Phenomenology of isospin-symmetry breaking with vector mesons

Péter Kovács [*] and György Wolf [†]
Institute for Particle and Nuclear Physics, Wigner Research Centre for Physics, 1121 Budapest, Hungary

Nora Weickgenannt
Institut de Physique Théorique, Université Paris Saclay, CEA, CNRS, F-91191 Gif-sur-Yvette, France and Institute for Theoretical Physics, Goethe University, Max-von-Laue-Strasse 1, D-60438 Frankfurt am Main, Germany

Dirk H. Rischke [‡]
Institute for Theoretical Physics, Goethe University, Max-von-Laue-Strasse 1, D-60438 Frankfurt am Main, Germany and Helmholtz Research Academy Hesse for FAIR, Campus Riedberg, Max-von-Laue-Strasse 12, D-60438 Frankfurt am Main, Germany

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We study the effect of isospin-symmetry breaking in the framework of the extended linear σ model in vacuum. In this model, several particles mix with each other at tree level, due to the three nonzero scalar condensates (nonstrange, strange, isospin). We resolve these mixings with the help of various field transformations. We compute all possible meson mixings and decay widths at tree level and perform a χ^2 fit to PDG data. A very good fit is found if we exclude the (very small ~130 keV) $\omega \to \pi\pi$ decay. We also investigate the violation of Dashen's theorem.

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I. INTRODUCTION

Understanding the meson mass spectrum is a fundamental task in particle physics. In principle, the QCD Lagrangian contains all relevant information, but the strong coupling becomes large in the low-energy regime, such that QCD is not solvable by perturbative methods. In this regime, quarks and gluons are confined in hadrons, which become the relevant degrees of freedom. Therefore, a possible solution is to use effective models for the hadronic degrees of freedom, which obey the same global symmetries as QCD [1], but do not contain gauge bosons. Here, all interactions are expressed by vertices of the hadronic fields. One of these effective models is the extended linear σ model (eLSM) for three flavors, which was discussed assuming isospin symmetry at zero temperature and baryochemical potential in Ref. [2] and later studied at finite temperature and/or finite baryochemical potential in Refs. [3–5]. It successfully describes the meson masses and decay widths at tree level and agrees well with lattice-QCD data at finite temperature [3]. However, isospin symmetry is broken in nature, i.e., the masses of up and down quarks are different (see, e.g., Ref. [6]). Consequently, the masses of charged and neutral mesons of the same flavor are slightly different. Note that the difference is not only due to the strong interaction, but also due to the electromagnetic interaction.

Isospin-symmetry breaking (IB) or isospin violation [7] was investigated from different aspects in the literature, like in connection with charge-symmetry breaking [8,9]. The latter can be seen in charge-conjugate systems as, e.g., in proton-proton and neutron-neutron binary systems, and is caused by different mechanisms like $\rho - \omega$ mixing [10], the nucleon mass-difference effect in one-pion exchange interactions [11], or isospin-violating meson-nucleon coupling constants [12]. The effect of $\rho - \omega$ mixing on the nuclear symmetry energy was investigated in Ref. [13], and its relation to low-energy pion-nucleon scattering was studied in Ref. [14]. Another important topic is the mechanism of production of light scalar mesons, such as $f_0(980)$, which is explained by the $f_0(980) - a_0^0(980)$ mixing [15–19]. This mixing, more precisely, the triple mixing of the f_0^L and f_0^H scalar-isoscalar and a_0^0 scalar-isovector states, naturally arises in effective theories including isospin-symmetry

[*]kovacs.peter@wigner.hu
[†]wolf.gyorgy@wigner.hu
[‡]drischke@th.physik.uni-frankfurt.de

breaking in the scalar sector. Similarly, in the pseudoscalar sector, there is the $\pi^0-\eta-\eta'$ mixing, which was investigated in Ref. [20]. The effects of IB were also thoroughly examined in chiral perturbation theory (ChPT) [21–29], in the $\phi\to\omega\pi^0$ isospin-violating decay [30], and in connection with vector form factors [31].

Lattice-QCD calculations have recently also reached such a precision that isospin-symmetry-breaking effects become important, e.g., in the case of the precise determination of leptonic and semileptonic decay rates [32]. These kinds of investigations can shed some light on the observed deviation from unitarity in the first row of the Cabibbo-Kobayashi-Maskawa (CKM) matrix (see, e.g., Refs. [32,33]). Different mass splittings were also examined on the lattice [34–37]. Recently, IB effects were also investigated in connection with the hadronic vacuum polarization [38,39] and the nuclear matrix element of Fermi β decays [40].

In this paper, we model phenomenologically the violation of isospin symmetry in the eLSM to describe these mass differences and also differences of charged and neutral decay widths. It should be mentioned that a similar effective model was already investigated in Ref. [41] in some detail. Here, however, we pursue a more thorough analysis with current experimental data taken from Ref. [42]. We resolve all the various mixings arising between different meson nonets and within each nonet. As it was already mentioned, the charged and neutral masses differ not only due to the $u-d$ quark mass difference, but also because of electromagnetic interactions. We take these electromagnetic contributions effectively into account through additive terms to the masses of the charged fields in the different nonets.

Dashen's theorem [43,44] states that, in the chiral limit, i.e., when the quark masses are zero, the following holds in the pseudoscalar sector:

$$(m_{\pi^\pm}-m_{\pi^0})|_{\rm em}=(m_{K^\pm}-m_{K^0})|_{\rm em},\qquad m_{\pi^0}|_{\rm em}=0,\qquad m_{K^0}|_{\rm em}=0. \tag{1}$$

However, there are also corrections to Dashen's theorem. In Ref. [45], using ChPT, the authors find only moderate deviations from Dashen's theorem, while in more recent works the deviation seems much more significant [46,47]. We perform fits to Particle Data Group (PDG) data studying a scenario where Dashen's theorem is valid, as well as various scenarios where the latter is violated. It should be noted also that, in addition to the $q\bar{q}$ scalar vacuum expectation values (VEVs) considered here, other four-quark scalar VEVs are also possible, which is beyond the scope of the current investigation. The inclusion of the latter would result in additional mixing among the $q\bar{q}$, four-quark, and scalar glueball states, which are investigated in detail in Refs. [48–53]. Our expectation is that the mixing with the four-quark nonet has a smaller impact on the pseudoscalars and a larger impact on the scalars [48]. The mixing between the scalar $q\bar{q}$ and four-quark sectors might improve the fit for the f_0 masses and decay widths, but this would need further investigation.

This paper is organized as follows. In Sec. II, the eLSM is introduced, its tree-level mixing terms are presented, and a collection of transformations is shown to resolve these various mixings. In Secs. III and IV, the physical masses and decay widths are summarized. Section V is dedicated to the description of the fitting procedure and to the acquired results. Conclusions are given in Sec. VI. Appendix A lists the explicit expressions for the squared-mass matrix elements, Appendix B gives a derivation of the f_{π^0} decay constant from the partially conserved axial current (PCAC) relation, Appendix C collects the detailed formulas for the tree-level decay widths, and Appendix D contains tables with the detailed results of the fits.

II. THE MODEL

A. Lagrangian with broken isospin

According to Ref. [2], the eLSM Lagrangian is given by

$$\begin{aligned}\mathcal{L}=&\,{\rm Tr}[(D_\mu\Phi)^\dagger(D^\mu\Phi)]-m_0^2{\rm Tr}(\Phi^\dagger\Phi)-\lambda_1[{\rm Tr}(\Phi^\dagger\Phi)]^2-\lambda_2{\rm Tr}(\Phi^\dagger\Phi)^2+{\rm Tr}[H(\Phi+\Phi^\dagger)]+c_1(\det\Phi-\det\Phi^\dagger)^2\\&-\frac{1}{4}{\rm Tr}(L^{\mu\nu}L_{\mu\nu}+R^{\mu\nu}R_{\mu\nu})+{\rm Tr}\left[\left(\frac{m_1^2}{2}+\Delta\right)(L^\mu L_\mu+R^\mu R_\mu)\right]+i\frac{g_2}{2}({\rm Tr}\{L_{\mu\nu}[L^\mu,L^\nu]\}+{\rm Tr}\{R_{\mu\nu}[R^\mu,R^\nu]\})\\&+\frac{h_1}{2}{\rm Tr}(\Phi^\dagger\Phi){\rm Tr}(L^\mu L_\mu+R^\mu R_\mu)+h_2{\rm Tr}(\Phi^\dagger L^\mu L_\mu\Phi+R^\mu\Phi^\dagger\Phi R_\mu)+2h_3{\rm Tr}(L_\mu\Phi R^\mu\Phi^\dagger)\\&+g_3[{\rm Tr}(L_\mu L_\nu L^\mu L^\nu)+{\rm Tr}(R_\mu R_\nu R^\mu R^\nu)]+g_4[{\rm Tr}(L_\mu L^\mu L_\nu L^\nu)+{\rm Tr}(R_\mu R^\mu R_\nu R^\nu)]\\&+g_5{\rm Tr}(L_\mu L^\mu){\rm Tr}(R_\nu R^\nu)+g_6[{\rm Tr}(L_\mu L^\mu){\rm Tr}(L_\nu L^\nu)+{\rm Tr}(R_\mu R^\mu){\rm Tr}(R_\nu R^\nu)],\end{aligned}\tag{2}$$

where

$$
\begin{aligned}
D^\mu\Phi &\equiv \partial^\mu\Phi - ig_1(L^\mu\Phi - \Phi R^\mu) - ieA_e^\mu[T_3,\Phi],\\
L^{\mu\nu} &\equiv \partial^\mu L^\nu - \partial^\nu L^\mu - ieA_e^\mu[T_3,L^\nu] + ieA_e^\nu[T_3,L^\mu],\\
R^{\mu\nu} &\equiv \partial^\mu L^\nu - \partial^\nu L^\mu - ieA_e^\mu[T_3,R^\nu] + ieA_e^\nu[T_3,R^\mu].
\end{aligned}
$$

