Gauge invariance, gauge field quantization, and renormalization in autoregularization

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Autoregularization, a new divergence-free framework for calculating scattering amplitudes, uses a Lorentz-invariant scale harvested from the kinematics of a scattering process to regularize the amplitude of the process [N. Prabhu, J. Phys. Commun. **7**, 115002 (2023).]. Preliminary validation studies show that autoregularization's predictions are in good agreement with experimental data—across several scattering processes and a wide range of energy scales. Further, *ab initio* tree-level calculation of the vacuum energy density of the free fields in the Standard Model, using autoregularization, is shown to yield a value that is smaller than the current estimate of the cosmic critical density. In this paper, we prove that the scattering amplitudes in QED, calculated using autoregularization, are gauge invariant. Our proof, which is valid both for autoregularization and current theory, is stronger in that it shows the amplitude of every Feynman diagram is gauge invariant in contrast to previous proofs, which establish gauge invariance only for sum of amplitudes of Feynman diagrams of a process. Next, we show that—unlike in the standard quantization framework, which requires modification of both the quantization framework itself as well as the Lagrangian in order to quantize gauge fields in covariant gauge—in autoregularization the gauge field in QED can be quantized, in covariant gauge, without modifying the standard quantization procedure or the Lagrangian and without introducing the ghost field. Finally, we illustrate renormalization based on autoregularization up to 1-loop in φ^4 theory. Since perturbative corrections are finite in autoregularization, the counterterms are not designed to remove divergences but to implement renormalization prescriptions at every order of perturbation. We also derive the renormalization group equation (RGE). Unlike in some regularization schemes (such as dimensional regularization), in which the physical meaning of the fictitious scale introduced by regularization is unclear, in autoregularization the scale in RGE has a transparent physical meaning—it is the Lorentz-invariant kinematic scale of the scattering process of interest. The increasing simplifications resulting within autoregularization and the agreement between its predictions and experimental data, together with the underlying thermodynamic argument, which shows that the framework is essential for a complete description of quantum fields, all converge to suggest that autoregularization provides the proper framework for the description of quantum fields.

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I. INTRODUCTION

In contrast to known regularization schemes, which introduce an arbitrary energy scale into renormalization, autoregularization [1] uses the energy scale harvested from the kinematics of a scattering process to regularize the amplitude of the process. Autoregularization is based on a new thermodynamic view that an interacting quantum field—which can exchange energy and particles with other quantum fields—can be regarded as a "system" that is in thermal and diffusive equilibrium with a "reservoir", made of the other quantum fields that coinhabit spacetime.

The current view of a quantum field is based on a hypothesis, proposed by Heisenberg and Pauli [2] in 1929–1930, that a quantum field can be regarded[1] as "...a dynamical system amenable to Hamiltonian treatment...." It is of historical interest that Dirac immediately objected to applying the "Hamiltonian treatment" to the field rather than to individual oscillators[2] [3]. Dirac's objections

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[1]The quoted text is from [3].

[2]Apart from his objections based on physical arguments, Dirac also cautioned about the "mathematical difficulties" that would result from the hypothesis [3].

notwithstanding, the current quantum field theory (QFT) is based on the Heisenberg-Pauli framework (HPF).

Autoregularization assumes that a quantum field can be regarded as a dynamical system, as suggested by Heisenberg and Pauli [2]. The behavior of any large dynamical system—such as an interacting quantum field—is governed not only by the laws of local microscopic interactions—encoded in the Lagrangian of a theory—but also by *additional* statistical laws, which are not encoded in the Lagrangian but *emerge* due to collective interactions among the large[3] number of modes of the system and the reservoir. Such emergent statistical laws, Landau and Lifshitz write [4], "are of a different kind" and "cannot in any way be reduced to purely mechanical laws"; that is, they do not follow from the Lagrangian formalism, but need to be added explicitly on top of the Lagrangian formalism if one seeks to obtain a complete description of interacting quantum fields.[4]

The HPF does not incorporate such additional emergent statistical laws. Hence, the current QFT, although it has been remarkably successful, is likely an *underconstrained* description of quantum fields, with the missing constraints manifesting as varied wrinkles in HPF, including the divergences that plague the framework. Imposing the additional emergent constraints on field fluctuations, as autoregularization does, is expected to preserve the successes of QFT, while ironing out the wrinkles in QFT. We start with a brief overview of autoregularization.

We can partition the set of interacting quantum fields that inhabit spacetime into a "system" comprising the quantum field of interest and a "reservoir" made of all the other quantum fields that coinhabit spacetime. Creation or annihilation of particles of the quantum field can then be viewed, from a thermodynamic perspective, as flow of particles into or out of the system. Since a quantum field can exchange both particles and energy with the reservoir, it can be regarded as a system that is in thermal and diffusive equilibrium with the reservoir.[5] The behavior of a system that is in thermal and diffusive equilibrium with a reservoir is well described by the grand canonical distribution (GCD), derived by J. Willard Gibbs [5].

The hallmark of GCD is that it states that a system's fluctuations are exponentially suppressed. Specifically, GCD states that the probability of a fluctuation that puts the system in a state with energy E and particle number n is proportional to the so-called Gibbs factor $e^{-(E-n\mu)/\tau}$, where τ and μ are the temperature and chemical potential that characterize the thermal and diffusive equilibrium between the system and the reservoir. Such exponential suppression of fluctuations is not predicted by the Lagrangian either in the classical or quantum theory and must be added on top of the Lagrangian formalism to fully describe the system's (quantum field's) behavior.

The above Gibbs factor however is not Lorentz invariant. The exponential suppression of field fluctuations, encoded in the above frame-dependent Gibbs factor, is achieved in autoregularization by scaling the creation and annihilation operators in the free field expansion with Lorentz-invariant "Gibbs factors", which reduce to the form of the frame-dependent Gibbs factors at large E. The details of autoregularization are discussed in [1] and summarized in Appendix A.

In preliminary validation studies of autoregularization, it was used to calculate the amplitudes of several scattering processes, and the results were found to be in good agreement with the experimental data (over a broad range of energy scales from $\lesssim$MeV to $\gtrsim$200 GeV) [1].[6]

Scaling the creation and annihilation operators with Gibbs factors—hereafter called *Gibbs scaling*—has several consequences. First, the exponential suppression of high-energy field fluctuations,[7] due to Gibbs scaling, eliminates divergences in scattering amplitudes at all orders of perturbation theory.[8]

A second consequence of Gibbs scaling is that *ab initio* calculation of the energy density of vacuum fluctuations of the free fields in the Standard Model, using autoregularization, yields a value that is less than the current estimate of cosmic critical density, potentially solving the cosmological constant problem [1].

[3]Infinite number of modes in the case where the system-reservoir complex is a set of interacting quantum fields.

[4]An analog in classical mechanics is that the observed Maxwell-Boltzmann distribution in ideal gas cannot be explained using only Newton's laws of mechanics. To derive Maxwell-Boltzmann distribution, one needs to add an additional statistical law, not contained in Newtonian formalism, that at equilibrium the entropy of the system is maximized.

[5]We assume that the timescale over which equilibrium is restored when the system is perturbed is much smaller than the time resolution of our instruments. Hence, we assume that we observe only the equilibrium or near-equilibrium behavior of the system.

[6]Specifically, the 1-loop corrections to the electron's gyromagnetic ratio and the Lamb shift, calculated using autoregularization, are in good agreement with experimental data. The 1-loop calculation of the QCD coupling constant showed that autoregularization also predicts asymptotic freedom in QCD. The running of the fine structure constant, computed at 1-loop using autoregularization, was shown to be in good agreement with the prediction of cutoff regularization. The results of the tree-level calculations of Compton scattering and pair annihilation were also shown to be in good agreement with experimental data.

[7]Also the exponential suppression of low-energy fluctuations of massless boson fields; see (A3).

[8]As examples, in Appendixes B and E, we show explicitly that the propagator and a 1-loop diagram in φ^4 theory, which are, respectively, quadratically and logarithmically divergent in HPF, are finite in autoregularization.

A third consequence of Gibbs scaling is that it breaks the equivalence between the equal-time commutation relations (ETCR) imposed on the field and its conjugate momentum in HPF and the commutation relations imposed on the creation and annihilation operators. Thus, either ETCR or commutation relations on creation and annihilation operators can be retained, and the other must be abandoned. Autoregularization retains the commutation relations on the creation and annihilation operators and abandons the ETCR on the field and its conjugate momentum. In a sense, the choice represents reversion to the Diracian approach, placing primacy on the individual oscillators, rather than the field, as the dynamical degrees of freedom that are to be quantized. As we show in Sec. III, the choice leads to enormous simplification in the quantization of the gauge field in QED.

Against the above backdrop, in this paper, we establish additional key properties of the autoregularization framework. Specifically, in Sec. I, we present a proof of gauge invariance of the scattering amplitudes in QED within autoregularization. The proof we present is valid for both autoregularization and the HPF and is stronger than the corresponding proof in HPF, in which gauge invariance is established only for sum of Feynman diagrams with a given set of external legs [6,7]. On the other hand, in Sec. I, we prove that every Feynman diagram is individually gauge invariant in autoregularization.

The ETCR that are imposed on a field and its conjugate momentum in HPF severely complicate the gauge field quantization in QED. The complications, which are often imputed to the redundant degrees of freedom in the gauge field rather than to the ETCR, are summarized in Sec. II. The complications have previously led to several nontrivial modifications of the canonical quantization procedure, such as the modification of the Lagrangian, introduction of ghost fields, imposing the gauge condition not as an operator condition but in a weaker form thereby abandoning Maxwell's equations in operator form, partitioning the Fock space by fiat into "physical" and "unphysical" states in covariant gauge, or in Coulomb gauge modifying the ETCR themselves.

In Sec. III, we describe the gauge field quantization in QED in covariant gauge using autoregularization. Since ETCR are not imposed in autoregularization, none of the complications that plague gauge field quantization in HPF arises in autoregularization. The canonical quantization procedure in autoregularization applies without modification to gauge field quantization. The redundant degrees of freedom decouple naturally from the theory. Unlike in HPF, Maxwell's equations follow as Euler-Lagrange equations, in operator form, in covariant gauge in autoregularization. The Fock space has a positive semidefinite metric. The Hamiltonian has a nonnegative expectation value on the entire Fock space, which does not need to be partitioned into "physical" and "unphysical" states. Unlike in HPF, the quantization of gauge field in autoregularization does not require introduction of the unphysical ghost fields. The enormous simplification that results from abandoning ETCR shows that the difficulties previously faced in gauge field quantization are to be imputed to ETCR and not the redundant degrees of freedom in the gauge field, reinforcing Dirac's objection to HPF [3].

Finally, in Sec. IV, we illustrate renormalization based on autoregularization. Unlike in HPF, the perturbative corrections to scattering amplitudes are all finite at every order in autoregularization. Hence, the counterterms and the bare parameters are finite as well, in autoregularization; the counterterms are designed not to cancel divergences but to implement the renormalization prescriptions. We derive the renormalization group equation (RGE). Unlike in HPF, in which RGE expresses independence of the correlation function with respect to the fictitious scale introduced during regularization, the RGE in autoregularization represents the evolution of the correlation function with the kinematic scale of the scattering process.

II. GAUGE INVARIANCE OF THE *S*-MATRIX IN QED

We establish the gauge invariance of the S-matrix amplitude of a general scattering process in QED,

$$e^-_{k_1} + \cdots + e^-_{k_m} + e^+_{q_1} + \cdots + e^+_{q_n} + \gamma_{l_1} + \cdots \gamma_{l_r} \to e^-_{\tilde{k}_1} + \cdots + e^-_{\tilde{k}_{\tilde{m}}} + e^+_{\tilde{q}_1} + \cdots + e^+_{\tilde{q}_{\tilde{n}}} + \gamma_{\tilde{l}_1} + \cdots \gamma_{\tilde{l}_{\tilde{r}}}, \tag{1}$$

in which m electrons, n positrons, and r photons scatter to $\tilde{m}$ electrons $\tilde{n}$ positrons and $\tilde{r}$ photons. The momenta of the particles are shown as subscripts. For brevity, we use the following notation:

$$\begin{gathered} k \coloneqq k_1, \ldots, k_m, \quad q \coloneqq q_1, \ldots, q_n, \quad l \coloneqq l_1, \ldots, l_r, \quad \tilde{k} \coloneqq \tilde{k}_1, \ldots, \tilde{k}_{\tilde{m}}, \quad \tilde{q} \coloneqq \tilde{q}_1, \ldots, \tilde{q}_{\tilde{n}}, \quad \tilde{l} \coloneqq \tilde{l}_1, \ldots, \tilde{l}_{\tilde{r}} \\ \bar{\Psi}(x) \coloneqq \bar{\psi}(x_1)\ldots\bar{\psi}(x_m), \quad \Psi(y) \coloneqq \psi(y_1)\ldots\psi(y_n), \quad x \coloneqq x_1, \ldots, \quad x_m, y \coloneqq y_1, \ldots, y_n \\ \Psi(\tilde{x}) \coloneqq \psi(\tilde{x}_1)\ldots\psi(\tilde{x}_{\tilde{m}}), \quad \bar{\Psi}(\tilde{y}) \coloneqq \bar{\psi}(\tilde{y}_1)\ldots\bar{\psi}(\tilde{y}_{\tilde{n}}), \quad \tilde{x} \coloneqq \tilde{x}_1, \ldots, \tilde{x}_{\tilde{m}}, \quad \tilde{y} \coloneqq \tilde{y}_1, \ldots, \tilde{y}_{\tilde{n}} \\ A_{[\alpha][\tilde{\alpha}]}(z, \tilde{z}) \coloneqq A_{\alpha_1}(z_1)\ldots A_{\alpha_r}(z_r) A_{\tilde{\alpha}_1}(\tilde{z}_1)\ldots A_{\tilde{\alpha}_{\tilde{r}}}(\tilde{z}_{\tilde{r}}), \quad z \coloneqq z_1, \ldots, z_r, \quad \tilde{z} \coloneqq \tilde{z}_1, \ldots, \tilde{z}_{\tilde{r}}, \end{gathered} \tag{2}$$

$$G_{[\alpha][\tilde{\alpha}]}(x,\tilde{x},y,\tilde{y},z,\tilde{z}) = \langle 0|T\Psi(\tilde{x})\Psi(y)\bar{\Psi}(x)\bar{\Psi}(\tilde{y})A_{[\alpha][\tilde{\alpha}]}(z,\tilde{z})|0\rangle. \tag{3}$$

