# Erratum: Real time quarkonium transport coefficients in open quantum systems from Euclidean QCD [Phys. Rev. D 108, 054024 (2023)]

Bruno Scheihing-Hitschfeld and Xiaojun Yao

(Received 28 February 2024; published 13 May 2024)

DOI: 10.1103/PhysRevD.109.099902

In Eq. (8), there should be additional normalization factors given by $N_c/\mathrm{Tr}[U(-i\beta-\infty,-\infty)e^{-\beta H}]$. The correct expressions are

$$\kappa_{\mathrm{fund}}=\frac{g^2}{3N_c}\mathrm{Re}\int \mathrm{d}t\langle \mathrm{Tr_c}[U(-\infty,t)E_i(t)U(t,0)E_i(0)U(0,-\infty)]\rangle_{T,Q}$$
$$\gamma_{\mathrm{fund}}=\frac{g^2}{3N_c}\mathrm{Im}\int \mathrm{d}t\langle \mathrm{Tr_c}[U(-\infty,t)E_i(t)U(t,0)E_i(0)U(0,-\infty)]\rangle_{T,Q},$$

where the subscript $T$ in the expectation value denotes that the state on which this expectation value is calculated is a thermal density matrix, while the subscript $Q$ means that this thermal state contains a static external color charge in the fundamental representation, e.g., a heavy quark. Mathematically, this expectation value is defined as $\langle O\rangle_{T,Q}\equiv N_c\mathrm{Tr}[U(-i\beta-\infty,-\infty)Oe^{-\beta H}]/\mathrm{Tr}[U(-i\beta-\infty,-\infty)e^{-\beta H}]$ and thus is different from that in Eq. (1).

In Eq. (15), there should be an overall minus sign in the second, third and fourth line because the electric field in Euclidean signature differs from the electric field in Minkowski signature by a factor of $i$, induced by the definitions $\tau=it$ and $A_0(0)=iA_4(0)$.

Equation (35) should read

$$\rho_{\mathrm{adj}}^{++}(\omega)\overset{\omega\gg T}{=}\frac{g^2T_F(N_c^2-1)\omega^3}{3\pi N_c}\left\{1+\frac{g^2}{(2\pi)^2}\left[\left(\frac{11N_c}{12}-\frac{N_f}{6}\right)\ln\left(\frac{\mu^2}{4\omega^2}\right)+N_c\left(\frac{149}{36}-\frac{\pi^2}{6}+\frac{\pi^2}{2}\mathrm{sgn}(\omega)\right)-\frac{5N_f}{9}\right]\right\}+\mathcal{O}(g^6).$$

The correction lies in the $\pi^2$ terms in the innermost parenthesis that multiplies $N_c$ and an overall factor of 2. This follows immediately from the known results at $\omega>0$ and the $\omega$-even part we found in Eq. (20).

We note that this $\omega$-even part is exactly the difference at next-to-leading order (NLO) between the spectral functions $\rho_{\mathrm{adj}}^{++}(\omega)$ and $\rho_{\mathrm{fund}}(\omega)$. Indeed, the difference $\Delta\rho$ between spectral functions we found in this work in Eq. (20) reproduces the difference between the transport coefficients $\gamma_{\mathrm{adj}}$ and $\gamma_{\mathrm{fund}}$ present in [1]. However, a more detailed examination reveals that the $\omega$-odd term proportional to $\pi^2$ we report here, which is the same for both $\rho_{\mathrm{adj}}^{++}$ and $\rho_{\mathrm{fund}}$, and the previous result of the Euclidean QCD calculation of $\rho_{\mathrm{fund}}$ in [2] differ. To be explicit, we report that the term proportional to $\pi^2$ in $\rho_{\mathrm{fund}}$ should be $-\pi^2/6$ instead of $-2\pi^2/3$ (a factor of $1/4$ smaller than in [2], where it appears as $-8\pi^2/3$ and should be $-2\pi^2/3$ in their normalization convention). In Appendix A we show a cross-check of the correctness of our result and in Appendix B we indicate where the calculation of [2] goes wrong.

We thank Mikko Laine for invaluable discussions that helped us determine the origin of the discrepancy between the results for $\rho_{\mathrm{fund}}$ and $\rho_{\mathrm{adj}}^{++}$ present in the literature.

