modified gravity, in the presence of rotation. In Appendix B we leave a few comments on the form of the external potential assumed in the main body of this paper. Appendix C elaborates the homogeneity of a particular differential equation governing the rotation induced component of the solution to the MLEE for a rotating polytrope.

II. ANALYTIC DERIVATION FOR THE DENSITY PROFILE OF A SLOWLY ROTATING STELLAR OBJECT IN MODIFIED GRAVITY THEORIES

In this section, we present the analytical formalism to incorporate slow rotation in any modified gravity theories in general. We would be specifically following the approach of [76]. Therefore, we will consider the rotation of the object to be along the z axis of the 3D Cartesian coordinate system, with the uniform angular speed denoted by ω. In polar coordinates $\{r, \mu, \phi\}$, where r is the radial coordinate, and $\mu(=\cos\vartheta)$, ϕ is the angular coordinates, the equations of mechanical equilibrium are as follows:

$$\frac{\partial P}{\partial r} = \rho \frac{\partial V}{\partial r} + \rho\omega^2 r(1-\mu^2), \tag{2.1}$$

$$\frac{\partial P}{\partial \mu} = \rho \frac{\partial V}{\partial \mu} - \rho\omega^2 r^2 \mu \tag{2.2}$$

with ϕ being neglected on account of axial symmetry and abiding by the convention of [76], V is chosen to be the negative of the gravitational potential energy. Although no additional terms due to modified gravity theories appear explicitly in Eqs. (2.1) and (2.2), the potential V inherently captures the effect of MG. However, in modified gravity theories, the Poisson equation gets modified, as can be seen, for example, from [29,101,102]

$$\nabla^2 V = -4\pi G\rho + LV_{\mathrm{mod}}(r,\mu). \tag{2.3}$$

In the above Eq. (2.3), we refer to the modification term $LV_{\mathrm{mod}}(r,\mu)$ as a general function, without mentioning its actual form; the form is going to be different for different classes of modified gravity theories. The μ dependence in the modification term is induced by the rotation; i.e., in the absence of rotation the modification term will solely depend upon the radial coordinate r,[2] leading to Eq. (1.12). Therefore, regarding the matter description, we take the total pressure P inside a rotating stellar object to be related to its density $\rho = \rho(r,\mu)$ by means of a polytropic relation:

$$P(r,\mu) = K\rho^{1+\frac{1}{n}}. \tag{2.4}$$

We emphasize the fact that the pressure and density are functions of both the radial as well as the angular coordinate in the presence of rotation-induced asymmetry. Analogously to the nonrotating case, we can introduce the dimensionless variables Θ and ξ, such that

$$\rho = \rho_c\Theta^n, \qquad r = r_c\xi \quad \text{with } r_c^2 = \frac{K(n+1)\rho_c^{(\frac{1}{n}-1)}}{4\pi G} \tag{2.5}$$

where Θ is a function of both ξ and μ. It should be noted that Eq. (2.5) is similar to Eq. (1.6), other than $\theta(\xi)$ being replaced by $\Theta(\xi,\mu)$.

To go further, let us propose the following.

Proposition. Let $\theta = \theta(\xi)$ be a solution of the modified Lane-Emden equation (1.16) of the nonrotating polytrope (1.1) and $\Theta = \Theta(\xi,\mu)$ of the rotating one (2.4) in the polar coordinates. Then, the Lane-Emden equation for a rotating polytrope with the uniform angular speed ω in modified gravity is given by

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial\Theta}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial\Theta}{\partial\mu}\right) = v + g_{\mathrm{mod}}(\xi,\mu) - \Theta^n \tag{2.6}$$

with $\Theta(0)=1$ and $\Theta'(0)=0$ as boundary conditions; $v=\omega^2/2\pi G\rho_c$ is a dimensionless parameter, which is a measure of the outward centrifugal force compared to the self-gravity of the rotating polytrope, while $g_{\mathrm{mod}}(\xi,\mu) = LV_{\mathrm{mod}}/4\pi G\rho_c$ is the dimensionless modification term depending on a given theory of gravity in general.

[2]For example in beyond Horndeski, the density perturbation to the Friedmann-Robertson-Walker (FRW) metric is in general spherically symmetric in the absence of rotation.

Proof. The Poisson equation (2.3) in polar coordinates takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial V}{\partial \mu}\right) = -4\pi G\rho + LV_{\mathrm{mod}}(r,\mu). \tag{2.7}$$

Using the mechanical equilibrium equations (2.1) and (2.2), along with the polytropic EOS (2.4) and Eq. (2.5), the Poisson equation (2.7) reduces to the modified Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial \Theta}{\partial \xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \Theta}{\partial \mu}\right) = v + g_{\mathrm{mod}}(\xi,\mu) - \Theta^n.$$

Therefore, we have demonstrated the general form of the MLEE for the rotating polytrope (2.6). The boundary conditions $\Theta(0)=1$, $\Theta'(0)=0$ follow from the physicality of the density and pressure profiles at the center; i.e., they will fall from their central maximum value to zero at the surface. ∎

Note that for a given central density, the parameter v, being the measure of the strength of rotation, will be the expansion parameter for certain functions and solutions in the sequel. Hereafter, such an exact form of the modified Lane-Emden equation in the case of rotation allows us to study rotating objects. Usually, one solves this equation numerically. However, we may also try to get an analytic handle on the solution—this is particularly useful because one can track the effects of modified gravity and easily compare it with the Newtonian case [76]. Moreover, it allows us to distinguish the modifications introduced by a given theory of gravity from the other effects, like the ones coming from, e.g., microphysics or other processes which happen in the stellar and substellar interiors. Therefore, let us propose the following theorem:

