Relativistic bulk rheology: From neutron star mergers to viscous cosmology

L. Gavassino[1,*] and Jorge Noronha[2,†]

[1]*Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37211, USA*
[2]*Illinois Center for Advanced Studies of the Universe and Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois, 61801, USA*

(Received 10 June 2023; revised 12 January 2024; accepted 16 April 2024; published 28 May 2024)

We develop the first causal and stable theory of a bulk-viscous relativistic pseudoplastic (or dilatant) fluid. This new formalism brings to light the rheological properties of several relativistic physical systems. Neutron star collisions can behave as a relativistic pseudoplastic material with viscous properties dictated by the nonconservation of lepton currents due to weak decay. Two-temperature relativistic plasmas, such as those surrounding supermassive galactic black holes, are predominantly pseudoplastic. Our framework can also be employed to construct novel viscous models for the evolution of the Universe with pseudoplastic or dilatant features.

DOI: 10.1103/PhysRevD.109.096040

I. INTRODUCTION

Rheology is the study of how matter responds to deformation [1–3]. Its main task is to determine relationships of the form $\Pi = \Pi[\theta]$, which express a certain stress component Π as a functional of the deformation rate θ that caused such stress. Rheology emerged as a branch of fluid mechanics one century ago motivated by the observation that many real-world fluids (the so-called "non-Newtonian fluids" [4]) defy the canonical Navier-Stokes description [5]. It was soon realized that all fluids are fundamentally non-Newtonian, and Navier-Stokes hydrodynamics is just the leading-order truncation of any (fluid-type) rheological relation $\Pi[\theta]$. Such truncation is allowed only at small spacetime gradients relative to some intrinsic scales of the fluid. Non-Newtonian corrections appear if such intrinsic scales are comparable to those associated with the variation of the hydrodynamic variables [6–8].

Since most engineering-related flows (e.g., the Couette flow [9]) are incompressible, nonrelativistic rheology mainly focuses on shear phenomena. However, in relativity, the situation is profoundly different. Many relativistic flows (e.g., Bjorken flow [10] in the context of heavy-ion collisions) experience large expansion rates. More importantly, systems whose description relies on relativistic fluid dynamics, namely the quark-gluon plasma formed in heavy-ion collisions [11], neutron star merger simulations [12], relativistic plasmas surrounding black holes [13], and viscous cosmology [14], require careful treatment of bulk (i.e., expansion-induced) viscosity, and are expected to explore dynamical regimes where the simple Navier-Stokes truncation [5,15] cannot be applied without displaying causality violation and unphysical instabilities [16]. Therefore, one needs a causal and stable relativistic theory of bulk-viscous rheology that allows us to express the bulk stress Π in terms of the relativistic expansion rate $\theta = \nabla_\mu u^\mu$, where u^μ is the fluid's 4-velocity. Here, we set the foundations of such a theory and discuss a few relevant applications. This work sheds new light on open questions concerning viscous hydrodynamics in neutron star mergers [17–22], heavy-ion collisions (e.g., concerning nonlinear causality [23–25], cavitation [26–29], attractors [30–33]), and cosmology [14,34–54]. Our metric has signature $(-+++)$ and $c = k_B = 1$. When convenient, the notation $\dot{X} = u^\mu \nabla_\mu X$ is also adopted.

II. STATEMENT OF THE PROBLEM

The simplest rheological model for fluids is the Navier-Stokes constitutive relation, $\Pi = -\zeta\theta$, where $\zeta > 0$ is a linear susceptibility (independent from θ), known as the bulk viscosity coefficient [5]. This model follows [55] from assuming a quasistationary process and a small θ (compared to some intrinsic scale of the system) expansion. Real-world flows may break both these assumptions. If the first assumption is violated, the fluid is called *viscoelastic* [2]. In a *pseudoplastic* fluid, the second assumption is violated [3]. The associated constitutive relations are, respectively [57],

$$\text{Viscoelastic}: \tau\dot{\Pi} + \Pi = -\zeta\theta, \tag{1}$$

$$\text{Pseudoplastic}: \Pi = -f\theta, \tag{2}$$

where $\tau > 0$ is a transport coefficient (the intrinsic scale), independent from θ, while f is an arbitrary function of θ.

*lorenzo.gavassino@vanderbilt.edu
†jn0508@illinois.edu

Physically, a viscoelastic fluid is a material whose stress exhibits a delay in the response to time-dependent deformation rates. The term reflects the fact that, in the high-frequency limit, (1) reduces to Hooke's law of elasticity, $\dot{\Pi} \propto \theta$ [62]. A pseudoplastic fluid is a material whose bulk viscosity coefficient, now defined as $\zeta = -\Pi/\theta$, changes as a function of the deformation rate. The term comes from the fact that, usually, such a $\zeta(\theta)$ decreases with θ so that the induced stress is relatively small at large deformation rates. As discussed below, both effects are present in neutron-star matter, QCD critical dynamics, relativistic plasmas surrounding black holes, and viscous cosmology.