## APPENDIX A: VERIFICATION OF THE DIFFERENCE BETWEEN $\rho^{++}_{\mathrm{adj}}$ AND $\rho_{\mathrm{fund}}$

The value of the $\pi^2$ term in $\rho^{++}_{\mathrm{adj}}$ at $\omega > 0$ can be traced back to [3] and was further verified in [4] to be $\pi^2/3$. If we further know that its $\omega$-even part is given by our results, i.e., $\frac{\pi^2}{2}\mathrm{sgn}(\omega)$, then it follows that the value of these terms for general $\omega$ is given by $-\pi^2/6 + \mathrm{sgn}(\omega)\pi^2/2$. Since a cross-check already exists for the result at $\omega > 0$ with both real and imaginary time calculations, here we cross-check the value of the difference $\rho^{++}_{\mathrm{adj}} - \rho_{\mathrm{fund}}$ with an imaginary time calculation.

The imaginary-time difference we calculate here is

$$\Delta G(\tau) \equiv G_{\mathrm{adj}}(\tau) - G_{\mathrm{fund}}(\tau), \tag{A1}$$

which can be brought into a rather succinct form after expanding the Wilson lines to linear order in $A$ [which is sufficient to get the difference up to $\mathcal{O}(g^4)$]:

$$\Delta G(\tau) = -\frac{g^3 T_F}{6N_c} f^{abc} \left\langle \mathcal{T}_E (\partial_4 A_i^a(\tau) - \partial_i A_4^a(\tau)) \int_0^\beta \mathrm{d}\tau' A_4^c(\tau')(\partial_4 A_i^b(0) - \partial_i A_4^b(0)) \right\rangle_{\mathcal{O}(g)}. \tag{A2}$$

The subscript $\mathcal{O}(g)$ indicates that only the tree-level 3-gluon vertex contributes, and $\mathcal{T}_E$ denotes Euclidean time ordering (bigger imaginary time arguments are implicitly pushed to the left of the expression). It is interesting to see that the Matsubara zero mode of the gauge field appears explicitly in these expressions.

A direct calculation in dimensional regularization (DR), introducing Feynman parameters when appropriate, leads to

$$\Delta\tilde{G}(k_n) = -i\frac{g^4 C_F N_c}{3} k_n^3 \times (k_n^2)^{D-4} \times \left\{ \frac{\Gamma(\frac{3-D}{2})^2}{2(4\pi)^{D-1}} + (D-2)\frac{\Gamma(4-D)}{(4\pi)^{D-1}} \int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{\left[\frac{1-y+xy}{\sqrt{y(1-y+yx(1-x))}}\right]^{D-4}}{\sqrt{y}(1-y+yx(1-x))^{3/2}} \right\}, \tag{A3}$$

where all of the dependence on $k_n$ is outside the curly bracket, and the terms inside the curly bracket simply correspond to a numerical prefactor.

One can show that $\Delta\rho(\omega) = 2\mathrm{Im}\{\Delta\tilde{G}(k_n)\}_{k_n \to -i(\omega+i0^+)}$. Performing the analytic continuation $k_n \to -i(\omega + i0^+)$ means that we obtain

$$-ik_n^3 \times (k_n^2)^{D-4} \to \omega^3 \times (-\omega^2 - i\omega 0^+)^{D-4} = \omega^3 |\omega|^{2D-8} \times e^{-i\pi \mathrm{sgn}(\omega)(D-4)}, \tag{A4}$$

where the analytic continuation from $k_n$ to $\omega$ is taken by continuously deforming $k_n$ starting from the real axis into the imaginary axis, without actually crossing the imaginary axis (i.e., without crossing the negative $k_n^2$ axis). Its imaginary part is

$$\mathrm{Im}\{\omega^3|\omega|^{2D-8} \times e^{-i\pi\mathrm{sgn}(\omega)(D-4)}\} = \pi(4-D)\mathrm{sgn}(\omega)\omega^3|\omega|^{2D-8} + \mathcal{O}((D-4)^3). \tag{A5}$$

It then follows that the difference between spectral functions in the limit $D \to 4$ is purely determined by the divergent contribution to $\Delta\tilde{G}(k_n)$, i.e., the last term inside the curly bracket of (A3). In the limit, $(4-D)\Gamma(4-D) \to 1$, and we may set $D = 4$ elsewhere. The integral over Feynman parameters gives a simple result,

$$\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \frac{1}{\sqrt{y}(1-y+yx(1-x))^{3/2}} = 2\pi, \tag{A6}$$

with which

$$\Delta\rho(\omega) = \lim_{D\to 4} 2\mathrm{Im}\{\Delta\tilde{G}(-i(\omega + i0^+))\} = \frac{g^2 C_F \omega^3}{3\pi} \frac{g^2}{(2\pi)^2} N_c \frac{\pi^2}{2} \mathrm{sgn}(\omega), \tag{A7}$$

just as we obtained via our real time calculation.

