Tolman-Ehrenfest effect for an ideal gas in a background of time-independent electric, magnetic, and gravitational fields

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The statistical mechanics of an ideal gas of point particles moving in a time independent background metric with $g_{0j} \neq 0$ is investigated. An explicit calculation shows that when there is no background electrostatic or magnetostatic field the thermodynamic pressure, energy density, and thermally averaged energy-momentum tensor depend on temperature and chemical potential only through the ratios $T_0/\sqrt{g_{00}}$ and $\mu_0/\sqrt{g_{00}}$. A background magnetostatic field does not change this, however with a background electrostatic field the previous results are multiplied by a factor $\exp(-eA_0/T_0)$, which is an exception to the strict Tolman-Ehrenfest rule because the system is open.

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I. INTRODUCTION

The Tolman-Ehrenfest effect [1–3] originated with the observation that classical electromagnetic radiation in thermal equilibrium with matter in a static background metric ($\partial_0 g_{\mu\nu} = 0$ and $g_{0j} = 0$) would have a thermally averaged energy-momentum tensor whose temperature dependence always occurs in the ratio $T_0/\sqrt{g_{00}}$, where T_0 is spacetime independent. There are a number of arguments [4–10] that support the Tolman-Ehrenfest result for the static case $g_{0j} = 0$. Reference [10] specifically treats the ideal gas.

There are arguments that the Tolman-Ehrenfest effects is valid when $g_{0j} \neq 0$ [11–19]. A special example in this category is that of a rotating Minkowski reference frame [18,19] in which $g_{t\phi}$ is nonzero but there is no curvature.

The Schwarzschild metric illustrates the issue. In the original (t, r, θ, ϕ) coordinates the metric is static:

$$(ds)^2 = \left(1 - \frac{2GM}{r}\right)(dt)^2 - \frac{(dr)^2}{1 - 2GM/r} - r^2 d\Omega^2 \quad (1.1)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The thermodynamic functions for an ideal gas in this background metric will depend on T_0 only through the ratio $T_0/\sqrt{1 - 2GM/r}$. A change to the outgoing Eddington-Finkelstein time coordinate [20,21]

$$t' = t - 2GM \ln(r - 2GM) \quad (1.2)$$

changes the form of the line element to

$$(ds)^2 = \left(1 - \frac{2GM}{r}\right)(dt')^2 + \frac{4GM}{r} dt' dr - \left(1 + \frac{2GM}{r}\right)(dr)^2 - r^2 d\Omega^2. \quad (1.3)$$

Since $g_{t'r} \neq 0$ this metric is not static but stationary. It will follow from the calculation described herein that an ideal gas of point particles in this metric will be the same function of $T_0/\sqrt{1 - 2GM/r}$.

The more important situations are those in which the metric is stationary and cannot be changed to static by a coordinate transformation. Though in this paper the metric is not required to satisfy the Einstein field equations, there are two familiar stationary metrics that do solve the field equations and are not coordinate equivalent to a static metric: the Kerr metric describing an uncharged, but rotating black hole; and the Kerr-Newmann metric describing a charged, rotating black hole [22]. In Boyer-Lindquist coordinates (t, r, θ, ϕ) both metrics have $g_{t\phi} \neq 0$. The Kerr-Newman black hole is surrounded by a static electric field and a static magnetic field. There are many other stationary metrics that solve the field equations [23]. A limitation of the present analysis is that equilibrium statistical mechanics, whether done in the canonical or the grand canonical ensemble, applies to situations in which the total number of particles is conserved. Thus if the metric has an event singularity statistical mechanics is only applicable well outside the event horizon.

The Kerr-Newman example motivates extending the investigation to include arbitrary background electrostatic and/or magnetostatic fields and this leads to a specific exception to the Tolman-Ehrenfest effect. The exception can be illustrated by a simple example. Consider an ideal gas of

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nonrelativistic point particles with no gravitational field. In kinetic theory the distribution function for such a gas in thermal equilibrium is

$$f_0 = n_0\left(\frac{2\pi}{mT_0}\right)^{3/2} e^{-\mathbf{p}^2/2mT_0}. \tag{1.4}$$

The particle density $n_0 = \int[d^3p/(2\pi)^3]f_0$ is spatially uniform. Suppose the particles have charge e and a time-independent external electric and magnetic field is imposed. After equilibrium is achieved the new distribution function should satisfy the Vlasov equation [24]

$$\frac{\partial f}{\partial t} + v^j\frac{\partial f}{\partial x^j} + e(\mathbf{E}+\mathbf{v}\times\mathbf{B})_j\frac{\partial f}{\partial p_j} = 0. \tag{1.5}$$

In a complete analysis **E** and **B** would be the sum of the external fields and the internal fields; the internal fields being determined self-consistently by solving Maxwell's equations with the charge density and current density computed from the distribution function f. If the external fields are much stronger than the internal fields produced by the charged particles then **E** and **B** may taken as the external fields, with time-independent scalar and vector potentials A_0 and **A**. The solution to the Vlasov equations is

$$f(\mathbf{x},\mathbf{p}) = n_0\left(\frac{2\pi}{mT_0}\right)^{3/2} e^{-[(\mathbf{p}-e\mathbf{A})^2/2m+eA_0]/T_0} \tag{1.6}$$

where A_0 and **A** are evaluated at the position $\mathbf{x}(t)$ of the particle and f has no explicit time dependence. The particle density is

$$n(\mathbf{x}) = \int\frac{d^3p}{(2\pi)^3}f(\mathbf{x},\mathbf{p}) = n_0e^{-eA_0(\mathbf{x})/T_0}. \tag{1.7}$$

The factor $\exp[-eA_0/T_0]$ will appear in other thermodynamic functions: pressure, energy density, entropy density. (The gauge transformation that eliminates A_0 would make **A** time-dependent and the distribution function would no longer satisfy the Vlasov equation.) It will turn out that in the presence of a stationary gravitational field the same factor $\exp[-eA_0/T_0]$ will occur, whereas the Tolman-Ehrenfest expectation would be a factor $\exp[-eA_0\sqrt{g_{00}}/T_0]$. This is not in conflict with derivations of the Tolman-Ehrenfest effect [4–18] which rely on $(T^{\mu\nu})_{;\mu} = 0$.

Throughout the discussion the particles are thermalized; the internal electric and magnetic fields produced by the charged particles are neglected and A_0, **A** are unthermalized, external potentials.

