Effective potential between static sources in quenched light-front Yukawa theory

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(Received 30 December 2023; accepted 11 April 2024; published 10 May 2024)

We compute a nonperturbative effective potential between two static fermions in light-front Yukawa theory as a Hamiltonian eigenvalue problem. Fermion pair production is suppressed, to make possible an exact analytic solution in the form of a coherent state of bosons that form clouds around the sources. The effective potential is essentially an interference term between individual clouds. The model is regulated with Pauli-Villars bosons and fermions, to achieve consistent quantization and renormalization of masses and couplings. This extends earlier work on scalar Yukawa theory where Pauli-Villars regularization did not play a central role. The key result is that the nonperturbative solution restores rotational symmetry even though the light-front formulation of Yukawa theory, with its preferred axis, appears antithetical to such a symmetry.

DOI: 10.1103/PhysRevD.109.096010

I. INTRODUCTION

The current understanding of quantum chromodynamics (QCD) as the theory of the strong interactions is that it provides for the confinement of quarks and gluons. Calculations within lattice QCD [1,2] have confirmed this [3–5] as have analytic calculations [6–8],[1] not to mention the experimental evidence for the absence of free quarks and gluons. What has not been established is the derivation of confinement within a light-front quantization of the theory [11–16]. This is something that must be nonperturbative or at least an all-orders resummation of perturbation theory.

We wish to explore how such a nonperturbative light-front calculation might be done.[2] Here we extend earlier work [21][3] on quenched scalar Yukawa theory to include fermion sources. The basic approach is to consider the effective potential for two static sources as a function of their separation, computed from the change in the eigenenergy of the system relative to the energy of two well-separated sources. This involves renormalization of the source mass and, as we will show, renormalization of the coupling to bosons. Regularization is provided by the inclusion of Pauli-Villars (PV) fermions and bosons.[4] The PV fermions provide a convenient simplification of the light-front quantization by eliminating what are known as instantaneous fermion terms from the light-front Hamiltonian; this is what makes the analytic solution possible. The PV bosons regulate the self-energy corrections to the fermion mass.

Some will be concerned that a method based on PV regularization cannot be extended to QCD, there being a general prejudice against PV regularization of non-Abelian gauge theories. However, as shown in [16], a consistent PV regularization can be constructed. The key is that the definition of the gauge transformation must be extended

[1]For earlier discussion of static potentials in QCD, see Refs. [9,10].

[2]For recent work on a perturbative calculation of a light-front effective potential, see Ref. [17]. They use the renormalization group procedure for effective particles (RGPEP) to compute a light-front effective potential in QCD and then use it in nonperturbative calculations. For alternative analyses of effective light-front potentials, see Refs. [18] and [19], and for a light-front analysis of a particle interaction with a static potential, see Ref. [20].

[3]In [21] there are two typographical errors. The factor g/m in Eq. (3.27) should be $g/(2m)$, and the $(q^+)^2$ in the numerator of Eq. (3.29) should be q^+.

[4]For earlier uses of PV regularization in nonperturbative light-front calculations, see Ref. [22] for Yukawa theory and [23–27] for quantum electrodynamics.

when the Lagrangian is extended to include the PV fields. In other words, there does remain a gauge invariance of the QCD Lagrangian when PV fields are included. In addition, there is a Becchi-Rouet-Stora-Tyutin (BRST) symmetry when the gauge is fixed covariantly.

Our definition of light-front coordinates [28] is to take $x^\pm = t \pm z$, with x^+ as the light-front time, and $\vec{x}_\perp = (x, y)$ in the transverse. The conjugate light-front energy is $p^- = E - p_z$, and the light-front momentum is $\underline{p} = (p^+ \equiv E + p_z, \vec{p}_\perp \equiv (p_x, p_y))$. The mass-shell condition $p^2 = m^2$ becomes $p^- = (\vec{p}_\perp^2 + m^2)/p^+$. Further details can be found in the review [16].

For static sources, fixed in a lab frame, the eigenstates are no longer eigenstates of light-front energy or momentum. The ordinary momentum is zero, including the z component. We therefore seek eigenstates of the ordinary energy with use of the operator $\mathcal{E} \equiv \frac{1}{2}(\mathcal{P}^- + \mathcal{P}^+)$ but do so in terms of light-front coordinates. Such a choice is also motivated by the fact that the definition of an effective potential is in the dependence of the ordinary energy on the source separation. Most calculations in light-front-quantized theories need not make this distinction because they use a basis where the light-front momentum is held fixed. With P_z no longer conserved, holding P^+ fixed is not possible.

For comparison, we do consider a variational analysis of $\mathcal{P}^-$ alone. This is most easily done after the calculations done for $\mathcal{E}$, in that one can simply remove terms associated with $\mathcal{P}^+$. This is carried out in Appendix C. The key result of this analysis is that an unregulated divergence appears due to the lack of P^+ conservation; integrals over individual longitudinal momenta have no upper bound. Introduction of an arbitrary cutoff would destroy the rotational symmetry of the effective potential.

The parameters of the Yukawa Lagrangian are renormalized by fixing the mass of the dressed fermion state at a physical value m and by requiring the effective potential to be of the standard Yukawa form $-\frac{g^2}{4\pi R} e^{-\mu R}$, where R is the source separation, μ the renormalized boson mass, and g the physical coupling. For the quenched case considered here, the boson mass in the Lagrangian is not actually renormalized. The form of the effective potential is obtained, including its rotational symmetry.

In Sec. II we summarize the structure of light-front Yukawa theory with PV regularization and of the Hamiltonian eigenvalue problem for static fermions. The solution for a single source is developed in Sec. III and for a double source in Sec. IV. We summarize the results in Sec. V and leave some details to appendixes.

II. LIGHT-FRONT YUKAWA THEORY

The Lagrangian for PV-regulated Yukawa theory is [16][5]

$$\mathcal{L} = \sum_k r_k \left[\frac{1}{2} (\partial_\mu \phi_k)^2 - \frac{1}{2} \mu_k^2 \phi_k^2 \right] + \sum_i s_i \bar{\psi}_i (i\gamma^\mu \partial_\mu - m_i) \psi_i - g_0 \sum_{ijk} \beta_i \beta_j \xi_k \bar{\psi}_i \psi_j \phi_k, \tag{2.1}$$

where $k = i = j = 0$ correspond to physical fields and positive integers to PV fields. The r_k and s_i are metric signatures, with $r_0 = s_0 = 1$ for the physical fields, $r_1 = s_1 = -1$ for the first PV fields, and the remainder (if any) determined by the constraints necessary for a consistent theory. The factors of β_i and ξ_k provide for adjustments of the relative couplings of the PV fields, with $\beta_0 = \xi_0 = 1$ as a definition of g_0 as the bare coupling for physical fields. The regularization of loops is provided by the constraints [16]

$$\sum_k r_k \xi_k^2 = 0 \quad \text{and} \quad \sum_i s_i \beta_i^2 = 0, \tag{2.2}$$

because one or the other of these combinations appears for each line in a loop when summed over the PV contributions. The r_k and s_i factors come from the propagator, and the ξ_k and β_i come from the vertices at the ends of the line. The leading UV divergence then involves these sums and is canceled by the constraints (2.2). For example, a boson line carrying momentum q contributes $\sum_k r_k \xi_k^2 / q_\perp^2 + \mathcal{O}(\sum_k r_k m_k^2 \xi_k^2 / q_\perp^4)$, and the leading term is zero. This cancellation will be seen explicitly in the next section. The nonperturbative calculation is effectively a resummation of the regulated loop expansion.

The mode expansions for the boson fields are

$$\phi_k(x) = \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}} \left[a_k(\underline{p}) e^{-ip\cdot x} + a_k^\dagger(\underline{p}) e^{ip\cdot x} \right], \tag{2.3}$$

with the nonzero commutation relations being

$$[a_k(\underline{p}), a_{k'}^\dagger(\underline{p}')] \equiv r_k \delta_{kk'} \delta(p^+ - p'^+) \delta(\vec{p}_\perp - \vec{p}'_\perp). \tag{2.4}$$

The factor $r_k = \pm 1$ fixes the sign of the metric for the field.

[5] In Eq. (223) of [16] the Yukawa Lagrangian is expressed explicitly in terms of separate fields rather than a more general sum. Here instead we present it in a form analogous to the QED Lagrangian in Eq. (131).