The relativistic theory for viscoelasticity goes under the name of Israel-Stewart theory [63,64]. Born as an approximation of kinetic theory [65], its accuracy as an (almost [66]) universal rheological model has been recently established systematically, both within a thermodynamic [67,68] and a linear-response framework [69]. On the other hand, a relativistic theory for pseudoplasticity is still missing. A straightforward implementation of (2) in a relativistic hydrodynamic model would result in causality violation and ultraviolet instabilities. Even combining (1) and (2) into $\tau\dot{\Pi} + \Pi = -f(\theta)$, the resulting system of equations would not be quasilinear [70], leading to insurmountable difficulties in establishing causality and well-posedness of the initial value problem [71]. Moreover, without a systematic procedure for computing $f(\theta)$ from microphysics, the applicability of rheological concepts in concrete relativistic systems remains hypothetical. Below, we present a simple mathematical procedure that allows one to rewrite previously existing frameworks in rheological form, automatically including viscoelasticity and pseudoelasticity. The resulting hydrodynamic theory can be easily proven causal, stable, strongly hyperbolic, and thermodynamically consistent. Furthermore, general formulas are given to compute the rheological transport coefficients directly from microphysics.

III. GENERAL MODELING

Our starting point is a simple observation: most of the relativistic systems where bulk viscosity is important, including our main systems of interest (neutron-star matter, QCD near the critical point, ionized plasmas, and cosmological fluids), can be modeled within the same framework, which we summarize below.

We consider the case where four effective fields parametrize the macroscopic state of the fluid: $\{u^\mu, \rho, n, \phi\}$, representing the 4-velocity, the rest-frame energy density, the rest-frame baryon density, and a nonequilibrium excursion parameter, respectively. The first three fields are the usual fluid dynamical variables whose existence reflects a corresponding conservation law (momentum, energy, and baryon number). The scalar field ϕ is an observable that, due to the expansion of the fluid, is driven out of local equilibrium. Usually, ϕ reflects the existence of a weakly broken conservation law [72], e.g., a chemical reaction [76]. Since here we only focus on bulk viscosity, we assume that the stress-energy tensor $T^{\mu\nu}$ and the baryon current n^μ are isotropic in the local rest frame so that [77],

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P g^{\mu\nu}, \qquad n^\mu = n u^\mu, \tag{3}$$

where $P(\rho, n, \phi)$ is the nonequilibrium rheological pressure and $g^{\mu\nu}$ is the (arbitrary) spacetime metric. The equations of motion of the fluid are, therefore,

$$\begin{aligned}
u^\mu \nabla_\mu u_\nu &= -(g^\mu{}_\nu + u^\mu u_\nu)\frac{\nabla_\mu P}{\rho + P},\\
u^\mu \nabla_\mu \rho &= -(\rho + P)\nabla_\mu u^\mu,\\
u^\mu \nabla_\mu n &= -n \nabla_\mu u^\mu,\\
u^\mu \nabla_\mu \phi &= -K \nabla_\mu u^\mu - F.
\end{aligned} \tag{4}$$

The first three equations are the conservation laws, $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu n^\mu = 0$. The last equation models dissipation. Its structure directly follows from isotropy and the assumption that ϕ is the only nonequilibrium degree of freedom. The coefficient $K(\rho, n, \phi)$ is called the compressibility and $F(\rho, n, \phi)$ is the returning force. Therefore, to completely specify the systems we are interested in, we only need to know how to express $\{P, K, F\}$ in terms of $\{\rho, n, \phi\}$.

Finally, the reader may note a similarity between (4) and the Hydro + effective theory framework, which was introduced in Ref. [78] to describe a near-hydrodynamic system with an additional mode that is parametrically slower than the other modes. This is not a coincidence, given the broad regime of applicability of Hydro+ [78].

A. Bulk rheology

Let us recast (4) into a rheological model for bulk viscosity, which also accounts for pseudoplasticity. First, we note that the equilibrium value $\phi_{\text{eq}}(\rho, n)$ of the nonconserved mode ϕ (at given ρ and n) can be computed from the relation $F(\rho, n, \phi_{\text{eq}}(\rho, n)) = 0$ [78]. This follows from the fact that the equilibrium state is stationary and nondeforming so that $0 = u^\mu \nabla_\mu \phi = -F$ at equilibrium. Then, given the expression for the nonequilibrium pressure, $P(\rho, n, \phi)$, we can define the equilibrium pressure and the bulk scalar as

$$\begin{aligned}
P_{\text{eq}}(\rho, n) &= P(\rho, n, \phi_{\text{eq}}(\rho, n)),\\
\Pi(\rho, n, \phi) &= P(\rho, n, \phi) - P(\rho, n, \phi_{\text{eq}}(\rho, n)),
\end{aligned} \tag{5}$$

so that the energy-momentum tensor takes the usual bulk-viscous form,

$$T^{\mu\nu} = (\rho + P_{\text{eq}} + \Pi)u^\mu u^\nu + (P_{\text{eq}} + \Pi)g^{\mu\nu}. \tag{6}$$

Inverting the second equation of (5), we obtain a relation of the form $\phi = \phi(\rho, n, \Pi)$. Clearly, $\phi_{\text{eq}}(\rho, n) = \phi(\rho, n, 0)$

and, since F vanishes at equilibrium, $F_\Pi = F/\Pi$ is nonsingular at $\Pi = 0$ as it converges to $\partial_\Pi F(\Pi = 0)$. Expressing the last equation of (4) in terms of Π, we obtain a relation that resembles Israel-Stewart theory (though here the equation is valid also for large $\Pi/P_{\rm eq}$),

$$\tau_\Pi u^\mu \nabla_\mu \Pi + \Pi = -\zeta \nabla_\mu u^\mu, \tag{7}$$

where $\tau_\Pi(\rho, n, \Pi)$ and $\zeta(\rho, n, \Pi)$ are now also rheological functions of Π, given by

$$\begin{aligned} \tau_\Pi &= \frac{1}{F_\Pi} \frac{\partial \phi}{\partial \Pi}\bigg|_{\rho,n}, \\ \zeta &= -\frac{1}{F_\Pi}\left[(\rho + P_{\rm eq} + \Pi)\frac{\partial \phi}{\partial \rho}\bigg|_{n,\Pi} + n\frac{\partial \phi}{\partial n}\bigg|_{\rho,\Pi} - K\right]. \end{aligned} \tag{8}$$

No approximation has been made above, meaning that (7) is mathematically equivalent to (4) (the equivalence also holds in curved spacetime).