The central problem is that the non-vanishing of $g_{0\ell}$ makes the Hamiltonian for a point particle moving on the geodesics of a time-independent background metric rather complicated. It will be shown in Sec. II that the particle Hamiltonian in the absence of a background electrostatic or magnetostatic field is

$$\bar{H} = \frac{1}{g^{00}}\left[\sqrt{g^{00}(m^2-g^{ij}p_ip_j)+(g^{0\ell}p_\ell)^2}+g^{0\ell}p_\ell\right]. \tag{1.8}$$

Here $p_j(t)$ are the canonical momenta and the metric components are evaluated at the particle position $x^i(t)$. Though covariant metric components $g_{\alpha\beta}$ do not appear in $\bar{H}$, the minimum value of $\bar{H}$ occurs at $p_j = -mg_{j0}/\sqrt{g_{00}}$ and the minimum energy is $m\sqrt{g_{00}}$.

Outline. Section II derives the Hamiltonian $\bar{H}$ and incorporates a time-independent background of static electric and magnetic fields. The Hamiltonian leads to the partition function and the thermodynamic pressure in the grand canonical ensemble in terms of two parameters T_0 and a chemical potential μ_0. The pressure due to the particles is of the form $e^{\beta_0(\mu_0-eA_0)}\bar{P}$.

Section III computes $\bar{P}$ explicitly and after using a particular addition theorem for Bessel functions the final result is

$$\bar{P} = \frac{m^2}{2\pi^2}\left(\frac{T_0}{\sqrt{g_{00}}}\right)^2K_2\left(m\frac{\sqrt{g_{00}}}{T_0}\right) \tag{1.9}$$

with no dependence on $g_{0\ell}$ or g_{jk}.

Section IV contains the computation of the thermally averaged energy-momentum tensor. The particle contribution has the perfect fluid form

$$\langle T^{\mu\nu}_{\rm part}\rangle = e^{\beta_0(\mu_0-eA_0)}[U^\mu U^\nu(\bar{\rho}+\bar{P}) - g^{\mu\nu}\bar{P}]. \tag{1.10}$$

Section V discusses the low temperature limit and how to change from classical Boltzman statistics to Bose or Fermi statistics.

Appendix A details the change of integration variables from the canonical momenta to Euclidean momenta that makes possible the integration in Sec. III.

Appendix B proves a result used in Sec. IV, namely that the thermally averaged energy-momentum tensor may be calculated from the variational derivative of the partition function with respect to the time-independent metric:

$$\frac{\delta\ln Z}{\delta g_{\mu\nu}} = -\frac{\beta_0\sqrt{g}}{2}[\langle T^{\mu\nu}_{\rm part}\rangle + T^{\mu\nu}_{\rm field}]. \tag{1.11}$$

Because $g_{0j}\neq 0$ it is not trivial to compute $\mathcal{H}_{\rm field}$ in terms of the canonical momenta π^j and the covariant fields F_{jk}. The result is displayed in (2.23). The derivative of this Hamiltonian density with respect to the metric, keeping π^j and F_{jk} fixed, yields the usual Hilbert energy-momentum tensor $T^{\mu\nu}_{\rm field}$ when the momenta are re-expressed in terms of field strengths.

Appendix C gives specific results for the energy density, number density, and entropy density. At low temperature the entropy density is given by a local form of the Sackur-Tetrode equation.

Greek letters run over 0, 1, 2, 3; Latin letters over 1, 2, 3. The metric signature is $(+---)$ and $g = |\det(g_{\mu\nu})|$; and $\hbar = c = 1$.

II. HAMILTONIAN AND PARTITION FUNCTION

The first step is to derive the expression (2.31) for the thermodynamic pressure. The starting point is the action for N particles in a background comprised of a time-independent metric plus electrostatic and magnetostatic fields [25]

$$S = -\sum_{n=1}^{N}\left\{m\int\sqrt{g_{\mu\nu}(x_n)dx_n^\mu dx_n^\nu} + e\int A_\mu(x_n)dx_n^\mu\right\} - \int d^4x\frac{\sqrt{g}}{4}F_{\mu\nu}(x)F^{\mu\nu}(x). \tag{2.1}$$

The particle coordinates x_n^j depend on the time coordinate x_n^0 or equivalently on the proper time $d\tau_n = g_{\mu\nu}(x_n)dx_n^\mu dx_n^\nu$. With $g_{0j} \neq 0$ the proper time is not time reversal invariant and the particle motion is not time-reversal invariance.

A. Hamiltonian for N particles

The Lagrangian for N particles is a sum of identical terms

$$L_N = \sum_{n=1}^{N} L(x_n^i, v_n^j)$$

where $v_n^j = dx^j/dx_n^0$ and each term has the form

$$L(x^i, v^j) = -m\sqrt{g_{00} + 2g_{j0}v^j + g_{jk}v^j v^k} - e(A_0 + A_j v^j).$$

The metric and the vector potential are evaluated at the position of the particle $x^k(t)$. The canonical momentum is

$$p_j = \frac{\partial L}{\partial v^j} = \frac{-m(g_{j0} + g_{jk}v^k)}{\sqrt{g_{00} + 2g_{0i}v^i + g_{i\ell}v^i v^\ell}} - eA_j. \tag{2.2}$$

In terms of the velocity $v^j = dx^j/dt$ this means that

$$H_{\mathrm{part}} = p_j v^j - L = \frac{m(g_{00} + g_{0j}v^j)}{\sqrt{g_{00} + 2g_{0i}v^i + g_{i\ell}v^i v^\ell}} + eA_0. \tag{2.3}$$

To express the velocity in terms of the momentum it is convenient to define

$$p'_j = p_j + eA_j. \tag{2.4}$$

Note that p_j depends only on t and A_j depends on $x^i(t)$. Equation (2.2) may be inverted to express the velocity in terms of momenta

$$v^j = \frac{g^{j0}}{g^{00}} - \frac{c^{jk}p'_k}{\sqrt{g^{00}[m^2 - c^{i\ell}p'_i p'_\ell]}}. \tag{2.5}$$

The matrix c^{jk} is given by

$$c^{jk} = g^{jk} - \frac{g^{j0}g^{k0}}{g^{00}} \tag{2.6}$$

and satisfies

$$g_{ij}c^{jk} = \delta_i^k. \tag{2.7}$$

(For the special case in which g_{ij} is diagonal, one can calculate $g^{\mu\nu}$ by Cramer's rule and show that c^{jk} is diagonal and $c^{jj} = 1/g_{jj}$.) The Hamiltonian for each particle is

$$H_{\mathrm{part}} = \bar{H} + eA_0$$
$$\bar{H} = \frac{1}{g^{00}}\left[\sqrt{g^{00}(m^2 - c^{i\ell}p'_i p'_\ell)} + g^{0\ell}p'_\ell\right]. \tag{2.8}$$

The minimum of $\bar{H}$ is $m\sqrt{g_{00}}$ and occurs at $p'_j = -mg_{j0}/\sqrt{g_{00}}$. For small velocity $\bar{H}$ may be expanded as

$$\bar{H} = m\sqrt{g_{00}}\left\{1 - \frac{g_{ij}v^i v^j}{2g_{00}} + 2\left[\frac{g_{0\ell}v^\ell}{g_{00}}\right]^2 + \mathcal{O}(v^3)\right\} \tag{2.9}$$

which shows that for small velocity if $g_{0\ell}v^\ell \neq 0$ the energy is higher than for a static metric.