The fields are in Heisenberg representation. The S-matrix amplitude of the above process is

$$\mathcal{S} = \prod_{a=1}^{r}\mathcal{M}_{-}^{\alpha_a}(l_a,s_a;z_a)\prod_{b=1}^{\tilde{r}}\mathcal{M}_{+}^{\tilde{\alpha}_b}(\tilde{l}_b,\tilde{s}_b;\tilde{z}_b)\prod_{i=1}^{\tilde{m}}\mathcal{D}_{+}(\tilde{k}_i,\tilde{w}_i;\tilde{x}_i)\prod_{j=1}^{n}\mathcal{D}_{-}(q_j,h_j;y_j)$$
$$\times G_{[\alpha][\tilde{\alpha}]}(x,\tilde{x},y,\tilde{y},z,\tilde{z})\prod_{c=1}^{m}\overleftarrow{\mathcal{D}}_{-}(k_c,w_c;x_c)\prod_{d=1}^{\tilde{n}}\overleftarrow{\mathcal{D}}_{+}(\tilde{q}_d,\tilde{h}_d;\tilde{y}_d), \tag{4}$$

where $\mathcal{M}_{\pm}^{\mu}, \mathcal{D}_{\pm}, \overleftarrow{\mathcal{D}}_{\pm}$ are the amputation operators,

$$\mathcal{M}_{\pm}^{\alpha}(l,s;z) = \frac{i\eta^{ss'}\epsilon^{\alpha}(l,s')}{g_p(l)\sqrt{Z_p}}\int d^4z e^{\pm ilz}\Box_z,$$
$$\mathcal{D}_{-}(q,h;y) = i\frac{\bar{v}(q,h)}{g_e(q)\sqrt{Z_e}}\int d^4y e^{-iqy}(i\partial\!\!\!/_y - m), \qquad \mathcal{D}_{+}(\tilde{k},\tilde{w};\tilde{x}) = -i\frac{\bar{u}(\tilde{k},\tilde{w})}{g_e(\tilde{k})\sqrt{Z_e}}\int d^4\tilde{x}e^{i\tilde{k}\tilde{x}}(i\partial\!\!\!/_{\tilde{x}} - m),$$
$$\overleftarrow{\mathcal{D}}_{-}(k,w;x) = i\int d^4x(i\overleftarrow{\partial\!\!\!/}_x + m)e^{-ikx}\left[\frac{u(k,w)}{g_e(k)\sqrt{Z_e}}\right],$$
$$\overleftarrow{\mathcal{D}}_{+}(\tilde{q},\tilde{h};\tilde{y}) = -i\int d^4\tilde{y}(i\overleftarrow{\partial\!\!\!/}_{\tilde{y}} + m)e^{i\tilde{q}\tilde{y}}\left[\frac{v(\tilde{q},\tilde{h})}{g_e(\tilde{q})\sqrt{Z_e}}\right]. \tag{5}$$

Z_p, Z_e are the normalization constants for photons and fermions, u and v are the usual Dirac spinors, $s, \tilde{s}, w, \tilde{w}, h, \tilde{h}$ are the polarizations, and g_p and g_e are the Gibbs factors for photons and fermions, respectively.

Before proceeding, we note that the correlation function $G_{[\alpha][\tilde{\alpha}]}(x,\tilde{x},y,\tilde{y},z,\tilde{z})$ is invariant under a global gauge transformation,

$$\psi \to e^{ie\alpha_0}\psi, \qquad \bar{\psi} \to e^{-ie\alpha_0}\bar{\psi} \qquad A_\mu \to A_\mu, \tag{6}$$

where α_0 is a real constant. The global gauge invariance follows by observing that, from Wick's theorem, the perturbative expansion of $G_{[\alpha][\tilde{\alpha}]}$ vanishes unless the number of ψ fields in $G_{[\alpha][\tilde{\alpha}]}$, namely, $\tilde{m}+n$ equals the number of $\bar{\psi}$ fields, namely, $m+\tilde{n}$. Further, every interaction vertex is globally gauge invariant.

An immediate consequence of the global gauge invariance is that in a local gauge transformation,

$$\psi(x) \to e^{ie\epsilon\alpha(x)}\psi(x), \qquad \bar{\psi}(x) \to e^{-ie\epsilon\alpha(x)}\bar{\psi}(x), \qquad A_\mu(x) \to A_\mu - \epsilon\partial_\mu\alpha(x), \tag{7}$$

in which the small parameter ϵ has been included for bookkeeping, we can assume that

$$\alpha(0) = 0. \tag{8}$$

If $\alpha(0) \neq 0$, then we can write

$$e^{ie\epsilon\alpha(x)} = e^{ie\epsilon(\alpha(x)-\alpha(0))}e^{ie\epsilon\alpha(0)} = e^{ie\epsilon\bar{\alpha}(x)}e^{ie\alpha_0}, \qquad \bar{\alpha}(x) \coloneqq \alpha(x) - \alpha(0), \qquad \alpha_0 \coloneqq \epsilon\alpha(0).$$

Thus, the local gauge transformation in (7) can be implemented by first applying the global gauge transformation corresponding to $e^{ie\alpha_0}$, which leaves $G_{[\alpha][\tilde{\alpha}]}$ invariant, followed by a local gauge transformation corresponding to $e^{ie\epsilon\bar{\alpha}(x)}$, where $\bar{\alpha}(0) = 0$.

The perturbative expansion of $\mathcal{S}$ contains connected as well as disconnected Feynman diagrams. Accordingly, we write

$$\mathcal{S} = \mathcal{S}^{(c)} + \mathcal{S}^{(d)}, \tag{9}$$

where $\mathcal{S}^{(c)}$ denotes the sum over all connected Feynman diagrams with $E = m+n+r+\tilde{m}+\tilde{n}+\tilde{r}$ external legs and $\mathcal{S}^{(d)}$ the sum over the remaining (disconnected) diagrams.

$\mathcal{S}^{(c)}$, in turn, can be written as

$$\mathcal{S}^{(c)} = \sum_{D_b} \mathcal{S}_b^{(c)}, \qquad \mathcal{S}_b^{(c)} = (2\pi)^4 \delta^4(p) \mathcal{M}_b^{(c)}, \quad \text{where}$$
$$p = \sum_{j=1}^{r} l_j + \sum_{j=1}^{m} k_j + \sum_{j=1}^{n} q_j - \sum_{j=1}^{\tilde{r}} \tilde{l}_j - \sum_{j=1}^{\tilde{m}} \tilde{k}_j - \sum_{j=1}^{\tilde{n}} \tilde{q}_j, \tag{10}$$

and the sum $\sum_{D_b}$ is over all connected diagrams $D_1, D_2, \ldots$ that have E external legs corresponding to the scattering process (1). The dependence of $\mathcal{M}_b^{(c)}$ on the external momenta $k, q, l, \tilde{k}, \tilde{q}, \tilde{l}$ is not shown explicitly for brevity. In the following argument, we show that each of the $\mathcal{M}_1^{(c)}, \mathcal{M}_2^{(c)}, \ldots$ is separately gauge invariant.

We choose an arbitrary connected diagram, denoted D_b, and establish the gauge invariance of the corresponding $\mathcal{M}_b^{(c)}$. A straightforward calculation shows that $\mathcal{M}_b^{(c)}$, the amputated amplitude of the connected diagram D_b, can be written in the following form:

$$\mathcal{M}_b^{(c)} = \kappa^{[\alpha][\tilde{\alpha}]} \bar{V} \bar{U} \mathcal{A}_{[\alpha][\tilde{\alpha}]}^{(b)}(k, q, l, \tilde{k}, \tilde{q}, \tilde{l}) U V,$$
$$\kappa^{[\alpha][\tilde{\alpha}]} = \prod_{j=1}^{l} \left[\frac{\eta^{s_j s'_j} \epsilon^{\alpha_j}(\vec{l_j}, s'_j) g_p(l_j)}{\sqrt{Z_p}} \right] \prod_{j=1}^{\tilde{l}} \left[\frac{\eta^{\tilde{s}_j \tilde{s}'_j} \epsilon^{\tilde{\alpha}_j}(\vec{\tilde{l}_j}, \tilde{s}'_j) g_p(\tilde{l}_j)}{\sqrt{Z_p}} \right],$$
$$U := \prod_{j=1}^{m} \frac{u(k_j, w_j) g_e(k_j)}{\sqrt{Z_e}}, \quad \bar{U} := \prod_{j=1}^{\tilde{m}} \frac{\bar{u}(\tilde{k}_j, \tilde{w}_j) g_e(\tilde{k}_j)}{\sqrt{Z_e}},$$
$$V := \prod_{j=1}^{\tilde{n}} \frac{v(\tilde{q}_j, \tilde{h}_j) g_e(\tilde{q}_j)}{\sqrt{Z_e}}, \quad \bar{V} := \prod_{j=1}^{n} \frac{\bar{v}(q_j, h_j) g_e(q_j)}{\sqrt{Z_e}}. \tag{11}$$

Using (10), we *define* $\mathcal{M}_b^{(c)}$, the amputated amplitude of the connected diagram D_b, as the coefficient of $(2\pi)^4 \delta^4(p)$ in $\mathcal{S}_b^{(c)}$. The same definition applies to the gauge transform as well. Specifically, if $\bar{\mathcal{S}}_b^{(c)}$ is the gauge transform of $\mathcal{S}_b^{(c)}$, then the gauge-transformed amputated amplitude of D_b, namely, $\bar{\mathcal{M}}_b^{(c)}$ is the coefficient of $(2\pi)^4 \delta^4(p)$ in $\bar{\mathcal{S}}_b^{(c)}$,

$$\bar{\mathcal{S}}_b^{(c)} = (2\pi)^4 \delta^4(p) \bar{\mathcal{M}}_b^{(c)} + \mathcal{B}, \tag{12}$$

where $\mathcal{B}$ does not contain a factor of $\delta^4(p)$ and hence does not contribute to the gauge-transformed amputated amplitude. To prove gauge invariance, we show that $\bar{\mathcal{M}}_b^{(c)} = \mathcal{M}_b^{(c)}$.

Consider the infinitesimal gauge transformation,

$$\psi(x) \to \psi'(x) = (1 + i\epsilon e \alpha(x)) \psi(x),$$
$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \epsilon \partial_\mu \alpha(x), \tag{13}$$

where the small parameter ϵ has been included for bookkeeping.

In the perturbative expansion of $G_{[\alpha][\tilde{\alpha}]}$, let $G_{[\alpha][\tilde{\alpha}]}^{(b)}$ denote the term that corresponds to the connected diagram D_b with E external legs. As the result of the infinitesimal gauge transformation (13),

$$G_{[\alpha][\tilde{\alpha}]}^{(b)} \to \bar{G}_{[\alpha][\tilde{\alpha}]}^{(b)} = G_{[\alpha][\tilde{\alpha}]}^{(b)} + \delta G_{[\alpha][\tilde{\alpha}]}^{(b)},$$
$$\mathcal{S}_b^{(c)} \to \bar{\mathcal{S}}_b^{(c)} = \mathcal{S}_b^{(c)} + \delta \mathcal{S}_b^{(c)}, \tag{14}$$

where $\mathcal{S}_b^{(c)}$ and $\bar{\mathcal{S}}_b^{(c)}$ are obtained by applying the amputation operators, shown in (5), to $G_{[\alpha][\tilde{\alpha}]}^{(b)}$ and $\bar{G}_{[\alpha][\tilde{\alpha}]}^{(b)}$, respectively.

First, since α is a c-number field, and $\langle 0|T(\partial_\mu \alpha) A_\nu|0\rangle = 0$, the photon propagator $\langle 0|T A_\mu A_\nu|0\rangle$ is gauge invariant at $O(\epsilon)$. Therefore, we can replace all transformed photon fields A'_μ in $\bar{G}_{[\alpha][\tilde{\alpha}]}^{(b)}$, at $O(\epsilon)$, with the corresponding untransformed fields A_μ.

Second, we note the $j^\mu(x) = \bar{\psi}(x)\gamma^\mu \psi(x)$ is invariant under the gauge transformation (13) at $O(\epsilon)$. Hence, the transformed fields ψ' and $\bar{\psi}'$ occurring in the transformed interaction Hamiltonian term $(j')^\mu A'_\mu$ within $\bar{G}_{[\alpha][\tilde{\alpha}]}^{(b)}$ can be replaced by the corresponding untransformed fields ψ and $\bar{\psi}$, respectively.

Thus, at $O(\epsilon)$, the only transformed fields left in $\bar{G}_{[\alpha][\tilde{\alpha}]}^{(b)}$ are $\bar{\Psi}'(x), \Psi'(y), \Psi'(\tilde{x}), \bar{\Psi}'(\tilde{y})$, which are the transforms of the fields $\bar{\Psi}(x), \Psi(y), \Psi(\tilde{x}), \bar{\Psi}(\tilde{y})$ defined in (2).

At $O(\epsilon)$, we have

$$\delta G_{[\alpha][\tilde{\alpha}]}^{(b)}(x, \tilde{x}, y, \tilde{y}, z, \tilde{z}) = i\epsilon e \left[\sum_{j=1}^{n} \alpha(y_j) + \sum_{j=1}^{\tilde{m}} \alpha(\tilde{x}_j) - \sum_{j=1}^{m} \alpha(x_j) - \sum_{j=1}^{\tilde{n}} \alpha(\tilde{y}_j) \right] \times G_{[\alpha][\tilde{\alpha}]}^{(b)}(x, \tilde{x}, y, \tilde{y}, z, \tilde{z}). \tag{15}$$

$\delta \mathcal{S}_b^{(c)}$, shown in (14), is obtained by the action of the amputation operators, shown in (5), on $\delta G_{[\alpha][\tilde{\alpha}]}^{(b)}$ above.