Equation (7) is the relativistic theory for bulk rheology we were looking for. Indeed, viscoelasticity is automatically accounted for by the relaxation time, and all the available formulations of Israel-Stewart bulk viscosity (e.g., DNMR [65], or Hiscock-Linblom [64]) correspond to particular choices of $\tau_\Pi(\Pi)$ and $\zeta(\Pi)$ (see Supplemental Material [79]). However, the theory can also describe pseudoplasticity. In fact, suppose that the system has reached an attractor state [30], where $\tau_\Pi u^\mu \nabla_\mu \Pi$ can be replaced with some function $h(\rho, n, \Pi, \nabla_\mu u^\mu)$. Then, we have an equation of the form $h(\Pi, \nabla_\mu u^\mu) + \Pi = -\zeta(\Pi)\nabla_\mu u^\mu$. If we isolate Π, we obtain a (fully nonlinear) expression $\Pi = -f(\nabla_\mu u^\mu)$, which can be interpreted as a late-time pseudoplastic constitutive relation. This further develops the idea of effective transport coefficients that encode the contribution from an infinite number of gradients, previously investigated in relativistic systems undergoing highly symmetric flows [31,80–83]. In this context, the results of [31] show that the hydrodynamic attractor found in kinetic and holographic systems undergoing Bjorken flow is in the pseudoplastic regime. We note, however, that the dependence of ζ with gradients discussed above holds, in different forms, for arbitrary flows (also in curved spacetime). Furthermore, causality and stability issues are automatically solved since (7) gives rise to a fluid model that is thermodynamically consistent, symmetric hyperbolic, and causal in the fully nonlinear regime (see Supplemental Material [79]).

IV. BULK-VISCOUS RHEOLOGY OF NEUTRON STAR MERGERS

We now argue that the ultradense matter formed in neutron star mergers must have bulk-viscous rheological properties. Current state-of-the-art simulations of neutron star mergers (see, e.g., [22]) solve (4) coupled to Einstein's equations. Assuming npe matter in the neutrino transparent regime, ϕ corresponds to the charge fraction $Y = n_e/n$ (n_e is the electron density), which only changes by weak-interaction decays of neutrons and protons. This gives $K = 0$ [84,85] and $F = \Gamma_\nu/n$, where Γ_ν are the weak-interaction rates, which include standard leakage schemes [86,87] as well as direct and modified Urca net rates [88,89]. In this case, $P = P(\rho, n, Y)$ can be determined directly from the underlying model for the equation of state away from beta equilibrium. Then, our "dictionary relations" (8) can be used to directly determine the transport coefficients for neutron-star matter far from beta equilibrium and define its rheological properties. Indeed, the first study of this kind has been carried out in [90], where our transport coefficients in (8) were evaluated numerically using a realistic equation of state compatible with astrophysical constraints. Therefore, the exact mathematical equivalence between (4) and (7), established here, conclusively shows for the first time that the matter formed in neutron star collisions is intrinsically viscous, going beyond the linear response analyses of [19,91,92]. This general result is valid for arbitrary equations of state, rates, and dynamical spacetimes. Whether or not such viscous effects can be measured using gravitational waves is still under debate [21,22,93–97].

A. An analytical model

To further discuss the physics behind the mathematical mapping leading to (8) and how pseudoplasticity affects the dynamics of the bulk stress, we consider below an oversimplified toy model for neutron-star matter, where the mapping can be carried out analytically. Inspired by the neutron-star models adopted in [20,92], we consider the following relations [98]:

$$P = n^2 e^{-Y}, \qquad F = (n^{-1/2} e^Y - 1)C^{-1}, \tag{9}$$

where $C > 0$ is a constant. This choice of P and F reproduces some qualitative features that neutron-star matter is expected to have. For example, P decreases with Y at constant n [90]. Additionally, the equilibrium fraction $Y_{\rm eq} = \ln\sqrt{n}$ (computed from the requirement that $F_{\rm eq} = 0$) increases with n, as predicted by nuclear models [20]. In the Supplementary Material [79], we verify that the resulting equations of motion are indeed causal, strongly hyperbolic [99], and thermodynamically consistent arbitrarily far from beta equilibrium, besides being covariantly stable [100,101], both thermodynamically and hydrodynamically [102–104]. Using (5), we obtain the equilibrium pressure and the bulk stress, $P_{\rm eq} = n^{3/2}$ and $\Pi = n^2 e^{-Y} - n^{3/2}$. From this, the expressions for the transport coefficients (8) follow immediately:

$$\tau_\Pi = C, \qquad \zeta = \left[\frac{n^{3/2}}{2} + 2\Pi\right] C. \tag{10}$$

One can see that ζ exhibits a nontrivial dependence on Π. Note that (10) holds for arbitrarily large values of Π.