Theorem. If $g_{\mathrm{mod}}(\xi,\mu)$ can be expanded in terms of the Legendre functions $P_l(\mu)$'s as

$$g_{\mathrm{mod}}(\xi,\mu) = g_{\mathrm{mod}0}(\xi) + v\left\{\bar{\tilde{g}}_{\mathrm{mod}}(\xi) + \sum_{j=1}^{\infty}\bar{\bar{\tilde{g}}}_{\mathrm{mod}j}(\xi)P_j(\mu)\right\}, \tag{2.8}$$

where $g_{\mathrm{mod}0}(\xi)$ is the nonrotating part, with $\bar{\tilde{g}}_{\mathrm{mod}}(\xi)$, and $\bar{\bar{\tilde{g}}}_{\mathrm{mod}j}(\xi)$ being the rotation induced ones, then the solution Θ of the modified Lane-Emden equation in the presence of slow rotation is

$$\Theta(\xi,\mu) = \theta(\xi) + v[\psi_0(\xi) + A_2\psi_2(\xi)P_2(\mu)], \tag{2.9}$$

where ψ_0 and ψ_2 satisfy the following equations:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) = -n\theta^{n-1}\psi_0 + 1 + \bar{\tilde{g}}_{\mathrm{mod}}(\xi), \tag{2.10}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) = \left(\frac{6}{\xi^2} - n\theta^{n-1}\right)\psi_2 + \frac{\bar{\bar{\tilde{g}}}_{\mathrm{mod}2}(\xi)}{A_2}, \tag{2.11}$$

with $\psi_0(0)=0=\psi_0'(0)$ and $\psi_2(0)=0=\psi_2'(0)$ being the respective boundary conditions, and

$$A_2 = -\frac{5}{6}\frac{\xi_1^2}{[3\psi_2(\xi_1)+\xi_1\psi_2'(\xi_1)]}, \tag{2.12}$$

with ξ_1 being the first zero of $\theta(\xi)$ while $'$ denotes derivative with respect to ξ.

Proof. Let us make the following choice for the modification term g_{mod}:

$$g_{\mathrm{mod}}(\xi,\mu) = g_{\mathrm{mod}0}(\xi) + v\tilde{g}_{\mathrm{mod}}(\xi,\mu), \tag{2.13}$$

where $g_{\mathrm{mod}0}(\xi)$ is the standard modification term coming from modified gravity theories in the nonrotating scenario Eq. (1.16), while $\tilde{g}_{\mathrm{mod}}(\xi,\mu)$ is the correction term appearing in the modified gravity theories when rotation is taken into consideration. The above expansion will enable us to extract out terms in orders of v, in the subsequent calculations, as we will see shortly.

In order to find a solution to Eq. (2.6), we assume the following form for Θ:

$$\Theta(\xi,\mu) = \theta(\xi) + v\Psi(\xi,\mu) + v^2\Phi(\xi,\mu) \tag{2.14}$$

where θ is the nonrotating solution, with Ψ and Φ being the rotation induced correction terms. We are considering slow rotation where the effects arising from ω^4 can be neglected. Therefore, we consistently work only up to the first order in v. Putting Eq. (2.14) in Eq. (2.6), the $O(v^0)$ gives back Eq. (1.16) as expected, while $O(v)$ gives the following equation:

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}\left(\xi^2\frac{\partial \Psi}{\partial \xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial \Psi}{\partial \mu}\right) = -n\theta^{n-1}\Psi + 1 + \tilde{g}_{\mathrm{mod}}(\xi,\mu). \tag{2.15}$$

Now, for a given stellar object (i.e., fixing n) and a theory of modified gravity in the nonrotating scenario [i.e., knowing the form of $g_{\mathrm{mod}0}(\xi)$], we know the solution $\theta(\xi)$ from Eq. (1.16). Therefore, all we need to find is the solution $\Psi(\xi,\mu)$ from Eq. (2.15), in order to obtain the complete solution Θ. For that, we assume the following form for Ψ:

$$\Psi(\xi,\mu) = \psi_0(\xi) + \sum_{j=1}^{\infty}A_j\psi_j(\xi)P_j(\mu) \tag{2.16}$$

where A_j's are normalizing constants and $P_j(\mu)$ corresponds to the Legendre function of index j, satisfying the Legendre differential equation

$$\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial P_j}{\partial\mu}\right)+j(j+1)P_j=0. \quad (2.17)$$

At this point it is crucial to highlight the importance of explicitly introducing the normalizing constants only with ψ_j's and not with ψ_0 and θ [see Eq. (2.14)]. As we will shortly come across the differential equations governing the functions ψ_0, and ψ_j's, in the later part of this section, it will be evident that for the given boundary conditions, the solution ψ_0 will be determined uniquely, while each ψ_j remains undetermined by an arbitrary multiplicative constant. The arbitrary constants are labeled as normalizing constants for a reason as will be justified shortly. Similarly, from Eq. (1.16) we see the solution θ is also unique; thus, we do not introduce any normalizing constant with it.

Substituting Eq. (2.16) in Eq. (2.15) and using Eq. (2.17) we get

$$0=\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)+n\theta^{n-1}\psi_0-1\right]-\tilde{g}_{\mathrm{mod}}(\xi,\mu)+\sum_{j=1}^{\infty}A_j\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)-\left(\frac{j(j+1)}{\xi^2}-n\theta^{n-1}\right)\psi_j\right]P_j(\mu). \quad (2.18)$$

From the above equation, it is clearly seen that the modification term $\tilde{g}_{\mathrm{mod}}$ couples to the independent terms associated with the linearly independent Legendre functions, and thus forbids their complete extraction and equating them to zero. One possible way of averting the situation is by having $\tilde{g}_{\mathrm{mod}}$ of this particular form:

$$\tilde{g}_{\mathrm{mod}}(\xi,\mu)=\bar{\bar{g}}_{\mathrm{mod}}(\xi)+\sum_{j=1}^{\infty}\bar{\bar{\bar{g}}}_{\mathrm{mod}j}(\xi)P_j(\mu). \quad (2.19)$$

Upon using Eq. (2.19) in Eq. (2.18) and equating coefficients of the linearly independent Legendre functions, we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right)=-n\theta^{n-1}\psi_0+1+\bar{\bar{g}}_{\mathrm{mod}}(\xi) \quad (2.20)$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right)=\left(\frac{j(j+1)}{\xi^2}-n\theta^{n-1}\right)\psi_j+\frac{\bar{\bar{\bar{g}}}_{\mathrm{mod}j}(\xi)}{A_j}. \quad (2.21)$$