Consider the term $\alpha(y_1) G_{[\alpha][\tilde{\alpha}]}^{(b)}(x, \tilde{x}, y, \tilde{y}, z, \tilde{z})$. Since α is a real-valued function, it can be expanded as

$$\alpha(y_1) = \int \frac{d^4 s}{(2\pi)^4} \{\tilde{\alpha}(s) e^{-isy_1} + \tilde{\alpha}^*(s) e^{isy_1}\}. \tag{16}$$

The action of the amputation operators, shown in (5), on $\alpha(y_1) G_{[\alpha][\tilde{\alpha}]}^{(b)}$ yields, after a straightforward calculation, $\delta \mathcal{S}_{b;q_1}^{(c)}$ given by

$$\delta\mathcal{S}^{(c)}_{b;q_1} = \kappa^{[\alpha][\tilde{\alpha}]}\bar{V}\,\bar{U}[-(\not{q}_1 + m)]\left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right]\bar{\mathcal{A}}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q', l, \tilde{k}, \tilde{q}, \tilde{l})\{\tilde{\alpha}(-p) + \tilde{\alpha}^*(p)\}UV, \tag{17}$$

where

$$\bar{\mathcal{A}}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q', l, \tilde{k}, \tilde{q}, \tilde{l}) = \left[\frac{1}{g_e(q_1)^2}\right]\mathcal{A}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q', l, \tilde{k}, \tilde{q}, \tilde{l}) \qquad q' \coloneqq q_1 - p, q_2, \ldots q_n. \tag{18}$$

Since $\mathcal{S}^{(c)}_b$ in (10) contains a factor of $\delta^4(p)$, we can—treating $k, q_2, \ldots, q_n, l, \tilde{k}, \tilde{q}, \tilde{l}$ as the independent variables—rewrite $\mathcal{A}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q, l, \tilde{k}, \tilde{q}, \tilde{l})$ in (11) in terms of q' as

$$\mathcal{M}^{(c)}_b = \kappa^{[\alpha][\tilde{\alpha}]}\bar{V}\,\bar{U}\,\mathcal{A}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q', l, \tilde{k}, \tilde{q}, \tilde{l})UV, \tag{19}$$

where, as defined above, $q' = q_1 - p, q_2, \ldots, q_n$. Using (17) and (19), we have

$$\begin{aligned}\mathcal{S}^{(c)}_b + \delta\mathcal{S}^{(c)}_{b;q_1} &= \kappa^{[\alpha][\tilde{\alpha}]}\bar{V}\,\bar{U}[(2\pi)^4\delta^4(p)g_e(q_1)^2 - \Delta(p, q_1)]\bar{\mathcal{A}}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q', l, \tilde{k}, \tilde{q}, \tilde{l})UV,\\ \Delta(p, q_1) &= (\not{q}_1 + m)\left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right]\{\tilde{\alpha}(-p) + \tilde{\alpha}^*(p)\}.\end{aligned} \tag{20}$$

Since $\bar{\mathcal{M}}^{(c)}_b$ is the coefficient of the factor $(2\pi)^4\delta^4(p)$ in $\mathcal{S}^{(c)}_b + \delta\mathcal{S}^{(c)}_{b;q_1}$, we need to extract the component of $\delta^4(p)$ in $\Delta(p, q_1)$. We do so in Appendix C, in which we have shown that

$$[\tilde{\alpha}(-p) + \tilde{\alpha}^*(p)]\left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right] = \alpha(0)\left[\frac{g_e(q_1)^2}{-\not{q}_1 - m + i\epsilon}\right](2\pi)^4\delta^4(p) + \Omega(p, q_1),$$

where $\Omega(p, q_1)$ has terms that contain derivatives of $\delta^4(p)$ and $\delta^4(p - q_1)$ and no factor of $\delta^4(p)$. Therefore, from (12), we have

$$\bar{\mathcal{M}}^{(c)}_b = \kappa^{[\alpha][\tilde{\alpha}]}\bar{V}\,\bar{U}\left[g_e(q_1)^2 - (\not{q}_1 + m)\alpha(0)\left[\frac{g_e(q_1)^2}{-\not{q}_1 - m + i\epsilon}\right]\right]\bar{\mathcal{A}}^{(b)}_{[\alpha][\tilde{\alpha}]}(k, q', l, \tilde{k}, \tilde{q}, \tilde{l})UV. \tag{21}$$

But from (8), $\alpha(0) = 0$, and the second term within the bracket, which contains the factor of $\alpha(0)$, vanishes at every $\epsilon \neq 0$. Using (21), (18), and (19), we have

$$\bar{\mathcal{M}}^{(c)}_b = \mathcal{M}^{(c)}_b, \tag{22}$$

showing that the amplitude of the diagram D_b is gauge invariant.

Although the above argument focused on the connected diagram D_b, it is important to note that the argument did not rely on any special feature of D_b other than that it is a connected diagram. Hence, the above argument can be applied to any connected diagram and thus establishes the gauge invariance of the amputated amplitude of every connected diagram in QED. In particular, it can be applied to diagrams $D_1, D_2, \ldots$ to establish the gauge invariance of $\mathcal{M}^{(c)}_1, \mathcal{M}^{(c)}_2, \ldots$ and hence the gauge invariance of $\mathcal{S}^{(c)}$.

Next, we consider the $\mathcal{S}^{(d)}$ in (9). $\mathcal{S}^{(d)}$ can be written as the sum of terms, where each term is a product of factors with each factor corresponding to a connected diagram. The above argument establishes the gauge invariance of each connected diagram and thus the gauge invariance of the product of connected diagrams. It follows that $\mathcal{S}^{(d)}$ is gauge invariant as well, completing the proof of gauge invariance of the S-matrix in QED.

III. CONSEQUENCES OF EQUAL-TIME COMMUTATION RELATIONS

As is well-known, imposing equal-time commutation relations (ETCR) on the gauge field and its conjugate in QED forces the quantization procedure for the gauge field to deviate significantly from the standard field quantization procedure. The complications that arise in the quantization of the gauge field and the several modifications that are made to address the complications are imputed not to the

ETCR but to the unphysical degrees of freedom in the gauge field. The ETCR on a field and its conjugate were first postulated by Heisenberg and Pauli [2].

In this section, we summarize the modifications that are made in the standard quantization procedure to accommodate the ETCR in two popular gauges—the Coulomb gauge and the Lorenz (covariant) gauge.

Since the momentum conjugate to A^0, namely, π^0 vanishes, A^0 is a nondynamical field. The Euler-Lagrange equation for A^0, however, imposes a condition on the gauge field, namely, the Gauss' law,

$$\nabla \cdot \vec{E} = 0, \tag{23}$$

which must be satisfied by the quantized gauge field.

In Lorenz (covariant) gauge, manifest Lorentz covariance is maintained by imposing ETCR on all four of the field components as

$$[A_\mu(\vec{x}, t), \pi_\nu(\vec{y}, t)] = i\eta_{\mu\nu}\delta^3(\vec{x} - \vec{y}). \tag{24}$$

The ETCR immediately leads to two contradictions. First, the ETCR for A^0 is not satisfied since $\pi^0 = 0$. Second, the ETCR for $\vec{A}$ contradicts Gauss' law (23). To fix the first contradiction, the Lagrangian is modified as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \to \mathcal{L}' = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{25}$$

While the modified Lagrangian repairs the problem of vanishing π^0, the theory described by $\mathcal{L}'$ is different from the original theory for the gauge field. For example, Maxwell's equations do not arise as Euler-Lagrange equations from $\mathcal{L}'$. To recover the original theory, one imposes the Lorenz (covariant) gauge condition,

$$\partial_\mu A^\mu = 0. \tag{26}$$

However, imposing (26) as an operator condition reverses the modification shown in (25). Therefore, a second *ad hoc* patch is applied by insisting that (26) holds not as an operator condition but in a weaker form as described later. Since (26) is not imposed in operator form, the Euler-Lagrange equations derived from $\mathcal{L}'$ are not Maxwell's equations.

The modification of the Lagrangian, shown in (25), repairs the problem in the ETCR of A^0, but creates a new problem that did not exist in $\mathcal{L}$—the new Lagrangian $\mathcal{L}'$ is not gauge invariant.

The gauge dependence of $\mathcal{L}'$ is repaired by applying a third *ad hoc* patch that involves introducing an unphysical ghost field. So $\mathcal{L}'$ is modified further by adding a term containing ghost field to get $\mathcal{L}''$. The ghost field is then stipulated to transform, under a gauge transformation, in exactly the right manner to recover gauge invariance of $\mathcal{L}''$ [8].

The ETCR on all the four components of A^μ together with the expansion of the free field as

$$A^\mu = \int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))}\sum_{s=0}^{3}\epsilon^\mu(\vec{k}, s) \times \{\hat{\mathbf{a}}(\vec{k}, s)e^{-ikx} + \hat{\mathbf{a}}^\dagger(\vec{k}, s)e^{ikx}\}$$

constrains the creation and annihilation operators to obey the commutation relations,

$$[\hat{\mathbf{a}}(\vec{k}, s), \hat{\mathbf{a}}^\dagger(\vec{k}', s')] = -\eta^{ss'}(2\omega(\vec{k}))(2\pi)^3\delta^3(\vec{k} - \vec{k}'). \tag{27}$$

The negative sign in (27) gives rise to another problem—it makes the metric of the Fock space and the expectation value of the Hamiltonian operator indefinite (not positive semidefinite) [9].

The negative expectation value of the Hamiltonian, and the restriction that the gauge condition (26) cannot be imposed as an operator condition, are addressed with the patch proposed by Gupta and Bleuler [10]. By fiat, a state in the Fock space is declared as a "physical" state if it satisfies the condition

$$\partial_\mu A^{(+)\mu}|\psi\rangle = 0, \tag{28}$$

as suggested by Gupta and Bleuler [10]. The $\partial_\mu A^{(+)\mu}$ represents the positive energy component. States in the Fock space that do not satisfy (28) are declared as "unphysical" states and, by fiat, excluded. The constraint (28) eliminates the negative expectation value of the Hamiltonian for the "physical" states and implements the gauge condition (26) in a weaker form as

$$\langle\psi|\partial_\mu A^\mu|\psi\rangle = 0, \qquad \psi\text{: physical state}. \tag{29}$$

Implementing the gauge condition (26) in the weaker form as (29), and not as an operator equation, implies that the operator $\mathcal{L}'$ remains different from the original Lagrangian operator $\mathcal{L}$ of the original theory, and Maxwell's equations do not emerge as Euler-Lagrange equations from $\mathcal{L}'$.

In Coulomb gauge, manifest Lorentz covariance is sacrificed by stipulating

$$\nabla \cdot \vec{A} = 0, \tag{30}$$

and the ETCR are imposed only on the three dynamical fields in $\vec{A}$. However, since the momentum conjugate to A^i is $\vec{\pi} = -E^i$, the ETCR,

$$[A^i(\vec{x}, t), \pi^j(\vec{y}, t)] = i\delta^{ij}\delta^3(\vec{x} - \vec{y}), \tag{31}$$

contradicts both Gauss' law (23) as well as the gauge condition (30). For example, when the operator $\partial/\partial y^j$ acts on (31), the left-hand side must vanish due to Gauss' law, but the right-hand side $i\partial/\partial y^j \delta^3(\vec{x}-\vec{y})$ does not. The stress between ETCR (31) on the one hand and Gauss' law (23) and gauge condition (30) on the other is resolved by either abandoning (31) or by abandoning (23) and (30) as operator constraints.

In one *ad hoc* modification, the δ function on the right-hand side of (31) is replaced with the so-called transverse delta function $\delta^3_{\rm tr}(\vec{x}-\vec{y})$, which is constructed to satisfy $\partial/\partial y^j \delta^3_{\rm tr}(\vec{x}-\vec{y}) = \partial/\partial x^j \delta^3_{\rm tr}(\vec{x}-\vec{y}) = 0$. That is, the ETCR (31) is replaced by the modified ETCR,

$$[A^i(\vec{x},t), \pi^j(\vec{y},t)] = i\delta^{ij}\delta^3_{\rm tr}(\vec{x}-\vec{y}), \tag{32}$$

which conforms with Gauss' law and the gauge condition.

The alternative is to make a different *ad hoc* modification by abandoning Gauss' law and gauge condition as operator conditions, implementing them instead in a weaker form [9]. The Fock space would then need to be partitioned into the set of "physical" states and "unphysical" states, along the lines followed in Gupta-Bleuler formalism.

In summary, imposing ETCR on the gauge field and its conjugate leads to several downstream consequences such as violation of Gauss' law, indefinite metric on the Fock space, and negative expectation value for the Hamiltonian. To preserve the ETCR, the above deficiencies are repaired with a sequence of *ad hoc* patches, such as modification of the Maxwell Lagrangian, abandoning Maxwell's equations in the operator form, introduction of "unphysical" ghost field, stipulating, by fiat, certain states in the Fock space as "unphysical", or in a somewhat extreme step changing the canonical ETCR itself as in (32).

IV. QUANTIZATION OF THE QED GAUGE FIELD IN AUTOREGULARIZATION

As discussed in the previous section, imposing ETCR on the gauge field and its conjugate in QED hinders the quantization of the gauge field. Several *ad hoc* modifications of the canonical quantization procedure, and even the ETCR itself, are used to overcome the hurdles created by the ETCR. The hurdles are often imputed to the redundant degrees of freedom in the gauge field [9] rather than their actual source—the ETCR.

Autoregularization does not impose ETCR on the field and its conjugate and only postulates commutation relations on the creation and annihilation operators. Therefore, as we show below, the gauge field can be quantized in autoregularization without any modification of the canonical quantization procedure, despite the redundant degrees of freedom in the gauge field. We present gauge field quantization based on autoregularization in the covariant gauge. As we show, Gauss' law, Maxwell's equations, and the gauge condition hold in the operator form. Gauge invariance is preserved without having to introduce the unphysical ghost field. The Fock space and the expectation value of the Hamiltonian are not indefinite. All the states in the Fock space are physical states. The redundant degrees of freedom decouple naturally from the theory.