B. Pseudoplasticity as an attractor

We can show that neutron-star matter modeled by (10) is pseudoplastic. To this end, let us solve (7) along the worldline of a fluid element (i.e., along an integral curve of u^μ), parametrized with the proper time t. Assuming a constant expansion rate $\nabla_\mu u^\mu$, and using the transport coefficients (10), we find

$$\frac{\Pi(t)}{P_{\rm eq}(t)} = Ae^{-(1+\frac{1}{2}\tau_\Pi\nabla_\mu u^\mu)t/\tau_\Pi} - \frac{\frac{1}{2}\tau_\Pi\nabla_\mu u^\mu}{1+\frac{1}{2}\tau_\Pi\nabla_\mu u^\mu}, \quad (11)$$

where A is an integration constant. We see that if $\tau_\Pi\nabla_\mu u^\mu > -2$, then the system admits a late-time attractor (the solution with $A = 0$), where Π is a nonlinear function of $\nabla_\mu u^\mu$. We can express this late-time constitutive relation in Navier-Stokes form, $\Pi = -\zeta_{\rm res}\nabla_\mu u^\mu$, by defining a resummed bulk viscosity coefficient (see Fig. 1),

$$\zeta_{\rm res} = \frac{\zeta_{\rm NS}}{1+\frac{1}{2}\tau_\Pi\nabla_\mu u^\mu}, \quad (12)$$

where $\zeta_{\rm NS} = \zeta(\Pi = 0)$ is the transport coefficient that would enter the Navier-Stokes model. As can be seen, if the fluid expands (i.e., $\nabla_\mu u^\mu > 0$), the effective viscosity is reduced, and it tends to zero when $\nabla_\mu u^\mu \to +\infty$. This is the standard signature of pseudoplasticity. If, however, the fluid is compressed (i.e., $\nabla_\mu u^\mu < 0$), then the effective viscosity becomes much larger than the Navier-Stokes viscosity. In rheology, this is the hallmark of dilatant behavior [3]. For $\tau_\Pi\nabla_\mu u^\mu < -2$, the general solution (11) does not have a late-time attractor.

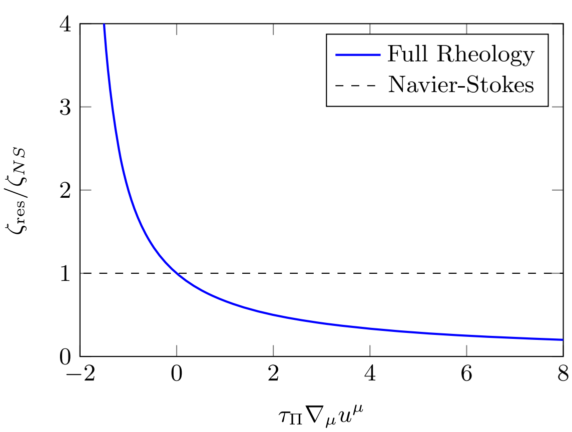

FIG. 1. Pseudoplastic features of our neutron-star matter toy model (9) (blue) compared with the Navier-Stokes prediction (dashed). The resummed bulk viscosity coefficient $\zeta_{\rm res}$ (rescaled by $\zeta_{\rm NS} = P_{\rm eq}\tau_\Pi/2$) is plotted as a function of the expansion rate $\nabla_\mu u^\mu$ (in units of τ_Π^{-1}).

C. Impact of pseudoplasticity

If a bulk-viscous fluid undergoes small quasiperiodic oscillations of frequency ω around thermodynamic equilibrium, the (approximate) damping time of the oscillation due to bulk dissipation is [20] $t_{\rm damp} = 2c_s^2(\rho + P_{\rm eq}) \times (1+\omega^2\tau_\Pi^2)/\zeta\omega^2$, where c_s is the speed of sound. Keeping ω fixed, and treating $t_{\rm damp}$ as a function of the reaction rate intensity C^{-1} [see (9) and (10)], $t_{\rm damp}$ has an absolute minimum when $\tau_\Pi = \omega^{-1}$. For this reason, it has been argued that in neutron star mergers, bulk viscosity should have the strongest impact in resonant regions [105], where the relaxation timescale of β-reactions equals the timescale of the hydrodynamic evolution. However, since such an estimate is carried out in the linear regime, it can only account for viscoelastic effects, and it completely neglects pseudoplasticity. Interestingly, we see from Fig. 1 that pseudoplastic corrections are largest not at $\tau_\Pi\nabla_\mu u^\mu = 1$ (as the resonance argument might suggest) but in the limit as $\tau_\Pi \to +\infty$, i.e., when dissipation is negligible. This explains why, in recent simulations of neutron-star migration [20] (which is a highly nonlinear process), rheological effects were shown to be the largest when beta reactions are suppressed, i.e., for $u^\mu\nabla_\mu Y \approx 0$.

V. TWO-TEMPERATURE PLASMAS

Due to the high temperatures (around 10^7–10^{13} K) achieved in black hole accretion, hydrogen becomes fully ionized [106]. Since the ee and pp inelastic collisions are much faster than ep inelastic collisions [107], the electron and proton gases achieve kinetic equilibrium at two different temperatures [108]. This allows us to rigorously model the plasma using the formalism presented in this work, where the additional nonequilibrium variable ϕ can be identified with the electron pressure. Considering ultrarelativistic electrons and nonrelativistic protons (with mass set to unity, for simplicity), one can straightforwardly derive the constitutive relations from thermodynamics (see Supplementary Material, Sec. [79]),

$$P = \frac{2}{3}(\rho - n) - \phi, \qquad K = \frac{4}{3}\phi,$$
$$F = \frac{1}{C}\left[\phi - \frac{2}{9}(\rho - n)\right], \quad (13)$$

where $C > 0$ is a constant for simplicity. The exact bulk transport coefficients in the rheological representation are

$$P_{\rm eq} = \frac{4}{9}(\rho - n), \qquad \tau_\Pi = C, \qquad \zeta = \frac{2C}{81}(\rho - n + 63\Pi). \quad (14)$$