The fermion field satisfies the Euler-Lagrange equation

$$s_i(i\gamma^\mu\partial_\mu - m_i)\psi_i - g_0\sum_{jk}\beta_i\beta_j\xi_k\phi_k\psi_j = 0. \tag{2.5}$$

To separate the dynamical part $\psi_{i+} \equiv \frac{1}{2}\gamma^0\gamma^+\psi_i$ from the constrained part $\psi_{i-} \equiv \frac{1}{2}\gamma^0\gamma^-\psi_i$, we project this Euler-Lagrange equation using $\frac{1}{2}\gamma^0\gamma^\pm$ to yield the following two equations:

$$s_i i\partial_+\psi_{i+} - s_i(-i\vec{\alpha}_\perp\cdot\partial_\perp + \beta m_i)\psi_{i-} - g_0\sum_{jk}\beta_i\beta_j\xi_k\phi_k\psi_{j-} = 0 \tag{2.6}$$

and

$$s_i i\partial_-\psi_{i-} - s_i(-i\vec{\alpha}_\perp\cdot\partial_\perp + \beta m_i)\psi_{i+} - g_0\sum_{jk}\beta_i\beta_j\xi_k\phi_k\psi_{j+} = 0, \tag{2.7}$$

with $\gamma^\mu = (\beta, \beta\vec{\alpha})$. Multiplication of the constraint equation (2.7) by $s_i\beta_i$ and a sum over i eliminates the interaction,[6] such that the constrained part of the summed fermion field $\psi \equiv \sum_i \beta_i\psi_i$ satisfies

$$i\partial_-\psi_- - \left(-i\vec{\alpha}_\perp\cdot\partial_\perp\psi_+ + \beta\sum_i\beta_i m_i\psi_{i+}\right) = 0. \tag{2.8}$$

This is just the constraint equation for a free fermion. We can then construct the Hamiltonian from the free-fermion mode expansion

$$\psi_i(x) = \int \frac{d\underline{p}}{\sqrt{16\pi^3 p^+}}\sum_{s=\pm 1/2}\left[b_{is}(\underline{p})u_{is}(\underline{p})e^{-ip\cdot x} + d_{is}^\dagger(\underline{p})v_{is}(\underline{p})e^{ip\cdot x}\right], \tag{2.9}$$

with

$$u_{is}(\underline{p}) \equiv \frac{1}{\sqrt{p^+}}[p^+ + \vec{\alpha}_\perp\cdot\vec{p}_\perp + \beta m_i]\chi_s,$$
$$v_{is}(\underline{p}) \equiv \frac{1}{\sqrt{p^+}}[p^+ + \vec{\alpha}_\perp\cdot\vec{p}_\perp - \beta m_i]\chi_{-s} \tag{2.10}$$

and

$$\chi_{+\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \qquad \chi_{-\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}. \tag{2.11}$$

The nonzero anticommutators are

$$\{b_{is}(\underline{p}), b_{js'}^\dagger(\underline{p}')\} = s_i\delta_{ij}\delta_{ss'}\delta(\underline{p}-\underline{p}'),$$
$$\{d_{is}(\underline{p}), d_{js'}^\dagger(\underline{p}')\} = s_i\delta_{ij}\delta_{ss'}\delta(\underline{p}-\underline{p}'), \tag{2.12}$$

and $s_i = \pm 1$ sets the metric.

Instantaneous fermion interactions do not appear. However, the physics of these interactions has not been lost; they are present implicitly and restored explicitly in the limit of infinite PV masses, which shrinks a PV-fermion exchange to a contact interaction [25].

The light-front Hamiltonian density is then

$$\mathcal{H} = \sum_k r_k\left[\frac{1}{2}(\vec{\partial}_\perp\phi_k)^2 + \frac{1}{2}\mu_k^2\phi_k^2\right] + \sum_i s_i\left[\psi_{i+}^\dagger(i\vec{\alpha}_\perp\cdot\vec{\partial}_\perp - \beta m_i)\frac{1}{i\partial_-}(i\vec{\alpha}_\perp\cdot\vec{\partial}_\perp - \beta m_i)\psi_{i+}\right] + g_0\sum_{ijk}\beta_i\beta_j\xi_k\phi_k\overline{\psi}_i\psi_j. \tag{2.13}$$

The Hamiltonian

$$\mathcal{P}^- \equiv \int d\underline{x}:\mathcal{H}:|_{x^+=0} = \mathcal{P}_{0a}^- + \mathcal{P}_{0f}^- + \mathcal{P}_{\text{n.p.}}^- + \mathcal{P}_{\text{pair}}^- \tag{2.14}$$

is specified by

$$\mathcal{P}_{0a}^- = \sum_k r_k\int d\underline{q}\,\frac{\mu_k^2 + q_\perp^2}{q^+}a_k^\dagger(\underline{q})a_k(\underline{q}), \tag{2.15}$$

$$\mathcal{P}_{0f}^- = \sum_i s_i\int d\underline{p}\,\frac{m_i^2 + \vec{p}_\perp^2}{p^+}\times\sum_s[b_{is}^\dagger(\underline{p})b_{is}(\underline{p}) + d_{is}^\dagger(\underline{p})d_{is}(\underline{p})], \tag{2.16}$$

$$\mathcal{P}_{\text{n.p.}}^- = g_0\sum_{ijk}\beta_i\beta_j\xi_k\int d\underline{x}\int\frac{d\underline{q}}{\sqrt{16\pi^3 q^+}}\left[a_k(\underline{q})e^{-i\underline{q}\cdot\underline{x}} + a_k^\dagger(\underline{q})e^{i\underline{q}\cdot\underline{x}}\right]\int\frac{d\underline{p}_1 d\underline{p}_2}{16\pi^3}\sum_s\left\{\left[\left(\frac{m_i}{p_1^+} + \frac{m_j}{p_2^+}\right)b_{is}^\dagger(\underline{p}_1)b_{js}(\underline{p}_2)\right.\right.$$

[6] In [16], the analogous process for QED contains an error in the line above Eq. (138). The factors $(-1)^i\sqrt{\beta_i}$ should be replaced with $s_i\beta_i$.

$$+\left(\frac{\sqrt{2}\vec{\epsilon}_{-2s}\cdot\vec{p}_{1\perp}}{p_1^+}+\frac{\sqrt{2}\vec{\epsilon}^{*}_{2s}\cdot\vec{p}_{2\perp}}{p_2^+}\right)b^\dagger_{is}(\underline{p}_1)b_{j,-s}(\underline{p}_2)\bigg]e^{i(\underline{p}_1-\underline{p}_2)\cdot\underline{x}}+\left[\left(\frac{m_i}{p_1^+}+\frac{m_j}{p_2^+}\right)d^\dagger_{js}(\underline{p}_2)d_{is}(\underline{p}_1)\right.$$
$$\left.-\left(\frac{\sqrt{2}\vec{\epsilon}_{-2s}\cdot\vec{p}_{1\perp}}{p_1^+}+\frac{\sqrt{2}\vec{\epsilon}^{*}_{2s}\cdot\vec{p}_{2\perp}}{p_2^+}\right)d^\dagger_{js}(\underline{p}_2)d_{i,-s}(\underline{p}_1)\right]e^{i(\underline{p}_2-\underline{p}_1)\cdot\underline{x}}\bigg\}, \tag{2.17}$$

and

$$\mathcal{P}^-_{\rm pair}=g_0\sum_{ijk}\beta_i\beta_j\xi_k\int d\underline{x}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[a_k(\underline{q})e^{-i\underline{q}\cdot\underline{x}}+a^\dagger_k(\underline{q})e^{i\underline{q}\cdot\underline{x}}\right]\int\frac{d\underline{p}_1d\underline{p}_2}{16\pi^3}\sum_s\left\{\left[\left(\frac{m_i}{p_1^+}-\frac{m_j}{p_2^+}\right)b^\dagger_{is}(\underline{p}_1)d^\dagger_{j,-s}(\underline{p}_2)\right.\right.$$
$$+\left(\frac{\sqrt{2}\vec{\epsilon}_{2s}\cdot\vec{p}_{1\perp}}{p_1^+}+\frac{\sqrt{2}\vec{\epsilon}^{*}_{-2s}\cdot\vec{p}_{2\perp}}{p_2^+}\right)b^\dagger_{is}(\underline{p}_1)d^\dagger_{j,s}(\underline{p}_2)\bigg]e^{i(\underline{p}_1+\underline{p}_2)\cdot\underline{x}}+\left[\left(\frac{m_i}{p_1^+}-\frac{m_j}{p_2^+}\right)b_{js}(\underline{p}_2)d_{i,-s}(\underline{p}_1)\right.$$
$$\left.-\left(\frac{\sqrt{2}\vec{\epsilon}_{-2s}\cdot\vec{p}_{1\perp}}{p_1^+}+\frac{\sqrt{2}\vec{\epsilon}^{*}_{2s}\cdot\vec{p}_{2\perp}}{p_2^+}\right)b_{js}(\underline{p}_2)d_{i,s}(\underline{p}_1)\right]e^{-i(\underline{p}_1+\underline{p}_2)\cdot\underline{x}}\bigg\}, \tag{2.18}$$

with $\sqrt{2}\vec{\epsilon}_{2s}\equiv-(2s,i)$ a two-dimensional, transverse vector [29], where s is the spin index and $i=\sqrt{-1}$. The integral over $\underline{x}$ has been left undone in the interaction terms, to accommodate the source wave packets introduced in the next sections. For the quenched theory, the term $\mathcal{P}^-_{\rm pair}$ is, of course, dropped. Also, we will limit our work to the fermion sector, and antifermion terms in $\mathcal{P}^-_{0f}$ and $\mathcal{P}^-_{\rm n.p.}$ will play no role.