The functions ψ_0 and ψ_j's satisfy the following boundary conditions:

$$\psi_0(0)=0;\quad \psi_0'(0)=0;\quad \psi_j(0)=0;\quad \psi_j'(0)=0. \quad (2.22)$$

Now, for a given theory of modified gravity, upon knowing the explicit forms of $\bar{\bar{g}}_{\mathrm{mod}}(\xi)$ and $\bar{\bar{\bar{g}}}_{\mathrm{mod}j}(\xi)$, one can solve for ψ_0 and ψ_j from the above, Eqs. (2.20) and (2.21). We see that while the solution ψ_0 is unique, ψ_j's remain undetermined by a multiplicative constant, i.e., if ψ_j is a solution then $A_j\psi_j$ is also a solution, which, however, is not the case for ψ_0. While this observation is apparent in standard Newtonian gravity, it needs some elaboration for the wide class of modified gravity theories we considered here (see Appendix C). However, these constants are not completely arbitrary as it seems; each A_j's unique expression, for a given solution ψ_j, can be predicted by exploiting the condition of continuity of the potential and its derivative at the stellar surface. Therefore, the correct solution stands out to be $A_j\psi_j$, and it is in this spirit A_j's are called normalizing constants. At this point, it is important to note that solving Eq. (2.21) for ψ_j seems improbable due to the existence of A_j, which is not known beforehand. However, in a broad class of modified gravity theories the modification term $\bar{\bar{\bar{g}}}_{\mathrm{mod}j}(\xi)$ depends upon the density and thus on Θ in such a way that it inherently carries a factor of A_j, and thus cancels out the A_j in the denominator; one can then solve for ψ_j. In sequel, we will show with examples that it does happen for certain classes of modified gravity theories.

Assuming for the time being that there exist modified gravity theories for which one can solve for ψ_0 and ψ_j's using the above set of equations (2.20) and (2.21), we are still left with determining the unknown normalizing constants A_j's. For that we will first determine V in terms of A_j's using the hydrostatic equilibrium condition and the polytropic equation of state, following the approach of [78]. Now, at the stellar surface, this V must correspond to a physically viable generic form of the potential exterior (say, V_{ext}) to the object. This necessitates equating V with V_{ext} as well as the radial derivative of V with that of V_{ext}. Upon doing this, we will obtain the coefficients A_j's in terms of the known solutions. We explicitly develop the formalism as follows.

The hydrostatic equilibrium condition for a rotating fluid in presence of gravity can be represented as

$$\nabla V+\nabla\left(\frac{1}{2}|\vec{\omega}\times\vec{r}|^2\right)=\frac{1}{\rho}\nabla P. \quad (2.23)$$

Now, using the polytropic EOS and taking the scalar product of the above equation with $d\vec{r}$ we integrate from the pole to any point (r,μ) within the stellar object.

$$\int_{(r,\mu)}^{\mathrm{pole}}dV+\int_{(r,\mu)}^{\mathrm{pole}}d\left(\frac{1}{2}|\vec{\omega}\times\vec{r}|^2\right)=\frac{K(n+1)}{n}\int_{(r,\mu)}^{\mathrm{pole}}\rho^{\frac{1}{n}-1}d\rho. \quad (2.24)$$

After a little bit of algebra, we then obtain

$$V = R\left[\Theta - \frac{1}{6}v(\xi^2 - P_2(\mu)\xi^2)\right] + \text{const}, \tag{2.25}$$

where $R \coloneqq (n+1)K\rho_c^{\frac{1}{n}}$ and the "const" above represents the potential at the pole (see Appendix A for an alternative derivation of the potential). The normalizing constants A_j's contained within Θ are still unknown. The A_j's now get determined by implementing the condition of continuity of V as well as its radial derivative at the stellar surface. For that we assume a particular form of the gravitational potential exterior to the stellar surface

$$V_{\text{ext}} = R\left[\frac{C_0}{\xi} + v\sum_{j=1}^{\infty}\frac{C_j}{\xi^{j+1}}P_j(\mu)\right] + \text{const}, \tag{2.26}$$

where $C_i, i = \{0, 1, \ldots\}$ are arbitrary constants. At this point we would like to comment on the validity of such a form, before delving further. We know that in many modified gravity theories, the gravitational potential exterior to a stellar object contains a Yukawa type correction term, which is responsible for the fifth force. Interestingly, such a correction term is not present in the above form for the external potential. The reason being the screening mechanisms associated to such modified gravity theories, which screens the fifth force at small scales, thereby recovering Einstein's general relativity (see Appendix B for a discussion on this). Now, as a next step we match the interior and exterior potentials and their derivatives at the stellar surface. However, since a rotating stellar object is oblate, it does not have a single well-defined radius defining the stellar surface. On the other hand, analytic values of the stellar radii, corresponding to different angular coordinates (let us call them angular stellar radii), can only be obtained if one has the complete solution Θ. Unfortunately, without knowing A_j's one cannot obtain Θ and thus the angular stellar radii. Thus with the spirit of analytic formalism, we choose the first zero ξ_1 of the Emden's function corresponding to the nonrotating modified gravity scenario as the point where we enforce the continuity of the potential and its radial derivative in the rotating scenario i.e.,

$$V|_{\xi_1} = V_{\text{ext}}|_{\xi_1}, \qquad \left.\frac{\partial V}{\partial \xi}\right|_{\xi_1} = \left.\frac{\partial V_{\text{ext}}}{\partial \xi}\right|_{\xi_1}. \tag{2.27}$$

Since we are considering slow rotation, where the degree of oblateness is not high, this approximation is well justified. Now, implementing the aforementioned condition Eq. (2.27) we get

$$\begin{aligned} A_j &= 0 = C_j \quad \forall \;\; j \neq 2; \\ A_2 &= -\frac{5}{6}\frac{\xi_1^2}{[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]}, \end{aligned} \tag{2.28}$$

where A_2 is obtained after eliminating nonzero C_2. Again, the form of A_2 is the same as in the standard Newtonian gravity, although the effects of modified gravity theory are encoded in ξ_1 and ψ_2 implicitly. We note here, that had we considered the normalized ψ_j's from the first place, without introducing the A_j's explicitly, then we would have obtained conditions on the ψ_j's by the similar approach as described above. To be specific we would have arrived at the following condition for the only nonzero ψ_2:

$$\xi_1\psi_2'(\xi_1) + 3\psi_2(\xi_1) + \frac{5}{6}\xi_1^2 = 0. \tag{2.29}$$

A careful inspection tells us that this is exactly what one would have arrived at by putting A_2 in Eq. (2.28) as unity. This is completely justified because considering a normalized ψ_2 amounts to $A_2 = 1$.