The free gauge field is expanded in autoregularization as

$$A^\mu(x) = \int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))} g_p(\vec{k}) \left\{ \sum_{s=0}^{3} \epsilon^\mu(\vec{k},s)[\hat{\mathbf{a}}(\vec{k},s)e^{-i\underline{k}x} + \hat{\mathbf{a}}^\dagger(\vec{k},s)e^{i\underline{k}x}] \right\}, \tag{33}$$

with the creation and annihilation operators satisfying the canonical commutation relations stipulated for generic boson fields,

$$[\hat{\mathbf{a}}(\vec{k},s), \hat{\mathbf{a}}^\dagger(\vec{k}',s')] = \delta_{s,s'}(2\omega(\vec{k}))(2\pi)^3\delta^3(\vec{k}-\vec{k}'), \quad s,s' = 0,1,2,3. \tag{34}$$

The polarization vectors satisfy Lorentz-invariant normalization,

$$\epsilon^\mu(\vec{k},s)\epsilon_\mu(\vec{k},s') = \eta^{ss'}. \tag{35}$$

We choose the polarization vectors as follows:

$$\epsilon^0(\vec{k},0) = (1,0,0,0), \qquad \epsilon^1(\vec{k},1) = (0,\hat{e}_1), \\ \epsilon^2(\vec{k},2) = (0,\hat{e}_2), \qquad \epsilon^3(\vec{k},3) = (0,\hat{k}), \qquad \hat{e}_1 \times \hat{e}_2 = \hat{k}. \tag{36}$$

As mentioned earlier we focus on Lorenz (covariant) gauge. In autoregularization, ETCR are not imposed on the gauge field and its conjugate. As a result, we do not have to, and do not, modify the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{37}$$

by adding a gauge-fixing term, as is done in HPF. Consequently, Maxwell's equations (Euler-Lagrange equations) are satisfied in operator form. Since the Lagrangian (37) is unmodified, it remains gauge invariant. Therefore, in autoregularization, one does not need to introduce ghost fields to preserve gauge invariance of the lagrangian. Further, there is no hurdle in autoregularization to imposing the Lorenz (covariant) gauge condition (26) in operator form, obviating the contrived partition of the Fock space into "physical" and "unphysical" states as in the Gupta-Bleuler formalism.

Using (33) and (36), the gauge condition (26) yields

$$\int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))} g_p(\vec{k})\omega(\vec{k})[-ie^{-i\underline{k}x}\{\hat{\mathbf{a}}(\vec{k},0) - \hat{\mathbf{a}}(\vec{k},3)\} + ie^{i\underline{k}x}\{\hat{\mathbf{a}}^\dagger(\vec{k},0) - \hat{\mathbf{a}}^\dagger(\vec{k},3)\}] = 0. \tag{38}$$

Applying the operator $\int d\vec{x} e^{i\underline{q}x} \overleftrightarrow{\partial}_0$ to the left- and right-hand sides of (38), where $\underline{q}^2 = 0$, we obtain

$$g_p(\vec{q})\omega(\vec{q})\{\hat{\mathbf{a}}(\vec{q},0) - \hat{\mathbf{a}}(\vec{q},3)\} = 0,$$

from which we conclude[9] that

$$\hat{\mathbf{a}}(\vec{q},0) = \hat{\mathbf{a}}(\vec{q},3). \tag{39}$$

Similarly, applying $\int d\vec{x} e^{-i\underline{q}x} \overleftrightarrow{\partial}_0$ to the left- and right-hand sides of (38) shows that

$$\hat{\mathbf{a}}^\dagger(\vec{q},0) = \hat{\mathbf{a}}^\dagger(\vec{q},3). \tag{40}$$

An immediate consequence of (39), (40), and (34) is that both $\hat{\mathbf{a}}^\dagger(\vec{q},0)$ and $\hat{\mathbf{a}}^\dagger(\vec{q},3)$ create states of zero norm when they act on vacuum. For example,

$$\begin{aligned}\langle 0|\hat{\mathbf{a}}(\vec{q},0)\hat{\mathbf{a}}^\dagger(\vec{q},0)|0\rangle &= \langle 0|\hat{\mathbf{a}}(\vec{q},3)\hat{\mathbf{a}}^\dagger(\vec{q},0)|0\rangle \\ &= \langle 0|\hat{\mathbf{a}}^\dagger(\vec{q},0)\hat{\mathbf{a}}(\vec{q},3)|0\rangle = 0.\end{aligned}$$

A similar argument shows that states created by $\hat{\mathbf{a}}^\dagger(\vec{q},3)$ also have zero norm. But the only vector in the Hilbert space that has zero norm is the zero vector [11]. Since the space of single particle states form a Hilbert space, we conclude that

$$\hat{\mathbf{a}}^\dagger(\vec{q},0)|0\rangle = \hat{\mathbf{a}}^\dagger(\vec{q},3)|0\rangle = 0. \tag{41}$$

That is, the gauge condition leads to the remarkable conclusion that not only the annihilation operators but also *the creation operators of the scalar and longitudinal modes annihilate vacuum!* The creation operators $\hat{\mathbf{a}}^\dagger(\vec{q},0)$ and $\hat{\mathbf{a}}^\dagger(\vec{q},3)$ do not produce single particle states, and thus, the redundant degrees of freedom naturally decouple from the theory.

From (39), (40), and (33), it follows that

$$\begin{aligned}\vec{E} &= -\nabla A^0 - \partial_0 \vec{A} \\ &= i\int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))}\omega(\vec{k})g_p(k)\left[\sum_{s=1}^{2}\vec{\epsilon}(\vec{k},s)\hat{\mathbf{a}}(\vec{k},s)e^{-ikx}\right. \\ &\quad \left. - \sum_{s=1}^{2}\vec{\epsilon}(\vec{k},s)\hat{\mathbf{a}}^\dagger(\vec{k},s)e^{ikx}\right], \\ \vec{B} &= \nabla\times\vec{A} \\ &= i\int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))}\omega(\vec{k})g_p(k) \\ &\quad \times\{[\vec{\epsilon}(\vec{k},2)\hat{\mathbf{a}}(\vec{k},1) - \vec{\epsilon}(\vec{k},1)\hat{\mathbf{a}}(\vec{k},2)]e^{-ikx} \\ &\quad - [\vec{\epsilon}(\vec{k},2)\hat{\mathbf{a}}^\dagger(\vec{k},1) - \vec{\epsilon}(\vec{k},1)\hat{\mathbf{a}}^\dagger(\vec{k},2)]e^{ikx}\}.\end{aligned} \tag{42}$$

The scalar and longitudinal creation and annihilation operators do not appear in $\vec{E}$ and $\vec{B}$. Since $\vec{\epsilon}(\vec{k},s)\cdot\vec{k} = 0$ for $s = 1, 2$, it follows from (42) that Gauss' law $\nabla\cdot\vec{E} = 0$ is satisfied in the operator form.

Using (42), the Hamiltonian [[12], Eq. (5.36)] can be written as

$$\begin{aligned}H &= \frac{1}{2}\int d^3x\{\vec{E}^2 + \vec{B}^2\} \\ &= \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))}\omega(\vec{k})g_p^2(k)\sum_{s=1}^{2}\{\hat{\mathbf{a}}(\vec{k},s)\hat{\mathbf{a}}^\dagger(\vec{k},s) \\ &\quad + \hat{\mathbf{a}}^\dagger(\vec{k},s)\hat{\mathbf{a}}(\vec{k},s)\}.\end{aligned}$$

Using the occupation number basis of the Fock space, formed by the eigenvectors of the number operator of the form [13]

$$|n_1, n_2, \ldots\rangle \coloneqq \prod_i \prod_{j=0}^{3} \frac{1}{(n_i!)^{1/2}}(a_i^\dagger)^{n_i}|0\rangle, \tag{43}$$

where the index i labels the momentum eigenstates[10] in the single-particle Hilbert space, we see that the Hamiltonian is positive semidefinite.

Using the occupation number basis of the Fock space and the commutation relations (34), one can also verify that all the states in the Fock space have non-negative norm.

In summary, we have shown that the gauge field in QED can be quantized in Lorenz (covariant) gauge with a straightforward application of the canonical quantization procedure in autoregularization. In contrast, the gauge field quantization in the same gauge in HPF (Heisenberg-Pauli framework) is a relatively convoluted procedure, as

[9] $g_p(\vec{q}) \neq 0$, but $\omega(\vec{q}) = 0$ for $\vec{q} = 0$. So strictly, we can conclude (26) only for $\vec{q} \neq 0$. Stipulating that (26) holds even when $\vec{q} = 0$ is not inconsistent with the gauge condition (26), and hence, we take it to hold for $\vec{q} = 0$ as well.

[10] Counting single-particle states $\hat{\mathbf{a}}^\dagger(\vec{k},s)|0\rangle$, $s = 0, 1, 2, 3$ as four different momentum eigenstates.

summarized in Sec. II. Specifically, Maxwell's equations are satisfied as Euler-Lagrange equations in operator form in autoregularization but are not satisfied in HPF. Second, the gauge condition (26) can be applied in operator form in autoregularization but only in a weaker form (29) in HPF. Third, the unphysical modes—the scalar and longitudinal modes—decouple naturally from the theory in autoregularization, as shown in (41). In contrast, in HPF, the Fock space is partitioned, by fiat, into "physical" and "unphysical" states to force the decoupling of the unphysical modes. Fourth, the Lagrangian remains gauge invariant in autoregularization obviating the need to introduce the ghost field; in contrast, the loss of gauge invariance due to the addition of a gauge-fixing term in HPF necessitates the introduction of the contrived ghost field in HPF. Fifth, the Fock space has no negative-norm states in autoregularization, whereas the single-particle states with scalar photons have negative norm in HPF. Finally, the expectation value of the Hamiltonian is naturally positive semidefinite in autoregularization. In contrast, curing the indefiniteness of the expectation value of the Hamiltonian in HPF hinges on the imposition of the Gupta-Bleuler condition (28) by fiat.

The differences between autoregularization and HPF can be imputed to the ETCR, which are imposed in HPF and omitted in autoregularization. In a sense, it can be argued that the convoluted modifications in the quantization procedure in HPF, necessitated by ETCR, reinforce Dirac's objection to the Heisenberg-Pauli framework.

V. RENORMALIZATION USING AUTOREGULRIZATION

In this section, we describe renormalization based on autoregularization. We also derive the renormalization group equation (RGE) based on autoregularization. The derivation is markedly different from the corresponding derivation in other regularization schemes, because the perturbative corrections in autoregularization are finite at all orders. Consequently, in renormalization based on autoregularization, the counterterms are designed not to cancel divergences but to implement renormalization prescriptions, and all the counterterms are finite as well. Also, unlike the other regularization schemes, which introduce a fictitious energy scale, the energy scale used in autoregularization has a transparent physical meaning—it is the kinematic scale of the scattering process of interest. Thus, the RGE depends on the scattering process under consideration. For these reasons, the renormalization procedure and the derivation of the RGE rely on different arguments than those used by other regularization procedures.

We present the details of the renormalization procedure and the derivation of the RGE using the φ^4 theory. Similar arguments can be used for other Lagrangians. The discussion is organized as follows. In Sec. V A, we describe the renormalization of φ^4 theory up to 1-loop to illustrate the renormalization procedure based on autoregularization. In Sec. V B, we derive the renormalization group equation based on autoregularization.

A. Renormalization of φ^4 theory at 1-loop

We use on-shell renormalization. That is, the renormalization (subtraction) prescriptions for the parameters are derived from physical measurements of the parameters.

Consider the Lagrangian of the φ^4 theory,

$$\mathcal{L}_p = \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m_p^2\varphi^2 - \frac{\lambda_p}{4!}\varphi^4, \tag{44}$$

where m_p is the physical self-energy (mass) of a particle measured in its rest frame. A particle at rest can be thought of as scattering from an on-shell in-state to an on-shell out-state, with the energy scale of the scattering process given by $\tau = m_p$.

λ_p represents the measured physical strength of the coupling in a 2-particle scattering process in which two incoming on-shell particles, with vanishing 3-momenta, scatter to two outgoing on-shell particles with vanishing 3-momenta, in the center-of-momentum frame of the scattering process. The energy scale of such a scattering process is $\tau = 2m_p$.

As is well-known, the full propagator in momentum space can be written as

$$G_f^{(2)}(k) = G_0^{(2)}(k) + G_0^{(2)}(k)\sum_{n=1}^{\infty}(\Sigma(k)G_0^{(2)}(k))^n,$$

where $G_0^{(2)}(k)$ is the propagator at $O(\lambda_p^0)$, and $\Sigma(k)$ represents the sum of amputated, 1-particle irreducible diagrams. Recalling that

$$G_0^{(2)}(k) = \frac{ig_\varphi^2(k; m_p, \tau)}{k^2 - m_p^2 + i\epsilon}, \tag{45}$$

we can formally sum the series to obtain

$$G_f^{(2)}(k) = \frac{ig_\varphi^2(k; m_p, \tau)}{k^2 - m_p^2 + i\epsilon - ig_\varphi^2(k; m_p, \tau)\Sigma(k)}, \tag{46}$$

where g_φ is the Gibbs factor for a massive scalar boson. Representing Σ at $O(\lambda_p)$ as Σ_1, we have

$$\Sigma_1(k; \lambda_p, m_p, \tau) = \frac{\lambda_p}{2}\int\frac{d^4q}{(2\pi)^4}\frac{g_\varphi^2(q; m_p, \tau)}{q^2 - m_p^2 + i\epsilon}.$$

We note that the correction Σ_1 depends on the scale τ. Second, as we have showed in Corollary 1 in Appendix B, Σ_1 is finite in autoregularization, unlike in the other regularization schemes in which it diverges quadratically.