The light-front momentum operator is $\mathcal{P}^+=\mathcal{P}^+_a+\mathcal{P}^+_f$ with

$$\mathcal{P}^+_a=\sum_k r_k\int d\underline{q}\,q^+a^\dagger_k(\underline{q})a_k(\underline{q}) \tag{2.19}$$

and

$$\mathcal{P}^+_f=\sum_i s_i\int d\underline{p}\,p^+\sum_s[b^\dagger_{is}(\underline{p})b_{is}(\underline{p})+d^\dagger_{is}(\underline{p})d_{is}(\underline{p})]. \tag{2.20}$$

We can then define the ordinary energy operator

$$\mathcal{E}=\frac{1}{2}(\mathcal{P}^-+\mathcal{P}^+). \tag{2.21}$$

In the next sections we explore eigenstates of $\mathcal{E}$ that are associated with one or two static fermion sources.

III. SINGLE STATIC SOURCE

We first consider a single source at $\pm\vec{R}/2$ to establish the renormalization of the fermion mass. The static fermion is described by a wave packet centered at $\underline{p}=(m,\vec{0}_\perp)$ in momentum space and at $\underline{x}=(\mp R_z,\pm\vec{R}_\perp/2)$ in coordinate space on the $x^+=0$ slice.[7] The static fermion state with spin s and PV type i is then

$$|F^\pm_{is}\rangle=\int_{p^+>0}d\underline{p}F^\pm(\underline{p})b^\dagger_{is}(\underline{p})|0\rangle_f, \tag{3.1}$$

with the function $F^\pm$ peaked at $\underline{p}=(m,\vec{0}_\perp)$ and its Fourier transform $\psi^\pm(\underline{x})$ peaked at $(\mp R_z,\pm\vec{R}_\perp/2)$. Here $|0\rangle_f$ is the vacuum annihilated by b_{is}. The physical and PV fermions are all static at the same location and at the same light-front momentum.

The Fourier transform is defined as

$$F^\pm(\underline{p})=\int\frac{d\underline{x}}{\sqrt{16\pi^3}}e^{i\underline{p}\cdot\underline{x}}\psi^\pm(\underline{x}). \tag{3.2}$$

For the inverse, where the p^+ integration is limited to positive values, we take advantage of the narrow peak in $F^\pm$ to extend the p^+ integral to $-\infty$

$$\psi^\pm(\underline{x})=\int\frac{d\underline{p}}{\sqrt{16\pi^3}}e^{-i\underline{p}\cdot\underline{x}}F^\pm(\underline{p}). \tag{3.3}$$

For a static source at $\underline{x}=(\mp R_z,\pm\vec{R}_\perp/2)$ we require that

$$|\psi^\pm(\underline{x})|^2\to\delta(x^-\pm R_z)\delta(\vec{x}_\perp\mp\vec{R}_\perp/2). \tag{3.4}$$

The common normalization

[7] The translation of the longitudinal coordinate, between a lab-fixed frame and light-front coordinates, is illustrated in Fig. 1 of [21]. We have $z=\pm R_z/2$ fixed and $x^+=t+z=0$; therefore, the ordinary time is $t=\mp R_z/2$ and the light-front spatial coordinate is $x^-=t-z=\mp R_z$.

$$1 = \int d\underline{x}|\psi^\pm(\underline{x})|^2 = \int d\underline{p}|F^\pm(\underline{p})|^2 \tag{3.5}$$

is fixed by requiring the indefinite norm

$$\langle F_{is}^\pm|F_{is}^\pm\rangle = \int d\underline{p}'d\underline{p}F^{\pm *}(\underline{p}')F^\pm(\underline{p})s_i\delta(\underline{p}'-\underline{p}) = s_i\int d\underline{p}|F^\pm(\underline{p})|^2 = s_i. \tag{3.6}$$

From this static-fermion state, we build a fermion state dressed by a cloud of bosons in a coherent state as the ansatz for the energy eigenstate:

$$|G^\pm F^\pm; s\rangle \equiv \sum_i C_i^\pm |G_i^\pm\rangle|F_{is}^\pm\rangle \tag{3.7}$$

with

$$|G_i^\pm\rangle \equiv \sqrt{Z_i^\pm}\left[\prod_k \exp\left(\int d\underline{q}G_{ki}^\pm(\underline{q})a_k^\dagger(\underline{q})\right)\right]|0\rangle_a \tag{3.8}$$

and $|0\rangle_a$ the vacuum annihilated by a_k. Because the spin-flip terms in $\mathcal{P}^-_{\rm n.p.}$ are proportional to $\vec{p}_\perp$ and the static state requires $\langle\vec{p}_\perp\rangle$ to be zero, the eigenstate is diagonal in spin, and the coefficient $C_i^\pm$ and the functions $G_{ki}^\pm$ are independent of s. The $\sqrt{Z_i^\pm}$ are normalization factors for the coherent state, given by

$$Z_i^\pm = \exp\left(-\sum_k r_k\int d\underline{q}|G_{ki}^\pm(\underline{q})|^2\right). \tag{3.9}$$

This is then to be the solution to

$$\mathcal{E}|G^\pm F^\pm; s\rangle = E^\pm|G^\pm F^\pm; s\rangle, \tag{3.10}$$

with $E^\pm = m$ for the ground state, which is the state of interest. Each term in the sum over i is also an eigenstate of the boson annihilation operators a_k, as is always the case for a coherent state:

$$a_k(\underline{q})|G_i^\pm\rangle = r_kG_{ki}^\pm(\underline{q})|G_i^\pm\rangle. \tag{3.11}$$

We begin with a projection of the eigenvalue problem onto a static fermion of type i

$$s_i\langle F_{is}^\pm|\mathcal{E}|G^\pm F^\pm;s\rangle = E^\pm s_i\langle F_{is}^\pm|G^\pm F^\pm;s\rangle = E^\pm C_i^\pm|G_i^\pm\rangle. \tag{3.12}$$

This can be reduced with use of the following projections for individual terms in $\mathcal{E} = \frac{1}{2}(\mathcal{P}^-_{0f} + \mathcal{P}^+_f + \mathcal{P}^-_{0a} + \mathcal{P}^+_a + \mathcal{P}^-_{\rm n.p.})$:

$$s_i\langle F_{is}^\pm|\frac{1}{2}(\mathcal{P}^-_{0f}+\mathcal{P}^+_f)|G^\pm F^\pm;s\rangle = \frac{1}{2}C_i^\pm\int d\underline{p}\left[\frac{m_i^2+p_\perp^2}{p^+}+p^+\right]|F_i^\pm(\underline{p})|^2|G_i^\pm\rangle = C_i^\pm\left(\frac{m_i^2}{2m}+\frac{m}{2}\right)|G_i^\pm\rangle, \tag{3.13}$$

$$s_i\langle F_{is}^\pm|\frac{1}{2}(\mathcal{P}^-_{0a}+\mathcal{P}^+_a)|G^\pm F^\pm;s\rangle = \frac{1}{2}C_i^\pm\sum_k\int d\underline{q}\left[\frac{\mu_k^2+q_\perp^2}{q^+}+q^+\right]a_k^\dagger(\underline{q})G_{ki}^\pm(\underline{q})|G_i^\pm\rangle, \tag{3.14}$$

and

$$s_i\langle F_{is}^\pm|\frac{1}{2}\mathcal{P}^-_{\rm n.p.}|G^\pm F^\pm;s\rangle = \frac{1}{2}g_0\beta_i\sum_{jk}s_j\beta_j\xi_kC_j^\pm\frac{m_i+m_j}{m} \times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[r_kG_{kj}^\pm(\underline{q})e^{\pm i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}+a_k^\dagger(\underline{q})e^{\mp i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\right]|G_j^\pm\rangle. \tag{3.15}$$

Details of the reduction for the $\mathcal{P}^-_{\rm n.p.}$ projection can be found in Appendix A. With the combination of all of these terms, the projected single-source eigenvalue problem becomes

$$C_i^\pm\left(\frac{m_i^2}{2m}+\frac{m}{2}\right)|G_i^\pm\rangle + \frac{1}{2}C_i^\pm\sum_k\int d\underline{q}\left[\frac{\mu_k^2+q_\perp^2}{q^+}+q^+\right]a_k^\dagger(\underline{q})G_{ki}^\pm(\underline{q})|G_i^\pm\rangle + \frac{1}{2}g_0\beta_i\sum_{jk}s_j\beta_j\xi_kC_j^\pm\frac{m_i+m_j}{m} \times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[r_kG_{kj}^\pm(\underline{q})e^{\pm i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}+a_k^\dagger(\underline{q})e^{\mp i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\right]|G_j^\pm\rangle = E^\pm C_i^\pm|G_i^\pm\rangle. \tag{3.16}$$

For this to hold, the coefficient of $a_k^\dagger(\underline{q})$ must be zero, to remove states with additional particles from the left-hand side:

$$0 = \frac{1}{2}C_i^\pm\left[\frac{\mu_k^2+q_\perp^2}{q^+}+q^+\right]G_{ki}^\pm(\underline{q})|G_i^\pm\rangle + \frac{1}{2}g_0\beta_i\sum_j s_j\beta_j\xi_k C_j^\pm\frac{m_i+m_j}{m} \times\frac{1}{\sqrt{16\pi^3 q^+}}e^{\mp i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}|G_j^\pm\rangle. \tag{3.17}$$