B. The equation of motion

Though equilibrium statistical mechanics does not employ the equation of motion it is important to check that the Hamiltonian does give the correct equation of motion for the particles. Hamilton's first equation

$$v^j = \frac{\partial H_{\mathrm{part}}}{\partial p_j} \tag{2.10}$$

reproduces (2.5). Hamilton's second equation

$$-\frac{dp_j}{dt} = \frac{\partial H_{\mathrm{part}}}{\partial x^j} \tag{2.11}$$

will yield the equation of motion. It is convenient to employ the proper velocity

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{2.12}$$

with components

$$u^j = \frac{v^j}{\sqrt{g_{00} + 2g_{0i}v^i + g_{i\ell}v^i v^\ell}} \tag{2.13}$$

$$u^0 = \frac{1}{\sqrt{g_{00} + 2g_{0i}v^i + g_{i\ell}v^i v^\ell}}. \tag{2.14}$$

Since $u_\alpha = g_{\alpha\mu}u^\mu$, reference to (2.2) and (2.3) shows that

$$p_j = -mu_j - eA_j \tag{2.15}$$

$$H_{\rm part} = mu_0 + eA_0. \tag{2.16}$$

Hamilton's second equation (2.11) becomes

$$m\frac{du_j}{dt} + e\frac{dA_j}{dt} = m\frac{\partial u_0}{\partial x^j} + e\frac{\partial A_0}{\partial x^j}. \tag{2.17}$$

A simple way to compute the spatial derivative of u_0 is to apply $\partial/\partial x^j$ to $m^2 = g^{\mu\nu}u_\mu u_\nu$, which results in

$$u^0\frac{\partial u_0}{\partial x^j} = -\frac{1}{2}(\partial_j g^{\mu\nu})u_\mu u_\nu + eu^k\partial_j A_k. \tag{2.18}$$

Multiplying (2.17) by u^0 and using $u^0(d/dt) = d/d\tau$ gives

$$m\left[\frac{du_j}{d\tau} - \frac{1}{2}(\partial_j g_{\mu\nu})u^\mu u^\nu\right] = eF_{j\alpha}u^\alpha. \tag{2.19}$$

As yet there are only three equations. To obtain the fourth contract (2.19) with u^j and use $u^j(du_j/d\tau) = -u^0(du_0/d\tau)$ to obtain

$$m\frac{du_0}{d\tau} = eF_{0j}u^j. \tag{2.20}$$

The four components of (2.19) and (2.20) are summarized by

$$m\left(\frac{du_\lambda}{d\tau} - \frac{1}{2}(\partial_\lambda g_{\mu\nu})u^\mu u^\nu\right) = eF_{\lambda\alpha}u^\alpha. \tag{2.21}$$

The contravariant form of this equation is

$$m\left(\frac{du^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu}g_{\mu\nu}u^\mu u^\nu\right) = eF^{\lambda\alpha}u_\alpha, \tag{2.22}$$

which is the correct equation of motion for a particle in a background electric/magnetic field [26]. Solutions for the particle trajectories are obtained in [27].

C. Hamiltonian for the background electric/magnetic field

The Hamiltonian for the background electromagnetic field is also needed. The canonical Hamiltonian density given by Noether's theorem in terms of the canonical momenta $\pi^j = \partial\mathcal{L}_{\rm field}/\partial(\partial_0 A_j)$ and canonical fields F_{jk} is

$$\mathcal{H}_{\rm field} = -\frac{\pi^j\pi^k g_{jk}}{2g^{00}\sqrt{g}} + \frac{\pi^j F_{jk}g^{k0}}{g^{00}} + \frac{\sqrt{g}}{4}F_{jk}F_{\ell m}c^{j\ell}c^{km} + \pi^j\partial_j A_0 \tag{2.23}$$

as shown in Appendix C.

D. Partition function for the ideal gas

In a gas of N particles the Hamiltonian for each particle is of the form (2.8) and is a function of each particle's contravariant position coordinates x^1, x^2, x^3 and its covariant momenta p_1, p_2, p_3. The Hamiltonian for N particles is the sum of the single particle Hamiltonians:

$$H_N = \sum_{n=1}^{N} H_{\rm part}(x^i_n, p^n_j) \tag{2.24}$$

and the total Hamiltonian is $H_N + \int d^3x\mathcal{H}_{\rm field}$. The Boltzman factor $e^{-\beta_0 H_N}$ is a product of N exponentials. The integration of each factor $e^{-\beta_0 H_{\rm part}}$ over its six-dimensional phase space is the same and so the partition function for N particles is

$$Z_N = \frac{1}{N!}\left[\int\frac{d^3x d^3p}{(2\pi)^3}e^{-\beta_0 H_{\rm part}}\right]^N e^{-\beta_0 H_{\rm field}}. \tag{2.25}$$

(Because of the convention $\hbar \to 1$ the denominator $h^3 = (2\pi\hbar)^3 \to (2\pi)^3$.)

In the grand canonical ensemble the partition function depends on the chemical potential μ_0:

$$Z = \sum_{N=0}^{\infty}(e^{\beta_0\mu_0})^N Z_N, \tag{2.26}$$

and therefore

$$\ln Z = \int\frac{d^3x d^3p}{(2\pi)^3}e^{-\beta_0(H_{\rm part}-\mu_0)} - \beta_0 H_{\rm field}. \tag{2.27}$$

The momentum integration variable may be shifted from the canonical p_j to p'_j

$$p'_j = p_j + eA_j; \tag{2.28}$$

this removes A_j from the partition function as expected from the Bohr-Van Leeuwen theorem [28].

The partition function is directly related to the thermodynamic pressure [29]

$$\ln Z = \beta_0 \int d^3x \sqrt{g} P. \tag{2.29}$$

Therefore

$$P = P_{\rm part} - \frac{1}{\sqrt{g}} \mathcal{H}_{\rm field}, \tag{2.30}$$

where the pressure exerted by the gas of particles is

$$P_{\rm part} = \frac{T_0}{\sqrt{g}} \int \frac{d^3 p'}{(2\pi)^3} e^{-\beta_0 (H_{\rm part} - \mu_0)}. \tag{2.31}$$

The next step is to perform this integration.