The scalar nonet Φ is given by

$$
\Phi \equiv \Phi_S + \Phi_{PS} = \sum_{a=0}^{8}(S_a + iP_a)T_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{(\sigma_N+a_0^0)+i(\eta_N+\pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & K_0^{\star +} + iK^+ \\ a_0^- + i\pi^- & \frac{(\sigma_N-a_0^0)+i(\eta_N-\pi^0)}{\sqrt{2}} & K_0^{\star 0} + iK^0 \\ K_0^{\star -} + iK^- & \bar{K}_0^{\star 0} + i\bar{K}^0 & \sigma_S + i\eta_S \end{pmatrix}, \tag{3}
$$

while the left- and right-handed vector nonets L^μ and R^μ are defined as

$$
L^\mu \equiv V^\mu + A^\mu \equiv \sum_{a=0}^{8}(V_a^\mu + A_a^\mu)T_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\omega_N+\rho^0}{\sqrt{2}} + \frac{f_{1N}+a_1^0}{\sqrt{2}} & \rho^+ + a_1^+ & K^{\star +} + K_1^+ \\ \rho^- + a_1^- & \frac{\omega_N-\rho^0}{\sqrt{2}} + \frac{f_{1N}-a_1^0}{\sqrt{2}} & K^{\star 0} + K_1^0 \\ K^{\star -} + K_1^- & \bar{K}^{\star 0} + \bar{K}_1^0 & \omega_S + f_{1S} \end{pmatrix}^\mu, \tag{4a}
$$

$$
R^\mu \equiv V^\mu - A^\mu \equiv \sum_{a=0}^{8}(V_a^\mu - A_a^\mu)T_a = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\omega_N+\rho^0}{\sqrt{2}} - \frac{f_{1N}+a_1^0}{\sqrt{2}} & \rho^+ - a_1^+ & K^{\star +} - K_1^+ \\ \rho^- - a_1^- & \frac{\omega_N-\rho^0}{\sqrt{2}} - \frac{f_{1N}-a_1^0}{\sqrt{2}} & K^{\star 0} - K_1^0 \\ K^{\star -} - K_1^- & \bar{K}^{\star 0} - \bar{K}_1^0 & \omega_S - f_{1S} \end{pmatrix}^\mu. \tag{4b}
$$

The fields H and Δ are defined as

$$
H = \sum_{i=0,3,8}\zeta_i T_i = \frac{1}{2}\mathrm{diag}(\zeta_N + \zeta_3, \zeta_N - \zeta_3, \sqrt{2}\zeta_S), \tag{5a}
$$

$$
\Delta = \sum_{i=0,3,8}\Delta_i T_i = \mathrm{diag}(\delta_u, \delta_d, \delta_s), \tag{5b}
$$

with T_a, $a \in \{0, \ldots, 8\}$, being the generators of U(3). It is worth noting that, in the matrices above and throughout the article, the N–S (nonstrange-strange) basis is used instead of the 0–8 basis, which for a generic field $\xi_a \in (S_a, P_a, V_a^\mu, A_a^\mu, H_a, \Delta_a)$ is defined as

$$
\xi_N = \frac{1}{\sqrt{3}}(\sqrt{2}\xi_0 + \xi_8), \qquad \xi_S = \frac{1}{\sqrt{3}}(\xi_0 - \sqrt{2}\xi_8). \tag{6}
$$

If the fields $\zeta_{N/S/3}$ are nonvanishing, chiral symmetry is explicitly broken. In particular, for $\zeta_3 \neq 0$ (and also for $\delta_3 \equiv \delta_u - \delta_d \neq 0$) the isospin symmetry is violated, which is the situation in nature. In this case, all scalar-isoscalar fields σ_N, σ_S, and a_0^0 can have nonzero vacuum expectation values denoted as $\phi_{N/S} \equiv \langle\sigma_{N/S}\rangle$ and $\phi_3 \equiv \langle a_0^0\rangle$. The condensates ϕ_N, ϕ_S, ϕ_3 can be considered as order parameters for the chiral phase transition at finite temperature. Their values at zero temperature and at tree level are determined by minimizing the classical potential $V_{\mathrm{cl}}(\phi_N, \phi_S, \phi_3)$, which can be read off the Lagrangian (2) after shifting the scalar-isoscalar fields by their expectation values,

$$
\sigma_N \to \phi_N + \sigma_N, \tag{7a}
$$

$$
\sigma_S \to \phi_S + \sigma_S, \tag{7b}
$$

$$
a_0^0 \to \phi_3 + a_0^0. \tag{7c}
$$

The explicit form of the classical potential reads

$$
\begin{aligned}
V_{\mathrm{cl}}(\phi_N, \phi_S, \phi_3) = {} & \frac{m_0^2}{2}[\phi_N^2 + \phi_S^2 + (\phi_3)^2] \\
& + \frac{\lambda_1}{4}[\phi_N^2 + \phi_S^2 + (\phi_3)^2]^2 \\
& + \frac{\lambda_2}{4}\left[\frac{\phi_N^4}{2} + 3\phi_N^2(\phi_3)^2 + \frac{(\phi_3)^4}{2} + \phi_S^4\right] \\
& - \zeta_N\phi_N - \zeta_S\phi_S - \zeta_3\phi_3.
\end{aligned} \tag{8}
$$

From the stationary points of V_{cl}, i.e., from the conditions $\partial V_{\mathrm{cl}}/\partial\phi_{N/S/3} = 0$, the fields $\zeta_{N/S/3}$ are derived as

$$\zeta_N = \phi_N\left\{m_0^2+\lambda_1[\phi_N^2+\phi_S^2+(\phi_3)^2]+\frac{\lambda_2}{2}[\phi_N^2+3(\phi_3)^2]\right\}, \tag{9a}$$

$$\zeta_S = \phi_S\{m_0^2+\lambda_1[\phi_S^2+\phi_N^2+(\phi_3)^2]+\lambda_2\phi_S^2\}, \tag{9b}$$

$$\zeta_3 = \phi_3\left\{m_0^2+\lambda_1[\phi_N^2+\phi_S^2+(\phi_3)^2]+\frac{\lambda_2}{2}[3\phi_N^2+(\phi_3)^2]\right\}. \tag{9c}$$

As can be seen, $\zeta_{N/S/3} \propto \phi_{N/S/3}$, i.e., if a field $\zeta_{N/S/3}$ is zero, there is a solution where the corresponding condensate $\phi_{N/S/3}=0$. However, there is another solution where $\zeta_{N/S/3}=0$ and $\phi_{N/S/3}\neq 0$ and this solution corresponds to the physical point.

B. Tree-level masses and mixing terms

After spontaneous symmetry breaking there will be various mixing terms, namely, those between different nonets and those within a given nonet. The latter ones are the off-diagonal elements of the squared-mass matrices and are given in Appendix A. The former kind of mixing terms relate certain fields of the axial-vector/vector nonets with those of the pseudoscalar/scalar nonets and read

$$\begin{aligned}\mathcal{L}_{Nmix} = &-g_1 i\phi_3(\rho^{-\mu}\partial_\mu a_0^+ - \rho^{+\mu}\partial_\mu a_0^-) - i\frac{g_1}{2}\left(\phi_N+\phi_3-\sqrt{2}\phi_S\right)(K^{*-\mu}\partial_\mu K_0^{\star+} - K^{\star+\mu}\partial_\mu K_0^{\star-})\\ &- i\frac{g_1}{2}\left(\phi_N-\phi_3-\sqrt{2}\phi_S\right)(\bar{K}^{\star0\mu}\partial_\mu K^0 - K^{\star0\mu}\partial_\mu\bar{K}^0) - g_1\phi_N(a_1^{-\mu}\partial_\mu\pi^+ + a_1^{+\mu}\partial_\mu\pi^-)\\ &-\frac{g_1}{2}\left(\phi_N+\phi_3+\sqrt{2}\phi_S\right)(K_1^{-\mu}\partial_\mu K^+ + K_1^{+\mu}\partial_\mu K^-) - \frac{g_1}{2}\left(\phi_N-\phi_3+\sqrt{2}\phi_S\right)(\bar{K}_1^{0\mu}\partial_\mu K^0 + K_1^{0\mu}\partial_\mu\overline{K^0})\\ &- g_1\sqrt{2}\phi_S f_{1S}^\mu\partial_\mu\eta_S - g_1\left[f_{1N}^\mu(\phi_N\partial_\mu\eta_N+\phi_3\partial_\mu\pi^0) + a_1^{0\mu}(\phi_3\partial_\mu\eta_N+\phi_N\partial_\mu\pi^0)\right].\end{aligned} \tag{10}$$

In order to calculate the tree-level meson masses, these mixing terms must be eliminated, i.e., the mass matrices have to be diagonalized. First we deal with mixings between nonets, then continue with the two-state mixings in the $N-3$ sectors of the vector and axial-vector nonets, and finally we resolve the three-state mixings in the $N-3-S$ sectors of the scalar and pseudoscalar nonets.