The equilibrium equation of state for the pressure has an adiabatic index $13/9$ [106], in agreement with simulations

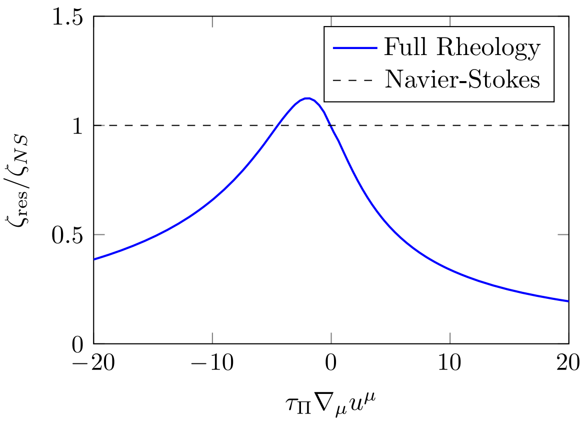

FIG. 2. Pseudoplastic features of a two-temperature plasma (9) (blue) compared with the Navier-Stokes prediction (dashed). The resummed bulk viscosity coefficient $\zeta_{\rm res}$ [rescaled by $\zeta_{\rm NS} = 2\tau_\Pi(\rho - n)/81$] is plotted as a function of the expansion rate $\nabla_\mu u^\mu$ (in units of τ_Π^{-1}).

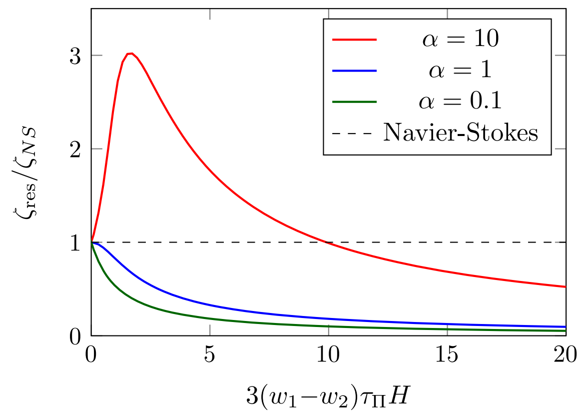

FIG. 3. Pseudoplastic features of an expanding two-component universe undergoing dissipative energy transfers, as predicted by equation (18).

of M 87 jets [109]. Working out the late-time attractor of a plasma undergoing uniform expansion leads to the resummed bulk viscosity coefficient (see Fig. 2),

$$\zeta_{\rm res} = -\frac{9 + \tau_\Pi \nabla_\mu u^\mu - 3\sqrt{9 + 2\tau_\Pi \nabla_\mu u^\mu + (\tau_\Pi \nabla_\mu u^\mu)^2}}{4(\tau_\Pi \nabla_\mu u^\mu)^2} 9\zeta_{\rm NS}. \tag{15}$$

We note that two-temperature plasmas are predominantly pseudoplastic. Appreciable deviations from Navier-Stokes appear when $\tau_\Pi \nabla_\mu u^\mu$ is of order 10, i.e., the nearly collisionless regime considered in black hole accretion simulations [110,111].

VI. BULK VISCOUS COSMOLOGY

As the Universe cools down, the relevant degrees of freedom vary, changing the thermodynamic conditions, the degree of ionization, and the radiation-matter ratio [112,113]. Thus, viscous effects in the expanding Universe can be modeled as equilibration processes between the dominant equation of state in a certain era and the dominant equation of state in the next era [114,115]. One can qualitatively describe each transition era-by-era through a simple two-fluid model with four free parameters, which gives us a rough estimate of the pseudoplastic features of the Universe at the transition.

Consider a cosmological fluid comprised of two interacting components, with energy densities ρ_1 and ρ_2 and pressures $P_1 = w_1\rho_1$ and $P_2 = w_2\rho_2$, with constant $w_1 > w_2 > 0$. The interaction between the two components takes the form of a dissipative energy exchange, which drives the system towards local thermodynamic equilibrium. For clarity, we assume the equilibrium condition is $\rho_1 = \alpha\rho_2$, for some constant $\alpha > 0$. Then, the conglomerate fluid can be described using the results of this work, with nonequilibrium mode $\phi = \rho_2$, and constitutive relations,

$$P = w_1\rho + (w_2 - w_1)\phi, \qquad K = (1 + w_2)\phi,$$
$$F = \left[\phi - \frac{\rho}{1+\alpha}\right] C^{-1}, \tag{16}$$

where, again, $C > 0$ is assumed constant. The exact bulk transport coefficients in the rheological representation are

$$P_{\rm eq} = \frac{\alpha w_1 + w_2}{\alpha + 1}\rho \qquad \tau_\Pi = C,$$
$$\frac{\zeta}{C} = \frac{\rho\alpha(w_1 - w_2)^2}{(1+\alpha)^2} + \left[1 + \frac{w_1 + \alpha w_2}{1+\alpha}\right]\Pi. \tag{17}$$