A slight rearrangement yields an implicit expression for $G_{ki}^\pm$

$$C_i^\pm G_{ki}^\pm(\underline{q})|G_i^\pm\rangle = -\frac{g_0\beta_i\xi_k}{\sqrt{16\pi^3q^+}}\frac{e^{\mp i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}}{\frac{\mu_k^2+q_\perp^2}{q^+}+q^+} \times\sum_j s_j\beta_j C_j^\pm\frac{m_i+m_j}{m}|G_j^\pm\rangle. \tag{3.18}$$

The eigenvalue problem (3.16) reduces to

$$C_i^\pm\left(\frac{m_i^2}{2m}+\frac{m}{2}\right)|G_i^\pm\rangle+\frac{1}{2}g_0\beta_i\sum_{jk}s_j\beta_j\xi_k C_j^\pm\frac{m_i+m_j}{m}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}r_kG_{kj}^\pm(\underline{q})e^{\pm i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}|G_j^\pm\rangle = E^\pm C_i^\pm|G_i^\pm\rangle. \tag{3.19}$$

On substitution of the expression (3.18) for $C_j^\pm G_{kj}^\pm(\underline{q})|G_j^\pm\rangle$, this becomes

$$C_i^\pm\left(\frac{m_i^2}{2m}+\frac{m}{2}\right)|G_i^\pm\rangle-\frac{g_0^2}{2}\mu I\beta_i\sum_j s_j\beta_j^2\frac{m_i+m_j}{m}\sum_{j'}s_{j'}\beta_{j'}C_{j'}^\pm\frac{m_j+m_{j'}}{m}|G_{j'}^\pm\rangle = E^\pm C_i^\pm|G_i^\pm\rangle, \tag{3.20}$$

with I the dimensionless self-energy integral

$$I\equiv\int\frac{d\underline{q}}{16\pi^3\mu}\sum_k\frac{r_k\xi_k^2}{(q^+)^2+q_\perp^2+\mu_k^2} \tag{3.21}$$

and μ the physical mass of the boson. The constraint $\sum_k r_k\xi_k^2=0$ on the ξ_k factors makes I finite.

This defines an $n_f\times n_f$ matrix problem

$$\left(\frac{m_0^2}{2m}+\frac{m}{2}\right)C_i^\pm|G_i^\pm\rangle+\sum_j V_{ij}C_j^\pm|G_j^\pm\rangle = E^\pm C_i^\pm|G_i^\pm\rangle, \tag{3.22}$$

where n_f is the number of fermion types and

$$V_{ij} = -\frac{g_0^2}{2}\mu I\beta_i s_j\beta_j\sum_{j'}s_{j'}\beta_{j'}^2\frac{m_i+m_{j'}}{m}\frac{m_{j'}+m_j}{m}. \tag{3.23}$$

Now we convert the Fock-space equation (3.22) into an algebraic equation by projecting it onto $\langle G_i^\pm|$

$$\left(\frac{m_0^2}{2m}+\frac{m}{2}\right)C_i^\pm+\sum_j V_{ij}\zeta_{ij}^\pm C_j^\pm = E^\pm C_i^\pm, \tag{3.24}$$

given the overlap integrals

$$\zeta_{ij}^\pm = \langle G_i^\pm|G_j^\pm\rangle = \sqrt{Z_iZ_j}\exp\left(-\sum_k r_k\int d\underline{q}\,G_{ki}^{\pm *}(\underline{q})G_{kj}^\pm(\underline{q})\right), \tag{3.25}$$

with $\zeta_{ii}^\pm=1$ and $\zeta_{ij}^{\pm *}=\zeta_{ji}^\pm$. The nontrivial $\zeta_{ij}^\pm$ can be computed from nonlinear equations for self-consistency with the solution for $G_{ki}^\pm$, which arises in the projection of (3.18) onto $\langle G_i^\pm|$:

$$C_i^\pm G_{ki}^\pm(\underline{q}) = -\frac{g_0\beta_i\xi_k}{\sqrt{16\pi^3q^+}}\frac{e^{\mp i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}}{\frac{\mu_k^2+q_\perp^2}{q^+}+q^+} \times\sum_j s_j\beta_j C_j^\pm\frac{m_i+m_j}{m}\zeta_{ij}^\pm. \tag{3.26}$$

The two equations (3.24) and (3.25) must then be solved simultaneously, with G_{ki} in (3.25) given by (3.26). The only $\vec{R}$ dependence appears in the exponentials which cancel in (3.25), leaving the equations independent of $\vec{R}$. Therefore, the ground state determines the physical fermion mass $m=E^\pm$ independent of the source location $\pm\vec{R}/2$, and the eigenvalue problem (3.24) provides the renormalization of the bare mass m_0 implicitly by giving m as a function of m_0. We then use these solutions to construct a solution for the two-source case in the next section. For this purpose, an explicit solution of the (3.24)–(3.25) system will not be needed.

IV. TWO STATIC SOURCES

To compute the effective potential between two sources a distance R apart, we place them at $\underline{x} = (\mp R_z, \pm\vec{R}_\perp/2)$ and construct the eigenstate of the ordinary energy $\mathcal{E}$. The effective potential is then the difference between the eigenvalue and the total rest mass $2m$, with m specified by the single-source problem solved in the previous section. Following the case of scalar Yukawa theory [21], we construct an ansatz for the eigenstate as a product of single-source solutions:

$$|G^+G^-F^+F^-;s_1s_2\rangle \equiv \sum_{ij} C_{ij}|G_i^+F_{is_1}^+\rangle|G_j^-F_{js_2}^-\rangle. \tag{4.1}$$

This is to be a solution of

$$\mathcal{E}|G^+G^-F^+F^-;s_1s_2\rangle = E|G^+G^-F^+F^-;s_1s_2\rangle. \tag{4.2}$$

We proceed as before with a projection onto the static fermion states

$$s_is_j\langle F_{js_2}^-|\langle F_{is_1}^+|\mathcal{E}|G^+G^-F^+F^-;s_1s_2\rangle = Es_is_j\langle F_{js_2}^-|\langle F_{is_1}^+|G^+G^-F^+F^-;s_1s_2\rangle. \tag{4.3}$$

Cross terms between F^+ and F^- do not contribute because they are proportional to

$$\int d\underline{p}F^{+*}(\underline{p})F^-(\underline{p}) = \int \frac{d\underline{x}}{16\pi^3}\psi^{+*}(\underline{x})\psi^-(\underline{x}), \tag{4.4}$$

and the second integral is zero from the lack of overlap between narrow wave packets centered apart. Again, the spin-flip terms of $\mathcal{P}^-_{\text{n.p.}}$ do not contribute, being proportional to the transverse fermion momentum $\vec{p}_\perp$ for which the expectation value is zero. The right-hand side of (4.3) is

$$Es_is_j\langle F_{js_2}^-|\langle F_{is_1}^+|G^+G^-F^+F^-;s_1s_2\rangle = EC_{ij}|G_i^+\rangle|G_j^-\rangle. \tag{4.5}$$

The projected terms in $\mathcal{E}$ for the left-hand side are

$$\begin{aligned}
&s_is_j\langle F_{js_2}^-|\langle F_{is_1}^+|\frac{1}{2}(\mathcal{P}^-_{0f}+\mathcal{P}^+_f)|G^+G^-F^+F^-;s_1s_2\rangle \\
&= \frac{1}{2}C_{ij}\int d\underline{p}\left\{\left[\frac{m_i^2+p_\perp^2}{p^+}+p^+\right]|F_i^+(\underline{p})|^2+\left[\frac{m_j^2+p_\perp^2}{p^+}+p^+\right]|F_j^+(\underline{p})|^2\right\}|G_i^+\rangle|G_j^-\rangle \\
&= C_{ij}\left(\frac{m_i^2}{2m}+\frac{m_j^2}{2m}+m\right)|G_i^+\rangle|G_j^-\rangle,
\end{aligned} \tag{4.6}$$

$$s_is_j\langle F_{js_2}^-|\langle F_{is_1}^+|\frac{1}{2}(\mathcal{P}^-_{0a}+\mathcal{P}^+_a)|G^+G^-F^+F^-;s_1s_2\rangle = \frac{1}{2}C_{ij}\sum_k\int d\underline{q}\left[\frac{\mu_k^2+q_\perp^2}{q^+}+q^+\right]a_k^\dagger(\underline{q})a_k(\underline{q})|G_i^+\rangle|G_j^-\rangle, \tag{4.7}$$

and

$$\begin{aligned}
&s_is_j\langle F_{js_2}^-|\langle F_{is_1}^+|\frac{1}{2}\mathcal{P}^-_{\text{n.p.}}|G^+G^-F^+F^-;s_1s_2\rangle \\
&= \frac{1}{2}g_0\sum_k\xi_k\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left\{\beta_i\sum_{i'}s_{i'}\beta_{i'}C_{i'j}\left[a_k(\underline{q})e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}+a_k^\dagger(\underline{q})e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\right]\frac{m_i+m_{i'}}{m}|G_{i'}^+\rangle|G_j^-\rangle\right. \\
&\left.+\beta_j\sum_{j'}s_{j'}\beta_{j'}C_{ij'}\left[a_k(\underline{q})e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}+a_k^\dagger(\underline{q})e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\right]\frac{m_j+m_{j'}}{m}|G_i^+\rangle|G_{j'}^-\rangle\right\}.
\end{aligned} \tag{4.8}$$

Details of this last projection are again in Appendix A. The boson annihilation operators in (4.7) and (4.8) can be replaced with use of $a_k(\underline{q})|G_i^+\rangle|G_j^-\rangle = r_k[G_{ki}^+(\underline{q})+G_{kj}^-(\underline{q})]|G_i^+\rangle|G_j^-\rangle$, which is the two-source extension of (3.11), where a coherent state is an eigenstate of the annihilation operator.