Thus the purpose of the counterterms in renormalization based on autoregularization is *not to cancel divergences* but to impose the renormalization prescription at every order of perturbation theory. The two renormalization prescriptions we stipulate are as follows:

RP_1: The full propagator $G_f^{(2)}(k)$ of a particle at rest $[k=(m_p,0,0,0)]$ must coincide with the propagator of a free field of mass m_p, namely, $G_0^{(2)}(k)$. That is,

$$G_f^{(2)}(k;m_p,\tau)|_{\tau=m_p}=G_0^{(2)}(k;m_p,\tau)|_{\tau=m_p},\qquad k=(m_p,0,0,0). \tag{47}$$

RP_2: The renormalized coupling constant, measured in a scattering process in which two incoming particles at rest scatter to two outgoing particles at rest, must be λ_p.

To implement the first prescription, we add a counterterm,

$$\delta\mathcal{L}_1=-\frac{1}{2}\lambda_p\Delta_1\varphi^2, \tag{48}$$

which modifies the propagator (46) at $O(\lambda_p)$ as

$$G_f^{(2)}(k)=\frac{ig_\varphi^2(k;m_p,\tau)}{k^2-m_p^2+i\epsilon-ig_\varphi^2(k;m_p,\tau)[\Sigma_1-i\lambda_p\Delta_1]}.$$

Setting $\tilde{k}=(m_p,0,0,0)$, the first prescription RP_1 is

$$[\Sigma_1-i\lambda_p\Delta_1]|_{k=\tilde{k}}=0,$$

which yields

$$\Delta_1(\tau)=\left(\frac{i}{2}\right)\int\frac{d^4q}{(2\pi)^4}\frac{g_\varphi^2(q;m_p,\tau)}{q^2-m_p^2+i\epsilon}. \tag{49}$$

The prescription (49) is manifestly Lorentz invariant owing to the Lorentz invariance of the Gibbs factor. From Corollary 1 (Appendix B) we also note that $\Delta_1(\tau)$ is finite, unlike in other regularization schemes.

Next, consider the amputated 4-point function $G^{(4)}(k_1,k_2,k_3,k_4)$ defined as

$$(2\pi)^4\delta^4\left(\sum_{j=1}^4k_j\right)G^{(4)}(k_1,k_2,k_3,k_4)=\int\prod_{j=1}^4\frac{d^4x_je^{ik_jx_j}(\Box_{x_j}+m_p^2)}{ig_\varphi^2(k_j;m_p,\tau)}\langle 0|T\{\varphi(x_1)\ldots\varphi(x_4)\}|0\rangle,$$

where φ is in Heisenberg representation.

Adding terms up to $O(\lambda_p^2)$, a simple calculation shows

$$\begin{aligned}G_2^{(4)}(k_1,k_2,k_3,k_4)=-i\lambda_p+(-i\lambda_p)^2\int\frac{d^4p}{(2\pi)^4}[&G_0^{(2)}(p)G_0^{(2)}(k_1+k_2-p)\\&+G_0^{(2)}(p)G_0^{(2)}(k_1+k_3-p)+G_0^{(2)}(p)G_0^{(2)}(k_1+k_4-p)].\end{aligned}$$

To impose the second renormalization prescription, we add the counterterm

$$\delta\mathcal{L}_2=-\lambda_p^2\left(\frac{\Delta_2}{4!}\right)\varphi^4, \tag{50}$$

which modifies the $G_2^{(4)}\to\tilde{G}_2^{(4)}=G_2^{(4)}-i\lambda_p^2\Delta_2$. That is,

$$\begin{aligned}\tilde{G}_2^{(4)}(k_1,k_2,k_3,k_4)=-i\lambda_p^2\Delta_2-i\lambda_p+(-i\lambda_p)^2\int\frac{d^4p}{(2\pi)^4}[&G_0^{(2)}(p)G_0^{(2)}(k_1+k_2-p)\\&+G_0^{(2)}(p)G_0^{(2)}(k_1+k_3-p)+G_0^{(2)}(p)G_0^{(2)}(k_1+k_4-p)].\end{aligned}$$

The second renormalization prescription stipulates that for $k_1=k_2=k_3=k_4=(m_p,0,0,0)=:q$,

$$\tilde{G}_2^{(4)}(q,q,q,q)=-i\lambda_p.$$

That is,

$$\Delta_2(\tau)=3i\int\frac{d^4p}{(2\pi)^4}\left[\frac{ig_\varphi^2(p;m_p,\tau)}{p^2-m_p^2+i\epsilon}\right]\left[\frac{ig_\varphi^2(2q-p;m_p,\tau)}{(2q-p)^2-m_p^2+i\epsilon}\right]. \tag{51}$$

From Corollary 3 (Appendix E), it follows that $\Delta_2(\tau)$ is finite. If the Gibbs factors are set to unity, then Δ_2 diverges logarithmically.

The above sample calculations illustrate the renormalization procedure based on autoregularization. As the calculations illustrate, the perturbative corrections and the counterterms are finite. Next, we turn to the derivation of the renormalization group equation.

B. Renormalization group equation

In this section, we derive the renormalization group equation based on autoregularization. We note that the bare Lagrangian can be written in terms of the bare parameters and fields as

$$\mathcal{L}_b = \mathcal{L}_b^{(b)} = \frac{1}{2}\partial_\mu\varphi_b\partial^\mu\varphi_b - \frac{1}{2}m_b^2\varphi_b^2 - \frac{\lambda_b}{4!}\varphi_b^4. \quad (52)$$

The bare Lagrangian can also be written in terms of physical parameters and fields using the multiplicative renormalization constants $Z_\varphi(\tau;\lambda_p,m_p), Z_m(\tau;\lambda_p,m_p)$, and $Z_\lambda(\tau;\lambda_p,m_p)$,

$$m_b = Z_m^{1/2}Z_\varphi^{-1/2}m_p, \qquad \varphi_b = Z_\varphi^{1/2}\varphi_p \qquad \lambda_b = Z_\lambda Z_\varphi^{-2}\lambda_p \quad (53)$$

as

$$\mathcal{L}_b = \mathcal{L}_b^{(p)} = \frac{1}{2}Z_\varphi\partial_\mu\varphi_p\partial^\mu\varphi_p - \frac{1}{2}Z_m m_p^2\varphi_p^2 - Z_\lambda\frac{\lambda_p}{4!}\varphi_p^4. \quad (54)$$

The renormalization constants can be calculated using the counterterms. For example, at 1-loop, from (48) and (51), we have

$$Z_m = 1 + \frac{\lambda_p\Delta_1}{m_p^2},$$

and from (50) and (51), we have

$$Z_\lambda = 1 + \lambda_p\Delta_2.$$

The wave function renormalization constant Z_φ does not receive a correction at 1-loop—the order to which we have shown renormalization calculation in Sec. VA. However, Z_φ does receive a correction at 2-loop; see [9]. The superscripts b and p indicate whether $\mathcal{L}_b$ is written in terms of the bare or physical quantities.

We define the n-point function $G^{(n)}(k_1,\ldots,k_n)$ in momentum space as

$$(2\pi)^4\delta^4\left(\sum_{i=1}^n k_i\right)G_p^{(n)}(k_1,\ldots,k_n;\lambda_p,m_p,\tau) = \int\prod_{j=1}^n d^4x_j e^{ik_jx_j}\langle 0|T\{\varphi_p(x_1)\ldots\varphi_p(x_n)\}|0\rangle, \quad (55)$$

where φ_p is in the Heisenberg representation. The parameters λ_p, m_p are shown explicitly to indicate $\langle 0|T\varphi_p(x_1)\ldots\varphi_p(x_n)0\rangle$ is evaluated in a perturbative expansion using the Lagrangian $\mathcal{L}_b^{(p)}$ written in terms of the physical parameters as shown in (54).

Similarly,

$$(2\pi)^4\delta^4\left(\sum_{i=1}^n k_i\right)G_b^{(n)}(k_1,\ldots,k_n;\lambda_b,m_b,\tau) = \int\prod_{j=1}^n d^4x_j e^{ik_jx_j}\langle 0|T\{\varphi_b(x_1)\ldots\varphi_b(x_n)\}|0\rangle, \quad (56)$$

where the parameters λ_b, m_b indicate that $\langle 0|T\varphi_b(x_1)\ldots\varphi_b(x_n)|0\rangle$ is evaluated in a perturbative expansion using the bare Lagrangian $\mathcal{L}_b$ written in terms of the bare parameters as in (52). The vacuum $|0\rangle$ shown in both (55) and (56) is the vacuum of the same interacting theory corresponding to the Lagrangian $\mathcal{L}_b$.

Since $\varphi_b(x) = Z_\varphi^{1/2}\varphi_p(x)$,

$$\langle 0|T\varphi_b(x_1)\ldots\varphi_b(x_n)|0\rangle = Z_\varphi^{n/2}\langle 0|T\varphi_p(x_1)\ldots\varphi_p(x_n)|0\rangle, \quad (57)$$

and thus,

$$G_b^{(n)}(k_1,\ldots,k_n;\lambda_b,m_b,\tau) = Z_\varphi^{n/2}G_p^{(n)}(k_1,\ldots,k_n;\lambda_p,m_p,\tau), \quad (58)$$

or using (53), we have

$$G_p^{(n)}(k_1,\ldots,k_n;\lambda_p,m_p,\tau) = Z_\varphi^{-n/2}G_b^{(n)}(k_1,\ldots,k_n;Z_\lambda Z_\varphi^{-2}\lambda_p,Z_m^{1/2}Z_\varphi^{-1/2}m_p,\tau). \quad (59)$$

In $G_p^{(n)}(k_1,\ldots,k_n;\lambda_p,m_p,\tau)$, we scale momenta on both the external legs (k_i) and internal legs (p_j) of a Feynman diagram, the mass m_p, and τ as

$$k_i \to q_i = k_i/\tau, \qquad p_j \to p'_j = p_i/\tau,$$
$$m_p \to \mu_p = m_p/\tau, \qquad \tau \to 1 = \tau/\tau,$$

where q_i, p'_j, μ_p are dimensionless variables.

Noting that for a generic momentum l, $g_\varphi^2(l,m_p,\tau) = g_\varphi^2(l/\tau,m_p/\tau,1)$ straightforward counting[11] shows that

[11]In φ^4 theory, a diagram with V vertices and n external legs has $I = 2V - n/2$ internal legs and $L = I - V + 1$ loops. The $I + n$ propagators contribute $= \tau^{-4V-n}$, and the L integral measures contribute $\tau^{4V-2n+4}$ to the counting, giving an overall factor of τ^{4-3n}.

$$G_p^{(n)}(k_1, \dots, k_n; \lambda_p, m_p, \tau) \\ = \tau^{4-3n} (\tilde{G})_p^{(n)}(q_1, \dots, q_n; \lambda_p, \mu_p, 1), \tag{60}$$

with the understanding that the momentumlike variables on external and internal legs and the loop integration variables are all dimensionless. We can rewrite the above identity[12] as

$$(\tilde{G})_p^{(n)}(q_1, \dots, q_n; \lambda_p, \mu_p, 1) \\ = \tau^{3n-4} G_p^{(n)}(\tau q_1, \dots, \tau q_n; \lambda_p, \tau\mu_p, \tau). \tag{61}$$

Instead of considering λ_p and m_p as the τ-independent free variables, we can regard λ_p and μ_p to be the τ-independent free variables, making $m_p = \tau\mu_p$, τ-dependent. The bare Lagrangian (54) can be written in terms of λ_p and μ_p as

$$\mathcal{L}_b = \tilde{Z}_\varphi \frac{1}{2} \partial^\mu \varphi_p \partial_\mu \varphi_p - \tilde{Z}_m \frac{1}{2} (\mu_p)^2 \varphi_p^2 - \tilde{Z}_\lambda \frac{\lambda_p}{4!} \varphi_p^4,$$

with

$$\tilde{Z}_m(\tau; \lambda_p, \mu_p) \coloneqq \tau^2 Z_m(\tau; \lambda_p, \tau\mu_p), \\ \tilde{Z}_\varphi(\tau; \lambda_p, \mu_p) \coloneqq Z_\varphi(\tau; \lambda_p, \tau\mu_p), \\ \tilde{Z}_\lambda(\tau; \lambda_p, \mu_p) \coloneqq Z_\lambda(\tau; \lambda_p, \tau\mu_p).$$

Then, the left-hand side of (61) is independent of τ, and therefore,

$$\frac{d}{d\tau} (\tilde{G})_p^{(n)}(q_1, \dots, q_n; \lambda_p, \mu_p, 1) = 0. \tag{62}$$

Using (59) and (61), we can rewrite (62) as

$$\frac{d}{d\tau} [\tau^{3n-4} Z_\varphi^{-n/2} G_b^{(n)}(\tau q_1, \dots, \tau q_n; Z_\lambda Z_\varphi^{-2} \lambda_p, Z_m^{1/2} Z_\varphi^{-1/2} \tau\mu_p, \tau)] \\ = 0. \tag{63}$$

As shown in Appendix F, differentiating in (63), we obtain the renormalization group equation,

$$\left\{ (3n-4) + (n/2)\gamma_a + \gamma_\lambda \lambda_p \frac{\partial}{\partial \lambda_p} + \gamma_\mu \mu_p \frac{\partial}{\partial \mu_p} \right. \\ \left. + k_i^a \frac{\partial}{\partial k_i^a} + \tau \frac{\partial}{\partial \tau} \right\} G_b^{(n)}(k_1, \dots, k_n; K_\lambda \lambda_p; K_m \tau\mu_p, \tau) = 0, \tag{64}$$

where γ_a, γ_λ, γ_μ are defined in (F5), and

$$K_\lambda(\lambda_p, \mu_p, \tau) = Z_\lambda Z_\varphi^{-2}, \qquad K_m(\lambda_p, \mu_p, \tau) = Z_m^{1/2} Z_\varphi^{-1/2}.$$

The renormalization constants Z_λ, Z_m, and Z_φ can be calculated order by order in perturbation theory as shown in Sec. V A. It should be noted that, unlike in other regularization schemes, the renormalization constants are finite, at every order of perturbation, in autoregularization. The one important feature of autoregularization is that the scale τ has a transparent physical meaning—it is the Lorentz-invariant kinematic scale of the scattering process of interest.