One can solve these equations in a Friedmann-Lemaître-Robertson-Walker background [112] with constant Hubble parameter $H > 0$. The resulting late-time attractor gives us a resummed bulk viscosity coefficient, which can be expressed in terms of $y = 3(w_1 - w_2)\tau_\Pi H$ as follows (see Fig. 3):

$$\frac{\zeta_{\rm res}}{\zeta_{\rm NS}} = \frac{1+\alpha}{2\alpha y^2}\Big[\alpha(y-1) - (y+1) + \sqrt{(1+\alpha)[\alpha(y-1)^2 + (y+1)^2]}\Big]. \tag{18}$$

For $\alpha > 1$, the Universe exhibits a dilatant behavior for not too large values of $\tau_\Pi H$. For $\alpha \leq 1$, it is always pseudoplastic, namely $\zeta_{\rm res} < \zeta_{\rm NS}$. Failure to account for these effects may lead to overestimating (or underestimating, in the dilatant case) the impact of viscous effects in the cosmological evolution.

VII. CONCLUSIONS

We constructed the first causal and stable theory of rheological bulk-viscous systems in relativity. This paves the way for systematically investigating the novel rheological properties displayed by relativistic systems. Our formalism is employed to show that neutron star mergers are intrinsically bulk-viscous systems with rheological pseudoplastic properties. Relativistic pseudoplasticity is also predicted to emerge in two-temperature relativistic plasmas surrounding supermassive galactic black holes. Our framework can be employed to formulate new viscous cosmological models with pseudoplastic features.

ACKNOWLEDGMENTS

L. G. is partially supported by a Vanderbilt Seeding Success Grant. J. N. is partially supported by the U.S. Department of Energy, Office of Science, Office for Nuclear Physics under Award No. DE-SC0021301. We thank M. Disconzi and E. Most for reading the manuscript and providing useful comments. L. G. thanks Andrew Chael for a deeply insightful discussion on two-temperature plasmas, and Gabriel Soares Rocha for interesting conversations about the relationship between rheology and attractors.

[1] A. Y. Malkin and A. I. Isayev, *RHEOLOGY: Concepts, Methods, and Applications*, 2nd ed. (ChemTec Publishing, Toronto, 2012).
[2] W. N. Findley, J. S. Lai, and K. Onaran, *Creep and Relaxation on Nonlinear Viscoelastic Materials* (Dover Publications, New York, NY, 1976).
[3] J. Steffe, *Rheological Methods in Food Process Engineering*, 2nd ed. (Freeman Press, San Francisco, 1996).
[4] C. Tropea, A. Yarin, and J. Foss, *Springer Handbook of Experimental Fluid Mechanics* (Springer, New York, 2007).
[5] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics—Volume 6 (Corse of Theoretical Physics)*, 2nd ed. (Butterworth-Heinemann, 1987), p. 552.
[6] J. Frenkel, *Kinetic Theory of Liquids*, 2nd ed. (Dover Publications, New York, 1955).
[7] M. Baggioli, M. Vasin, V. Brazhkin, and K. Trachenko, Phys. Rep. **865**, 1 (2020), gapped momentum states.
[8] T. Andrade, M. Baggioli, and O. Pujolàs, Phys. Rev. D **100**, 106014 (2019).
[9] L. Landau and E. Lifshitz, *Fluid Mechanics*, v. 6, 2nd ed. (Pergamon Press, New York, 1987).
[10] J. D. Bjorken, Phys. Rev. D **27**, 140 (1983).
[11] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2019).
[12] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, New York, 2013).
[13] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), Astrophys. J. Lett. **930**, L16 (2022).
[14] I. Brevik, O. Gron, J. de Haro, S. D. Odintsov, and E. N. Saridakis, Int. J. Mod. Phys. **D26**, 1730024 (2017).
[15] C. Eckart, Phys. Rev. **58**, 919 (1940).
[16] W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[17] M. G. Alford, M. Braby, and A. Schmitt, J. Phys. G **35**, 115007 (2008).
[18] L. Gavassino, M. Antonelli, and B. Haskell, Classical Quantum Gravity **38**, 075001 (2021).
[19] T. Celora, I. Hawke, P. C. Hammond, N. Andersson, and G. L. Comer, Phys. Rev. D **105**, 103016 (2022).
[20] G. Camelio, L. Gavassino, M. Antonelli, S. Bernuzzi, and B. Haskell, Phys. Rev. D **107**, 103032 (2023).
[21] E. R. Most, S. P. Harris, C. Plumberg, M. G. Alford, J. Noronha, J. Noronha-Hostler, F. Pretorius, H. Witek, and N. Yunes, Mon. Not. R. Astron. Soc. **509**, 1096 (2021).
[22] E. R. Most, A. Haber, S. P. Harris, Z. Zhang, M. G. Alford, and J. Noronha, arXiv:2207.00442.
[23] F. S. Bemfica, M. M. Disconzi, V. Hoang, J. Noronha, and M. Radosz, Phys. Rev. Lett. **126**, 222301 (2021).
[24] C. Plumberg, D. Almaalol, T. Dore, J. Noronha, and J. Noronha-Hostler, Phys. Rev. C **105**, L061901 (2022).
[25] C. Chiu and C. Shen, Phys. Rev. C **103**, 064901 (2021).
[26] G. Torrieri and I. Mishustin, Phys. Rev. C **78**, 021901 (2008).
[27] K. Rajagopal and N. Tripuraneni, J. High Energy Phys. 03 (2010) 018.
[28] G. S. Denicol, C. Gale, and S. Jeon, Proc. Sci. CPOD2014 (**2015**) 033.
[29] M. Byres, S. H. Lim, C. McGinn, J. Ouellette, and J. L. Nagle, Phys. Rev. C **101**, 044902 (2020).
[30] M. P. Heller and M. Spalinski, Phys. Rev. Lett. **115**, 072501 (2015).
[31] P. Romatschke, Phys. Rev. Lett. **120**, 012301 (2018).
[32] M. Strickland, J. Noronha, and G. Denicol, Phys. Rev. D **97**, 036020 (2018).
[33] J. Jankowski and M. Spaliński, Prog. Part. Nucl. Phys. **132**, 104048 (2023).
[34] T. Padmanabhan and S. M. Chitre, Phys. Lett. **120A**, 433 (1987).
[35] W. A. Hiscock and J. Salmonson, Phys. Rev. D **43**, 3249 (1991).
[36] D. Pavon, J. Bafaluy, and D. Jou, Classical Quantum Gravity **8**, 347 (1991).
[37] M. Zakari and D. Jou, Phys. Rev. D **48**, 1597 (1993).
[38] R. Maartens, Classical Quantum Gravity **12**, 1455 (1995).
[39] W. Zimdahl, Phys. Rev. D **53**, 5483 (1996).
[40] J. C. Fabris, S. V. B. Goncalves, and R. R. de Sa, Gen. Relativ. Gravit. **38**, 495 (2006).