III. CALCULATION OF THE PARTICLE PRESSURE

The terms in $H_{\rm part}$ involving μ_0 and eA_0 have no momentum dependence and so

$$P_{\rm part} = e^{\beta_0(\mu_0 - eA_0)} \bar{P} \tag{3.1}$$

$$\bar{P} = \frac{T_0}{\sqrt{g}} \int \frac{d^3 p'}{(2\pi)^3} e^{-\beta_0 \bar{H}}. \tag{3.2}$$

The dependence on μ_0 may be written

$$\beta_0 \mu_0 = \frac{\sqrt{g_{00}}}{T_0} \frac{\mu_0}{\sqrt{g_{00}}}. \tag{3.3}$$

and is in agreement with Oscar Klein's argument [30] that both T_0 and μ_0 will always occur divided by $\sqrt{g_{00}}$. The factor $\exp[-eA_0/T_0]$ in (3.1) is an exception to the Tolman-Ehrenfest rule and will remain in the final result. This section will show that $\bar{P}$ is given by (3.25) and is a function only of the ratio $T_0/\sqrt{g_{00}}$.

A. Euclidean momenta

To compute $\bar{P}$ it is necessary to change the integration variables from p'_j with metric g_{jk} into Euclidean momenta k_a. The details of this transformation are given in Appendix A. The Hamiltonian becomes

$$\bar{H} = \frac{1}{\sqrt{g^{00}}} \left[\sqrt{m^2 + \mathbf{k}^2} + \mathbf{s} \cdot \mathbf{k} \right], \tag{3.4}$$

where $\mathbf{s}$ is a Euclidean vector with length squared

$$\mathbf{s}^2 = g_{0j} g^{j0} = 1 - g_{00} g^{00}. \tag{3.5}$$

The minimum of $\bar{H}$ occurs at $k_a = -m s_a/\sqrt{1 - s^2}$ and the value of the minimum is still $m\sqrt{g_{00}}$. The change in integration variables requires

$$d^3 p' = \sqrt{g g^{00}} d^3 k, \tag{3.6}$$

which leads to

$$\bar{P} = T_0 \sqrt{g^{00}} \int \frac{d^3 k}{(2\pi)^3} e^{-\beta_0 \bar{H}}. \tag{3.7}$$

B. Dimensionless variables

The change to a dimensionless integration variable $\mathbf{u} = \mathbf{k}/m$ and introduction of a dimensionless parameter

$$z = \frac{m}{T_0 \sqrt{g^{00}}}. \tag{3.8}$$

converts the exponent in the integrand of (3.2) to

$$\beta_0 \bar{H} = z[\sqrt{1 + \mathbf{u}^2} + \mathbf{s} \cdot \mathbf{u}]. \tag{3.9}$$

The pressure integral is

$$\bar{P}(s, z) = \frac{m^4}{z} \int \frac{d^3 u}{(2\pi)^3} e^{-\beta_0 \bar{H}}. \tag{3.10}$$

It appears that $\bar{P}$ is a function of the two variables s and z. To show that it actually depends only on a particular combination of these two variables requires performing the integration. The angular integration gives

$$\bar{P}(s, z) = \frac{m^4}{2\pi^2 s z^2} \int_0^\infty u du\, e^{-z\sqrt{1+u^2}} \sinh[szu]. \tag{3.11}$$

Next expand the sinh in an infinite series:

$$\bar{P}(s, z) = \frac{m^4}{2\pi^2 z} \sum_{\ell=0}^{\infty} \frac{(sz)^{2\ell}}{(2\ell+1)!} \int_0^\infty du\, e^{-z\sqrt{1+u^2}} u^{2\ell+2}. \tag{3.12}$$

The necessary integrals are modified Bessel functions of the second kind [31]:

$$\int_0^\infty du\, e^{-z\sqrt{1+u^2}} u^{2\ell+2} = \frac{\Gamma(\ell + \frac{3}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{z}\right)^{\ell+1} K_{\ell+2}(z) \tag{3.13}$$

and the pressure becomes

$$\bar{P}(s, z) = \frac{m^4}{8\pi^2} \sum_{\ell=0}^{\infty} \frac{s^{2\ell}}{\ell!} \left(\frac{z}{2}\right)^{\ell-2} K_{\ell+2}(z). \tag{3.14}$$

By using the asymptotic value of $K_{\ell+2}(z)$ either for low temperature ($z \gg 1$) or for high temperature ($z \ll 1$) one can perform the sum on ℓ and obtain the leading term and the sub-leading term in either regime and confirm the Tolman-Ehrenfest effect to that order. This suggests that the infinite series representation can be simplified.

C. Simpler expression for $\bar{P}$

It will turn out that the entire series is only a function of the single variable

$$z\sqrt{1-s^2} = m\frac{\sqrt{g_{00}}}{T_0}. \tag{3.15}$$

To show this define the derivative combination

$$\mathcal{D} = z\frac{\partial}{\partial z} + \left(\frac{1}{s} - s\right)\frac{\partial}{\partial s}, \tag{3.16}$$

with the property

$$\mathcal{D}(z\sqrt{1-s^2}) = 0. \tag{3.17}$$

Application of $\mathcal{D}$ to (3.14) gives

$$\begin{aligned}\mathcal{D}\bar{P}(s,z) = \frac{m^4}{8\pi^2}\sum_{\ell=0}^{\infty}\Bigg[&\frac{s^{2\ell}}{\ell!}2\ell\left(\frac{1}{s^2}-1\right)\left(\frac{z}{2}\right)^{\ell-2}K_{\ell+2}(z)\\ &+\frac{s^{2\ell}}{\ell!}(\ell-2)\left(\frac{z}{2}\right)^{\ell-2}K_{\ell+2}(z)\\ &+\frac{s^{2\ell}}{\ell!}\left(\frac{z}{2}\right)^{\ell-2}z\frac{d}{dz}K_{\ell+2}(z)\Bigg].\end{aligned} \tag{3.18}$$

The last line is simplified by using the identity [31]

$$z\frac{d}{dz}K_{\ell+2}(z) = (\ell+2)K_{\ell+2}(z) - zK_{\ell+3}(z). \tag{3.19}$$