VI. CONCLUSION

We have presented a proof of gauge invariance of the QED S-matrix that is stronger than previous proofs. Our argument—valid within both autoregularization and the current QFT—shows that the amplitude of every contributing Feynman diagram is separately gauge invariant; in contrast, the previous arguments establish gauge invariance of a sum of amplitudes of the contributing Feynman diagrams. We also showed that autoregularization yields a new method for quantizing the QED gauge field in covariant gauge that is significantly simpler than the standard Gupta-Bleuler method. In the new gauge quantization method, ghost fields are not needed. The unphysical modes decouple naturally. The QED Lagrangian does not have to be modified. The expectation value of the Hamiltonian is positive semidefinite. The Fock space has a nonnegative norm and does not have to be partitioned into "physical" and "unphysical" states, nor gauge condition imposed in a weak form. Finally, we illustrated renormalization at 1-loop in autoregularization using the φ^4 theory and derived the renormalization group equation (RGE). The RGE we derive has a straightforward physical interpretation—it describes the evolution of the correlation function with the kinematic scale of the scattering process of interest.

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APPENDIX A: OVERVIEW OF AUTOREGULARIZATION

In this appendix, we present a summary of autoregularization, which is described in greater detail in [1]. Consider a scattering process $\mathcal{P}$ in which m incoming particles with momenta $p_1, \dots p_m$ scatter to n outgoing particles with momenta $q_1, \dots, q_n$,

$$\mathcal{P}: \quad p_1 + \dots + p_m \to q_1 + \dots + q_n. \tag{A1}$$

For simplicity, we assume that one of the particles participating in the scattering process is described by a scalar field φ of mass $m > 0$.

In autoregularization, the free-field expansion of φ is taken to be

[12] We get $3n - 4$ as the scaling exponent, instead of $n - 4$ obtained in vertex functions because, unlike in the vertex functions, the propagators of external legs are included in $G_p^{(n)}(k_1, \dots, k_n; \lambda_p, m_p, \tau)$.

$$\varphi(x;\mathcal{P}) = \int \frac{d\vec{k}}{(2\pi)^3(2\omega(\vec{k}))} g_\varphi(k;\mathcal{P})[\hat{\mathbf{a}}(\vec{k})e^{-ikx} + \hat{\mathbf{a}}^\dagger(\vec{k})e^{ikx}], \tag{A2}$$

where $g_\varphi(k;\mathcal{P})$ is Lorentz-invariant *Gibbs factor*, which depends on the process $\mathcal{P}$. The Gibbs factors are defined as [[1], Eq. (6)],

$$g_\psi(k;\mathcal{P}) = \begin{cases} \left[\frac{d_\psi}{e^{(E_\psi(k)-\mu_\psi)/\tau}-1}\right]^{1/4} & \psi\colon \text{ massive boson} \\ \left[\frac{d_\psi}{e^{(E_\psi(k)-\mu_\psi)/\tau}+1}\right]^{1/4} & \psi\colon \text{ massive fermion}, \\ \left[\frac{d_\psi}{e^{(E_\psi(k)/\tau+\tau/E_\psi(k)}-1}\right]^{1/4} & \psi\colon \text{ massless boson} \end{cases} \tag{A3}$$

where d_ψ is the number of degrees of freedom of the ψ field, defined in [[1], Sec. 2.2]. $E_\psi(k)$ is a Lorentz-invariant that reduces to $\omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}$ in the center-of-momentum (CM) frame. The construction of $E_\psi(k)$ is described in [[1], Sec. 2.1]. μ_ψ is a Lorentz-invariant chemical potential of ψ, defined in [1, Eq. (7)]. τ is the Lorentz invariant kinematic scale of the process $\mathcal{P}$, defined in [1, Eq. (8)] and reproduced in (A4) below:

$$\tau_{\mathcal{P}} = \max_{\mathcal{I},\mathcal{J}}\left\{\sqrt{|k_\mu k^\mu|}\,|k = \sum_{i\in\mathcal{I}} p_i - \sum_{j\in\mathcal{J}} q_j\right\}, \tag{A4}$$

where $\mathcal{I}\subseteq\{1,\ldots,m\}, \mathcal{J}\subseteq\{1,\ldots,n\}, 0<|\mathcal{I}|+|\mathcal{J}|<m+n$. When the process $\mathcal{P}$ is clear from the context, we abbreviate $g_\psi(k;\mathcal{P})$ to $g_\psi(k)$ and omit the subscript in $\tau_{\mathcal{P}}$.

The free-field expansion in (A2) satisfies the Klein-Gordon equation. From (A3), we note that g_ψ^4 resembles the Fermi-Dirac (Bose-Einstein) distribution if ψ is a massive fermion (boson). An extra $\tau/E_\psi(k)$ term has been added to the exponent in the Gibbs factor of the massless boson to suppress the IR divergence; the validity of the extra term, and of the form of all the Gibbs factors, is supported by the preliminary tests of autoregularization presented in [1]. We also note that g_ψ^4 resembles the classical Gibbs factor for $E_\psi(k)/\tau \gg 1$.

Since the Gibbs factor depends on $\mathcal{P}$, the free-field expansion in (A2) also depends on the scattering process $\mathcal{P}$ representing a departure from previous formalism, which relied on process-independent free-field expansions. The calculation of the scattering amplitudes is thus customized to the scattering process under consideration.

The Gibbs factor in the free-field expansion (A2) breaks the equivalence between equal-time commutation relations (ETCR) imposed on the field and its conjugate momentum on the one hand and the commutation relations on the creation and annihilation operators on the other. In autoregularization, the standard commutation relations are imposed on the creation and annihilation operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^\dagger$, and the ETCR on the field and its momentum are omitted. Thus, the individual oscillators are quantized and not the field, representing a reversion to the Diracian approach.

Straightforward calculations show that the Gibbs factor in (A2) modifies the propagator. Thus, we have

$$\begin{aligned} D(x-y) &= \langle 0|T\varphi(x)\varphi(y)|0\rangle \\ &= \lim_{\epsilon\to 0}\left[i\int\frac{d^4k}{(2\pi)^4}\left\{\frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon}\right\}g_\varphi(k;\mathcal{P})^2\right]. \end{aligned}$$

Similarly, factors of g_ψ^2 appear in the propagator when ψ is a massive fermion or a massless boson. Finally, the Gibbs factors lead to a straightforward modification of the LSZ formalism [[1], Sec. 2.4].

APPENDIX B: FINITENESS OF THE PROPAGATOR

We show that the propagator of the φ^4 theory, $D(x-y)$, which is quadratically divergent in the Heisenberg-Pauli framework (HPF), is finite in autoregularization.

The propagator is given by

$$D(x-y) = \langle 0|T\varphi(x)\varphi(y)|0\rangle = \lim_{\epsilon\to 0} i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}g_\varphi^2(k)}{k^2-m^2+i\epsilon},$$

where $g_\varphi(k)$ is the Gibbs factor, which we abbreviate as $g(k)$ hereafter. Since $D(x-y)$ is Lorentz invariant, we can choose a convenient frame for the calculation. We choose the center-of-momentum frame of the scattering process of interest in which $g(k)$ depends only on $\vec{k}$ and is given by

$$g(k) = \left[\frac{2}{e^{\omega(\vec{k})/\tau}-1}\right]^{1/4} = \left[\frac{e^{-\tilde{\omega}(r)/2\tau}}{\sinh(\tilde{\omega}(r)/2\tau)}\right]^{1/4},$$

where $\omega(\vec{k}) = \sqrt{|\vec{k}|^2 + m^2}$, $\tilde{\omega}(r) = \sqrt{r^2+m^2}$, $r = |\vec{k}|$, and τ is the kinematic scale of the scattering process of interest as defined in (A4). For a massive scalar field, which has no conserved charge, the chemical potential $\mu_\varphi = 0$. With some abuse of notation, we write $g(k)$ as $g(\vec{k})$ hereafter. Then,

$$\begin{aligned} D(x-y) &= i\int\frac{d\vec{k}}{(2\pi)^4}e^{i\vec{k}\cdot(\vec{x}-\vec{y})} \\ &\quad\times g(\vec{k})^2\lim_{\epsilon\to 0}\left[\int_0^\infty dk^0\frac{e^{-ik^0(x^0-y^0)}}{(k^0)^2-(\omega(\vec{k})-i\epsilon)^2}\right]. \end{aligned}$$

The k^0 integral is evaluated using contour integration, closing the contour with a lower semicircle if $x^0 > y^0$ and the upper semicircle if $x^0 \le y^0$. Performing the contour integration we obtain

$$D(x-y) = \frac{1}{16\pi^3}\int d\vec{k}\left[\frac{g(\vec{k})^2}{\omega(\vec{k})}\right]e^{\pm i\omega(\vec{k})|x^0-y^0|+i\vec{k}\cdot(\vec{x}-\vec{y})}.$$

Taking the absolute value and performing the angular integration, we have

$$|D(x-y)| \le \frac{1}{4\pi^2}\int_0^\infty drr^2\left[\frac{1}{\tilde{\omega}(r)}\right]\left[\frac{e^{-\tilde{\omega}(r)/2\tau}}{\sinh(\tilde{\omega}(r)/2\tau)}\right]^{1/2}.$$

Noting that $\tilde{\omega}(r) \ge r, \tilde{\omega}(r) \ge m, \sinh(\tilde{\omega}(r)/2\tau) \ge \tilde{\omega}(r)/2\tau \ge m/2\tau$, we have

$$|D(x-y)| \le \frac{\sqrt{\tau}}{2\pi^2\sqrt{2m}}\int_0^\infty drre^{-r/4\tau} = \frac{4\sqrt{2}(\tau)^{5/2}}{\pi^2\sqrt{m}} < \infty.$$

The above argument holds without modification if $x = y$. Therefore, we obtain the following corollary.

Corollary 1.

$$\left|\int\frac{d^4k}{(2\pi)^4}\frac{g(k)^2}{k^2-m^2+i\epsilon}\right| < \infty$$

We also obtain the following corollary for fermion propagator.

Corollary 2. The electron propagator,

$$S(x) = \lim_{\epsilon\to 0}\left[i\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}g_e(k)^2(\not{k}+m)}{k^2-m^2+i\epsilon}\right],$$

is finite, where

$$g_e(k)^2 = \left[\frac{2e^{-(\tilde{\omega}(\rho)-\mu_e)/2\tau}}{\cosh((\tilde{\omega}(\rho)-\mu_e)/2\tau)}\right]^{1/2}, \qquad \mu_e = \sqrt{\frac{9\alpha}{16\pi}}, \qquad \tilde{\omega}(\rho) \coloneqq [\rho^2+m^2]^{1/2}, \qquad \rho = |\vec{k}|.$$

$g_e(k)$ and μ_e are defined in (A3) and [1, Eq. (7)], respectively. α is the fine structure constant. $g_e(k)^2$ has the form shown above in the center-of-momentum frame.

Proof. We note

$$S(x) = i\gamma_0B^0(x;m) + i\gamma_jB^j(x;m) + iC(x;m),$$
$$B^\mu(x;m) = \int\frac{d^4k}{(2\pi)^4}\frac{k^\mu e^{-ikx}g_e(k)^2}{k^2-m^2+i\epsilon}, \qquad C(x;m) = m\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ikx}g_e(k)^2}{k^2-m^2+i\epsilon}$$

Since $g_e(k) = g_e(-k)$ in center-of-momentum frame, $B^\mu(0;m) = 0$. So in calculating B^μ, we assume $x \neq 0$. Performing k^0 integration, closing the contour using the upper (lower) semicircle, if $x^0 < 0$ ($x^0 > 0$), we have

$$B^0(x;m) = \pm\frac{i}{4\pi^2}\int d\rho\rho^2\tilde{g}_e(\rho)^2e^{\pm i\omega(\vec{k})x^0+i\vec{k}\cdot\vec{x}}.$$

Noting that $\cosh(\theta) \ge 1$, and $\tilde{\omega}(\rho) > \rho$,

$$|B^0(x;m)| < \frac{e^{\mu_e/4\tau}}{2\sqrt{2}\pi^2}\int_0^\infty d\rho\rho^2e^{-\tilde{\omega}(\rho)/4\tau} < \frac{e^{\mu_e/4\tau}}{2\sqrt{2}\pi^2}\int_0^\infty d\rho\rho^2e^{-\rho/4\tau} = \frac{32\sqrt{2}\tau^3e^{\mu_e/4\tau}}{\pi^2}.$$

Similarly, performing the k^0 contour integration in B^j and C, and noting $|k^j| < |\vec{k}| = \rho$, we have for $j = 1, 2, 3$,

$$B^j(x;m) = -\int\frac{d\vec{k}}{(2\pi)^4}g_e(k)^2k^j\left[\frac{2\pi i}{2\tilde{\omega}(|\vec{k}|)}\right]e^{\pm i\tilde{\omega}(|\vec{k}|)x^0+i\vec{k}\cdot\vec{x}},$$
$$|B^j(x;m)| \le \frac{e^{\mu_e/4\tau}}{8\sqrt{2}\pi^3}\int d\vec{k}\frac{|\vec{k}|e^{-\tilde{\omega}(|\vec{k}|)/4\tau}}{\tilde{\omega}(|\vec{k}|)} < \frac{e^{\mu_e/4\tau}}{2\sqrt{2}\pi^2}\int_0^\infty d\rho\rho^2e^{-\rho/4\tau} = \frac{32\sqrt{2}\tau^3e^{\mu_e/4\tau}}{\pi^2}.$$