Having obtained the A_j's, we now write the general form of the complete solution of the MLEE in the presence of slow rotation,

$$\Theta(\xi,\mu) = \theta(\xi) + v\left[\psi_0(\xi) - \frac{5}{6}\frac{\xi_1^2}{[3\psi_2(\xi_1) + \xi_1\psi_2'(\xi_1)]} \times \psi_2(\xi)P_2(\mu)\right]$$

where ψ_0 and ψ_2 are solutions to the following equations, obtained from Eqs. (2.20) and (2.21), respectively,

$$\begin{aligned} \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) &= -n\theta^{n-1}\psi_0 + 1 + \bar{\bar{g}}_{\text{mod}}(\xi) \\ \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_2}{d\xi}\right) &= \left(\frac{6}{\xi^2} - n\theta^{n-1}\right)\psi_2 + \frac{\bar{\bar{\bar{g}}}_{\text{mod}2}(\xi)}{A_2}. \end{aligned} \tag{2.30}$$

■

Recalling the discussion after Eq. (2.21), we see from Eq. (2.30) that in order to solve for ψ_2 one needs to know A_2, which itself depends upon the solution ψ_2. Therefore, unless the term $\bar{\bar{\bar{g}}}_{\text{mod}2}(\xi)$ depends upon density in such a way that it inherently carries a factor of A_2, which cancels the one in the denominator, one cannot solve for the complete solution in this analytic formalism. We, therefore, enlist the three conditions, which the generic correction term g_{mod} due to modified gravity theories should satisfy, in order to obtain the complete solution in this particular analytic formalism[3]:

[3]We note that in case this third property does not get satisfied for a certain class of modified gravity theories, one can employ a self-iterating numerical scheme to obtain a solution for such coupled equations. Developing such a scheme is nevertheless a daunting task, and we are not aware of such an attempt till now.

(1) $g_{\rm mod}$ can be expanded as in Eq. (2.13) to the first order in v.
(2) The $O(v)$ correction term $\tilde{g}_{\rm mod}$ can be expanded as in Eq. (2.19) in terms of Legendre functions.
(3) The $\bar{\bar{\bar{g}}}_{{\rm mod}j}$ term should be proportional to A_j.

In Sec. III we show that the aforementioned three conditions are indeed satisfied for a wide class of modified gravity theories.

III. ON THE GENERIC $g_{\rm mod}$ TERM

In this section, we show that one can carry out the above analytical formalism in several modified gravity theories in the literature. For that, we revisit the Poisson equation in the different classes of modified gravity theories given by (1.13)–(1.15) in the Introduction. From these forms of the Poisson equation, one realizes that we can write the modification term in general as

$$LV_{\rm mod}(r,\mu) = k_1\nabla^2(\bar{\alpha}(r)\rho) \tag{3.1}$$

where k_1 represents the overall constant comprising of fundamental constant G, numerical factors and the associated modified gravity parameter. The term $\bar{\alpha}(r)$ is, in general, a radial function, which, for example, takes up the constant value 1 for the Palatini $f(\mathcal{R})$ and EiBI, while it is r^2 for generalized beyond Horndeski. Converting the above equation into its nondimensional form and multiplying it with a factor of $1/4\pi G\rho_c$, we obtain

$$\tilde{k}_1\nabla^2_\xi(\alpha(\xi)\Theta^n) \tag{3.2}$$

where ∇^2_ξ is the dimensionless Laplacian, and $\tilde{k}_1$ is the overall constant appearing upfront, which, in general, depends on r_c, and the modified gravity parameter. The function $\alpha(\xi)$ is the nondimensional version of $\bar{\alpha}(r)$. To put things into perspective, let us mention that Eq. (3.1) corresponds to the $LV_{\rm mod}$ term in Eq. (2.3), while Eq. (3.2) represents the $g_{\rm mod}$ term in Eq. (2.6). Thus, we will investigate whether this generic $g_{\rm mod}$ term satisfies the three conditions mentioned in the Sec. II, which are required for our analytic formalism to go through.

Expressing Eq. (3.2) explicitly we have

$$\begin{aligned} g_{\rm mod}(\xi,\mu) &= \tilde{k}_1\nabla^2_\xi(\alpha(\xi)\Theta^n) \\ &= \tilde{k}_1\left[\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial(\alpha\Theta^n)}{\partial\xi}\right) + \frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial(\alpha\Theta^n)}{\partial\mu}\right)\right]. \end{aligned} \tag{3.3}$$

Then, using $\Theta^n = \theta^n + vn\theta^{n-1}\Psi$ [to the first order in v from Eq. (2.14)] along with Eq. (2.16) in Eq. (3.3), we obtain

$$g_{\rm mod}(\xi,\mu) = g_{\rm mod0}(\xi) + v\left\{\bar{\bar{g}}_{\rm mod}(\xi) + \sum_{j=1}^{\infty}\bar{\bar{\bar{g}}}_{{\rm mod}j}(\xi)P_j(\mu)\right\} \tag{3.4}$$

where

$$\begin{aligned} g_{\rm mod0}(\xi) = \frac{\tilde{k}_1}{\xi}\theta^{n-2}\Big\{&n(n-1)\xi\alpha\theta'^2 + \theta^2(2\alpha' + \xi\alpha'') \\ &+ n\theta(2(\alpha+\xi\alpha')\theta' + \xi\alpha\theta'')\Big\} \end{aligned} \tag{3.5}$$