[41] R. Colistete, J. C. Fabris, J. Tossa, and W. Zimdahl, Phys. Rev. D **76**, 103516 (2007).
[42] B. Li and J. D. Barrow, Phys. Rev. D **79**, 103521 (2009).
[43] A. Avelino and U. Nucamendi, J. Cosmol. Astropart. Phys. 04 (2009), 006.
[44] W. S. Hipolito-Ricaldi, H. Velten, and W. Zimdahl, J. Cosmol. Astropart. Phys. 06 (2009), 016.
[45] W. S. Hipolito-Ricaldi, H. Velten, and W. Zimdahl, Phys. Rev. D **82**, 063507 (2010).
[46] J.-S. Gagnon and J. Lesgourgues, J. Cosmol. Astropart. Phys. 09 (2011).
[47] O. F. Piattella, J. C. Fabris, and W. Zimdahl, J. Cosmol. Astropart. Phys. 05 (2011) 029.
[48] H. Velten and D. Schwarz, Phys. Rev. D **86**, 083501 (2012).
[49] H. Velten and D. Schwarz, J. Cosmol. Astropart. Phys. 09 (2011) 016.
[50] I. Brevik and O. Gron, in *Recent Advances in Cosmology*, edited by A. Travena and B. Soren (Nova Science Publishers, 2013).
[51] M. M. Disconzi, T. W. Kephart, and R. J. Scherrer, Phys. Rev. D **91**, 043532 (2015).
[52] M. M. Disconzi, T. W. Kephart, and R. J. Scherrer, Int. J. Mod. Phys. D **26**, 1750146 (2017).
[53] N. Cruz, E. González, and G. Palma, Gen. Relativ. Gravit. **52**, 62 (2020).
[54] N. Cruz, E. González, and G. Palma, Mod. Phys. Lett. A **36**, 2150032 (2021).
[55] A choice of hydrodynamic frame is also made, see [56].
[56] K. Kovtun, J. Phys. A **45**, 473001 (2012).
[57] Equation (1) is just the simplest model for viscoelasticity, known as the “Maxwell model” [8,58,62]. Other more complicated expressions are known (e.g., the Burgers model [59,60]). The most general time-dependent linear rheological relation takes the form $\Pi(t) = \int G(t-t')\theta(t')dt'$, where G is some linear-response Green’s function [61].
[58] L. Gavassino and M. Antonelli, Classical Quantum Gravity **40**, 075012 (2023).
[59] J. Málek, K. R. Rajagopal, and K. Tůma, Fluids **3**, 69 (2018).
[60] L. Gavassino, Classical Quantum Gravity **40**, 165008 (2023).
[61] G. S. Denicol, J. Noronha, H. Niemi, and D. H. Rischke, Phys. Rev. D **83**, 074019 (2011).
[62] L. Landau and E. Lifshitz, *Theory of Elasticity* (Pergamon Press, New York, 1970).
[63] W. Israel and J. M. Stewart, Ann. Phys. (Leipzig) **118**, 341 (1979).
[64] W. A. Hiscock and L. Lindblom, Ann. Phys. (Leipzig) **151**, 466 (1983).
[65] G. S. Denicol, H. Niemi, E. Molnár, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012).
[66] M. P. Heller, R. A. Janik, M. Spaliński, and P. Witaszczyk, Phys. Rev. Lett. **113**, 261601 (2014).
[67] L. Gavassino, M. M. Disconzi, and J. Noronha, arXiv:2302.03478.
[68] L. Gavassino, M. M. Disconzi, and J. Noronha, arXiv:2302.05332.
[69] D. Wagner and L. Gavassino, Phys. Rev. D **109**, 016019 (2024).
[70] Y. Choquet-Bruhat, *General Relativity and the Einstein Equations* (Oxford University Press, New York, 2009).
[71] M. M. Disconzi, arXiv:2308.09844.
[72] In response theory, a weakly broken conservation law is a gapped mode ϕ whose relaxation time τ_ϕ is parametrically slow compared to the relaxation time τ_{micro} of all other gapped modes [73,74]. Under this assumption, one can define a quasi-hydrodynamic regime, valid over the timescale $\tau_{\text{micro}} \ll t \lesssim \tau_\phi$, where the fluid exists in a quasi-equilibrium state, and ϕ plays the role of a generalized thermodynamic variable (see [75], p. 118).
[73] S. Grozdanov, A. Lucas, and N. Poovuttikul, Phys. Rev. D **99**, 086012 (2019).
[74] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. D **106**, 056010 (2022).
[75] L. Landau and E. Lifshitz, *Statistical Physics*, 3rd ed. (Pergamon Press, 1980).
[76] L. Gavassino and M. Antonelli, Front. Astron. Space Sci. **8**, 686344 (2021).
[77] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W.H. Freeman and Co., San Francisco, 1973).
[78] M. Stephanov and Y. Yin, Phys. Rev. D **98**, 036006 (2018).
[79] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.109.096040 contains some formal derivations that were omitted from the main text.
[80] G. S. Denicol and J. Noronha, Phys. Rev. D **97**, 056021 (2018).
[81] J.-P. Blaizot and L. Yan, Phys. Lett. B **780**, 283 (2018).
[82] A. Behtash, C. N. Cruz-Camacho, S. Kamata, and M. Martinez, Phys. Lett. B **797**, 134914 (2019).
[83] G. S. Denicol and J. Noronha, Nucl. Phys. **A1005**, 121748 (2021).
[84] B. Carter, *Covariant Theory of Conductivity in Ideal Fluid or Solid Media* (Springer-Verlag Berlin, 1989), Vol. 1385, p. 1.
[85] G. L. Comer and D. Langlois, Classical Quantum Gravity **11**, 709 (1994).
[86] M. H. Ruffert, H. T. Janka, and G. Schaefer, Astron. Astrophys. **311**, 532 (1996).
[87] S. Rosswog and M. Liebendörfer, Mon. Not. R. Astron. Soc. **342**, 673 (2003).
[88] D. G. Yakovlev, A. D. Kaminker, O. Y. Gnedin, and P. Haensel, Phys. Rep. **354**, 1 (2001).
[89] M. G. Alford, A. Haber, S. P. Harris, and Z. Zhang, Universe **7**, 399 (2021).
[90] Y. Yang, M. Hippert, E. Speranza, and J. Noronha, Phys. Rev. C **109**, 015805 (2024).
[91] L. Gavassino, M. Antonelli, and B. Haskell, Classical Quantum Gravity **38**, 075001 (2021).
[92] G. Camelio, L. Gavassino, M. Antonelli, S. Bernuzzi, and B. Haskell, Phys. Rev. D **107**, 103031 (2023).
[93] M. G. Alford, L. Bovard, M. Hanauske, L. Rezzolla, and K. Schwenzer, Phys. Rev. Lett. **120**, 041101 (2018).
[94] P. Hammond, I. Hawke, and N. Andersson, Phys. Rev. D **107**, 043023 (2023).
[95] M. Chabanov and L. Rezzolla, arXiv:2307.10464.
[96] P. L. Espino, P. Hammond, D. Radice, S. Bernuzzi, R. Gamba, F. Zappa, L. F. L. Micchi, and A. Perego, arXiv:2311.00031.
[97] J. L. Ripley, A. Hegade K. R., R. S. Chandramouli, Yunes, and Nicolas, arXiv:2312.11659.