1. Mixings between different nonets

In order to eliminate the mixings between different nonets as listed in Eq. (10), the (axial-)vector fields have to be shifted by appropriately chosen derivative terms of the (pseudo)scalar fields. Such a shift spoils the canonical normalization of the (pseudo)scalar fields. Consequently, the (pseudo)scalar fields must be rescaled by adequate wave function renormalization factors Z_i, which will subsequently result in the appearance of factors Z_i^2 in the expressions of the (pseudo)scalar squared masses. The situation is slightly more complicated in the $N-3-S$ sector of the axial-vector–pseudoscalar mixing. The transformations are found to be

$$\rho_\mu^\pm \to \tilde{\rho}_\mu^\pm + Z_{a_0^\pm} w_{\rho^\pm}\partial_\mu\tilde{a}_0^\pm, \tag{11a}$$

$$a_0^\pm \to Z_{a_0^\pm}\tilde{a}_0^\pm, \tag{11b}$$

$$K_\mu^{\star\pm} \to \tilde{K}_\mu^{\star\pm} + Z_{K_0^{\star\pm}} w_{K^{\star\pm}}\partial_\mu\tilde{K}_0^{\star\pm}, \tag{12a}$$

$$K_0^{\star\pm} \to Z_{K_0^{\star\pm}}\tilde{K}_0^{\star\pm}, \tag{12b}$$

$$K_\mu^{\star0,\bar{0}} \to \tilde{K}_\mu^{\star0,\bar{0}} + Z_{K_0^{\star0}} w_{K^{\star0,\bar{0}}}\partial_\mu\tilde{K}_0^{\star0,\bar{0}}, \tag{13a}$$

$$K_0^{\star0,\bar{0}} \to Z_{K_0^{\star0}}\tilde{K}_0^{\star0,\bar{0}}, \tag{13b}$$

$$a_{1\mu}^\pm \to \tilde{a}_{1\mu}^\pm + Z_{\pi^\pm} w_{a_1^\pm}\partial_\mu\tilde{\pi}^\pm, \tag{14a}$$

$$\pi^\pm \to Z_{\pi^\pm}\tilde{\pi}^\pm, \tag{14b}$$

$$K_{1\mu}^\pm \to \tilde{K}_{1\mu}^\pm + Z_{K^\pm} w_{K_1^\pm}\partial_\mu\tilde{K}^\pm, \tag{15a}$$

$$K^\pm \to Z_{K^\pm}\tilde{K}^\pm, \tag{15b}$$

$$K_{1\mu}^{0,\bar{0}} \to \tilde{K}_{1\mu}^{0,\bar{0}} + Z_{K^0} w_{K_1^0}\partial_\mu\tilde{K}^{0,\bar{0}}, \tag{16a}$$

$$K^{0,\bar{0}} \to Z_{K^0}\tilde{K}^{0,\bar{0}}, \tag{16b}$$

while the nonet mixing in the axial-vector–pseudoscalar $N-3-S$ sector can be resolved by

$$\begin{pmatrix} f_{1N} \\ a_1^0 \end{pmatrix}^\mu \to \begin{pmatrix} \tilde{f}_{1N} \\ \tilde{a}_1^0 \end{pmatrix}^\mu + \mathbb{W}\begin{pmatrix} \partial^\mu\tilde{\eta}_N \\ \partial^\mu\tilde{\pi}^0 \end{pmatrix}, \tag{17a}$$

$$f_{1S}^\mu \to f_1^{H,\mu} + w_{f_{1S}}\partial^\mu\tilde{\eta}_S, \tag{17b}$$

$$\eta_N \to \tilde{\eta}_N, \qquad \pi^0 \to \tilde{\pi}^0, \qquad \eta_S \to \tilde{\eta}_S, \tag{17c}$$

where

$$w_{\rho^\pm} = \pm i \frac{g_1 \phi_3}{m^2_{\rho^\pm}}, \tag{18a}$$

$$w_{K^{\star\pm}} = \pm i \frac{g_1(\phi_N + \phi_3 - \sqrt{2}\phi_S)}{2m^2_{K^{\star\pm}}}, \tag{18b}$$

$$w_{K^{\star 0,\bar{0}}} = \pm i \frac{g_1(\phi_N - \phi_3 - \sqrt{2}\phi_S)}{2m^2_{K^{\star 0}}}, \tag{18c}$$

$$w_{a_1^\pm} = \frac{g_1 \phi_N}{m^2_{a_1^\pm}}, \tag{18d}$$

$$w_{K_1^\pm} = \frac{g_1(\phi_N + \phi_3 + \sqrt{2}\phi_S)}{2m^2_{K_1^\pm}}, \tag{18e}$$

$$w_{K_1^0} = \frac{g_1(\phi_N - \phi_3 + \sqrt{2}\phi_S)}{2m^2_{K_1^0}}, \tag{18f}$$

$$w_{f_{1S}} = \frac{\sqrt{2}g_1 \phi_S}{m^2_{f_{1S}}}, \tag{18g}$$

$$\mathbb{W} \equiv \begin{pmatrix} w_\eta^f & w_\pi^f \\ w_\eta^a & w_\pi^a \end{pmatrix}, \tag{19a}$$

$$w_\eta^f = \frac{g_1}{\det \mathbb{M}_A^2}(\phi_N m^2_{a_1^0} - \phi_3 m^2_{f_{1N}a_1^0}), \tag{19b}$$

$$w_\pi^f = \frac{g_1}{\det \mathbb{M}_A^2}(-\phi_N m^2_{f_{1N}a_1^0} + \phi_3 m^2_{a_1^0}), \tag{19c}$$

$$w_\eta^a = \frac{g_1}{\det \mathbb{M}_A^2}(-\phi_N m^2_{f_{1N}a_1^0} + \phi_3 m^2_{f_{1N}}), \tag{19d}$$

$$w_\pi^a = \frac{g_1}{\det \mathbb{M}_A^2}(\phi_N m^2_{f_{1N}} - \phi_3 m^2_{f_{1N}a_1^0}), \tag{19e}$$

$$Z_{a_0^\pm} = \frac{m_{\rho^\pm}}{\sqrt{m^2_{\rho^\pm} - g_1^2(\phi_3)^2}}, \tag{20a}$$

$$Z_{K_0^{\star\pm}} = \frac{2m_{K^{\star\pm}}}{\sqrt{4m^2_{K^{\star\pm}} - g_1^2(\phi_N + \phi_3 - \sqrt{2}\phi_S)^2}}, \tag{20b}$$

$$Z_{K_0^{\star 0}} = \frac{2m_{K^{\star 0}}}{\sqrt{4m^2_{K^{\star 0}} - g_1^2(\phi_N - \phi_3 - \sqrt{2}\phi_S)^2}}, \tag{20c}$$

$$Z_{\pi^\pm} = \frac{m_{a_1^\pm}}{\sqrt{m^2_{a_1^\pm} - g_1^2\phi_N^2}}, \tag{20d}$$

$$Z_{K^\pm} = \frac{2m_{K_1^\pm}}{\sqrt{4m^2_{K_1^\pm} - g_1^2(\phi_N + \phi_3 + \sqrt{2}\phi_S)^2}}, \tag{20e}$$

$$Z_{K^0} = \frac{2m_{K_1^0}}{\sqrt{4m^2_{K_1^0} - g_1^2(\phi_N - \phi_3 + \sqrt{2}\phi_S)^2}}, \tag{20f}$$

and $\mathbb{M}_A^2$ is the squared-mass matrix in the $N-3$ sector of the axial-vector nonet.

On the right-hand sides of Eqs. (B11)–(C17), the transformed fields are denoted by a tilde. After substituting Eqs. (B11)–(C17) into the quadratic part of the Lagrangian, the tildes are dropped, except for the $\tilde{f}_{1N}$, $\tilde{a}_1^0$, $\tilde{\eta}_N$, $\tilde{\pi}^0$, $\tilde{\eta}_S$ fields, where additional transformations are needed (see the next sections).

2. Two-state mixings in the $N-3$ sector of (axial) vectors

The vector and axial-vector mass matrices in the $N-3$ sector are

$$\mathbb{M}_V^2 = \begin{pmatrix} m^2_{\omega_N} & m^2_{\omega_N\rho^0} \\ m^2_{\omega_N\rho^0} & m^2_{\rho^0} \end{pmatrix}, \tag{21a}$$

$$\mathbb{M}_A^2 = \begin{pmatrix} m^2_{f_{1N}} & m^2_{f_{1N}a_1^0} \\ m^2_{f_{1N}a_1^0} & m^2_{a_1^0} \end{pmatrix}, \tag{21b}$$

where the explicit form of the matrix elements $m^2_{\omega_N}$, $m^2_{\rho^0}$, $m^2_{\omega_N\rho^0}$, $m^2_{f_{1N}}$, $m^2_{a_1^0}$, $m^2_{f_{1N}a_1^0}$ are given by Eqs. (A3g)–(A3i) and (A4g)–(A4i), respectively. It is worth noting that, unlike in the scalar-pseudoscalar sector, in the case of the vector-axial vector sector there is no mixing term between the N–S and S–3 sectors at tree level, i.e., there are no $\phi-\omega$ or $\phi-\rho^0$ mixing terms in the case of the vectors and no $f_1^H - f_1^L$ and $f_1^H - a_1^0$ mixing terms in the case of the axial vectors. However, there is $\phi-\omega$ mixing, albeit small, see, e.g., Refs. [54–58]. In our tree-level case, this effect is missing, but since it is small, we do not introduce it by hand.

The matrices $\mathbb{M}^2_{V/A}$ can be diagonalized by orthogonal transformations,

$$\tilde{\mathbb{M}}^2{}_{V/A} = \mathbb{O}_{V/A}\mathbb{M}^2_{V/A}\mathbb{O}^T_{V/A}, \tag{22a}$$

$$\mathbb{O}_{V/A} = \begin{pmatrix} \cos\vartheta_{V/A} & \sin\vartheta_{V/A} \\ -\sin\vartheta_{V/A} & \cos\vartheta_{V/A} \end{pmatrix}. \tag{22b}$$

Consequently, the resulting eigenvalues and mixing angles are

$$m^2_{\omega/\rho^0} = \frac{1}{2}\left(m^2_{\omega_N} + m^2_{\rho^0} \pm \sqrt{(m^2_{\omega_N} - m^2_{\rho^0})^2 + 4m^4_{\omega_N\rho^0}}\right), \tag{23a}$$

$$m^2_{f_1^L/a_1^0} = \frac{1}{2}\left(m^2_{f_{1N}} + m^2_{a_1^0} \pm \sqrt{(m^2_{f_{1N}} - m^2_{a_1^0})^2 + 4m^4_{f_{1N}a_1^0}}\right), \tag{23b}$$

$$\tan(2\vartheta_V) = \frac{m^2_{\omega_N\rho^0}}{m^2_{\omega_N} - m^2_{\rho^0}}, \tag{23c}$$

$$\tan(2\vartheta_A) = \frac{m^2_{f_{1N}a_1^0}}{m^2_{f_{1N}} - m^2_{a_1^0}}. \tag{23d}$$

While the field transformations that should be performed in the Lagrangian are

$$\begin{pmatrix} \omega_N \\ \rho^0 \end{pmatrix}^\mu \to \mathbb{O}_V^T \begin{pmatrix} \omega \\ \rho^0 \end{pmatrix}^\mu, \tag{24a}$$

$$\begin{pmatrix} \tilde{f}_{1N} \\ \tilde{a}_1^0 \end{pmatrix}^\mu \to \mathbb{O}_A^T \begin{pmatrix} f_1^L \\ a_1^0 \end{pmatrix}^\mu, \tag{24b}$$

where, if we combine the second transformation with Eq. (17a) we end up with the following transformation for the original f_{1N} and a_1^0 axial-vector fields of the Lagrangian:

$$\begin{pmatrix} f_{1N} \\ a_1^0 \end{pmatrix}^\mu \to \mathbb{O}_A^T \begin{pmatrix} f_1^L \\ a_1^0 \end{pmatrix}^\mu + \begin{pmatrix} w_\eta^f & w_\pi^f \\ w_\eta^a & w_\pi^a \end{pmatrix} \begin{pmatrix} \partial^\mu \tilde{\eta}_N \\ \partial^\mu \tilde{\pi}^0 \end{pmatrix}. \tag{25}$$

After the diagonalization, the ω, ρ^0 vector and the f_1^L, a_1^0 axial-vector fields correspond to the physical $\omega(782)$, $\rho^0(770)$, $f_1(1280)$, and $a_1^0(1186)$ states, respectively.