## APPENDIX B: RESOLVING THE TENSION WITH PREVIOUS RESULTS

We now discuss the calculation in [2] of the terms proportional to $\pi^2$ and how they arrived at a different result. In short, the issue is that the IR regulators employed in [2] fundamentally alter the analytic structure of the integral to be calculated.

In [2], the integral structure that generates the $\pi^2$ terms is given by their Eq. (A.42),

$$\delta_{3m}\tilde{\mathcal{I}}_5 = \frac{32\pi^3}{(4\pi)^2}\int_{\mathbf{k}} \mathrm{Im}\left\{\int_0^{1/2} \mathrm{d}s \frac{k_n^2}{k_n^2+k^2}\frac{2k_n^2}{(2sk_n)^2+k^2}\left(\ln\frac{k_n^2+k^2}{k_n^2} - \ln(1-4s^2)\right)\right\}_{k_n\to -i\omega+0^+}, \tag{B1}$$

where we have omitted the DR scale $\mu$, and written the expression without the IR regulator $\lambda$ present in their work (they write $\frac{k_n^2}{k_n^2+k^2+\lambda^2}$ instead of $\frac{k_n^2}{k_n^2+k^2}$ next to the d$s$ integral sign). We have also written $2k_n^2$ in the place of $k_n^2-k^2$ (the numerator on the second fraction under the $s$ integral sign) because one can show that their difference will not lead to any terms proportional to $\pi^2$ in the result. Furthermore, we have multiplied their expression by $16\pi^3$ so that it contributes to $\rho_{\mathrm{fund}}$ as the numerical factor obtained from (B1) that multiplies $2g^4T_FN_c(N_c^2-1)\omega^3/(3(2\pi)^3)$. That is to say, the result of the multiplication of (B1) with $2g^4T_FN_c(N_c^2-1)\omega^3/(3(2\pi)^3)$ is an additive contribution to $\rho_{\mathrm{fund}}$. (Because of all of these changes we denote the first symbol as $\delta_{3m}$ instead of $\delta_3$.)

The calculation of [2] proceeds by introducing a regulator in the form of a mass term, then doing the analytic continuation, taking the imaginary part, and evaluating the integrals at the end. This would work if such a regulator did not change the positions of the poles relative to the branch cuts of the integrand, which, crucially, it does. If we view the integrand of Eq. (B1) as a function of $k_n$ in the complex plane, at each fixed $k$, then there are poles at $k_n=\pm ik$, branch cuts starting at $k_n=\pm ik$ and extending to $\pm i\infty$ due to the integration over $s$, and a branch cut between $k_n=\pm ik$ due to the explicit logarithm in the integrand. See Fig. 1 for a graphic representation.

Starting from this picture, introducing a regulator in the denominator of the first factor under the $s$ integral sign amounts to moving the positions of the poles into the branch cut generated by the integration over $s$. Since the analytic continuation is essentially a limit taken by starting in the $\mathrm{Re}(k_n)>0$ region of Fig. 1, it is crucial that the position of the poles relative to the