All terms proportional to $K_{\ell+2}$ combine:

$$\begin{aligned}\mathcal{D}\bar{P}(s,z) = &\frac{m^4}{8\pi^2}\sum_{\ell=1}^{\infty}\frac{s^{2\ell-2}}{(\ell-1)!}2\left(\frac{z}{2}\right)^{\ell-2}K_{\ell+2}(z)\\ &-\frac{m^4}{8\pi^2}\sum_{\ell=0}^{\infty}\frac{s^{2\ell}}{\ell!}\left(\frac{z}{2}\right)^{\ell-2}zK_{\ell+3}(z).\end{aligned} \tag{3.20}$$

The two series cancel and thus $\mathcal{D}\bar{P}(s,z) = 0$ and so $\bar{P}(s,z)$ is a function only of the single variable (3.15):

$$\bar{P}(s,z) = \frac{m^4}{8\pi^2}\Psi(z\sqrt{1-s^2}), \tag{3.21}$$

or equivalently

$$\sum_{\ell=0}^{\infty}\frac{s^{2\ell}}{\ell!}\left(\frac{z}{2}\right)^{\ell-2}K_{\ell+2}(z) = \Psi(z\sqrt{1-s^2}). \tag{3.22}$$

To obtain an explicit form for Ψ, set $s=0$

$$\left(\frac{2}{z}\right)^2 K_2(z) = \Psi(z). \tag{3.23}$$

This determines the function Ψ and so for $s \neq 0$

$$\bar{P}(s,z) = \frac{m^4}{2\pi^2}\frac{K_2(z\sqrt{1-s^2})}{[z\sqrt{1-s^2}]^2}. \tag{3.24}$$

In more physical variables

$$\bar{P} = \frac{m^2}{2\pi^2}\left(\frac{T_0}{\sqrt{g_{00}}}\right)^2 K_2\left(m\frac{\sqrt{g_{00}}}{T_0}\right). \tag{3.25}$$

(That the two expressions (3.14) and (3.25) for $\bar{P}$ are equal amounts to

$$\frac{1}{4}\sum_{\ell=0}^{\infty}\frac{s^{2\ell}}{\ell!}\left(\frac{z}{2}\right)^{\ell-2}K_{\ell+2}(z) = \frac{K_2(z\sqrt{1-s^2})}{[z\sqrt{1-s^2}]^2}, \tag{3.26}$$

which is a known addition theorem for Bessel functions [32].)

D. Energy density

For later purposes it is convenient to compute the quantity

$$\bar{\rho} = \frac{T_0}{\sqrt{g}}\int\frac{d^3p'}{(2\pi)^3}\bar{H}e^{-\beta_0\bar{H}} = T_0\frac{\partial\bar{P}}{\partial T_0} - \bar{P}. \tag{3.27}$$

Using (3.19) gives

$$\bar{\rho} + \bar{P} = \frac{m^3}{2\pi^2}\left(\frac{T_0}{\sqrt{g_{00}}}\right)K_3\left(m\frac{\sqrt{g_{00}}}{T_0}\right). \tag{3.28}$$

IV. THE ENERGY-MOMENTUM TENSOR

Appendix B shows that the thermally averaged energy-momentum tensor may be computed from the variation derivative of the partition function. Eqs. (B7) and (B24) give

$$\frac{\delta \ln Z}{\delta g_{\mu\nu}} = -\frac{\beta_0\sqrt{g}}{2}\left[\langle T^{\mu\nu}_{\rm part}\rangle + T^{\mu\nu}_{\rm field}\right]. \tag{4.1}$$

Since $\ln Z = \beta_0\int d^3x\sqrt{g}P$ this is equivalent to

$$\langle T^{\mu\nu}_{\rm part}\rangle + T^{\mu\nu}_{\rm field} = -\frac{2}{\sqrt{g}}\frac{\partial(\sqrt{g}P)}{\partial g_{\mu\nu}}. \tag{4.2}$$

Using $\sqrt{g}P$ from (2.30) gives

$$\langle T^{\mu\nu}_{\rm part}\rangle = -\frac{2}{\sqrt{g}}\frac{\partial(\sqrt{g}P_{\rm part})}{\partial g_{\mu\nu}} \tag{4.3}$$

$$T^{\mu\nu}_{\rm field} = \frac{2}{\sqrt{g}}\frac{\partial\mathcal{H}_{\rm field}}{\partial g_{\mu\nu}}. \tag{4.4}$$

A. Field EMT

In Appendix B, $\mathcal{H}_{\rm field}$ is expressed in terms of canonical momenta π^j and canonical fields F_{jk} with the result (B19). The partial derivative (4.4) is computed keeping π^j and F_{jk} fixed; when the result is re-expressed in terms of the fields the result is the usual Hilbert energy-momentum tensor:

$$T^{\mu\nu}_{\rm field} = -F^{\mu\alpha}F^{\nu\beta}g_{\alpha\beta} + \frac{g^{\mu\nu}}{4}F_{\alpha\beta}F^{\alpha\beta}. \tag{4.5}$$

B. Particle EMT

To implement Eq. (4.3) begin with $P_{\rm part} = e^{\beta_0(\mu_0 - eA_0)}\bar{P}$. Since $\bar{P}$ depends only on the g_{00} component of the metric the derivative in (4.3) is

$$\langle T^{\mu\nu}_{\rm part}\rangle = e^{\beta_0(\mu_0-eA_0)}\left[-2\delta^\mu_0\delta^\nu_0\frac{\partial\bar{P}}{\partial g_{00}} - g^{\mu\nu}\bar{P}\right] \tag{4.6}$$

As shown in (3.24) the g_{00} dependence of $\bar{P}$ occurs only through the ratio $T_0/\sqrt{g_{00}}$ and so a g_{00} derivative is equivalent to a T_0 derivative:

$$-2g_{00}\frac{\partial\bar{P}}{\partial g_{00}} = T_0\frac{\partial\bar{P}}{\partial T_0} = \bar{\rho} + \bar{P}. \tag{4.7}$$

Therefore

$$\langle T^{\mu\nu}_{\rm part}\rangle = e^{\beta_0(\mu_0-eA_0)}\left\{\delta^\mu_0\delta^\nu_0\frac{(\bar{\rho}+\bar{P})}{g_{00}} - g^{\mu\nu}\bar{P}\right\}. \tag{4.8}$$

The thermal average of the particle velocity is zero because $v^j = \partial H_{\rm part}/\partial p_j$:

$$\int\frac{d^3p'}{(2\pi)^3}v^j e^{-\beta_0(H_{\rm part}-\mu_0)} = 0. \tag{4.9}$$