3. Three-state mixings within the scalar nonet

There is a three-state mixing in the $N-3-S$ sector of the scalar nonet among the σ_N, a_0^0, and σ_S fields of the Lagrangian. This mixing can be resolved by a three-dimensional orthogonal transformation $\mathbb{O}_S$ resulting in the squared-mass eigenvalues $\lambda_{f_0^L}$, $\lambda_{a_0^0}$, and $\lambda_{f_0^H}$ and eigenstates f_0^L, a_0^0, and f_0^H, respectively. The symmetric scalar squared-mass mixing matrix is

$$\mathbb{M}_S^2 = \begin{pmatrix} m^2_{\sigma_N} & m^2_{\sigma_N a_0^0} & m^2_{\sigma_N\sigma_S} \\ m^2_{\sigma_N a_0^0} & m^2_{a_0^0} & m^2_{a_0^0\sigma_S} \\ m^2_{\sigma_N\sigma_S} & m^2_{a_0^0\sigma_S} & m^2_{\sigma_S} \end{pmatrix}, \tag{26}$$

where the explicit form of the matrix elements are given by Eqs. (A2d)–(A2i). The diagonalization can be written as

$$\mathbb{O}_S \mathbb{M}_S^2 \mathbb{O}_S^T = \tilde{\mathbb{M}}_S^2 \equiv \mathrm{diag}(\lambda_{S_l}, \lambda_{S_m}, \lambda_{S_h}), \tag{27a}$$

$$\text{requiring} \qquad \lambda_{S_l} \le \lambda_{S_m} < \lambda_{S_h}, \tag{27b}$$

where $\mathbb{O}_S$ and the eigenvalues are to be calculated numerically. The particle assignment of this sector is not as straightforward as for the others due to the fact that there are two a_0's and five f_0's below 2 GeV according to PDG data [42]. In this sector, the squared-mass eigenvalues and the field transformations are given by

$$m^2_{f_0^L} = \lambda_{S_l}, \qquad m^2_{a_0^0} = \lambda_{S_m}, \qquad m^2_{f_0^H} = \lambda_{S_h}, \tag{28a}$$

$$\begin{pmatrix} \sigma_N \\ a_0^0 \\ \sigma_S \end{pmatrix} \longrightarrow \mathbb{O}_S^T \begin{pmatrix} f_0^L \\ a_0^0 \\ f_0^H \end{pmatrix}. \tag{28b}$$

Here we tried all possible assignments and chose the one that resulted in the lowest χ^2, but due to the large error in this sector, this part of the fit is not very restrictive (see also Sec. V for the specific assignment).

4. Three-state mixings within the pseudoscalar nonet

Even after the transformations (17b), (17c), and (25) have been applied, there is still a three-state mixing within the $N-3-S$ sector of the pseudoscalar nonet, which concerns both the kinetic and the mass terms. The affected fields are $\mathbf{x}^T = (\tilde{\eta}_N, \tilde{\pi}^0, \tilde{\eta}_S)$. The relevant part of the Lagrangian reads

$$\mathcal{L}_{P_{N3S}} = \frac{1}{2}\partial^\mu \mathbf{x}^T \mathbb{D}_P \partial_\mu \mathbf{x} - \frac{1}{2}\mathbf{x}^T \mathbb{M}_P^2 \mathbf{x}, \tag{29}$$

where

$$\mathbb{D}_P = \left(\begin{array}{c|c} \mathbb{1} - g_1 \mathbb{W}^T \mathbb{N} & \mathbf{0} \\ \hline \mathbf{0} & 1 - \frac{2g_1^2\phi_S^2}{m^2_{f_{1S}}} \end{array}\right), \tag{30a}$$

$$\mathbb{N} = \begin{pmatrix} \phi_N & \phi_3 \\ \phi_3 & \phi_N \end{pmatrix}, \tag{30b}$$

$$\mathbb{M}_P^2 = \begin{pmatrix} m^2_{\eta_N} & m^2_{\eta_N\pi^0} & m^2_{\eta_N\eta_S} \\ m^2_{\eta_N\pi^0} & m^2_{\pi^0} & m^2_{\pi^0\eta_S} \\ m^2_{\eta_N\eta_S} & m^2_{\pi^0\eta_S} & m^2_{\eta_S} \end{pmatrix}, \tag{30c}$$

and $\mathbb{W}$ is defined in Eq. (19a), while the explicit expressions for the elements of the squared-mass matrix $\mathbb{M}_P^2$ are given by Eqs. (A1d)–(A1i). The matrix $\mathbb{D}_P$ is symmetric and can be diagonalized by a rotation in the $N-3$ plane,

$$\mathbb{O}_D = \begin{pmatrix} \cos\vartheta_D & \sin\vartheta_D & 0 \\ -\sin\vartheta_D & \cos\vartheta_D & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{31}$$

Consequently, $\mathcal{L}_{P_{N3S}}$ can be written as

$$\mathcal{L}_{P_{N3S}} = \frac{1}{2}\partial^\mu \mathbf{y}^T \tilde{\mathbb{D}}_P \partial_\mu \mathbf{y} - \frac{1}{2}\mathbf{y}^T \mathbb{M}_P'^2 \mathbf{y}, \qquad \mathbf{y} \equiv \mathbb{O}_D \mathbf{x}, \tag{32}$$

with

$$\tilde{\mathbb{D}}_P \equiv \mathbb{O}_D \mathbb{D}_P \mathbb{O}_D^T = \mathrm{diag}\left(\lambda_{D_1}, \lambda_{D_2}, 1 - \frac{2g_1^2\phi_S^2}{m_{f_{1S}}^2}\right), \tag{33a}$$

$$\lambda_{D_{1,2}} = \frac{(\mathrm{Tr}\mathbb{D}_P^{2\times 2} \pm \sqrt{(\mathrm{Tr}\mathbb{D}_P^{2\times 2})^2 - 4\det \mathbb{D}_P^{2\times 2}})}{2}, \tag{33b}$$

$$\mathbb{M}_P'^2 \equiv \mathbb{O}_D \mathbb{M}_P^2 \mathbb{O}_D^T. \tag{33c}$$

Here, $\mathbb{D}_P^{2\times 2} \equiv \mathbb{1} - g_1 \mathbb{W}^\mathsf{T}\mathbb{N}$ denotes the upper left (2×2) block of $\mathbb{D}_P$, which is assumed to be positive definite. Accordingly, we can define

$$Z_{P_N} = \frac{1}{\sqrt{\lambda_{D_1}}}, \qquad Z_{P_3} = \frac{1}{\sqrt{\lambda_{D_2}}}, \tag{34a}$$

$$Z_{\eta_S} = \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2\phi_S^2}}, \tag{34b}$$

$$\mathbb{Z}_P = \mathrm{diag}(Z_{P_N}, Z_{P_3}, Z_{\eta_S}), \qquad \mathbf{y}' = \mathbb{Z}_P^{-1}\mathbf{y}, \tag{34c}$$

which subsequently leads to

$$\mathcal{L}_{P_{N3S}} = \frac{1}{2}\partial^\mu \mathbf{y}'^T \partial_\mu \mathbf{y}' - \frac{1}{2}\mathbf{y}'^T \tilde{\mathbb{M}}_P^2 \mathbf{y}', \tag{35a}$$

$$\tilde{\mathbb{M}}_P^2 \equiv \mathbb{Z}_P \mathbb{M}_P'^2 \mathbb{Z}_P. \tag{35b}$$

Now the kinetic part has become canonical, and the symmetric matrix $\tilde{\mathbb{M}}_P^2$ can be diagonalized by an orthogonal transformation $\mathbb{O}_M$,

$$\mathbb{O}_M \tilde{\mathbb{M}}_P^2 \mathbb{O}_M^T = \mathrm{diag}(\lambda_{P_l}, \lambda_{P_m}, \lambda_{P_h}), \tag{36a}$$

$$\text{requiring} \qquad \lambda_{P_l} < \lambda_{P_m} < \lambda_{P_h}, \tag{36b}$$

where $\mathbb{O}_M$ and the eigenvalues are to be calculated numerically. The field transformations are given by

$$\begin{pmatrix} \eta_N \\ \pi^0 \\ \eta_S \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{\eta}_N \\ \tilde{\pi}^0 \\ \tilde{\eta}_S \end{pmatrix} = \mathbb{O}_D^T \mathbb{Z}_P \mathbb{O}_M^T \begin{pmatrix} \pi^0 \\ \eta \\ \eta' \end{pmatrix} \equiv \mathbb{O}_P \begin{pmatrix} \pi^0 \\ \eta \\ \eta' \end{pmatrix}. \tag{37}$$

The particle assignment of this sector is straightforward,

$$m_\pi^2 = \lambda_{P_l}, \qquad m_\eta^2 = \lambda_{P_m}, \qquad m_{\eta'}^2 = \lambda_{P_h}, \tag{38}$$

and the resulting field vector contains the physical π, η, η' fields,

$$\mathbb{O}_M \mathbf{y}' = \mathbf{y}^{\mathrm{ph}} \equiv (\pi^0, \eta, \eta')^T. \tag{39}$$