$$\begin{aligned} \bar{\bar{g}}_{\rm mod}(\xi) = \frac{\tilde{k}_1}{\xi}n\theta^{n-3}\Big\{&(n-2)(n-1)\xi\alpha\psi_0\theta'^2 \\ &+(n-1)\theta(2\theta'(\xi\psi_0\alpha' + \alpha(\psi_0+\xi\psi_0{}')) + \xi\alpha\psi_0\theta'') \\ &+\theta^2(2(\alpha+\xi\alpha')\psi_0{}' + \psi_0(2\alpha'+\xi\alpha'') + \xi\alpha\psi''_0)\Big\} \end{aligned} \tag{3.6}$$

$$\begin{aligned} \bar{\bar{\bar{g}}}_{{\rm mod}j}(\xi) = \frac{\tilde{k}_1}{\xi}A_j\Big[n\theta^{n-3}\Big\{&(n-2)(n-1)\xi\alpha\psi_j\theta'^2 \\ &+(n-1)\theta(2\theta'(\xi\psi_j\alpha' + \alpha(\psi_j+\xi\psi'_j)) + \xi\alpha\psi_j\theta'') \\ &+\theta^2(2(\alpha+\xi\alpha')\psi'_j + \psi_j(2\alpha'+\xi\alpha'') + \xi\alpha\psi''_j)\Big\} \\ &-\frac{1}{\xi}j(j+1)n\alpha\theta^{n-1}\psi_j\Big] \end{aligned} \tag{3.7}$$

where $'$ denotes first-order derivative with respect to ξ and $''$ denotes second-order one. From Eq. (3.7) we see that $\bar{\bar{\bar{g}}}_{{\rm mod}j} \propto A_j$ and is homogeneous of degree one in ψ_j. Therefore, we see that all the three conditions enlisted in the previous section get satisfied for these broad classes of modified gravity theories and hence one can use our analytical formalism in these theories.

At this stage, it is convenient to propose our hypothesis:

Corollary. In general, our analytical formalism can be used in any modified gravity theories, where the correction term of the corresponding Poisson equation contains density, its higher order radial derivatives, or its Laplacian.

IV. CONCLUSION

In this work, we have presented a general formalism to find the density profile of a slowly rotating stellar object in the presence of modified gravity. By adapting the formalism given in [76], we demonstrated a generic approach to incorporate modified gravity effects. We have shown three conditions that the additional modified gravity term arising in the Poisson equation needs to satisfy in order to abide by our formalism. First, the modified term is required to be expanded into a summation between nonrotating and rotating counterparts. Second, the rotating part should be further expanded in series involving Legendre functions. Finally, the coefficients appearing with the Legendre functions need to explicitly involve density, its derivative terms, or its Laplacian. We have undertaken three well known theories of modified gravity, i.e., scalar-tensor theories beyond Horndeski, Palatini $f(\mathcal{R})$ gravity, and Eddington-inspired Born-Infeld gravity, which are shown to satisfy all these three conditions.

An interesting aspect to note from our results is that in the presence of rotation, only the $j = 0$, 2 Legendre components of $LV_{\rm mod}$ (or $g_{\rm mod}$) contribute to the Poisson equation, although there have been no *a priori* restrictions on such a modification term. Physically it can be justified as follows. We expect the rotating polytrope to take up the shape of an oblate spheroid. Therefore, only the monopole and quadrupole terms should contribute to the solution Θ, the potential V, and thus to the modification term $g_{\rm mod}$ as well. This is usually the case in the Newtonian limit of Einstein's general relativity, so it is but natural to expect the same in the modified gravity theories as well, which for no reason should violate the axial symmetry.

As already mentioned, this work is focused on slow rotation, which is necessary to set the matching conditions, Eq. (2.27), at the first zero of a spherically symmetric nonrotating configuration θ. While it is valid for a slowly rotating polytrope, it prevents one from finding a solution for fast rotation. To overcome this limitation, one must follow a semianalytic approach, where fast rotation is achieved using multiple small increments of stellar rotation. The matching condition is reused iteratively at the first zero of the last updated rotating Emden's function Θ. To this end, we draw the reader's attention to the fact that the parameter v, being a measure of the ratio of outward centrifugal force and self-gravity, can incorporate fast rotation for higher central density and yet be small enough to neglect $\mathcal{O}(v^2)$ corrections. Let us also emphasize that the central density can increase when a theory parameter increases in the modification term—then, in modified gravity theories, the same v can correspond to larger rotation ω due to an increase in ρ_c. It is so because the parameter v we expand the solution Θ about includes the central density ρ_c^{-1}, lowering its value. Because of that fact, this approximation breaks down for a specific large value of v in Newtonian physics, while in modified gravity one can still consider more rapidly rotating objects.

To summarize, this work enables us to analytically obtain the density profile of a slowly rotating star by elegantly utilizing its axial symmetry. This is a stepping stone for further studies in modeling of rotating stellar and planetary objects in the presence of modified gravity. The provided formalism will allow us to find a complete solution of a further specified modified Lane-Emden equation. We will present the results on the overall rotating density profile in future work.

ACKNOWLEDGMENTS

A. W. acknowledges financial support from MICINN (Ministerio de Ciencia e Innovación , Spain) *Ayuda Juan de la Cierva—incorporacíon* 2020 No. IJC2020-044751-I. S. C. would like to thank Tapobrata Sarkar for useful discussions. The authors sincerely thank the anonymous referees for their constructive suggestions.

APPENDIX A: EFFECT OF MODIFIED GRAVITY ON THE POTENTIAL

In this appendix we derive the form of the interior potential V following the approach of [76]. This approach helps us explicitly identify the effects of the modified gravity on the potential. In contrast to the derivation in Sec. II, here we use the Poisson equation and equations of mechanical equilibrium as follows.