[98] We have set all dimensional constants to one for convenience. Furthermore, we have decided to neglect the temperature dependence of the pressure and the reaction rate to simplify the analysis.
[99] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics*, edited by L. Rezzolla and O. Zanotti (Oxford University Press, New York, 2013), ISBN-10: 0198528906; ISBN-13: 978-0198528906.
[100] L. Gavassino, Phys. Rev. X **12**, 041001 (2022).
[101] L. Gavassino, Phys. Lett. B **840**, 137854 (2023).
[102] L. Gavassino, Classical Quantum Gravity **38**, 21LT02 (2021).
[103] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. **128**, 010606 (2022).
[104] L. Gavassino, Classical Quantum Gravity **39**, 185008 (2022).
[105] M. G. Alford, L. Bovard, M. Hanauske, L. Rezzolla, and K. Schwenzer, Phys. Rev. Lett. **120**, 041101 (2018).
[106] S. T. S. Shapiro, *Black Holes, White Dwarfs,and Neutron Stars: The Physics of Compact Objects* (WILEY-VCH Verlag GmbH and Co. KGaA, Mörlenbach, 1983).
[107] P. M. Bellan, *Fundamentals of Plasma Physics* (Cambridge University Press, Cambridge, England, 2006).
[108] K. Akiyama *et al.* (Event Horizon Telescope Collaboration), Astrophys. J. Lett. **875**, L5 (2019).
[109] M. Mościbrodzka, H. Falcke, and H. Shiokawa, Astron. Astrophys. **586**, A38 (2016).
[110] M. Chandra, C. F. Gammie, F. Foucart, and E. Quataert, Astrophys. J. **810**, 162 (2015).
[111] F. Foucart, M. Chandra, C. F. Gammie, E. Quataert, and A. Tchekhovskoy, Mon. Not. R. Astron. Soc. **470**, 2240 (2017).
[112] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, 1st ed. (John Wiley & Sons, Inc., New York, 1972), p. 657.
[113] S. Dodelson, *Modern Cosmology* (Academic Press, Amsterdam, 2003).
[114] S. Weinberg, Astrophys. J. **168**, 175 (1971).
[115] N. Udey and W. Israel, Mon. Not. R. Astron. Soc. **199**, 1137 (1982).