The projected eigenvalue problem is

$$\begin{aligned}
&C_{ij}\left\{\frac{m_i^2}{2m}+\frac{m_j^2}{2m}+m+\frac{1}{2}\sum_k\int d\underline{q}\left[\frac{\mu_k^2+q_\perp^2}{q^+}+q^+\right]a_k^\dagger(\underline{q})r_k[G_{ki}^+(\underline{q})+G_{kj}^-(\underline{q})]\right\}|G_i^+\rangle|G_j^-\rangle \\
&+\frac{1}{2}g_0\sum_k\xi_k\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left\{\beta_i\sum_{i'}s_{i'}\beta_{i'}C_{i'j}\left[r_k[G_{ki'}^+(\underline{q})+G_{kj}^-(\underline{q})]e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\right.\right.
\end{aligned}$$

$$+a_k^\dagger(\underline{q})e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\big]\frac{m_i+m_{i'}}{m}|G_{i'}^+\rangle|G_j^-\rangle$$
$$+\beta_j\sum_{j'}s_{j'}\beta_{j'}C_{ij'}\big[r_k[G_{ki}^+(\underline{q})+G_{kj'}^-(\underline{q})]e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}$$
$$+a_k^\dagger(\underline{q})e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\big]\frac{m_j+m_{j'}}{m}|G_i^+\rangle|G_{j'}^-\rangle\Big\} = EC_{ij}|G_i^+\rangle|G_j^-\rangle. \quad (4.9)$$

The reduction of this eigenvalue problem is detailed in Appendix B.

From the reduction we obtain the effective potential $V_{\rm eff}\equiv E-2m$ as

$$V_{\rm eff}=-\frac{g_0^2}{2}\left(\sum_i\frac{s_i\beta_i^2m_i}{m}\right)^2\frac{e^{-\mu R}}{8\pi R}, \quad (4.10)$$

which is clearly rotationally invariant. We define the physical coupling g by a match to the standard form for the Yukawa potential $-\frac{g^2}{4\pi R}e^{-\mu R}$, which implies

$$g=g_0\sum_i\frac{s_i\beta_i^2m_i}{2m}. \quad (4.11)$$

We can, of course, have $g=g_0$ if we include 2 PV fermions and impose the additional constraint $\sum_i s_i\beta_i^2m_i=2m$.

V. SUMMARY

In this work we have thus obtained the standard, rotationally invariant Yukawa potential as the effective potential between two static sources in quenched, light-front Yukawa theory. The effective potential comes from the interference between the two boson clouds that dress the individual sources and is computed nonperturbatively. The rotational invariance exists despite the special status for the z axis in light-front quantization.

The key to our approach is to recognize the ordinary energy as the relevant quantity, both because momentum is not conserved when sources are static and because the effective potential should be defined in terms of this energy. Light-front energy combines energy and a momentum component, making it only indirectly related.

To carry out the calculation, we have introduced Pauli-Villars fermions and bosons. The PV fermions eliminate instantaneous interaction terms which would otherwise interfere with the construction of analytic solutions. The PV bosons regulate the infinite self-energy of the sources. The couplings of these are adjusted to satisfy constraints that guarantee the regularization, the correct mass and coupling renormalizations, and the removal of instantaneous fermion interactions from the light-front Hamiltonian. The instantaneous interactions are restored in the limit of infinite PV mass.

Given the successful derivation of a rotationally invariant potential in quenched Yukawa theory, the next step to be taken is to introduce pair production and annihilation of free fermions and their own accompanying PV counterparts. (The PV fermions associated with the static sources need to be separate because they are themselves static.) With pairs included in the basis, a coherent state solution will no longer be available as the full solution, and Fock-space methods must be invoked. To have an eigenvalue problem of finite size will then require truncation, either explicitly in Fock space or in the operator sense of the light-front coupled-cluster method [30]. The effects of pairs will include renormalization of the boson mass, renormalization of the static fermion coupling, and modifications of the form of the effective potential. At short separations, these modifications will be due to the charge renormalization and the screening that takes place. At large separations, pairs provide for the Yukawa analog of string breaking. Completion of this work in Yukawa theory will provide a useful reference for the calculation of effective potentials in QED and QCD.

APPENDIX A: PROJECTIONS FOR THE NO-PAIR CONTRIBUTION

We first consider the single-source case. Substitution of the definitions of the individual factors $|F_{is}^\pm\rangle$, $\mathcal{P}_{\rm n.p.}^-$, and $|G^\pm F^\pm;s\rangle$, as given in (3.1), (2.17), and (3.7), respectively, yields

$$s_i\langle F_{is}^\pm|\frac{1}{2}\mathcal{P}_{\rm n.p.}^-|G^\pm F^\pm;s\rangle=\frac{1}{2}g_0s_i\int d\underline{p}'F^{\pm*}(\underline{p}')_f\langle 0|b_{is}(\underline{p}')\sum_{i'jk}\beta_{i'}\beta_j\xi_k\int d\underline{x}\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[a_k(\underline{q})e^{-i\underline{q}\cdot\underline{x}}+a^\dagger(\underline{q})e^{i\underline{q}\cdot\underline{x}}\right]$$
$$\times\int\frac{d\underline{p}_1d\underline{p}_2}{16\pi^3}\left(\frac{m_{i'}}{p_1^+}+\frac{m_j}{p_2^+}\right)e^{i(\underline{p}_1-\underline{p}_2)\cdot\underline{x}}\sum_{s'}b_{i's'}^\dagger(\underline{p}_1)b_{js'}(\underline{p}_2)\sum_{i''}C_{i''}^\pm\int d\underline{p}F^\pm(\underline{p})b_{i''s}^\dagger(\underline{p})|0\rangle_f|G_{i''}^\pm\rangle. \quad (A1)$$

Given the contractions

$$b_{is}(\underline{p}')b^{\dagger}_{i's'}(\underline{p}_1) \to s_i\delta_{ii'}\delta_{ss'}\delta(\underline{p}'-\underline{p}_1) \tag{A2}$$

and

$$b_{js'}(\underline{p}_2)b^{\dagger}_{i''s}(\underline{p}) \to s_j\delta_{ji''}\delta_{s's}\delta(\underline{p}-\underline{p}_2), \tag{A3}$$

the Kronecker and Dirac deltas and the property $s_i^2 = 1$ can be used to reduce the expression to

$$s_i\langle F^{\pm}_{is}|\frac{1}{2}\mathcal{P}^{-}_{\text{n.p.}}|G^{\pm}F^{\pm};s\rangle = \frac{1}{2}g_0\beta_i\sum_{jk}s_j\beta_j\xi_k\int d\underline{x}\int\frac{d\underline{p}'}{\sqrt{16\pi^3}}F^{\pm*}(\underline{p}')e^{i\underline{p}'\cdot\underline{x}} \\ \times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[a_k(\underline{q})e^{-i\underline{q}\cdot\underline{x}}+a^{\dagger}(\underline{q})e^{i\underline{q}\cdot\underline{x}}\right]\int\frac{d\underline{p}}{\sqrt{16\pi^3}}F^{\pm}(\underline{p})e^{-i\underline{p}\cdot\underline{x}}\left(\frac{m_i}{p'^+}+\frac{m_j}{p^+}\right)C^{\pm}_j|G^{\pm}_j\rangle. \tag{A4}$$

The momentum-space wave function $F^{\pm}$ is peaked at $(m,\vec{0}_{\perp})$, which allows any factor of p^+ or p'^+ to be replaced with m. The two Fourier transforms of $F^{\pm}$ can be written in terms of the spatial wave function ψ in (3.3) and the product replaced by the coordinate-space delta functions of (3.4):

$$\int\frac{d\underline{p}'}{\sqrt{16\pi^3}}F^{\pm*}(\underline{p}')e^{i\underline{p}'\cdot\underline{x}}\int\frac{d\underline{p}}{\sqrt{16\pi^3}}F^{\pm}(\underline{p})e^{-i\underline{p}\cdot\underline{x}} = |\psi^{\pm}(\underline{x})|^2 \to \delta(x^-\pm R_z)\delta(\vec{x}_{\perp}\mp\vec{R}_{\perp}/2). \tag{A5}$$

Integration over $\underline{x}$ then gives

$$s_i\langle F^{\pm}_{is}|\frac{1}{2}\mathcal{P}^{-}_{\text{n.p.}}|G^{\pm}F^{\pm};s\rangle = \frac{1}{2}g_0\beta_i\sum_{jk}s_j\beta_j\xi_kC^{\pm}_j\frac{m_i+m_j}{m} \\ \times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[a_k(\underline{q})e^{\pm i(q^+R_z+\vec{q}_{\perp}\cdot\vec{R}_{\perp})/2}+a^{\dagger}(\underline{q})e^{\mp i(q^+R_z+\vec{q}_{\perp}\cdot\vec{R}_{\perp})/2}\right]|G^{\pm}_j\rangle. \tag{A6}$$

Use of (3.11) then yields the result given in (3.15).