It is worth noting that $\mathbb{O}_P$ is not an orthogonal transformation. Using Eqs. (25), (37), and (C17), finally the transformations of the f_{1N}^μ, $a_1^{0\mu}$, and f_{1S}^μ fields into physical fields are

$$\begin{pmatrix} f_{1N} \\ a_1^0 \end{pmatrix}^\mu \to \mathbb{O}_A^T \begin{pmatrix} f_1^L \\ a_1^0 \end{pmatrix}^\mu + \begin{pmatrix} w_\eta^f & w_\pi^f \\ w_\eta^a & w_\pi^a \end{pmatrix} \begin{pmatrix} \mathbb{O}_{P11} & \mathbb{O}_{P12} & \mathbb{O}_{P13} \\ \mathbb{O}_{P21} & \mathbb{O}_{P22} & \mathbb{O}_{P23} \end{pmatrix} \begin{pmatrix} \partial^\mu \pi^0 \\ \partial^\mu \eta \\ \partial^\mu \eta' \end{pmatrix}, \tag{40a}$$

$$f_{1S}^\mu \to f_1^{H,\mu} + w_{f_{1S}}(\mathbb{O}_{P31}\partial^\mu \pi^0 + \mathbb{O}_{P32}\partial^\mu \eta + \mathbb{O}_{P33}\partial^\mu \eta'). \tag{40b}$$

III. TREE-LEVEL PHYSICAL MASSES

After taking care of all the mixings, the squared-mass eigenvalues for the pseudoscalars, scalars, vectors, and axial vectors, respectively, are given by

$$M_{\pi^\pm}^2 = Z_{\pi^\pm}^2 m_{\pi^\pm}^2 + m_{\mathrm{em},P}^2, \tag{41a}$$

$$M_{K^\pm}^2 = Z_{K^\pm}^2 m_{K^\pm}^2 + m_{\mathrm{em},P}^2 + m_{\mathrm{em},P_K}^2, \tag{41b}$$

$$M_{K^0}^2 = Z_{K^0}^2 m_{K^0}^2, \tag{41c}$$

$$M_\eta^2 = m_\eta^2, \tag{41d}$$

$$M_{\pi^0}^2 = m_{\pi^0}^2, \tag{41e}$$

$$M_{\eta'}^2 = m_{\eta'}^2, \tag{41f}$$

$$M_{a_0^\pm}^2 = Z_{a_0^\pm}^2 m_{a_0^\pm}^2 + m_{\mathrm{em},S}^2, \tag{42a}$$

$$M_{K_0^{\star\pm}}^2 = Z_{K_0^{\star\pm}}^2 m_{K_0^{\star\pm}}^2 + m_{\mathrm{em},S}^2 + m_{\mathrm{em},S_K}^2, \tag{42b}$$

$$M_{K_0^{\star 0}}^2 = Z_{K_0^{\star 0}}^2 m_{K_0^{\star 0}}^2, \tag{42c}$$

$$M_{f_0^L}^2 = m_{f_0^L}^2, \tag{42d}$$

$$M^2_{a_0^0} = m^2_{a_0^0}, \tag{42e}$$

$$M^2_{f_0^H} = m^2_{f_0^H}, \tag{42f}$$

$$M^2_{\rho^\pm} = m^2_{\rho^\pm} + m^2_{\mathrm{em},V}, \tag{43a}$$

$$M^2_{K^{\star\pm}} = m^2_{K^{\star\pm}} + m^2_{\mathrm{em},V} + m^2_{\mathrm{em},V_K}, \tag{43b}$$

$$M^2_{K^{\star 0}} = m^2_{K^{\star 0}}, \tag{43c}$$

$$M^2_{\omega/\rho^0} = \frac{1}{2}\left(m^2_{\omega_N} + m^2_{\rho^0} \pm \sqrt{(m^2_{\omega_N} - m^2_{\rho^0})^2 + 4m^2_{\omega_N\rho^0}}\right), \tag{43d}$$

$$M^2_{\phi} = m^2_{\omega_S}, \tag{43e}$$

$$M^2_{a_1^\pm} = m^2_{a_1^\pm} + m^2_{\mathrm{em},A}, \tag{44a}$$

$$M^2_{K_1^\pm} = m^2_{K_1^\pm} + m^2_{\mathrm{em},A} + m^2_{\mathrm{em},A_K}, \tag{44b}$$

$$M^2_{K_1^0} = m^2_{K_1^0}, \tag{44c}$$

$$M^2_{f_1^L/a_1^0} = \frac{1}{2}\left(m^2_{f_{1N}} + m^2_{a_1^0} \pm \sqrt{(m^2_{f_{1N}} - m^2_{a_1^0})^2 + 4m^2_{f_{1N}a_1^0}}\right), \tag{44d}$$

$$M^2_{f_1^H} = m^2_{f_{1S}}, \tag{44e}$$

where the explicit expressions for the tree-level masses m_x^2 are given in Appendix A (squared-mass matrix elements), Sec. II B 3 (scalar $N-3-S$ squared-mass eigenvalues), and Sec. II B 4 (pseudoscalar $N-3-S$ squared-mass eigenvalues). Moreover, we introduced electromagnetic mass terms in each nonet, namely, $m^2_{\mathrm{em},S}$, $m^2_{\mathrm{em},P}$, $m^2_{\mathrm{em},V}$, $m^2_{\mathrm{em},A}$, and m^2_{em,S_K}, m^2_{em,P_K}, m^2_{em,V_K}, m^2_{em,A_K}, which are of the same order as the contribution from isospin-symmetry breaking. We will consider three different cases. In the first, we consider Dashen's theorem to be valid (see the introduction), so $m^2_{\mathrm{em},S_K} = m^2_{\mathrm{em},P_K} = m^2_{\mathrm{em},V_K} = m^2_{\mathrm{em},A_K} = 0$. This means an electromagnetic mass contribution for each charged particle in each sector. The second case is when Dashen's theorem is violated, but the additional electromagnetic contribution for the kaonic particles is the same in all sectors, $m^2_{\mathrm{em},S_K} = m^2_{\mathrm{em},P_K} = m^2_{\mathrm{em},V_K} = m^2_{\mathrm{em},A_K} \equiv m^2_{\mathrm{em},K}$. The third—and most general—case is where these contributions can be different in each sector.

IV. TREE-LEVEL DECAY WIDTHS

We start from the usual tree-level expression for two-body decay,

$$\Gamma_{A\to BC} = \frac{k}{8\pi M_A^2}|\mathcal{M}_{A\to BC}|^2, \tag{45}$$

where $k \equiv \sqrt{\vec{k}^2}$ is the absolute value of the three-momentum of the produced particles B, C in the rest frame of the decaying particle A and $\mathcal{M}_{A\to BC}$ is the tree-level matrix element of the process.

In a straightforward, but lengthy calculation we computed tree-level decay widths for the following processes:

(i) Vector-meson decays,

$$\rho^0 \to \pi^+\pi^-, \tag{46a}$$

$$\rho^- \to \pi^-\pi^0, \qquad \rho^- \to \pi^-\eta, \tag{46b}$$

$$\omega \to \pi^+\pi^-, \tag{46c}$$

$$\bar{K}^{\star 0} \to \pi^{0,+}K^{\bar{0},-}, \tag{46d}$$

$$K^{\star -} \to \pi^{0,-}K^{-,\bar{0}}, \tag{46e}$$

$$\Phi \to K^0\bar{K}^0, \qquad \Phi \to K^+K^-. \tag{46f}$$

(ii) Axial-vector-meson decays,

$$a_1^0 \to \rho^+\pi^-, \tag{47a}$$

$$a_1^- \to \pi^-\gamma, \qquad a_1^- \to \rho^{-,0}\pi^{0,-}, \tag{47b}$$

$$f_1^H \to K^{\star\pm,0,\bar{0}}K^{\mp,\bar{0},0}. \tag{47c}$$

(iii) Scalar-meson decays,

$$\bar{K}_0^{\star 0} \to \pi^{0,+}K^{\bar{0},-}, \tag{48a}$$

$$K_0^{\star -} \to \pi^{0,-}K^{-,\bar{0}}, \tag{48b}$$

$$a_0^0 \to \pi^0\eta, \quad a_0^0 \to \pi^0\eta', \quad a_0^0 \to K^{0,+}\bar{K}^{\bar{0},-}, \tag{48c}$$

$$a_0^- \to \pi^-\eta, \quad a_0^- \to \pi^-\eta', \quad a_0^- \to K^0K^-, \tag{48d}$$

$$f_0^{L/H} \to \pi^{0,+}\pi^{0,-}, \qquad f_0^{L/H} \to K^{0,+}K^{\bar{0},-}. \tag{48e}$$

It should be noted that above we only listed the negatively charged particle decays (in case of charged particles), because the charge-conjugated decays have the same decay width. Similarly, in the case of the kaonic particles, we gave only the decays of the conjugate particles, like $\bar{K}^{\star 0}$ and $\bar{K}_0^{\star 0}$, since their charge-conjugated partners ($K^{\star 0}$ and $K_0^{\star 0}$)

have the same decay width in the given channel. The explicit expressions for all decay widths can be found in Appendix C.

V. FIT AND RESULTS

As discussed in detail in Ref. [2], there are 13 unknown parameters in the Lagrangian in the case of isospin symmetry, namely, m_0^2, m_1^2, c_1, δ_S, g_1, g_2, ζ_N, ζ_S, λ_1, λ_2, h_1, h_2, and h_3.[1] In addition to the parameters, the condensates ϕ_N and ϕ_S are also unknown. However, the external fields ζ_N and ζ_S can be calculated using the field equations at $T = 0$ once ϕ_N and ϕ_S are known, thus instead of these fields, the condensates can be used in the fitting procedure. Consequently, the unknown parameters to be determined in the case of isospin symmetry are m_0^2, m_1^2, c_1, δ_S, g_1, g_2, ϕ_N, ϕ_S, λ_1, λ_2, h_1, h_2, and h_3. After determining these parameters, the fields ζ_N and ζ_S can be calculated via Eqs. (9a) and (9b).