Finally,

$$|C(x;m)| = \left|-im\int\frac{d\vec{k}}{16\pi^3}\frac{g_e(k)^2e^{\pm i\tilde{\omega}(|\vec{k}|)x^0+i\vec{k}\cdot\vec{x}}}{\tilde{\omega}(|\vec{k}|)}\right| < \frac{me^{\mu_e/4\tau}}{2\sqrt{2}\pi^2}\int_0^\infty d\rho\rho e^{-\rho/4\tau} = \frac{4\sqrt{2}m\tau^2e^{\mu_e/4\tau}}{\pi^2}.$$

■

APPENDIX C: COMPONENT OF $\delta^4(p)$ IN $\Delta(p, q_1)$

We recall from (20)

$$\Delta(p,q_1) = (\not{q}_1 + m)\left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right]\{\tilde{\alpha}(-p) + \tilde{\alpha}^*(p)\},$$

Our objective is to extract the component of $\delta^4(p)$ in $\Delta(p, q_1)$. Using the definition of α in (16), we have

$$\tilde{\alpha}(-p) + \tilde{\alpha}^*(p) = \int d^4x e^{-ipx}\alpha(x).$$

Expanding $\alpha(x)$ around $x = 0$, we have

$$\alpha(x) = \alpha(0) + \partial_\mu\alpha(0)x^\mu + \ldots.$$

Therefore, from (C1), we have

$$\begin{aligned}\tilde{\alpha}(-p) + \tilde{\alpha}^*(p) &= \alpha(0)(2\pi)^4\delta^4(p) + \partial_\mu\alpha(0)\int d^4x x^\mu e^{-ipx} + \ldots \\ &= \alpha(0)[(2\pi)^4\delta^4(p)] + i(2\pi)^4\partial_\mu\alpha(0)\left[\frac{\partial}{\partial p_\mu}\delta^4(p)\right] + \ldots = \alpha(0)(2\pi)^4\delta^4(p) + \beta(p),\end{aligned}$$

where $\beta(p)$ contains terms that have derivatives of $\delta^4(p)$. Therefore,

$$[\tilde{\alpha}(-p) + \tilde{\alpha}^*(p)]\left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right] = \alpha(0)\left[\frac{g_e(q_1)^2}{-\not{q}_1 - m + i\epsilon}\right](2\pi)^4\delta^4(p) + \left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right]\beta(p). \tag{C1}$$

We note that

$$\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon} = -i\int d^4x S(x)e^{(p-q_1)x}, \tag{C2}$$

where $S(x)$ is the electron propagator,

$$S(x) = i\not{B}(x;m) + iC(x;m), \qquad B^\mu = \int \frac{d^4k}{(2\pi)^4}\frac{k^\mu e^{-ikx}g_e(k)^2}{k^2 - m^2 + i\epsilon}, \qquad C = \int \frac{d^4k}{(2\pi)^4}\frac{me^{-ikx}g_e(k)^2}{k^2 - m^2 + i\epsilon}.$$

In the center-of-momentum frame of the process of interest, $g_e(k) = g_e(-k)$, and as a result,[13] $B^\mu(0;m) = 0$. From Corollary 2 in Appendix B, we know that $B(x;m)$ and $C(x;m)$ are finite.

Expanding $B^\mu(x;m)$ and $C(x;m)$ about $x = 0$, and recalling that $B^\mu(0;m) = 0$, we have

$$B^\mu(x;m) = \partial_\nu B^\mu(0;m)x^\nu + \ldots, \qquad C(x;m) = C(0;m) + \partial_\nu C(0;m)x^\nu + \ldots. \tag{C3}$$

Inserting (C3) into (C2), we obtain

$$\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon} = C(0;m)(2\pi)^4\delta^4(p-q_1) + [\partial_\nu\not{B}(0;m) + \partial_\nu C(0;m)][i(2\pi)^4]\left[\frac{\partial}{\partial p_\nu}\delta^4(p-q_1)\right] + \ldots, \tag{C4}$$

Inserting (C4) into (C1), we get

$$[\tilde{\alpha}(-p) + \tilde{\alpha}^*(p)]\left[\frac{g_e(p-q_1)^2}{\not{p} - \not{q}_1 - m + i\epsilon}\right] = \alpha(0)\left[\frac{g_e(q_1)^2}{-\not{q}_1 - m + i\epsilon}\right](2\pi)^4\delta^4(p) + C(0;m)(2\pi)^4\delta^4(p-q_1)\beta(p) + \Lambda(p,q_1),$$

[13]The integrand becomes an odd function.

where $\Lambda(p, q_1)$ contains terms that have derivatives of $\delta^4(p-q_1)$ and $\delta^4(p)$ and, as mentioned before, $\beta(p)$ contains terms that have derivatives of $\delta^4(p)$. In summary,

$$[\tilde{\alpha}(-p)+\tilde{\alpha}^*(p)]\left[\frac{g_e(p-q_1)^2}{\not{p}-\not{q}_1-m+i\epsilon}\right] = \alpha(0)\left[\frac{g_e(q_1)^2}{-\not{q}_1-m+i\epsilon}\right](2\pi)^4\delta^4(p)+\Omega(p,q_1), \quad \text{(C5)}$$

where $\Omega(p,q_1) = C(0;m)(2\pi)^4\delta^4(p-q_1)\beta(p)+\Lambda(p,q_1)$ contains terms that have derivatives of $\delta^4(p)$ and $\delta^4(p-q_1)$.

APPENDIX D: FINITENESS OF AN INTEGRAL

The following lemma is used in the proof of finiteness in Appendix E.

Lemma 1. Let $f(x)$ be a real-valued continuously differentiable function with bounded derivative in the interval $[a-\epsilon, a+\epsilon]$, where $a \in \mathbb{R}$ and $0 < \epsilon < \infty$. Then,

$$F := \int_{a-\epsilon}^{a+\epsilon} \frac{f(x)}{x-a} dx < \infty.$$

Proof. The integral F is defined in the sense of Cauchy principal value as

$$F = \lim_{\delta\to 0}[F_-(\delta)+F_+(\delta)], \qquad F_-(\delta) = \int_{a-\epsilon}^{a-\delta}\frac{f(x)}{x-a}dx,$$
$$F_+(\delta) = \int_{a+\delta}^{a+\epsilon}\frac{f(x)}{x-a}dx,$$

where $0<\delta<\epsilon$. Setting $a-x=y$ in $F_-(\delta)$ and $x-a=y$ in $F_+(\delta)$, we have

$$F = \lim_{\delta\to 0}\int_\delta^\epsilon \left[\frac{f(a+y)-f(a-y)}{y}\right]dy.$$

From Mean Value Theorem, there exists a $c(y) \in (a-y, a+y)$ for which

$$f(a+y)-f(a-y) = f'(c(y))(2y).$$

Therefore,

$$F = 2\lim_{\delta\to 0}\int_\delta^\epsilon f'(c(y))dy = 2\int_0^\epsilon f'(c(y))dy. \quad \text{(D1)}$$

Let

$$m = \min_{a-\epsilon\le x\le a+\epsilon} f'(x), \qquad M = \max_{a-\epsilon\le x\le a+\epsilon} f'(x). \quad \text{(D2)}$$

From the assumptions in the Lemma, m and M are finite. Using (D1) and (D2), we have

$$2m\epsilon \le F = 2\int_0^\epsilon f'(c(y))dy \le 2M\epsilon,$$

which completes the proof. ■

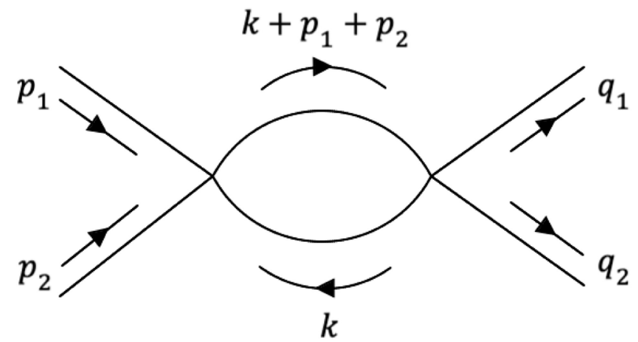

FIG. 1. A 1-loop diagram $\mathcal{D}$ of 2-particle scattering in φ^4 theory.

APPENDIX E: FINITENESS OF THE AMPLITUDE OF A 1-LOOP DIAGRAM

Figure 1 shows a 1-loop diagram $\mathcal{D}$ that arises in 2-particle scattering in the φ^4 theory[14] of a massive scalar field. We show that the amplitude of $\mathcal{D}$ is finite in autoregularization. In contrast, $\mathcal{D}$'s amplitude is logarithmically divergent in HPF.

The amputated amplitude of $\mathcal{D}$ in autoregularization is

$$A = (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\beta I,$$
$$\beta = \frac{\lambda^2 g(p_1)g(p_2)g(q_1)g(q_2)}{(\sqrt{Z})^4},$$

where

$$I = \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4} \times \left[\frac{g^2(k)g^2(k+p_1+p_2)}{(k^2-m^2+i\epsilon)((k+p_1+p_2)^2-m^2+i\epsilon)}\right], \quad \text{(E1)}$$

where the Gibbs factor g_φ has been abbreviated to g. The corresponding amplitude in HPF, obtained by setting the Gibbs factors to unity, is logarithmically divergent.

We show that the integral I is finite. Since I is Lorentz invariant, we can perform the calculation in any frame. We perform the calculation in the center-of-momentum frame $\mathcal{F}_{CM}$ in which $p_1+p_2 = (2\sqrt{a^2+m^2}, 0, 0, 0)$, where $a = |\vec{p}_1| = |\vec{p}_2|$. For brevity, we set $\chi := 2\sqrt{a^2+m^2}$. In $\mathcal{F}_{CM}$, $g(k)$ depends only on $\vec{k}$ (see [[1], Sec. 2.1]) and with some abuse of notation, we write $g(k) = g(\vec{k})$. Hence, in $\mathcal{F}_{CM}$, $g(k+p_1+p_2) = g(k) = g(\vec{k})$. So, we can write I as

[14] $\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$. $m>0$ and $0<\lambda\ll 1$.

$$I = \lim_{\epsilon\to 0}\int \frac{d\vec{k}}{(2\pi)^4} g(\vec{k})^4 \int dk^0 \left[\frac{1}{((k^0)^2 - (\omega(\vec{k}) - i\epsilon)^2)((k^0+\chi)^2 - (\omega(\vec{k}) - i\epsilon)^2)}\right],$$

where $\omega(\vec{k}) \coloneqq [\vec{k}^2 + m^2]^{1/2}$.

The k^0 integral can be evaluated with contour integration closing the contour using the upper semicircle. The integral on the semicircle vanishes (see [[14], Sec. 70]) yielding

$$\begin{aligned} I_0 &= \lim_{\epsilon\to 0}\int dk^0 \left[\frac{1}{((k^0)^2 - (\omega(\vec{k}) - i\epsilon)^2)((k^0+\chi)^2 - (\omega(\vec{k}) - i\epsilon)^2)}\right] \\ &= -\frac{\pi i}{\omega(\vec{k})\chi}\left[\frac{1}{\chi - 2\omega(\vec{k})} + \frac{1}{\chi + 2\omega(\vec{k})}\right]. \end{aligned}$$

I can be written as

$$\begin{aligned} I &= -\frac{\pi i}{(2\pi)^4\chi}\{I_- + I_+\}, \\ I_\pm &= \int d\vec{k}\left[\frac{g(\vec{k})^4}{\omega(\vec{k})(\chi \pm 2\omega(\vec{k}))}\right] = 4\pi\int_0^\infty dr \left[\frac{r^2\tilde{g}(r)^4}{\tilde{\omega}(r)(\chi \pm 2\tilde{\omega}(r))}\right], \end{aligned} \tag{E2}$$

where $r = |\vec{k}|$, τ is the scale of the process as defined in (A4), and

$$\tilde{\omega}(r) \coloneqq [r^2 + m^2]^{1/2}, \qquad \tilde{g}(r)^4 \coloneqq \left[\frac{2}{e^{\tilde{\omega}(r)/\tau} - 1}\right] = \frac{e^{-\tilde{\omega}(r)/2\tau}}{\sinh(\tilde{\omega}(r)/2\tau)}. \tag{E3}$$

Since

$$\tilde{\omega}(r) \geq m, \qquad \tilde{\omega}(r) \geq r, \qquad \chi \geq 2m, \qquad \sinh(\tilde{\omega}(r)/2\tau) \geq \sinh(m/2\tau), \tag{E4}$$

we have

$$\tilde{g}(r)^4 \leq \frac{e^{-r/2\tau}}{\sinh(m/2\tau)},$$

and

$$I_+ \leq \frac{\pi}{m^2\sinh(m/2\tau)}\int_0^\infty dr r^2 e^{-r/2\tau} = \frac{16\pi\tau^3}{m^2\sinh(m/2\tau)} < \infty. \tag{E5}$$

To prove the finiteness of I_- we rationalize the denominator in the integrand. Using the definitions of χ and $\tilde{\omega}(r)$, we obtain

$$I_- = -\pi\int_0^\infty dr \frac{f(r)}{(r-a)}, \qquad f(r) \coloneqq \left[\frac{r^2 e^{-\tilde{\omega}(r)/2\tau}(\chi + 2\tilde{\omega}(r))}{\tilde{\omega}(r)(r+a)\sinh(\tilde{\omega}(r)/2\tau)}\right]. \tag{E6}$$

We prove the finiteness of I_- by considering two cases: $a = 0$ and $a > 0$. In the trivial case $a = 0$, in which the incoming particles are at rest in $\mathcal{F}_{CM}$, $\chi = 2m$, and we have

$$\begin{aligned} |I_-| &= 2\pi\int_0^\infty dr\left[\frac{e^{-\tilde{\omega}(r)/2\tau}(m + \tilde{\omega}(r))}{\tilde{\omega}(r)\sinh(\tilde{\omega}(r)/2\tau)}\right] = I_-^{(1)} + I_-^{(2)}, \\ I_-^{(1)} &= 2\pi m\int_0^\infty dr\left[\frac{e^{-\tilde{\omega}(r)/2\tau}}{\tilde{\omega}(r)\sinh(\tilde{\omega}(r)/2\tau)}\right], \qquad I_-^{(2)} = 2\pi\int_0^\infty dr\left[\frac{e^{-\tilde{\omega}(r)/2\tau}}{\sinh(\tilde{\omega}(r)/2\tau)}\right]. \end{aligned}$$

Using the inequalities in (E4), we have

$$I_-^{(1)} \le \frac{2\pi m}{m\sinh(m/2\tau)}\int_0^\infty e^{-r/2\tau}dr = \frac{4\pi m\tau}{m\sinh(m/2\tau)} < \infty,$$
$$I_-^{(2)} \le \frac{2\pi}{\sinh(m/2\tau)}\int_0^\infty e^{-r/2\tau}dr = \frac{4\pi\tau}{\sinh(m/2\tau)} < \infty.$$

Therefore, when $a = 0$, we have $|I_-| = I_-^{(1)} + I_-^{(2)} < \infty$.