The normalized velocity vector of the ideal gas is therefore $U^\mu = \delta^\mu_0/\sqrt{g_{00}}$, which allows (4.8) to be expressed in the perfect fluid form

$$\begin{aligned}\langle T^{\mu\nu}_{\rm part}\rangle &= e^{\beta_0(\mu_0-eA_0)}\{U^\mu U^\nu(\bar{\rho}+\bar{P}) - g^{\mu\nu}\bar{P}\}\\ &= \{U^\mu U^\nu(\rho_{\rm part}+P_{\rm part}) - g^{\mu\nu}P_{\rm part}\}.\end{aligned} \tag{4.10}$$

Because of the external potential $A_0(\mathbf{x})$ the covariant divergence of $\langle T^{\mu\nu}_{\rm part}\rangle$ is not zero:

$$\langle T^{\mu\nu}_{\rm part}\rangle_{;\mu} = g^{\nu j}\frac{e}{T_0}(\partial_j A_0). \tag{4.11}$$

V. DISCUSSION

A. Low temperature example

The spatial dependence of $\bar{P}$ comes entirely from g_{00} and so $\bar{P}$ is constant on surfaces of constant g_{00}. The full particle pressure $P_{\rm part} = e^{\beta_0(\mu_0-eA_0)}\bar{P}$ has somewhat different isobaric surfaces if $A_0 \neq 0$. The result (3.25) for $\bar{P}$ may be evaluated when $Z = m\sqrt{g_{00}}/T_0 \gg 1$ using the asymptotic behavior $K_2(Z) \to e^{-Z}\sqrt{\pi/2Z}$:

$$P_{\rm part} = \frac{T_0}{\sqrt{g_{00}}}\left(\frac{mT_0}{2\pi\sqrt{g_{00}}}\right)^{3/2}e^{\beta_0(\mu_0-eA_0-m\sqrt{g_{00}})} + \dots. \tag{5.1}$$

The resulting thermodynamic functions (number density, entropy density, and energy density) are displayed in Appendix C. The gradient of the pressure is

$$\frac{\partial_j P_{\rm part}}{P_{\rm part}} = -\beta_0\left[m\frac{\partial_j g_{00}}{2g_{00}} + e\partial_j A_0\right] - \frac{5}{4}\frac{\partial_j g_{00}}{(g_{00})^{3/2}}. \tag{5.2}$$

For the Kerr-Newman metric in Boyer-Lindquist coordinates t, r, θ, ϕ both g_{00} and A_0 depend on r and θ. At large distances the θ dependence is negligible

$$g_{00} \approx 1 - \frac{2GM}{r} \tag{5.3}$$

$$A_0 \approx \frac{Q}{r} \tag{5.4}$$

and the pressure gradient is radial:

$$\frac{dP_{\rm part}}{dr} \approx n\left(-\frac{GMm}{r^2} + \frac{eQ}{r^2}\right) \tag{5.5}$$

after using $\beta_0 P_{\rm part} \approx n$ and neglecting the $\mathcal{O}(T_0/m)$ correction. The first term in the parenthesis is the gravitational force exerted on the particle by the central mass; the second term is the electrostatic force on the particle.

B. Quantum statistics

The calculations presented are for distinguishable point particles obeying Boltzman statistics. If instead the particles are indistinguishable and obey either Bose-Einstein or Fermi-Dirac statistics the particle pressure becomes

$$P_{\rm part} = -\frac{T_0}{\xi\sqrt{g}}\int\frac{d^3p}{(2\pi)^3}\ln[1 - \xi e^{-\beta_0(H_{\rm part}-\mu_0)}] \tag{5.6}$$

where $\xi = 1$ for Bose statistics and $\xi = -1$ for Fermi statistics. When expanded in a series

$$P_{\rm part} = \frac{T_0}{\xi\sqrt{g}} \sum_{b=1}^{\infty} \frac{(\xi e^{\beta_0(\mu_0 - eA_0)})^b}{b} \int \frac{d^3p}{(2\pi)^3} e^{-b\beta_0 \bar{H}} \quad (5.7)$$

the integral is the same as (3.2) except that β_0 is replaced by $b\beta_0$. The value of the integral may be read off from (3.25)

$$P_{\rm part} = \frac{m^2}{2\pi^2 \xi} \sum_{b=1}^{\infty} (\xi e^{\beta_0(\mu_0 - eA_0)})^b \left(\frac{T_0}{b\sqrt{g_{00}}}\right)^2 K_2\left(mb\frac{\sqrt{g_{00}}}{T_0}\right) \quad (5.8)$$

and satisfies the Tolman-Ehrenfest rule except for the eA_0 dependence. If the ensemble contains particles and antiparticles the total particle pressure requires adding to (5.8) another such series with μ_0 replaced by $-\mu_0$. A simple check of (5.8) is that of massless bosons with $\mu_0 = A_0 = 0$ in which case

$$P_{\rm part} = \frac{\pi^2}{90}\left(\frac{T_0}{\sqrt{g_{00}}}\right)^4. \quad (5.9)$$

APPENDIX A: CHANGE TO EUCLIDEAN MOMENTA

The calculation in Sec. III of the partition function requires integrating $\exp[-\beta_0 \bar{H}]$ with respect to the canonical momenta $dp_1 dp_2 dp_3$, where

$$\bar{H} = \frac{1}{g^{00}}\left[\sqrt{g^{00}(m^2 - c^{ij}p'_i p'_j)} + g^{0j}p'_j\right], \quad (A1)$$

with c^{ij} as defined in (2.6), and $p'_j = p_j + eA_j$. The spatial metric g_{jk} may be expanded at any point x^i in terms of three Euclidean frame vectors

$$g_{jk} = -\sum_{a=1}^{3} f_{(a)j} f_{(a)k}, \quad (A2)$$

which are the analogs of vierbeins in three-dimensional space. (The minus sign arises because the eigenvalues of g_{jk} are negative.) The contravariant form of the frame vectors is

$$f^i_{(a)} = -c^{ij} f_{(a)j}. \quad (A3)$$

Equations (A2), and (A3) imply

$$\sum_{a=1}^{3} f^i_{(a)} f_{(a)j} = \delta^i_j$$
$$f^j_{(a)} f_{(b)j} = \delta_{ab}$$
$$\sum_{a=1}^{3} f^i_{(a)} f^j_{(a)} = -c^{ij}. \quad (A4)$$

The quadratic term in $\bar{H}$ is

$$-c^{ij}p_i p_j = \sum_{a=1}^{3} (f^i_{(a)} p'_i)(f^j_{(a)} p'_j), \quad (A5)$$

which suggests introducing Euclidean momenta k_a

$$k_a = f^j_{(a)} p'_j. \quad (A6)$$

(This is not a canonical transformation: the defining Poisson bracket $\{x^i, p_j\} = \delta^i_j$ implies $\{x^i, k_a\} = f^i_{(a)}$.) $\bar{H}$ is now

$$\bar{H} = \frac{1}{g^{00}}\left[\sqrt{g^{00}(m^2 + \mathbf{k}^2)} + g^{0j}\sum_{a=1}^{3} f_{(a)j} k_a\right]. \quad (A7)$$