In the case of isospin-symmetry violation, there are several changes in the parameter set:

(i) The fields δ_u, δ_d, and δ_s, as well as m_1^2 appear in all the meson masses (see Appendix A) in the following three combinations:

$$\tilde{m}_1^2 = m_1^2 + \delta_u + \delta_d,$$
$$\tilde{\delta}_s = \delta_s - \frac{1}{2}(\delta_u + \delta_d),$$
$$\delta_3 = \delta_u - \delta_d.$$

Thus, instead of m_1^2 and δ_S ($\delta_N = \delta_u + \delta_d$ can be incorporated into m_1^2, similar to as in Ref. [2]), we can fit $\tilde{m}_1^2$ and $\tilde{\delta}_s$. All in all, instead of the parameters δ_u, δ_d, δ_s, and m_1^2, we have to fit only $\tilde{m}_1^2$, $\tilde{\delta}_s$, and δ_3.

(ii) There is a new condensate ϕ_3.

(iii) There are four, five, or eight electromagnetic mass contributions as it is described below Eq. (44e). In the first case, these are $m^2_{\mathrm{em},S}$, $m^2_{\mathrm{em},P}$, $m^2_{\mathrm{em},V}$, and $m^2_{\mathrm{em},A}$ (Dashen theorem-respecting scenario or DS), in the second case we have $m^2_{\mathrm{em},S}$, $m^2_{\mathrm{em},P}$, $m^2_{\mathrm{em},V}$, $m^2_{\mathrm{em},A}$, and $m^2_{\mathrm{em},K}$ (Dashen theorem-violating scenario I or DVS-I), while in the third case these are $m^2_{\mathrm{em},S}$, $m^2_{\mathrm{em},P}$, $m^2_{\mathrm{em},V}$, $m^2_{\mathrm{em},A}$, m^2_{em,S_K}, m^2_{em,P_K}, m^2_{em,V_K}, and m^2_{em,A_K} (Dashen theorem-violating scenario II or DVS-II). It is worth noting that during the fit an upper bound of 10 MeV was introduced, since a higher value for the electromagnetic correction to the mass is physically unrealistic.

(iv) Finally, there are two additional mass terms, δm_V^2 in the vector and δm_A^2 in the axial-vector sector, in order to generate a splitting between the ρ^0 and ω and similarly between the a_1^0 and f_1^L masses.

Altogether there are 21, 22, or 25 unknown parameters depending on the handling of the electromagnetic mass contributions (see above) in the isospin-symmetry broken case—compared to the 13 parameters in the isospin-symmetric case—namely, m_0^2, m_1^2, c_1, δ_S, δ_3, g_1, g_2, ϕ_N, ϕ_S, ϕ_3, λ_1, λ_2, h_1, h_2, h_3, $m^2_{\mathrm{em},S}$, $m^2_{\mathrm{em},P}$, $m^2_{\mathrm{em},V}$, $m^2_{\mathrm{em},A}$, δm_V^2, δm_A^2, and optionally $m^2_{\mathrm{em},K}$, or m^2_{em,S_K}, m^2_{em,P_K}, m^2_{em,V_K}, and m^2_{em,A_K}.

In order to determine these parameters, we calculated physical quantities at tree level and used a multiparametric χ^2 minimization method (MINUIT [59]) similar to Ref. [2]. The values for the physical quantities are taken from experiment, that is, from the PDG [42]. More precisely, we take only the mean value from the PDG, and in most cases we use an artificially increased error instead of the experimental value if the latter is smaller than a prescribed percentage (see below). We do this because some of the masses are known with very high precision, e.g., the mass of the η meson is known with 0.003% precision, and we do not expect such a phenomenological model to describe a mass with that high accuracy. Our expected accuracy is around several percent. Since the effect of isospin-symmetry breaking is about 3%, for instance, for the mass of the pion and below 1% for all the other quantities, we cannot simply fit the neutral and charged quantities separately (our increased error would be greater than the effect). Thus, we chose to fit instead the isospin-averaged neutral and charged quantities and their differences. We use a small artificial minimal error of 5% for the isospin-averaged masses of those particles that can be modeled very precisely within our model, i.e., the pseudoscalars and vector mesons. For the corresponding mass differences, we use a somewhat larger minimal error of 20%. The same minimal error is also used for the axial vectors and the scalar $K_0^\star$ and a_0, while the minimal error is 50% for the masses of the $f_0^{L/H}$ mesons, since the latter cannot be described very precisely within our model because it does not contain the other three isoscalar-scalar states. For the sake of completeness, we list here all the values used for our fit (it should be noted that the decay widths of the scalar $f_0^{L/H}$ fields are not used in the fit):

(i) Weak-decay constants. For the pion and kaon decay constants, we use [42]

$$f_\pi = 92.06 \pm 4.60 \text{ MeV}, \tag{49a}$$

$$f_K = 110.10 \pm 5.51 \text{ MeV}, \tag{49b}$$

where we have also applied the 5% minimal-error prescription. The decay constants are related to the condensates through the PCAC relations, which in our model leads to

[1] In Ref. [2] we have used h_{0N} and h_{0S} to denote the explicit symmetry-breaking parameters instead of ζ_N and ζ_S.

$$f_{\pi^\pm} = \frac{\phi_N}{Z_{\pi^\pm}}, \tag{50a}$$

$$f_{\pi^0} = (\mathbb{O}_P^{-1}{}_{11} + \mathbb{O}_P^{-1}{}_{12})(\phi_N + \phi_3) + \mathbb{O}_P^{-1}{}_{13}\phi_S, \tag{50b}$$

$$f_{K^\pm} = \frac{\phi_N + \phi_3 + \sqrt{2}\phi_S}{2Z_{K^\pm}}, \tag{50c}$$

$$f_{K^0} = \frac{\phi_N - \phi_3 + \sqrt{2}\phi_S}{2Z_{K^0}}, \tag{50d}$$

where the somewhat unusual expression for f_{π^0} is due to the mixing in the $N-3-S$ sector. Its explicit form is derived in Appendix B. We have fitted only the charged weak-decay constants, since from the combination of measurements and theory only this can be determined (see Ref. [42], part 72, Leptonic Decays of Charged Pseudoscalar Mesons),

$$f_\pi \overset{!}{=} f_{\pi^\pm}, \tag{51a}$$

$$f_K \overset{!}{=} f_{K^\pm}. \tag{51b}$$

It is worth noting that f_{π^0} and f_{K^0} will be predictions.

(ii) Pseudoscalar masses. For the charged particles, we use the isospin averages and differences,

$$\bar{M}_\pi = \frac{M_{\pi^0} + 2M_{\pi^\pm}}{3} = 138.04 \pm 6.90 \text{ MeV}, \tag{52a}$$

$$\Delta M_\pi = M_{\pi^0} - M_{\pi^\pm} = -4.59 \pm 0.92 \text{ MeV}, \tag{52b}$$

$$\bar{M}_K = \frac{M_{K^0} + M_{K^\pm}}{2} = 495.64 \pm 24.78 \text{ MeV}, \tag{52c}$$

$$\Delta M_K = M_{K^0} - M_{K^\pm} = 3.93 \pm 0.79 \text{ MeV}, \tag{52d}$$

$$M_\eta = 547.86 \pm 27.39 \text{ MeV}, \tag{52e}$$

$$M_{\eta'} = 957.78 \pm 47.89 \text{ MeV}. \tag{52f}$$

(iii) Scalar-meson masses. For the charged particles, we use only the isospin averages, since there are no data for the differences,

$$\bar{M}_{a_0} = \frac{M_{a_0^0} + 2M_{a_0^\pm}}{3} = 1474 \pm 294.8 \text{ MeV}, \tag{53a}$$

$$\bar{M}_{K_0^\star} = \frac{M_{K_0^{\star 0}} + M_{K_0^{\star\pm}}}{2} = 1425 \pm 285 \text{ MeV}, \tag{53b}$$

$$M_{f_0^L} = 1350 \pm 675 \text{ MeV}, \tag{53c}$$

$$M_{f_0^H} = 1733 \pm 867 \text{ MeV}. \tag{53d}$$

Here, a_0 is assigned to $a_0(1450)$ and $K_0^\star$ to $K_0^\star(1430)$.[2] For the two f_0 any two combinations from the five states $f_0(500)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ have been checked previously—i.e., in the isospin-symmetric case—and the assignment $f_0^L = f_0(1370)$, $f_0^H = f_0(1710)$ was found to be favored [2]. Thus, we started our minimization procedure with this assignment.

(iv) Vector-meson masses. For the charged particles, we use the isospin averages and differences,

$$\bar{M}_\rho = \frac{M_{\rho^0} + 2M_{\rho^\pm}}{3} = 775.16 \pm 38.76 \text{ MeV}, \tag{54a}$$

$$\Delta M_\rho = M_{\rho^0} - M_{\rho^\pm} = 0.15 \pm 0.57 \text{ MeV}, \tag{54b}$$

$$\bar{M}_{K^\star} = \frac{M_{K^{\star 0}} + M_{K^{\star\pm}}}{2} = 895.51 \pm 44.78 \text{ MeV}, \tag{54c}$$

$$\Delta M_{K^\star} = M_{K^{\star 0}} - M_{K^{\star\pm}} = 0.08 \pm 0.94 \text{ MeV}, \tag{54d}$$

$$M_\omega = 782.66 \pm 38.13 \text{ MeV}, \tag{54e}$$

$$M_\phi = 1019.46 \pm 50.97 \text{ MeV}. \tag{54f}$$

(v) Axial-vector masses. For the charged particles, we use only the isospin averages, since there are no data for the differences,

$$\bar{M}_{a_1} = \frac{M_{a_1^0} + 2M_{a_1^\pm}}{3} = 1230 \pm 246 \text{ MeV}, \tag{55a}$$

$$\bar{M}_{K_1} = \frac{M_{K_1^0} + M_{K_1^\pm}}{2} = 1253 \pm 250.6 \text{ MeV}, \tag{55b}$$

$$M_{f_1^L} = 1281.9 \pm 256.38 \text{ MeV}, \tag{55c}$$

$$M_{f_1^H} = 1426.3 \pm 285.26 \text{ MeV}. \tag{55d}$$

(vi) Vector-meson decays. We use

$$\bar{\Gamma}_{\rho\to\pi\pi} = \frac{\Gamma_{\rho^0\to\pi^+\pi^-} + 2\Gamma_{\rho^\pm\to\pi^\pm\pi^0}}{3} = 148.533 \pm 7.426 \text{ MeV}, \tag{56a}$$