When there are two sources, the projection needed is

$$s_is_j\langle F^{-}_{js_2}|\langle F^{+}_{is_1}|\frac{1}{2}\mathcal{P}^{-}_{\text{n.p.}}|G^+G^-F^+F^-;s_1s_2\rangle \\ = \frac{1}{2}s_is_j\,{}_f\langle 0|\int d\underline{p}_2'F^{-*}(\underline{p}_2')b_{js_2}(\underline{p}_2')\int d\underline{p}_1'F^{+*}(\underline{p}_1')b_{is_1}(\underline{p}_1')g_0\sum_{lmk}\beta_l\beta_m\xi_k\int d\underline{x} \\ \times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[a_k(\underline{q})e^{-i\underline{q}\cdot\underline{x}}+a^{\dagger}_k(\underline{q})e^{i\underline{q}\cdot\underline{x}}\right]\int\frac{d\underline{p}_1d\underline{p}_2}{16\pi^3}e^{i(\underline{p}_1-\underline{p}_2)\cdot\underline{x}}\left(\frac{m_l}{p_1^+}+\frac{m_m}{p_2^+}\right)\sum_s b^{\dagger}_{ls}(\underline{p}_1)b_{ms}(\underline{p}_2) \\ \times\sum_{i'j'}C_{i'j'}\int d\underline{p}_1''F^+(\underline{p}_1'')b^{\dagger}_{i's_1}(\underline{p}_1'')\int d\underline{p}_2''F^-(\underline{p}_2'')b^{\dagger}_{j's_2}(\underline{p}_2'')|0\rangle_f|G^+_{i'}\rangle|G^-_{j'}\rangle. \tag{A7}$$

Contraction of $b_{ms}(\underline{p}_2)$ with each of the rightmost creation operators yields

$$s_is_j\langle F^{-}_{js_2}|\langle F^{+}_{is_1}|\frac{1}{2}\mathcal{P}^{-}_{\text{n.p.}}|G^+G^-F^+F^-;s_1s_2\rangle \\ = \frac{1}{2}s_is_j\,{}_f\langle 0|\int d\underline{p}_2'F^{-*}(\underline{p}_2')b_{js_2}(\underline{p}_2')\int d\underline{p}_1'F^{+*}(\underline{p}_1')b_{is_1}(\underline{p}_1')g_0\sum_{lmk}\beta_l\beta_m\xi_k\int d\underline{x} \\ \times\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}}\left[a_k(\underline{q})e^{-i\underline{q}\cdot\underline{x}}+a^{\dagger}_k(\underline{q})e^{i\underline{q}\cdot\underline{x}}\right]\int\frac{d\underline{p}_1d\underline{p}_2}{16\pi^3}e^{i(\underline{p}_1-\underline{p}_2)\cdot\underline{x}}\left(\frac{m_l}{p_1^+}+\frac{m_m}{p_2^+}\right)\sum_s b^{\dagger}_{ls}(\underline{p}_1)$$

$$\times \sum_{i'j'} C_{i'j'} \left\{ \int d\underline{p}_1'' F^+(\underline{p}_1'') \int d\underline{p}_2'' F^-(\underline{p}_2'') s_m \delta_{ss_1} \delta_{mi'} \delta(\underline{p}_2 - \underline{p}_1'') b^\dagger_{j's_2}(\underline{p}_2'') \right.$$
$$\left. - \int d\underline{p}_1'' F^+(\underline{p}_1'') b^\dagger_{i's_1}(\underline{p}_1'') \int d\underline{p}_2'' F^-(\underline{p}_2'') s_m \delta_{ss_2} \delta_{mj'} \delta(\underline{p}_2 - \underline{p}_2'') \right\} |0\rangle_f |G^+_{i'}\rangle |G^-_{j'}\rangle. \tag{A8}$$

With use of the Kronecker and Dirac deltas, this becomes

$$s_i s_j \langle F^-_{js_2}| \langle F^+_{is_1}| \frac{1}{2} \mathcal{P}^-_{\text{n.p.}} |G^+ G^- F^+ F^-; s_1 s_2\rangle$$
$$= \frac{1}{2} s_i s_j \, {}_f\langle 0| \int d\underline{p}_2' F^{-*}(\underline{p}_2') b_{js_2}(\underline{p}_2') \int d\underline{p}_1' F^{+*}(\underline{p}_1') b_{is_1}(\underline{p}_1') g_0 \sum_{lk} \beta_l \xi_k \int d\underline{x}$$
$$\times \int \frac{d\underline{q}}{\sqrt{16\pi^3 q^+}} \left[a_k(\underline{q}) e^{-i\underline{q}\cdot\underline{x}} + a_k^\dagger(\underline{q}) e^{i\underline{q}\cdot\underline{x}} \right] \int \frac{d\underline{p}_1 d\underline{p}_2}{16\pi^3} e^{i(\underline{p}_1 - \underline{p}_2)\cdot\underline{x}}$$
$$\times \sum_{i'j'} C_{i'j'} \left\{ s_{i'} \beta_{i'} \left(\frac{m_l}{p_1^+} + \frac{m_{i'}}{p_2^+} \right) F^+(\underline{p}_2) \int d\underline{p}_2'' F^-(\underline{p}_2'') b^\dagger_{ls_1}(\underline{p}_1) b^\dagger_{j's_2}(\underline{p}_2'') \right.$$
$$\left. - s_{j'} \beta_{j'} \left(\frac{m_l}{p_1^+} + \frac{m_{j'}}{p_2^+} \right) \int d\underline{p}_1'' F^+(\underline{p}_1'') b^\dagger_{ls_2}(\underline{p}_1) b^\dagger_{i's_1}(\underline{p}_1'') F^-(\underline{p}_2) \right\} |0\rangle_f |G^+_{i'}\rangle |G^-_{j'}\rangle. \tag{A9}$$

The remaining contractions produce

$$s_i s_j \langle F^-_{js_2}| \langle F^+_{is_1}| \frac{1}{2} \mathcal{P}^-_{\text{n.p.}} |G^+ G^- F^+ F^-; s_1 s_2\rangle$$
$$= \frac{1}{2} s_i s_j \int d\underline{p}_2' F^{-*}(\underline{p}_2') \int d\underline{p}_1' F^{+*}(\underline{p}_1') g_0 \sum_{lk} \beta_l \xi_k \int d\underline{x} \int \frac{d\underline{q}}{\sqrt{16\pi^3 q^+}} \left[a_k(\underline{q}) e^{-i\underline{q}\cdot\underline{x}} + a_k^\dagger(\underline{q}) e^{i\underline{q}\cdot\underline{x}} \right] \int \frac{d\underline{p}_1 d\underline{p}_2}{16\pi^3} e^{i(\underline{p}_1 - \underline{p}_2)\cdot\underline{x}}$$
$$\times \sum_{i'j'} C_{i'j'} \left\{ s_{i'} \beta_{i'} \left(\frac{m_l}{p_1^+} + \frac{m_{i'}}{p_2^+} \right) F^+(\underline{p}_2) \int d\underline{p}_2'' F^-(\underline{p}_2'') \right.$$
$$\times [s_i \delta_{il} \delta(\underline{p}_1' - \underline{p}_1) s_j \delta_{jj'} \delta(\underline{p}_2' - \underline{p}_2'') - s_i \delta_{s_1 s_2} \delta_{ij'} \delta(\underline{p}_1' - \underline{p}_2'') s_j \delta_{s_2 s_1} \delta_{jl} \delta(\underline{p}_2' - \underline{p}_1)]$$
$$- s_{j'} \beta_{j'} \left(\frac{m_l}{p_1^+} + \frac{m_{j'}}{p_2^+} \right) \int d\underline{p}_1'' F^+(\underline{p}_1'') F^-(\underline{p}_2)$$
$$\left. \times [s_i \delta_{s_1 s_2} \delta_{il} \delta(\underline{p}_1' - \underline{p}_1) s_j \delta_{s_2 s_1} \delta_{ji'} \delta(\underline{p}_2' - \underline{p}_1'') - s_i \delta_{ii'} \delta(\underline{p}_1' - \underline{p}_1'') s_j \delta_{jl} \delta(\underline{p}_2' - \underline{p}_1)] \right\} |G^+_{i'}\rangle |G^-_{j'}\rangle. \tag{A10}$$