The Poisson equation (2.7) in ξ, μ variables takes the form (to the first order in v)

$$\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial V}{\partial\xi}\right)+\frac{1}{\xi^2}\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial V}{\partial\mu}\right) = -(n+1)K\rho_c^{\frac{1}{n}}\left[\theta^n+n\theta^{n-1}v\left\{\psi_0+\sum_{j=1}^{\infty}A_j\psi_j(\xi)P_j(\mu)\right\} - g_{\rm mod}(\xi,\mu)\right]. \tag{A1}$$

In order to solve for V in the above equation, we develop V in the form (to the first order in v)

$$V = U(\xi)+v\left\{V_0(\xi)+\sum_{j=1}^{\infty}V_j(\xi)P_j(\mu)\right\}, \tag{A2}$$

where U is the modified gravity potential of the nonrotating configuration. Upon using Eq. (A2) in Eq. (A1), and then equating the $O(v^0)$ component and coefficients of $P_j(\mu)$ in the $O(v)$ component, we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dU}{d\xi}\right) = -R\{\theta^n - g_{\rm mod0}(\xi)\} \tag{A3}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_0}{d\xi}\right) = -R\{n\theta^{n-1}\psi_0 - \bar{\bar{g}}_{\rm mod}(\xi)\} \tag{A4}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right) - \frac{j(j+1)}{\xi^2}V_j = -R\{n\theta^{n-1}A_j\psi_j - \bar{\bar{\bar{g}}}_{{\rm mod}j}(\xi)\}. \tag{A5}$$

In the above set of equations, we now explicitly see how the modified gravity affects the component functions—U, V_0, and V_j of the gravitational potential V. We get the analytic forms of these component functions, after a little bit of algebra.

Using Eq. (1.16) in Eq. (A3) we obtain

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dU}{d\xi}\right) = \frac{R}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) \tag{A6}$$

whereby we deduce

$$U = R\theta + \text{const} \tag{A7}$$

Although this analytic form looks exactly the same in the case of standard Newtonian gravity (see [76]), the difference lies in the fact that here θ, being the solution to Eq. (1.16), is implicitly carrying the information of the modified gravity theory under consideration. Such information is also included in R, as it depends on ρ_c, which is given by (1.8) and (1.11).

Using Eq. (2.20) in Eqs. (A4) and (2.21) in Eq. (A5) we obtain

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_0}{d\xi}\right) = R\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_0}{d\xi}\right) - 1\right], \tag{A8}$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{dV_j}{d\xi}\right) - \frac{j(j+1)}{\xi^2}V_j = RA_j\left[\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right) - \frac{j(j+1)}{\xi^2}\psi_j\right], \tag{A9}$$

whereby we deduce

$$V_0 = R\left(\psi_0 - \frac{1}{6}\xi^2\right) + \text{const}, \tag{A10}$$

$$V_j = R(A_j\psi_j + B_j\xi^j) + \text{const}, \tag{A11}$$

where B_j's are arbitrary constants appearing from the regular solution of Eq. (A9). After rearranging terms we get

$$V = R\left[\Theta + v\left\{\sum_{j=1}^{\infty} B_j\xi^j P_j(\mu) - \frac{1}{6}\xi^2\right\}\right], \tag{A12}$$

where once again we mention that although this analytic form looks exactly the same in the case of standard Newtonian gravity (see [76]), the information of modified gravity theory is carried implicitly by the solution Θ, as pointed out above.

After converting the first relation of Eq. (2.2) into its dimensionless form [by using Eqs.(2.4) and (2.5)], we substitute Eq. (A12) for V in the same. Equating coefficients of $P_j(\mu)$ we obtain

$$B_j = 0 \quad \forall \;\; j \neq 2; \qquad B_2 = \frac{1}{6}. \tag{A13}$$

Thus we have

$$V = R\left[\Theta - \frac{1}{6}v(\xi^2 - P_2(\mu)\xi^2)\right] + \text{const}. \tag{A14}$$

APPENDIX B: COMMENTS ABOUT THE FORM OF V_{ext} ASSUMED IN OUR PAPER

In many of the modified gravity theories, the gravitational potential outside the stellar object has a Yukawa-type modification term, which leads to the *fifth* force. However, one does not see the effects of the fifth force in table-top experiments because they are screened at small scales due to the screening mechanism associated with the theory, thereby recovering Einstein's general relativity. However, the modification in the gravitational interaction is present at large scales. Such screening is important, for example, in the solar system scales where the observations conform with the Einstein's general relativistic predictions while allowing for the modifications to explain accelerated expansion at large scales. There are several screening mechanisms, like symmetron screening and the Vainshtein screening mechanisms, to name a few. The form of the V_{ext} thus chosen is valid in the region where the screening is effective, while an appropriate modification term needs to be incorporated beyond a certain length scale where the screening fails. However, the radius beyond which the screening fails turns out to be extremely large compared to the radius of the stellar object; thus the correction terms to V_{ext} become subleading and negligibly small; so the form Eq. (2.26) we have chosen is well justified. As an example, in modified gravity theory equipped with the Vainshtein mechanism, the characteristic radius beyond which the modifications are present and inside which the modifications are screened turns out to be 100 pc for a solar mass star, which is 10^8 times larger compared to the solar radius.

APPENDIX C: EQUIVALENCE BETWEEN THE SOLUTIONS ψ_j AND $A_j\psi_j$

In this appendix we elaborate on the equivalence between the solutions ψ_j and $A_j\psi_j$, as mentioned in relation to Eq. (2.21)

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi_j}{d\xi}\right) = \left(\frac{j(j+1)}{\xi^2} - n\theta^{n-1}\right)\psi_j + \frac{\bar{\bar{\bar{g}}}_{\text{mod}\,j}(\xi)}{A_j}. \tag{C1}$$

To begin with, let us mention that in the standard Newtonian gravity the absence of the last term on the right-hand side of the above equation makes it apparent that if ψ_j is a solution then $A_j\psi_j$ will also be a solution to the same equation. However, it is not obvious in the case of modified gravity unless we decode the dependence of the modification term $\bar{\bar{\bar{g}}}_{\text{mod}\,j}(\xi)$ on the components of the solution Θ.

For the wide class of modified gravity theories considered in this paper, $\bar{\bar{\bar{g}}}_{\text{mod}\,j} \propto A_j$ and is homogeneous of degree one in ψ_j (see Sec. III). Therefore Eq. (C1) is actually a second-order homogeneous linear differential equation in ψ_j. Thus if ψ_j is a solution to Eq. (C1), then so must be $A_j\psi_j$.