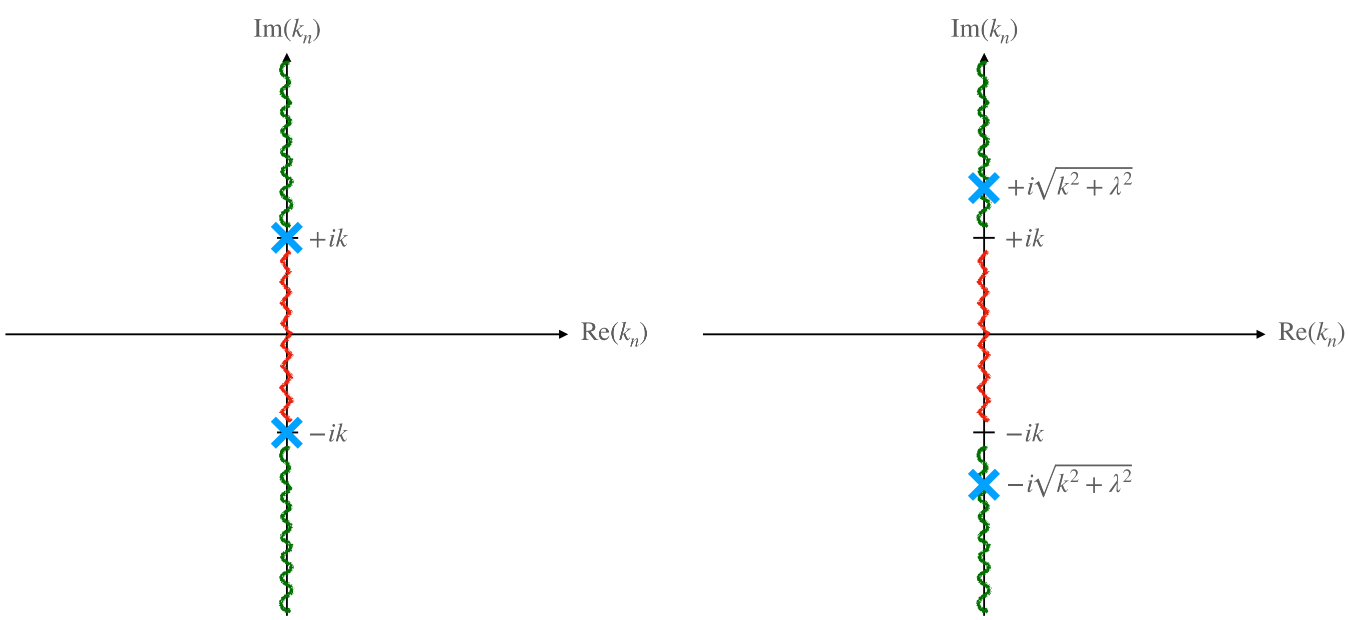


FIG. 1. Graphic representation of the pole structure of Eq. (B1) in the complex $k_n$ plane at fixed $k$. Poles are represented with blue crosses, the branch cut between $-ik$ and $+ik$ is represented by a red zigzag line, and the branch cuts above and below $\pm ik$ are represented by wavy green lines. The latter branch cut is induced by the presence of the Wilson line. Left: the pole structure without a regulator. Right: the pole structure with the regulator used in [2]. Without the branch cuts induced by the Wilson line, this regulator is not problematic because the branch cut denoted by a red zigzag line does not intersect the poles. However, with the branch cuts induced by the Wilson line, moving the poles in this manner qualitatively alters the pole structure because the contributions from the regions where $ik<\pm\mathrm{Im}(k_n)<i\sqrt{k^2+\lambda^2}$, $\mathrm{Re}(k_n)\approx 0$ will contribute with an opposite sign to the unregulated version.

branch cuts stays the same so that the analytic structure of the correlation function one intends to calculate stays unaltered. The regulator in [2] does not satisfy this requirement. Indeed, one can verify by a direct numerical calculation that

$$\delta_{3m}\tilde{\mathcal{I}}_5(\omega) = \frac{\pi^2}{3}\mathrm{Im}\left\{k_n^2\sqrt{k_n^2}\right\}_{k_n\to -i\omega+0^+}, \tag{B2}$$

as opposed to $(-2\pi^2/3)\mathrm{Im}\{k_n^2\sqrt{k_n^2}\}_{k_n\to -i\omega+0^+}$, which is what was found in [2].

Furthermore, there is an additional contribution that the calculation in Appendix A.4 of [2] did not consider, which to our knowledge was first calculated in [5] (see pages 154–160). It originates explicitly from the Matsubara zero mode. This contribution that was neglected in [2] corresponds to $-\pi^2/2$ in our normalization of the terms in the parenthesis with the prefactor $N_c$ in the expression for $\rho_{\mathrm{adj}}^{++}(\omega)$ (the corrected Eq. (35) at the beginning of this erratum). It then follows that the term proportional to $\pi^2$ in the sought result is $\pi^2/3 - \pi^2/2 = -\pi^2/6$, as claimed at the beginning of this erratum with regard to Eq. (35).

[1] A. M. Eller, J. Ghiglieri, and G. D. Moore, Thermal heavy quark self-energy from euclidean correlators, Phys. Rev. D **99**, 094042 (2019); Phys. Rev. D **102**, 039901(E) (2020).
[2] Y. Burnier, M. Laine, J. Langelage, and L. Mether, Colour-electric spectral function at next-to-leading order, J. High Energy Phys. 08 (2010) 094.
[3] M. Eidemuller and M. Jamin, QCD field strength correlator at the next-to-leading order, Phys. Lett. B **416**, 415 (1998).
[4] T. Binder, K. Mukaida, B. Scheihing-Hitschfeld, and X. Yao, Non-Abelian electric field correlator at NLO for dark matter relic abundance and quarkonium transport, J. High Energy Phys. 01 (2022) 137.
[5] A. M. Eller, The color-electric field correlator under gradient flow at next-to-leading order in quantum chromodynamics, Ph.D. thesis, Tech. U., Dortmund (main), Darmstadt, Tech. Hochsch., 2021.