The additional Kronecker and Dirac deltas, and the fact that $s_i^2 = 1$, reduce this to

$$s_i s_j \langle F^-_{js_2}| \langle F^+_{is_1}| \frac{1}{2} \mathcal{P}^-_{\text{n.p.}} |G^+ G^- F^+ F^-; s_1 s_2\rangle$$
$$= \frac{1}{2} g_0 \sum_k \xi_k \int d\underline{x} \int \frac{d\underline{q}}{\sqrt{16\pi^3 q^+}} \left[a_k(\underline{q}) e^{-i\underline{q}\cdot\underline{x}} + a_k^\dagger(\underline{q}) e^{i\underline{q}\cdot\underline{x}} \right] \int \frac{d\underline{p}_1 d\underline{p}_2}{16\pi^3} e^{i(\underline{p}_1 - \underline{p}_2)\cdot\underline{x}} \tag{A11}$$
$$\times \left\{ \sum_{i'} s_{i'} \beta_{i'} \left[C_{i'j} \beta_i \int d\underline{p}_2' |F^-(\underline{p}_2')|^2 F^{+*}(\underline{p}_1) F^+(\underline{p}_2) \left(\frac{m_i}{p_1^+} + \frac{m_{i'}}{p_2^+} \right) |G^+_{i'}\rangle |G^-_j\rangle \right. \right.$$
$$\left. - \delta_{s_1 s_2} C_{i'i} \beta_j \int d\underline{p}_1' F^{+*}(\underline{p}_1') F^-(\underline{p}_1') F^{-*}(\underline{p}_1) F^+(\underline{p}_2) \left(\frac{m_j}{p_1^+} + \frac{m_{i'}}{p_2^+} \right) |G^+_{i'}\rangle |G^-_i\rangle \right]$$
$$- \sum_{j'} s_{j'} \beta_{j'} \left[\delta_{s_1 s_2} C_{jj'} \beta_i \int d\underline{p}_2' F^{-*}(\underline{p}_2') F^+(\underline{p}_2') F^{+*}(\underline{p}_1) F^-(\underline{p}_2) \left(\frac{m_i}{p_1^+} + \frac{m_{j'}}{p_2^+} \right) |G^+_j\rangle |G^-_{j'}\rangle \right.$$
$$\left. \left. - C_{ij'} \beta_j \int d\underline{p}_1' |F^+(\underline{p}_1')|^2 F^{-*}(\underline{p}_1) F^-(\underline{p}_2) \left(\frac{m_j}{p_1^+} + \frac{m_{j'}}{p_2^+} \right) |G^+_i\rangle |G^-_{j'}\rangle \right] \right\}. \tag{A12}$$

The $\underline{p}_1'$ and $\underline{p}_2'$ integrations yield either unity for direct terms or zero for cross terms; the latter can be identified by the leading $\delta_{s_1 s_2}$. The integral in the direct terms is the normalization integral for $F^\pm$. The integral for the cross terms is zero because of zero overlap between the wave packets of the two sources. The integrals over $\underline{p}_1$ and $\underline{p}_2$ can be rewritten in terms of the Fourier transforms $\psi^\pm$, with factors of $\underline{p}_1'$ and $\underline{p}_2'$ replaced with $(m, \vec{0}_\perp)$ at the peak of $F^\pm$. This leaves factors of $|\psi^\pm(\underline{x})|^2$ which become $\delta(x^- \pm R_z)\delta(\vec{x}_\perp \mp \vec{R}_\perp/2)$. The integral over $\underline{x}$ can then be performed. The resulting expression is what is quoted in (4.8).

APPENDIX B: REDUCTION OF THE TWO-SOURCE EIGENVALUE PROBLEM

We begin from the statement of the full eigenvalue problem for two static sources as given in (4.9). The coefficient of the collected $a_k^\dagger(\underline{q})$ terms is

$$C_{ij}\frac{1}{2}\sum_k\left[\frac{\mu_k^2+q_\perp^2}{q^+}+q^+\right]r_k[G_{ki}^+(\underline{q})+G_{kj}^-(\underline{q})]|G_i^+\rangle|G_j^-\rangle + \frac{1}{2}g_0\xi_k\frac{1}{\sqrt{16\pi^3 q^+}}\left\{\beta_i\sum_{i'}s_{i'}\beta_{i'}C_{i'j}e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_i+m_{i'}}{m}|G_{i'}^+\rangle|G_j^-\rangle + \beta_j\sum_{j'}s_{j'}\beta_{j'}C_{ij'}e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_j+m_{j'}}{m}|G_i^+\rangle|G_{j'}^-\rangle\right\}. \tag{B1}$$

If we set $C_{ij}=C_i^+C_j^-$, this coefficient of $a_k^\dagger(\underline{q})$ is automatically zero, given the solution to the single-source case. This leaves the double-source eigenvalue problem (4.9) in the form

$$C_i^+C_j^-\left\{\frac{m_i^2}{2m}+\frac{m_j^2}{2m}+m\right\}|G_i^+\rangle|G_j^-\rangle+\frac{1}{2}g_0\sum_k\xi_k\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}} \times\left\{\beta_i\sum_{i'}s_{i'}\beta_{i'}C_{i'}^+C_j^-r_k[G_{ki'}^+(\underline{q})+G_{kj}^-(\underline{q})]e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_i+m_{i'}}{m}|G_{i'}^+\rangle|G_j^-\rangle + \beta_j\sum_{j'}s_{j'}\beta_{j'}C_i^+C_{j'}^-r_k[G_{ki}^+(\underline{q})+G_{kj'}^-(\underline{q})]e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_j+m_{j'}}{m}|G_i^+\rangle|G_{j'}^-\rangle\right\} = EC_i^+C_j^-|G_i^+\rangle|G_j^-\rangle. \tag{B2}$$

The combination $C_i^\pm G_{ki}^\pm(\underline{q})|G_i^\pm\rangle$ can be replaced in each appearance with use of (3.18)

$$C_i^+C_j^-\left\{\frac{m_i^2}{2m}+\frac{m_j^2}{2m}+m\right\}|G_i^+\rangle|G_j^-\rangle-\frac{1}{2}g_0\sum_k\xi_k\int\frac{d\underline{q}}{\sqrt{16\pi^3q^+}} \times\left\{\beta_i\sum_{i'}s_{i'}\beta_{i'}C_j^-r_ke^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_i+m_{i'}}{m}\frac{g_0\beta_{i'}\xi_k}{\sqrt{16\pi^3q^+}}\frac{e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}}{\frac{\mu_k^2+q_\perp^2}{q^+}+q^+}\sum_{j'}s_{j'}\beta_{j'}C_{j'}^+\frac{m_{i'}+m_{j'}}{m}|G_{j'}^+\rangle|G_j^-\rangle + \beta_i\sum_{i'}s_{i'}\beta_{i'}C_{i'}^+r_ke^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_i+m_{i'}}{m}|G_{i'}^+\rangle\frac{g_0\beta_j\xi_k}{\sqrt{16\pi^3q^+}}\frac{e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}}{\frac{\mu_k^2+q_\perp^2}{q^+}+q^+}\sum_{j'}s_{j'}\beta_{j'}C_{j'}^-\frac{m_j+m_{j'}}{m}|G_{j'}^-\rangle + \beta_j\sum_{j'}s_{j'}\beta_{j'}C_{j'}^-r_ke^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_j+m_{j'}}{m}\frac{g_0\beta_i\xi_k}{\sqrt{16\pi^3q^+}}\frac{e^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}}{\frac{\mu_k^2+q_\perp^2}{q^+}+q^+}\sum_{i'}s_{i'}\beta_{i'}C_{i'}^+\frac{m_i+m_{i'}}{m}|G_{i'}^+\rangle|G_{j'}^-\rangle + \beta_j\sum_{j'}s_{j'}\beta_{j'}C_i^+r_ke^{-i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}\frac{m_j+m_{j'}}{m}|G_i^+\rangle\frac{g_0\beta_{j'}\xi_k}{\sqrt{16\pi^3q^+}}\frac{e^{i(q^+R_z+\vec{q}_\perp\cdot\vec{R}_\perp)/2}}{\frac{\mu_k^2+q_\perp^2}{q^+}+q^+}\sum_{i'}s_{i'}\beta_{i'}C_{i'}^-\frac{m_{j'}+m_{i'}}{m}|G_{i'}^-\rangle\right\} = EC_i^+C_j^-|G_i^+\rangle|G_j^-\rangle. \tag{B3}$$

The $\underline{q}$ dependent factors can be combined into a factor that is either the self-energy integral I defined in (3.21), where the exponential factors cancel, or the integral

$$Y^{\pm}(\vec{R}) = \int \frac{d\underline{q}}{16\pi^3} \sum_k \frac{r_k \xi_k^2 e^{\pm i(q^+ R_z + \vec{q}_\perp \cdot \vec{R}_\perp)}}{(q^+)^2 + q_\perp^2 + \mu_k^2}. \tag{B4}$$