Define a Euclidean vector

$$s_a = \frac{g^{0j} f_{(a)j}}{\sqrt{g^{00}}} \quad (A8)$$

with length

$$s^2 = \sum_{a=1}^{3} [s_a]^2 = g_{0j} g^{0j} = 1 - g_{00} g^{00}. \quad (A9)$$

The Hamiltonian has the simple form

$$\bar{H} = \frac{1}{\sqrt{g^{00}}}\left[\sqrt{m^2 + \mathbf{k}^2} + \mathbf{s}\cdot\mathbf{k}\right]. \quad (A10)$$

The inverse of relation (A6) is

$$p'_j = \sum_{a=1}^{3} f_{(a)j} k_a, \quad (A11)$$

which gives for the Jacobian of the change from p'_j to Euclidean k_a

$$d^3p' = \sqrt{|\det(g_{ij})|}d^3k = \sqrt{\frac{g^{00}}{|\det(g^{\mu\nu})|}}d^3k$$
$$= \sqrt{gg^{00}}d^3k \tag{A12}$$

where $g = |\det(g_{\mu\nu})|$. These results yield (3.7).

APPENDIX B: $T^{\mu\nu}$ FOR PARTICLES AND BACKGROUND FIELD

1. Calculation of $T^{\mu\nu}_{\rm part}$ from $H_{\rm part}$

The energy-momentum tensor for the particles may be calculated using the Hilbert variational principle:

$$-\frac{\sqrt{g}}{2}T^{\mu\nu}_{\rm part}(x) = \frac{\delta}{\delta g_{\mu\nu}(x)}\int dx'^0 L_{\rm part}(x')$$
$$= \delta^3(\mathbf{x}-\mathbf{x}')\left[\frac{\partial L_{\rm part}}{\partial g_{\mu\nu}}\right]_{\partial_0 x^i, x^i} \tag{B1}$$

where the metric variation is performed before the equations of motion are imposed on the particle velocity and position. The following steps depend three facts: (i) $L_{\rm part}$ does not depend on the momentum p_j, (ii) $L_{\rm part} = p_j\partial_0 x^j - H_{\rm part}$ and $p_j\partial_0 x^j$ does not depend on the metric, (iii) $H_{\rm part}$ does not depend on the velocity $\partial_0 x^i$:

$$\left[\frac{\partial L_{\rm part}}{\partial g_{\mu\nu}}\right]_{\partial_0 x^i, x^i} = \left[\frac{\partial L_{\rm part}}{\partial g_{\mu\nu}}\right]_{\partial_0 x^i, x^i, p_j} \tag{B2}$$

$$= -\left[\frac{\partial H_{\rm part}}{\partial g_{\mu\nu}}\right]_{\partial_0 x^i, x^i, p_j} \tag{B3}$$

$$= -\left[\frac{\partial H_{\rm part}}{\partial g_{\mu\nu}}\right]_{x^i, p_j}. \tag{B4}$$

Therefore

$$\frac{\sqrt{g}}{2}T^{\mu\nu}_{\rm part}(x) = \delta^3(\mathbf{x}-\mathbf{x}')\left[\frac{\partial H_{\rm part}}{\partial g_{\mu\nu}}\right]_{x^i, p_j}. \tag{B5}$$

The partition function for the particles

$$\ln Z_{\rm part} = \int \frac{d^3x d^3p}{(2\pi)^3}e^{-\beta_0(H_{\rm part}-\mu_0)} \tag{B6}$$

has a variational derivative

$$\frac{\delta \ln Z_{\rm part}}{\delta g_{\mu\nu}(x')} = -\beta_0\frac{\sqrt{g}}{2}\int\frac{d^3p d^3x}{(2\pi)^3}T^{\mu\nu}_{\rm part}e^{-\beta_0(H_{\rm part}-\mu_0)} \tag{B7}$$

This is the starting point of Sec. IV.

Explicit form for $T^{\mu\nu}_{\rm part}$. It is not difficult to explicitly differentiate $H_{\rm part}$ with respect to $g_{\mu\nu}$ by considering the metric dependence of the covariant particle velocity u_α. From (2.15) u_j is independent of the metric because p_j and A_j are. The derivative of $g^{\alpha\beta}u_\alpha u_\beta = 1$ with respect to $g^{\mu\nu}$ yields

$$\frac{\partial u_0}{\partial g^{\mu\nu}} = -\frac{u_\mu u_\nu}{2u^0}. \tag{B8}$$

This gives the derivative of $H_{\rm part} = mu_0 + eA_0$. A change from the contravariant metric to the covariant metric gives

$$\sqrt{g}T^{\mu\nu}_{\rm part} = \delta^3(\mathbf{x}-\mathbf{x}')m\frac{u^\mu u^\nu}{u^0} \tag{B9}$$

as expected.

2. Calculation of $T^{\mu\nu}_{\rm field}$ from $\mathcal{H}_{\rm field}$

a. The canonical $\mathcal{H}_{\rm field}$

Because $g_{0j} \neq 0$ it is not trivial to compute $\mathcal{H}_{\rm field}$ in terms of the canonical momenta π^j and the magnetic field F_{jk}. The starting point is the Lagrange density

$$\mathcal{L}_{\rm field} = -\frac{\sqrt{g}}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}, \tag{B10}$$

which implies a canonical momentum

$$\pi^j = \frac{\partial\mathcal{L}_{\rm field}}{\partial(\partial_0 A_j)} = -\sqrt{g}F^{0j} \tag{B11}$$

and the Hamiltonian density

$$\mathcal{H}_{\rm field} = \pi^j\partial_0 A_j + \frac{\sqrt{g}}{4}F_{\alpha\beta}F^{\alpha\beta}$$
$$= \pi^j F_{0j} + \frac{\sqrt{g}}{4}F_{\alpha\beta}F^{\alpha\beta} + \pi^j\partial_j A_0. \tag{B12}$$

To express this in terms of canonical variables requires the identity

$$g_{jk}\pi^k = -\sqrt{g}F^{0k}g_{kj} = -\sqrt{g}g^{0\mu}F_{\mu j} \tag{B13}$$

which may be solved for F_{0j}:

$$F_{0j} = \frac{1}{g^{00}}\left[-\frac{g_{jk}\pi^k}{\sqrt{g}} + F_{j\ell}g^{\ell 0}\right]. \tag{B14}$$