[2] For the sake of completeness, we also checked the other options, when a_0 and $K_0^\star$ are assigned to $a_0(980)$ and $K_0^\star(700)$.

$$\Delta\Gamma_{\rho\to\pi\pi} = \Gamma_{\rho^0\to\pi^+\pi^-} - \Gamma_{\rho^\pm\to\pi^\pm\pi^0} = -1.7 \pm 1.6 \text{ MeV}, \tag{56b}$$

$$\bar{\Gamma}_{K^\star\to K\pi} = \frac{\Gamma_{\bar{K}^{\star 0}\to\pi^{0,+}K^{\bar{0},-}} + \Gamma_{K^{\star -}\to\pi^{0,-}K^{-,\bar{0}}}}{2} = 46.75 \pm 2.34 \text{ MeV}, \tag{56c}$$

$$\Delta\Gamma_{K^\star\to K\pi} = \Gamma_{\bar{K}^{\star 0}\to\pi^{0,+}K^{\bar{0},-}} - \Gamma_{K^{\star -}\to\pi^{0,-}K^{-,\bar{0}}} = 1.1 \pm 1.8 \text{ MeV}, \tag{56d}$$

$$\Gamma_{\omega\to\pi^+\pi^-} = 0.133 \pm 0.026 \text{ MeV}, \tag{56e}$$

$$\bar{\Gamma}_{\phi\to KK} = \frac{\Gamma_{\phi\to K^0\bar{K}^0} + \Gamma_{\phi\to K^+K^-}}{2} = 1.763 \pm 0.088 \text{ MeV}, \tag{56f}$$

$$\Delta\Gamma_{\phi\to KK} = \Gamma_{\phi\to K^0\bar{K}^0} - \Gamma_{\phi\to K^+K^-} = -0.646 \pm 0.129 \text{ MeV}. \tag{56g}$$

(vii) Axial-vector-meson decays. We use

$$\bar{\Gamma}_{a_1\to\rho\pi} = \frac{\Gamma_{a_1^0\to\rho^\pm\pi^\mp} + 2\Gamma_{a_1^\pm\to\rho^{\pm,0}\pi^{0,\pm}}}{3} = 425 \pm 175 \text{ MeV}, \tag{57a}$$

$$\Gamma_{a_1^\pm\to\pi^\pm\gamma} = 0.64 \pm 0.246 \text{ MeV}, \tag{57b}$$

$$\Gamma_{f_1^H\to K^{\star\pm,0,\bar{0}}K^{\mp,\bar{0},0}} = 43.60 \pm 8.72 \text{ MeV}. \tag{57c}$$

The $f_1^H \to K^\star K$ decay width is determined as described in Ref. [2].

(viii) Scalar-meson decays. We use

$$\Gamma_{a_0} = \bar{\Gamma}_{a_0\to K\bar{K}} + \bar{\Gamma}_{a_0\to\pi\eta} + \bar{\Gamma}_{a_0\to\pi\eta'} = 265 \pm 53 \text{ MeV}, \tag{58a}$$

$$\bar{\Gamma}_{K_0^\star\to K\pi} = \frac{\Gamma_{\bar{K}_0^{\star 0}\to\pi^{0,+}K^{\bar{0},-}} + \Gamma_{K_0^{\star -}\to\pi^{0,-}K^{-,\bar{0}}}}{2} = 270 \pm 80 \text{ MeV}, \tag{58b}$$

$$\Gamma_{f_0^L\to\pi^{0,+}\pi^{0,-}} = 250 \pm 125 \text{ MeV}, \tag{58c}$$

$$\Gamma_{f_0^L\to K^{0,+}K^{\bar{0},-}} = 150 \pm 75 \text{ MeV}, \tag{58d}$$

$$\Gamma_{f_0^H\to\pi^{0,+}\pi^{0,-}} = 20.2 \pm 10.1 \text{ MeV}, \tag{58e}$$

$$\Gamma_{f_0^H\to K^{0,+}K^{\bar{0},-}} = 87.7 \pm 43.9 \text{ MeV}, \tag{58f}$$

where

$$\bar{\Gamma}_{a_0\to K\bar{K}} = \frac{\Gamma_{a_0^0\to K^{0,+}K^{\bar{0},-}} + 2\Gamma_{a_0^\pm\to K^0K^\pm}}{3}, \tag{58g}$$

$$\bar{\Gamma}_{a_0\to\pi\eta} = \frac{\Gamma_{a_0^0\to\pi^0\eta} + 2\Gamma_{a_0^\pm\to\pi^\pm\eta}}{3}, \tag{58h}$$

$$\bar{\Gamma}_{a_0\to\pi\eta'} = \frac{\Gamma_{a_0^0\to\pi^0\eta'} + 2\Gamma_{a_0^\pm\to\pi^\pm\eta'}}{3}. \tag{58i}$$

The decay widths for $f_0^L = f_0(1370)$ are not taken from the PDG [42], but are instead estimates based on Refs. [42,60–62] (see also Ref. [2]).

At first, the best solution taken from Ref. [2] was used to check whether we reproduce the isospin-symmetric solution with the new numerical fitting algorithm, where we do not fit the isospin-symmetry-breaking quantities. With that parameter set, we got a solution with a reasonable $\chi^2 = 8.9$.

Then we initialized new parameter sets from 1.2×10^7 random points in the parameter space, set the error level for the isospin-symmetry-breaking quantities to 20%, and used the DVS-I. The best solution leads to $\chi^2 = 29.8$, where the largest contribution comes from $\Gamma_{\omega\to\pi\pi}$, which is about 25, while all other quantities give a rather small χ^2 contribution. (The detailed results of this fit and the corresponding parameter set are shown in Tables II and III in Appendix D). This may be due to the fact that $\Gamma_{\omega\to\pi\pi}$ is the only quantity that is directly proportional to ϕ_3^2 [cf. Eq. (C11)], so it is very sensitive to the value of ϕ_3, which—due to the fitting of other quantities—tends to be very small, $\sim\mathcal{O}(10^{-6}) - \mathcal{O}(10^{-5})$ GeV, compared to $\phi_{N/S}$. One might think that this problem with the $\omega \to \pi\pi$ decay could be solved by the omitted $\phi - \omega$ mixing. However, this would not work because the $\phi \to \pi\pi$ decay width is small $\sim 4.2 \times 10^{-5}$ MeV [42] and in addition its contribution to the $\omega \to \pi\pi$ decay width includes a $\cos\theta_V$ (with $\theta_V = 86.82°$, see Ref. [56]), which is $\sim 5 \times 10^{-2}$, leading to a total $\mathcal{O}(10^{-7})$ contribution that is much too small to correct the decay width of $\omega \to \pi\pi$ given by this calculation.

Next, the error level for the isospin-symmetry-breaking quantities was increased in several steps from 20% to 500% and the resulting χ^2 values are shown in Fig. 1 for the best solution in each case. If the error level for the isospin-symmetry-breaking quantities is increased to infinity, i.e., their χ^2 contributions are neglected, an isospin-symmetric solution is obtained. In this way a solution with $\chi^2 = 3.2$ can be achieved. It is somehow surprising that with this limiting procedure a better solution can be obtained than if we start from 1.2×10^7 random points in the isospin-symmetric case, which leads to a solution with $\chi^2 > 7$.

In every fit, the $\Gamma_{\omega\to\pi\pi}$ decay width produced the largest contribution to χ^2. Therefore, we also made fits in which the ω decay is omitted. In the case of DVS-I, this leads to a

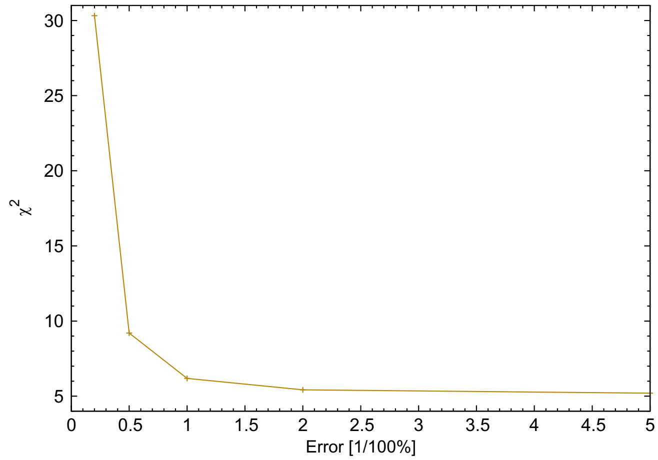

FIG. 1. χ^2 as a function of prescribed error starting from 1.2×10^7 random points in the parameter space.

solution with $\chi^2 = 6.5$ and $\chi^2_{\mathrm{red}} \equiv \chi^2/N_{\mathrm{d.o.f.}} = 0.6$ (see Tables II and III in Appendix D). It is worth noting that in this case the number of degrees of freedom $N_{\mathrm{d.o.f.}}$, which is the difference between the number of fitted quantities and the number of fit parameters, was 11.

We also investigated all possible scenarios [described in Sec. III below (44e)], namely, DS, DVS-I, and DVS-II, both with or without fitting the $\omega \to \pi\pi$ decay width. The resulting χ^2 and χ^2_{red} values are displayed in Table I. According to the reduced χ^2_{red} values, the quality of the fit is very similar for the DS and DVS-I, while the DVS-II is worse both with or without fitting the $\omega \to \pi\pi$ decay width. In the case without the $\omega \to \pi\pi$ fit, the χ^2_{red} is below 1 in every case, which testifies for very high-quality fits. Looking only at the χ^2_{red} values, the DS or DVS-I are favored over the DVS-II. According to Refs. [46,47], Dashen's theorem is significantly violated, favoring DVS-I, but this is not reflected in our fit. On the other hand, we can exclude DVS-II, since in this scenario—besides the larger χ^2_{red} values—the predictive power of our model decreases significantly. This is because in each sector the isospin averages and differences can be set with the help of the electromagnetic mass contributions, and the only constraints on isospin-symmetry violation come from the decay widths. The detailed results of the fits and the parameter values for the DS and DVS-I with and without the ω decay are shown in Tables II and III of Appendix D, respectively.

TABLE I. χ^2 and χ^2_{red} values for the different scenarios (see Sec. V), with or without fitting the ω decay width.