We then obtain

$$\begin{aligned}
&C_i^+ C_j^- \left\{ \frac{m_i^2}{2m} + \frac{m_j^2}{2m} + m \right\} |G_i^+\rangle |G_j^-\rangle - \frac{g_0^2}{2} \left\{ \beta_i C_j^- \mu I \sum_{i'} s_{i'} \beta_{i'}^2 \frac{m_i + m_{i'}}{m} \sum_{j'} s_{j'} \beta_{j'} C_{j'}^+ \frac{m_{i'} + m_{j'}}{m} |G_{j'}^+\rangle |G_j^-\rangle \right. \\
&\quad + \beta_i \beta_j Y^+(\vec{R}) \sum_{i'} s_{i'} \beta_{i'} C_{i'}^+ \frac{m_i + m_{i'}}{m} \sum_{j'} s_{j'} \beta_{j'} C_{j'}^- \frac{m_j + m_{j'}}{m} |G_{i'}^+\rangle |G_{j'}^-\rangle \\
&\quad + \beta_i \beta_j Y^-(\vec{R}) \sum_{j'} s_{j'} \beta_{j'} C_{j'}^- \frac{m_j + m_{j'}}{m} \sum_{i'} s_{i'} \beta_{i'} C_{i'}^+ \frac{m_i + m_{i'}}{m} |G_{i'}^+\rangle |G_{j'}^-\rangle \\
&\quad \left. + C_i^+ \beta_j \mu I \sum_{j'} s_{j'} \beta_{j'}^2 \frac{m_j + m_{j'}}{m} \sum_{i'} s_{i'} \beta_{i'} C_{i'}^- \frac{m_{j'} + m_{i'}}{m} |G_i^+\rangle |G_{i'}^-\rangle \right\} \\
&= E C_i^+ C_j^- |G_i^+\rangle |G_j^-\rangle.
\end{aligned} \tag{B5}$$

This can be rearranged to reveal parts directly related to the single-source problem

$$\begin{aligned}
&\left\{ C_i^+ \left(\frac{m_i^2}{2m} + \frac{m}{2} \right) |G_i^+\rangle - \frac{g_0^2}{2} \beta_i \mu I \sum_{i'} s_{i'} \beta_{i'}^2 \frac{m_i + m_{i'}}{m} \sum_{j'} s_{j'} \beta_{j'} C_{j'}^+ \frac{m_{i'} + m_{j'}}{m} |G_{j'}^+\rangle \right\} C_j^- |G_j^-\rangle \\
&\quad + C_i^+ |G_i^+\rangle \left\{ C_j^- \left(\frac{m_j^2}{2m} + \frac{m}{2} \right) |G_j^-\rangle - \frac{g_0^2}{2} \beta_j \mu I \sum_{j'} s_{j'} \beta_{j'}^2 \frac{m_j + m_{j'}}{m} \sum_{i'} s_{i'} \beta_{i'} C_{i'}^- \frac{m_{j'} + m_{i'}}{m} |G_{i'}^-\rangle \right\} \\
&\quad - \frac{g_0^2}{2} \beta_i \beta_j Y^+(\vec{R}) \sum_{i'} s_{i'} \beta_{i'} C_{i'}^+ \frac{m_i + m_{i'}}{m} \sum_{j'} s_{j'} \beta_{j'} C_{j'}^- \frac{m_j + m_{j'}}{m} |G_{i'}^+\rangle |G_{j'}^-\rangle \\
&\quad - \frac{g_0^2}{2} \beta_i \beta_j Y^-(\vec{R}) \sum_{j'} s_{j'} \beta_{j'} C_{j'}^- \frac{m_j + m_{j'}}{m} \sum_{i'} s_{i'} \beta_{i'} C_{i'}^+ \frac{m_i + m_{i'}}{m} |G_{i'}^+\rangle |G_{j'}^-\rangle \\
&= E C_i^+ C_j^- |G_i^+\rangle |G_j^-\rangle.
\end{aligned} \tag{B6}$$

According to (3.20), with $E^{\pm} = m$, the first curly bracket is simply $m C_i^+ |G_i^+\rangle$, and the second is $m C_j^- |G_j^-\rangle$. These two terms contribute $2m C_i^+ C_j^- |G_i^+\rangle |G_j^-\rangle$ to the equation and, when subtracted from both sides, leave the effective potential $V_{\rm eff}(\vec{R}) \equiv E - 2m$ determined by

$$-\frac{g_0^2}{2} \beta_i \beta_j Y(\vec{R}) \sum_{i'} s_{i'} \beta_{i'} C_{i'}^+ \frac{m_i + m_{i'}}{m} \sum_{j'} s_{j'} \beta_{j'} C_{j'}^- \frac{m_j + m_{j'}}{m} |G_{i'}^+\rangle |G_{j'}^-\rangle = V_{\rm eff} C_i^+ C_j^- |G_i^+\rangle |G_j^-\rangle, \tag{B7}$$

with $Y \equiv Y^+ + Y^-$.

To extract $V_{\rm eff}$, we define $|G^{\pm}\rangle \equiv \sum_i C_i^{\pm} s_i \beta_i |G_i^{\pm}\rangle$, multiply Eq. (B7) by $s_i \beta_i s_j \beta_j$, and sum over i and j

$$\begin{aligned}
&-\frac{g_0^2}{2} Y(\vec{R}) \sum_i s_i \beta_i^2 \sum_j s_j \beta_j^2 \left[\frac{m_i}{m} |G^+\rangle + \sum_{i'} s_{i'} \beta_{i'} C_{i'}^+ \frac{m_{i'}}{m} |G_{i'}^+\rangle \right] \\
&\times \left[\frac{m_j}{m} |G^-\rangle + \sum_{j'} s_{j'} \beta_{j'} C_{j'}^- \frac{m_{j'}}{m} |G_{j'}^-\rangle \right] = V_{\rm eff} |G^+\rangle |G^-\rangle.
\end{aligned} \tag{B8}$$

To simplify the result further, we use the constraint $\sum_i s_i \beta_i^2 = 0$, which eliminates all terms between the square brackets except for the product of the first in each. We can then equate coefficients of $|G^+\rangle |G^-\rangle$ to obtain

$$V_{\rm eff} = -\frac{g_0^2}{2} Y(\vec{R}) \left(\sum_i \frac{s_i \beta_i^2 m_i}{m} \right)^2. \tag{B9}$$

The integrals in Y are computed in [21]. They yield

$$Y(\vec{R}) = \sum_k r_k \xi_k^2 \frac{e^{-\mu_k R}}{8\pi R}. \tag{B10}$$

In the limit of infinite PV boson masses, with $r_0 = 1$, $\xi_0 = 1$, and $\mu_0 = \mu$, this reduces to $e^{-\mu R}/(8\pi R)$, and the effective potential is found to be

$$V_{\rm eff} = -\frac{g_0^2}{2}\left(\sum_i \frac{s_i \beta_i^2 m_i}{m}\right)^2 \frac{e^{-\mu R}}{8\pi R}. \tag{B11}$$

This is the expression given in (4.10).

APPENDIX C: VARIATION WITH RESPECT TO $\mathcal{P}^-$

If we vary the bound state with respect to the expectation value of $\mathcal{P}^-$ rather than $\mathcal{E} \equiv \frac{1}{2}(\mathcal{P}^- + \mathcal{P}^+)$, we find that the self-energy integral I, defined in (3.21), and the effective-potential integrals $Y^\pm$, defined in (B4), are replaced by

$$I = \int \frac{d^2 q_\perp}{16\pi^3 \mu} \sum_k \frac{r_k \xi_k^2}{q_\perp^2 + \mu_k^2} \int_0^\infty dq^+ \tag{C1}$$

and

$$Y^\pm(\vec{R}) = \int \frac{d^2 q_\perp}{16\pi^3} \sum_k \frac{r_k \xi_k^2 e^{\pm i \vec{q}_\perp \cdot \vec{R}_\perp}}{q_\perp^2 + \mu_k^2} \int_0^\infty dq^+ e^{\pm i q^+ R_z}, \tag{C2}$$

respectively. The absence of the $(q^+)^2$ terms in the denominators is due to the absence of $\mathcal{P}^-$ terms. Without $\mathcal{P}^-$ being part of the calculation, Eq. (3.16) is altered in such a way that the q^+ factor in the second term in the square bracket that multiplies $a_k^\dagger(\underline{q})$ is missing. It is this q^+ that appears in the denominator of $G_{ki}^\pm(\underline{q})$ and yields the missing $(q^+)^2$ terms in I and $Y^\pm$.

These new integrals are clearly divergent in a way that the PV regularization cannot handle. In standard light-front calculations, the q^+ integrals would be naturally cut off at the total longitudinal momomentum P^+ and not cause any difficulty. Here, however, P^+ is not conserved, and we would need to introduce an arbitrary cutoff that will destroy the rotational symmetry. We therefore find this to be another motivation for the use of $\mathcal{E}$ as the operator of interest rather than $\mathcal{P}^-$.

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