Next, we consider the case $a > 0$. Using (E6), we set

$$N(r) = r^2 e^{-\tilde{\omega}(r)/2\tau}(\chi + 2\tilde{\omega}(r)), \qquad D(r) = \tilde{\omega}(r)(r+a)\sinh(\tilde{\omega}(r)/2\tau). \tag{E7}$$

Evaluating[15] $N'(r)$ and $D'(r)$, we see that

$$f'(r) = \frac{D(r)N'(r) - N(r)D'(r)}{D(r)^2}$$

is continuous and bounded in $[a-\epsilon, a+\epsilon]$ for any $0 < \epsilon < a$. We pick a small ϵ, satisfying $0 < \epsilon < a$ and conclude, using Lemma 1, in Appendix D, that

$$I_\epsilon \coloneqq -\pi\int_{a-\epsilon}^{a+\epsilon}\frac{f(r)}{r-a}dr < \infty. \tag{E8}$$

Using (E6) we can write I_- as

$$I_- = I_L + I_\epsilon + I_R, \qquad I_L = -\pi\int_0^{a-\epsilon}\frac{f(r)}{r-a}dr, \qquad I_R = -\pi\int_{a+\epsilon}^\infty\frac{f(r)}{r-a}dr. \tag{E9}$$

We write $f(r)$ as

$$\begin{aligned} f(r) &= f_1(r) + f_2(r),\\ f_1(r) &= \frac{r^2 e^{-\tilde{\omega}(r)/2\tau}\chi}{\tilde{\omega}(r)(r+a)\sinh(\tilde{\omega}(r)/2\tau)}, \qquad f_2(r) = \frac{2r^2 e^{-\tilde{\omega}(r)/2\tau}}{(r+a)\sinh(\tilde{\omega}(r)/2\tau)}. \end{aligned} \tag{E10}$$

Using the bounds in (E4) and noting that $|r^2 - a^2| \ge \epsilon^2$ in $[0, a-\epsilon]$ and in $[a+\epsilon, \infty)$, we obtain

$$\left|\frac{f_1(r)}{r-a}\right| \le \left[\frac{\chi}{m\epsilon^2\sinh(m/2\tau)}\right]r^2 e^{-r/2\tau}, \qquad \left|\frac{f_2(r)}{r-a}\right| \le \left[\frac{2}{\epsilon^2\sinh(m/2\tau)}\right]r^2 e^{-r/2\tau}. \tag{E11}$$

Using (E9)–(E11) and setting

$$\xi = \left[\frac{\pi}{\epsilon^2\sinh(m/2\tau)}\right]\left[\frac{\chi}{m} + 2\right],$$

we obtain

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$$N'(r) = \frac{e^{-\tilde{\omega}(r)/2\tau}[4\tau r\tilde{\omega}(r)(\chi + 2\tilde{\omega}(r)) - r^3(\chi + 2\tilde{\omega}(r)) + 4r^3\tau]}{2\tau\tilde{\omega}(r)},$$
$$D'(r) = \frac{\sinh(\tilde{\omega}(r)/2\tau)\{2\tau r(r+a) + 2\tau\tilde{\omega}(r)^2\} + r\tilde{\omega}(r)(r+a)\cosh(\tilde{\omega}(r)/2\tau)}{2\tau\tilde{\omega}(r)}.$$

$$|I_L| \leq \xi \int_0^{a-\epsilon} r^2 e^{-r/2\tau} dr \leq \xi(16\tau^3) < \infty,$$
$$|I_R| \leq \xi \int_{a+\epsilon}^{\infty} r^2 e^{-r/2\tau} dr \leq \xi[2\tau\{8a\tau + 4a^2 + 8\tau^2\}] \times e^{-\epsilon/\tau} < \infty. \quad \text{(E12)}$$

From (E9), (E8), and (E12), it follows that I_- is finite. The finiteness of I then follows from (E2), (E5), and the finiteness of I_-.

Since $g(k) = g(-k)$, changing the variable of integration $k \to -k$ in the above argument, we get the following corollary.

Corollary 3.

$$\lim_{\epsilon\to 0} \int \frac{d^4k}{(2\pi)^4} \left[\frac{g^2(k)g^2(p_1+p_2-k)}{(k^2-m^2+i\epsilon)((p_1+p_2-k)^2-m^2+i\epsilon)}\right] < \infty.$$

APPENDIX F: RENORMALIZATION GROUP EQUATION

In this appendix, we present the calculations underlying the derivation of (64) from (63). For convenience, Eq. (63) is reproduced below:

$$\frac{d}{d\tau}[\tau^{3n-4} Z_\varphi^{-n/2} G_b^{(n)}(\tau q_1, \ldots, \tau q_n; Z_\lambda Z_\varphi^{-2}\lambda_p, Z_m^{1/2} Z_\varphi^{-1/2}\tau\mu_p, \tau)] = 0.$$

Differentiating in (63) and dividing throughout by $\tau^{3n-5} Z_\varphi^{-n/2}$, the equation becomes

$$(3n-4)G_b^{(n)} - (n/2)\tau\left(\frac{d\ln Z_\varphi}{d\tau}\right)G_b^{(n)} + \tau\frac{dG_b^{(n)}}{d\tau} = 0. \quad \text{(F1)}$$

We change the set of independent variables from the physical parameters $(\lambda_p, \mu_p, \tau_p)$ to the bare parameters (λ_b, m_b, τ_b). Although $\tau_p = \tau_b = \tau$, we have used different subscripts because different sets of variables are held constant in $\partial/\partial\tau_p$ and $\partial/\partial\tau_b$. In the above notation, the τ occurring in (F1) is τ_b. Further, we use the following definitions:

$$K_\lambda = Z_\lambda Z_\varphi^{-2} = \frac{\lambda_b}{\lambda_p}, \qquad K_m = Z_m^{1/2} Z_\varphi^{-1/2} = \frac{m_b}{\tau\mu_p}. \quad \text{(F2)}$$

Then, using the chain rule, we have

$$\begin{bmatrix} \frac{\partial}{\partial\lambda_p} \\ \frac{\partial}{\partial\mu_p} \\ \frac{\partial}{\partial\tau_p} \end{bmatrix} = \begin{bmatrix} \frac{\partial\lambda_b}{\partial\lambda_p} & \frac{\partial m_b}{\partial\lambda_p} & \frac{\partial\tau_b}{\partial\lambda_p} \\ \frac{\partial\lambda_b}{\partial\mu_p} & \frac{\partial m_b}{\partial\mu_p} & \frac{\partial\tau_b}{\partial\mu_p} \\ \frac{\partial\lambda_b}{\partial\tau_p} & \frac{\partial m_b}{\partial\tau_p} & \frac{\partial\tau_b}{\partial\tau_p} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial\lambda_b} \\ \frac{\partial}{\partial m_b} \\ \frac{\partial}{\partial\tau_b} \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} (K_{\lambda,\lambda_p}\lambda_p + K_\lambda) & K_{m,\lambda_p}\tau_p\mu_p & 0 \\ K_{\lambda,\mu_p}\lambda_p & (K_{m,\mu_p}\mu_p + K_m)\tau_p & 0 \\ K_{\lambda,\tau_p}\lambda_p & (K_{m,\tau_p}\tau_p + K_m)\mu_p & 1 \end{bmatrix}}_{A} \begin{bmatrix} \frac{\partial}{\partial\lambda_b} \\ \frac{\partial}{\partial m_b} \\ \frac{\partial}{\partial\tau_b} \end{bmatrix} = A \begin{bmatrix} \frac{\partial}{\partial\lambda_b} \\ \frac{\partial}{\partial m_b} \\ \frac{\partial}{\partial\tau_b} \end{bmatrix}.$$

where we have used the notation $K_{a,b} \coloneqq \frac{\partial K_a}{\partial b}$.

Then, denoting the (i,j)th element of the matrix A as A_{ij}, and assuming that A is invertible,

$$B \coloneqq A^{-1} = \frac{1}{\Delta}\begin{bmatrix} A_{22} & -A_{12} & 0 \\ -A_{21} & A_{11} & 0 \\ (A_{21}A_{32} - A_{31}A_{22}) & -(A_{11}A_{32} - A_{31}A_{12}) & \Delta \end{bmatrix},$$
$$\Delta = A_{11}A_{22} - A_{21}A_{12} = \tau_p[(K_{\lambda,\lambda_p}\lambda_p + K_\lambda)(K_{m,\mu_p}\mu_p + K_m) - K_{\lambda,\mu_p}\lambda_p K_{m,\lambda_p}\mu_p],$$

and

$$\begin{bmatrix} \frac{\partial}{\partial \lambda_b} \\ \frac{\partial}{\partial m_b} \\ \frac{\partial}{\partial \tau_b} \end{bmatrix} = A^{-1} \begin{bmatrix} \frac{\partial}{\partial \lambda_p} \\ \frac{\partial}{\partial \mu_p} \\ \frac{\partial}{\partial \tau_p} \end{bmatrix} = \begin{bmatrix} B_{11} \frac{\partial}{\partial \lambda_p} + B_{12} \frac{\partial}{\partial \mu_p} \\ B_{21} \frac{\partial}{\partial \lambda_p} + B_{22} \frac{\partial}{\partial \mu_p} \\ B_{31} \frac{\partial}{\partial \lambda_p} + B_{32} \frac{\partial}{\partial \mu_p} + \frac{\partial}{\partial \tau_p} \end{bmatrix}, \tag{F3}$$

where

$$\begin{aligned} B_{11} &= \left[\frac{(K_{m,\mu_p}\mu_p + K_m)\tau_p}{\Delta}\right], \qquad B_{12} = -\left[\frac{K_{m_p,\lambda_p}\tau_p\mu_p}{\Delta}\right], \\ B_{21} &= -\left[\frac{K_{\lambda,\mu_p}\lambda_p}{\Delta}\right], \qquad B_{22} = \left[\frac{(K_{\lambda,\lambda_p}\lambda_p + K_\lambda)}{\Delta}\right], \\ B_{31} &= \left[\frac{K_{\lambda,\mu_p}\lambda_p(K_{m,\tau}\tau_p + K_m)\mu_p - K_{\lambda,\tau_p}\lambda_p(K_{m,\mu_p}\mu_p + K_m)\tau_p}{\Delta^2}\right], \\ B_{32} &= -\left[\frac{(K_{\lambda,\lambda_p}\lambda_p + K_\lambda)(K_{m,\tau}\tau_p + K_m)\mu_p - K_{\lambda,\tau_p}\lambda_p K_{m_p,\lambda_p}\tau_p\mu_p}{\Delta^2}\right]. \end{aligned} \tag{F4}$$

Inserting (F3) into (F1), we obtain

$$\begin{aligned} \Big\{(3n-4)G_b^{(n)} - (n/2)\tau_p\left[\left(B_{31}\frac{\partial}{\partial \lambda_p} + B_{32}\frac{\partial}{\partial \mu_p} + \frac{\partial}{\partial \tau_p}\right)\ln(Z_\varphi)\right]G_b^{(n)} \\ + \tau_p\left[B_{31}\frac{\partial}{\partial \lambda_p} + B_{32}\frac{\partial}{\partial \mu_p} + \frac{\partial}{\partial \tau_p}\right]G_b^{(n)}\Big\} = 0. \end{aligned}$$

We drop the subscript in τ_p, understanding that we are working in $(\tau_p, \lambda_p, \mu_p)$ coordinates hereafter. Defining

$$\gamma_a \coloneqq -\tau\left(B_{31}\frac{\partial}{\partial \lambda_p} + B_{32}\frac{\partial}{\partial \mu_p} + \frac{\partial}{\partial \tau}\right)\ln(Z_\varphi), \qquad \gamma_\lambda \coloneqq \tau B_{31}/\lambda_p, \qquad \gamma_\mu \coloneqq \tau B_{32}/\mu_p, \tag{F5}$$

and recalling that $m_p = \tau\mu_p$, we obtain the equation

$$\left\{(3n-4) + (n/2)\gamma_a + \gamma_\lambda\lambda_p\frac{\partial}{\partial \lambda_p} + \gamma_\mu\mu_p\frac{\partial}{\partial \mu_p} + \tau\frac{\partial}{\partial \tau}\right\}G_b^{(n)}(\tau q_1, \ldots, \tau q_n; Z_\lambda Z_\varphi^{-2}\lambda_p; Z_m^{1/2}Z_\varphi^{-1/2}\tau\mu_p, \tau) = 0,$$

where Z_λ, Z_φ, and Z_m are as defined in (53). We revert to the variables k_i instead of q_i, to obtain the renormalization group equation,

$$\left\{(3n-4) + (n/2)\gamma_a + \gamma_\lambda\lambda_p\frac{\partial}{\partial \lambda_p} + \gamma_\mu\mu_p\frac{\partial}{\partial \mu_p} + k_i^a\frac{\partial}{\partial k_i^a} + \tau\frac{\partial}{\partial \tau}\right\}G_b^{(n)}(k_1, \ldots, k_n; K_\lambda\lambda_p; K_m\tau\mu_p, \tau) = 0,$$

where $K_\lambda(\lambda_p, \mu_p, \tau)$ and $K_m(\lambda_p, \mu_p, \tau)$ are defined in (F2).

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