[1] T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010).
[2] S. Capozziello and M. de Laurentis, Phys. Rep. **509**, 167 (2011).
[3] S. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011).
[4] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, Phys. Rep. **692**, 1 (2017).
[5] I. D. Saltas and I. Lopes, Phys. Rev. Lett. **123**, 091103 (2019).
[6] I. D. Saltas and J. Christensen-Dalsgaard, Astron. Astrophys. **667**, A115 (2022).
[7] A.-C. Davis, E. A. Lim, J. Sakstein, and D. J. Shaw, Phys. Rev. D **85**, 123006 (2012).
[8] J. Sakstein, Phys. Rev. D **88**, 124013 (2013).
[9] L. Bildsten, E. F. Brown, C. D. Matzner, and G. Ushomirsky, Astrophys. J. **482**, 442 (1997).
[10] G. Ushomirsky, C. D. Matzner, E. F. Brown, L. Bildsten, V. G. Hilliard, and P. C. Schroeder, Astrophys. J. **497**, 253 (1998).
[11] K. Koyama and J. Sakstein, Phys. Rev. D **91**, 124066 (2015).
[12] A. Burrows and J. Liebert, Rev. Mod. Phys. **65**, 301 (1993).
[13] U. Das and B. Mukhopadhyay, J. Cosmol. Astropart. Phys. 05 (2015) 045.
[14] R. Saito, D. Langlois, D. Yamauchi, S. Mizuno, and J. Gleyzes, in *Everything about Gravity: Proceedings of the Second LeCosPA International Symposium*, edited by C. Pisin (World Scientific Publishing Company, Singapore, 2016), pp. 398–403.
[15] O. Bertolami and H. Mariji, Phys. Rev. D **93**, 104046 (2016).
[16] I. D. Saltas, I. Sawicki, and I. Lopes, J. Cosmol. Astropart. Phys. 05 (2018) 028.
[17] A. V. Astashenok, S. D. Odintsov, and V. K. Oikonomou, Phys. Rev. D **106**, 124010 (2022).
[18] A. V. Astashenok, S. D. Odintsov, and V. K. Oikonomou, Symmetry **15**, 1141 (2023).
[19] D. J. Stevenson, Annu. Rev. Astron. Astrophys. **29**, 163 (1991).
[20] S. Auddy, S. Basu, and S. R. Valluri, Adv. Astron. **2016**, 574327 (2016).
[21] F. Birch, Phys. Rev. **71**, 809 (1947).
[22] L. H. Thomas, Proc. Cambridge Philos. Soc. **23**, 542 (1927).
[23] E. Fermi, Z. Phys. **48**, 73 (1928).
[24] P. A. M. Dirac, Proc. Cambridge Philos. Soc. **26**, 376 (1930).
[25] R. P. Feynman, N. Metropolis, and E. Teller, Phys. Rev. **75**, 1561 (1949).
[26] E. E. Salpeter and H. S. Zapolsky, Phys. Rev. **158**, 876 (1967).
[27] S. Seager, M. Kuchner, C. A. Hier-Majumder, and B. Militzer, Astrophys. J. **669**, 1279 (2007).
[28] G. P. Horedt, *Polytropes—Applications in Astrophysics and Related Fields* (Springer, Dordrecht, 2004), Vol. 306.
[29] M. Bañados and P. G. Ferreira, Phys. Rev. Lett. **105**, 011101 (2010).
[30] G. J. Olmo, D. Rubiera-Garcia, and A. Wojnar, Phys. Rep. **876**, 1 (2020).
[31] J. D. Toniato, D. C. Rodrigues, and A. Wojnar, Phys. Rev. D **101**, 064050 (2020).
[32] G. J. Olmo, D. Rubiera-Garcia, and A. Wojnar, Phys. Rev. D **104**, 024045 (2021).
[33] G. W. Horndeski, Int. J. Theor. Phys. **10**, 363 (1974).
[34] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, Phys. Rev. Lett. **114**, 211101 (2015).
[35] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, J. Cosmol. Astropart. Phys. 02 (2015) 018.
[36] A. I. Vainshtein, Phys. Lett. **39B**, 393 (1972).
[37] T. Kobayashi, Y. Watanabe, and D. Yamauchi, Phys. Rev. D **91**, 064013 (2015).
[38] M. Crisostomi and K. Koyama, Phys. Rev. D **97**, 021301 (2018).
[39] J. Beltrán Jiménez, L. Heisenberg, G. J. Olmo, and D. Rubiera-Garcia, Phys. Rep. **727**, 1 (2018).
[40] P. Banerjee, D. Garain, S. Paul, R. Shaikh, and T. Sarkar, Astrophys. J. **924**, 20 (2022).
[41] E. Mattanzi and A. Sulaksono, AIP Conf. Proc. **2234**, 040016 (2020).
[42] E. N. Saridakis, R. Lazkoz, V. Salzano, P. V. Moniz, S. Capozziello, J. Beltrán Jiménez, M. De Laurentis, and G. J. Olmo, *Modified Gravity and Cosmology: An Update by the CANTATA Network* (Springer International Publishing, Switzerland, 2021).
[43] A. Wojnar, Lect. Notes Phys. **1017**, 363 (2023).
[44] S. Chandrasekhar, Mon. Not. R. Astron. Soc. **95**, 207 (1935).
[45] R. K. Jain, C. Kouvaris, and N. G. Nielsen, Phys. Rev. Lett. **116**, 151103 (2016).
[46] S. Banerjee, S. Shankar, and T. P. Singh, J. Cosmol. Astropart. Phys. 10 (2017) 004.
[47] A. Wojnar, Int. J. Geom. Methods Mod. Phys. **18**, 2140006 (2021).
[48] I. H. Belfaqih, H. Maulana, and A. Sulaksono, Int. J. Mod. Phys. D **30**, 2150064 (2021).
[49] L. Sarmah, S. Kalita, and A. Wojnar, Phys. Rev. D **105**, 024028 (2022).
[50] S. Kalita and L. Sarmah, Phys. Lett. B **827**, 136942 (2022).
[51] S. Kalita and B. Mukhopadhyay, Astrophys. J. **909**, 65 (2021).
[52] S. Chowdhury and T. Sarkar, Astrophys. J. **884**, 95 (2019).
[53] J. Sakstein, Phys. Rev. Lett. **115**, 201101 (2015).
[54] J. Sakstein, Phys. Rev. D **92**, 124045 (2015).
[55] M. Crisostomi, M. Lewandowski, and F. Vernizzi, Phys. Rev. D **100**, 024025 (2019).
[56] G. J. Olmo, D. Rubiera-Garcia, and A. Wojnar, Phys. Rev. D **100**, 044020 (2019).
[57] A. S. Rosyadi, A. Sulaksono, H. A. Kassim, and N. Yusof, Eur. Phys. J. C **79**, 1030 (2019).
[58] S. Capozziello, M. de Laurentis, I. de Martino, M. Formisano, and S. D. Odintsov, Phys. Rev. D **85**, 044022 (2012).
[59] A. Wojnar, Phys. Rev. D **104**, 104058 (2021).
[60] A. Kozak and A. Wojnar, Phys. Rev. D **104**, 084097 (2021).
[61] A. Kozak and A. Wojnar, Int. J. Geom. Methods Mod. Phys. **19**, 2250157 (2022).