This allows the first term in (B12) to be expressed in terms of momentum and covariant field strength. For the second term in (B12) use the quantity

$$c^{\alpha\mu} = g^{\alpha\mu} - \frac{g^{\alpha 0} g^{\mu 0}}{g^{00}} \tag{B15}$$

to obtain

$$F_{\alpha\beta} F^{\alpha\beta} = F_{\alpha\beta} c^{\alpha\mu} c^{\beta\nu} F_{\mu\nu} + 2 \frac{F^{0\alpha} g_{\alpha\mu} F^{0\mu}}{g^{00}}. \tag{B16}$$

Because $c^{\alpha\mu}$ is zero if either $\alpha = 0$ or $\mu = 0$ and $F^{0\alpha}$ is zero if $\alpha = 0$ the identity simplifies to

$$F_{\alpha\beta} F^{\alpha\beta} = F_{jk} c^{j\ell} c^{km} F_{\ell m} + 2 \frac{F^{0j} g_{jk} F^{0k}}{g^{00}}. \tag{B17}$$

This allows the second term in (B12) to be expressed as

$$\frac{\sqrt{g}}{4} F_{\alpha\beta} F^{\alpha\beta} = \frac{\pi^j \pi^k g_{jk}}{2 g^{00} \sqrt{g}} + \frac{\sqrt{g}}{4} F_{jk} c^{j\ell} c^{km} F_{\ell m}. \tag{B18}$$

The resulting Hamiltonian density in terms of canonical variables is therefore

$$\mathcal{H}_{\rm field} = -\frac{\pi^j \pi^k g_{jk}}{2 g^{00} \sqrt{g}} + \frac{\pi^j F_{jk} g^{k0}}{g^{00}} + \frac{\sqrt{g}}{4} F_{jk} F_{\ell m} c^{j\ell} c^{km} + \pi^j \partial_j A_0. \tag{B19}$$

A simple check is that $\partial_0 A_j = \partial \mathcal{H}_{\rm field} / \partial \pi_j$.

b. The metric derivative of $\mathcal{H}_{\rm field}$

The canonical momenta π^j and the magnetic field F_{jk} do not depend on the metric. The metric dependence of $\mathcal{H}_{\rm field}$ due to $\sqrt{g}$ is easy to compute from (B19) and gives

$$\left[\frac{\partial \sqrt{g}}{\partial g^{\mu\nu}}\right] \frac{\partial \mathcal{H}_{\rm field}}{\partial \sqrt{g}} = -\frac{\sqrt{g}}{2} \left(\frac{g_{\mu\nu}}{4} F_{\alpha\beta} F^{\alpha\beta} \right) \tag{B20}$$

after using (B18) to convert canonical momentum back into fields. To compute the metric derivative with $\sqrt{g}$ fixed

$$\left. \frac{\partial \mathcal{H}_{\rm field}}{\partial g^{\mu\nu}} \right|_{\sqrt{g}} \tag{B21}$$

is more difficult and must be calculated for the separate cases $\mu\nu = 00, j0, jk$. After computing the metric derivatives it is necessary to use the relation

$$F^{0k} = g^{0\mu} F_{\mu j} c^{jk} \tag{B22}$$

which follows from (B13). The final result is

$$\frac{\partial \mathcal{H}_{\rm field}}{\partial g^{\mu\nu}} = -\frac{\sqrt{g}}{2} \left[-F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} + \frac{g_{\mu\nu}}{4} F_{\alpha\beta} F^{\alpha\beta} \right]. \tag{B23}$$

A change from contravariant $g^{\mu\nu}$ to covariant $g_{\mu\nu}$ and the relation $\ln Z_{\rm field} = -\beta_0 H_{\rm field}$ give

$$\frac{\delta \ln Z_{\rm field}}{\delta g_{\mu\nu}} = -\beta_0 \frac{\sqrt{g}}{2} T^{\mu\nu}_{\rm field}. \tag{B24}$$

APPENDIX C: THERMODYNAMIC FUNCTIONS

1. Exact relations

a. Thermodynamic quantities

The thermodynamic pressure due to the particles is

$$P_{\rm part} = \frac{T_0}{\sqrt{g}} \int \frac{d^3 p'}{(2\pi)^3} e^{-\beta_0 (H_{\rm part} - \mu_0)}. \tag{C1}$$

Differentiation gives

$$\mu_0 \frac{\partial P_{\rm part}}{\partial \mu_0} + T_0 \frac{\partial P_{\rm part}}{\partial T_0} = P_{\rm part} + \rho_{\rm part} + n \frac{e A_0}{\sqrt{g_{00}}}. \tag{C2}$$

where

$$\rho_{\rm part} = \frac{1}{\sqrt{g}} \int \frac{d^3 p'}{(2\pi)^3} \bar{H} e^{-\beta_0 (H_{\rm part} - \mu_0)}. \tag{C3}$$

The first term in (C2) is related to the number density:

$$\frac{n}{\sqrt{g_{00}}} = \frac{\partial P_{\rm part}}{\partial \mu_0} = \beta_0 P_{\rm part}. \tag{C4}$$

This is the ideal gas law

$$P_{\rm part} = n \frac{T_0}{\sqrt{g_{00}}} \tag{C5}$$

in local form. The second term in (C2) is related to the local entropy density

$$s = \sqrt{g_{00}} \frac{\partial P_{\rm part}}{\partial T_0} = \frac{1}{T} \left[P_{\rm part} + \rho_{\rm part} + n \frac{e A_0}{\sqrt{g_{00}}} - \mu n \right]. \tag{C6}$$

2. Low temperature regime

Using the approximation (5.1) the number density is

$$n(\mathbf{x}) = \left(\frac{m T_0}{2\pi \sqrt{g_{00}}} \right)^{3/2} e^{\beta_0 (\mu_0 - e A_0 - m \sqrt{g_{00}})}. \tag{C7}$$

The entropy density $s(\mathbf{x})$ can be computed from (C6) and afterward if the chemical potential is expressed in terms of the density the result is

$$s(\mathbf{x}) = n\left[\frac{5}{2} + \ln\left\{\frac{1}{n}\left(\frac{mT_0}{2\pi\sqrt{g_{00}}}\right)^{3/2}\right\}\right], \tag{C8}$$

which is the local form of the Sackur-Tetrode equation [33]. The energy density of the particles is

$$\rho_{\mathrm{part}}(\mathbf{x}) = n\left[m + \frac{3}{2}\frac{T_0}{\sqrt{g_{00}}}\right]. \tag{C9}$$

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