[62] A. Kozak and A. Wojnar, Universe **8**, 3 (2021).
[63] A. Kozak and A. Wojnar, Phys. Rev. D **108**, 044055 (2023).
[64] A. Wojnar, Phys. Rev. D **103**, 044037 (2021).
[65] A. Wojnar, Phys. Rev. D **102**, 124045 (2020).
[66] S. Chowdhury and T. Sarkar, J. Cosmol. Astropart. Phys. 05 (2021) 040.
[67] M. Guerrero, D. Rubiera-Garcia, and A. Wojnar, Eur. Phys. J. C **82**, 707 (2022).
[68] M. C. Straight, J. Sakstein, and E. J. Baxter, Phys. Rev. D **102**, 124018 (2020).
[69] D. A. Gomes and A. Wojnar, Eur. Phys. J. C **83**, 492 (2023).
[70] S. Chowdhury and T. Sarkar, Mon. Not. R. Astron. Soc. **523**, 518 (2023).
[71] M. Benito and A. Wojnar, Phys. Rev. D **103**, 064032 (2021).
[72] A. Kozak, K. Soieva, and A. Wojnar, Phys. Rev. D **108**, 024016 (2023).
[73] S. Kalita, L. Sarmah, and A. Wojnar, Universe **8**, 647 (2022).
[74] A. Wojnar, Phys. Rev. D **105**, 124053 (2022).
[75] S. Kalita, L. Sarmah, and A. Wojnar, Phys. Rev. D **107**, 044072 (2023).
[76] S. Chandrasekhar and E. A. Milne, Mon. Not. R. Astron. Soc. **93**, 390 (1933).
[77] J. J. Monaghan and I. W. Roxburgh, Mon. Not. R. Astron. Soc. **131**, 13 (1965).
[78] A. Kovetz, Astrophys. J. **154**, 999 (1968).
[79] G. B. Cook, S. L. Shapiro, and S. A. Teukolsky, Astrophys. J. **422**, 227 (1994).
[80] D. Kong, K. Zhang, and G. Schubert, Mon. Not. R. Astron. Soc. **448**, 456 (2015).
[81] S. Chowdhury, P. Banerjee, D. Garain, and T. Sarkar, Astrophys. J. **929**, 117 (2022).
[82] J. Ballot, F. Lignières, D. R. Reese, and M. Rieutord, arXiv:0912.1679.
[83] S. Yoshida, Res. Notes Am. Astron. Soc. **6**, 227 (2022).
[84] B. Reina, Mon. Not. R. Astron. Soc. **455**, 4512 (2016).
[85] F. Occhionero, Mem. Soc. Astron. Ital. **38**, 331 (1967), https://ui.adsabs.harvard.edu/abs/1967MmSAI..38..331O/abstract.
[86] D. D. Doneva, S. S. Yazadjiev, N. Stergioulas, and K. D. Kokkotas, Phys. Rev. D **88**, 084060 (2013).
[87] B. Kleihaus, J. Kunz, S. Mojica, and M. Zagermann, Phys. Rev. D **93**, 064077 (2016).
[88] F. M. da Silva, L. C. N. Santos, and C. C. Barros, Classical Quantum Gravity **38**, 165011 (2021).
[89] H. Komatsu, Y. Eriguchi, and I. Hachisu, Mon. Not. R. Astron. Soc. **237**, 355 (1989).
[90] V. Paschalidis and N. Stergioulas, Living Rev. Relativity **20**, 7 (2017).
[91] L. Herrera and N. O. Santos, Phys. Rep. **286**, 53 (1997).
[92] L. Herrera and W. Barreto, Phys. Rev. D **88**, 084022 (2013).
[93] L. Herrera and W. Barreto, Phys. Rev. D **87**, 087303 (2013).
[94] G. Abellán, E. Fuenmayor, and L. Herrera, Phys. Dark Universe **28**, 100549 (2020).
[95] V. Folomeev, Phys. Rev. D **97**, 124009 (2018).
[96] D. Deb, S. Ghosh, S. K. Maurya, M. Khlopov, and S. Ray, arXiv:1811.11797.
[97] M. F. Shamir and S. Zia, Eur. Phys. J. C **77**, 448 (2017).
[98] M. Farasat Shamir and M. Ahmad, Eur. Phys. J. C **77**, 674 (2017).
[99] Ö. Sert, F. Çeliktaş, and M. Adak, Eur. Phys. J. C **78**, 824 (2018).
[100] H. O. Silva, C. F. B. Macedo, E. Berti, and L. C. B. Crispino, Classical Quantum Gravity **32**, 145008 (2015).
[101] D. C. Rodrigues,, A. Hernandez-Arboleda, and A. Wojnar, Phys. Dark Universe **41**, 101230 (2023).
[102] P. Banerjee, D. Garain, S. Paul, R. Shaikh, and T. Sarkar, Astrophys. J. **910**, 23 (2021).