	With ω decay		Without ω decay	
Case	χ^2	χ^2_{red}	χ^2	χ^2_{red}
DS	31.6	2.4	6.5	0.6
DVS I	30.3	2.5	6.2	0.6
DVS II	31.3	3.5	6.0	0.8

We also checked if instead of omitting the $\omega \to \pi^+\pi^-$ decay, either the $\rho^0 \to \pi^+\pi^-$ or the $\rho^\pm \to \pi^\mp\pi^0$ decay is omitted, because ω mixes with ρ^0. In both cases, the χ^2 gets much worse, 22.7 and 23.2, respectively, while the χ^2_{red} value is 2.1 in both cases. So even if there is $\rho^0 - \omega$ mixing, omitting the ρ^0 decay will not improve the fit as much as omitting the ω decay.

According to Eqs. (50b) and (50d), we can calculate the neutral-pion and neutral-kaon decay constants that cannot be measured directly. In the case of the DVS-I without the ω decay, the mean values are

$$f_{\pi^0} = 109.00 \text{ MeV}, \tag{59a}$$

$$f_{K^0} = 109.55 \text{ MeV}. \tag{59b}$$

In our case, the difference $\Delta f_K = f_K^0 - f_K^+$ is very small ($\approx$5 keV) compared to a calculation [63] based on QCD sum rules, where it was found to be $\Delta f_K = 1.5$ MeV.

VI. CONCLUSION

In this paper, we have investigated isospin-symmetry-breaking effects in the framework of the eLSM at tree level. We have shown that the various particle mixings caused by the three nonzero condensates are quite complicated and can only be resolved by several field transformations. We have calculated the physical masses of the mesons at tree level. All possible tree-level decay widths are also determined, which leads to long expressions due to the many field transformations. We took existing experimental data on the considered mesons and determined the unknown parameters of the model by a χ^2 fitting procedure. In our approach, we also investigated the fulfillment of Dashen's theorem. For this purpose, we introduced an additional electromagnetic mass contribution in the kaonic sectors and analyzed the possible best fits. We found that there is no strong violation of Dashen's theorem at tree level in our approach, i.e., fits respecting or violating Dashen's theorem are of equally good quality. We found that the decay width of $\omega \to \pi\pi$ gave the largest contribution to the χ^2, showing that this decay cannot be described at tree level in this model due to the smallness of ϕ_3 required by the other isospin-symmetry-breaking channels.

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APPENDIX A: ELEMENTS OF THE MASS SQUARED MATRICES

The elements of the squared meson mass matrices at tree level after isospin-symmetry breaking—before the field shifts and orthogonal transformations in the $N-3-S$ sectors—are given as follows.

(i) Pseudoscalars,

$$m^2_{\pi^\pm} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \frac{\lambda_2}{2}(\phi_N^2 + 3\phi_3^2), \tag{A1a}$$

$$m^2_{K^\pm} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \lambda_2\left[-\frac{1}{\sqrt{2}}(\phi_N + \phi_3)\phi_S + \frac{(\phi_N + \phi_3)^2}{2} + \phi_S^2\right], \tag{A1b}$$

$$m^2_{K^0} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \lambda_2\left[-\frac{1}{\sqrt{2}}(\phi_N - \phi_3)\phi_S + \frac{(\phi_N - \phi_3)^2}{2} + \phi_S^2\right], \tag{A1c}$$

$$m^2_{\eta_N} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \frac{\lambda_2}{2}(\phi_N^2 + \phi_3^2) + c_1\phi_N^2\phi_S^2, \tag{A1d}$$

$$m^2_{\pi^0} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \frac{\lambda_2}{2}(\phi_N^2 + \phi_3^2) + c_1\phi_S^2\phi_3^2, \tag{A1e}$$

$$m^2_{\eta_S} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \lambda_2\phi_S^2 + \frac{c_1}{4}(\phi_N^2 - \phi_3^2)^2, \tag{A1f}$$

$$m^2_{\eta_N\pi^0} = m^2_{\pi^0\eta_N} = (\lambda_2 - c_1\phi_S^2)\phi_N\phi_3, \tag{A1g}$$

$$m^2_{\eta_N\eta_S} = m^2_{\eta_S\eta_N} = \frac{c_1}{2}\phi_N\phi_S(\phi_N^2 - \phi_3^2), \tag{A1h}$$

$$m^2_{\pi^0\eta_S} = m^2_{\eta_S\pi^0} = -\frac{c_1}{2}\phi_3\phi_S(\phi_N^2 - \phi_3^2). \tag{A1i}$$

(ii) Scalars,

$$m^2_{a_0^\pm} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \frac{\lambda_2}{2}(3\phi_N^2 + \phi_3^2), \tag{A2a}$$

$$m^2_{K_0^{\star\pm}} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \lambda_2\left[\frac{1}{\sqrt{2}}(\phi_N + \phi_3)\phi_S + \frac{(\phi_N + \phi_3)^2}{2} + \phi_S^2\right], \tag{A2b}$$

$$m^2_{K_0^{\star0}} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + \phi_S^2) + \lambda_2\left[\frac{1}{\sqrt{2}}(\phi_N - \phi_3)\phi_S + \frac{(\phi_N - \phi_3)^2}{2} + \phi_S^2\right], \tag{A2c}$$

$$m^2_{\sigma_N} = m_0^2 + \lambda_1(3\phi_N^2 + \phi_3^2 + \phi_S^2) + \frac{3\lambda_2}{2}(\phi_N^2 + \phi_3^2), \tag{A2d}$$

$$m^2_{a_0^0} = m_0^2 + \lambda_1(\phi_N^2 + 3\phi_3^2 + \phi_S^2) + \frac{3\lambda_2}{2}(\phi_N^2 + \phi_3^2), \tag{A2e}$$

$$m^2_{\sigma_S} = m_0^2 + \lambda_1(\phi_N^2 + \phi_3^2 + 3\phi_S^2) + 3\lambda_2\phi_S^2, \tag{A2f}$$

$$m^2_{\sigma_N a_0^0} = m^2_{a_0^0\sigma_N} = (2\lambda_1 + 3\lambda_2)\phi_N\phi_3, \tag{A2g}$$

$$m^2_{\sigma_N\sigma_S} = m^2_{\sigma_S\sigma_N} = 2\lambda_1\phi_N\phi_S, \tag{A2h}$$

$$m^2_{a_0^0\sigma_S} = m^2_{\sigma_S a_0^0} = 2\lambda_1\phi_3\phi_S. \tag{A2i}$$

(iii) Vectors,

$$m_{\rho^\pm}^2 = \tilde{m}_1^2 + \frac{\phi_N^2}{2}(h_1 + h_2 + h_3) + \frac{\phi_3^2}{2}(h_1 + h_2 - h_3 + 2g_1^2) + \frac{h_1}{2}\phi_S^2, \quad \text{(A3a)}$$

$$m_{K^{\star\pm}}^2 = \tilde{m}_1^2 + \tilde{\delta}_s + \frac{1}{2}\delta_3 + \frac{h_1}{2}(\phi_N^2 + \phi_3^2) + \frac{1}{4}(\phi_N + \phi_3)^2(h_2 + g_1^2) + \frac{\phi_S^2}{2}(h_1 + h_2 + g_1^2) \quad \text{(A3b)}$$

$$+ \frac{1}{\sqrt{2}}(\phi_N + \phi_3)\phi_S(h_3 - g_1^2), \quad \text{(A3c)}$$

$$m_{K^{*0}}^2 = \tilde{m}_1^2 + \tilde{\delta}_s - \frac{1}{2}\delta_3 + \frac{h_1}{2}(\phi_N^2 + \phi_3^2) + \frac{1}{4}(\phi_N - \phi_3)^2(h_2 + g_1^2) + \frac{\phi_S^2}{2}(h_1 + h_2 + g_1^2) \quad \text{(A3d)}$$

$$+ \frac{1}{\sqrt{2}}(\phi_N - \phi_3)\phi_S(h_3 - g_1^2), \quad \text{(A3e)}$$

$$m_{\omega_N}^2 = \tilde{m}_1^2 + \frac{\phi_N^2}{2}(h_1 + h_2 + h_3) + \frac{\phi_3^2}{2}(h_1 + h_2 + h_3) + \frac{h_1}{2}\phi_S^2 + \delta m_V^2, \quad \text{(A3f)}$$

$$m_{\rho^0}^2 = \tilde{m}_1^2 + \frac{\phi_N^2}{2}(h_1 + h_2 + h_3) + \frac{\phi_3^2}{2}(h_1 + h_2 + h_3) + \frac{h_1}{2}\phi_S^2 - \delta m_V^2, \quad \text{(A3g)}$$

$$m_{\omega_N\rho^0}^2 = (h_2 + h_3)\phi_N\phi_3 + \delta_3, \quad \text{(A3h)}$$

$$m_{\omega_S}^2 = \tilde{m}_1^2 + 2\tilde{\delta}_s + \frac{h_1}{2}(\phi_N^2 + \phi_3^2) + \phi_S^2\left(\frac{h_1}{2} + h_2 + h_3\right), \quad \text{(A3i)}$$

$$m_{\omega_N\omega_S}^2 = m_{\rho^0\omega_S}^2 = 0. \quad \text{(A3j)}$$

(iv) Axial-vectors,

$$m_{a_1^\pm}^2 = \tilde{m}_1^2 + \frac{\phi_N^2}{2}(h_1 + h_2 - h_3 + 2g_1^2) + \frac{\phi_3^2}{2}(h_1 + h_2 + h_3) + \frac{h_1}{2}\phi_S^2, \quad \text{(A4a)}$$

$$m_{K_1^\pm}^2 = \tilde{m}_1^2 + \tilde{\delta}_s + \frac{1}{2}\delta_3 + \frac{h_1}{2}(\phi_N^2 + \phi_3^2) + \frac{1}{4}(\phi_N + \phi_3)^2(h_2 + g_1^2) + \frac{\phi_S^2}{2}(h_1 + h_2 + g_1^2) \quad \text{(A4b)}$$

$$- \frac{1}{\sqrt{2}}(\phi_N + \phi_3)\phi_S(h_3 - g_1^2), \quad \text{(A4c)}$$

$$m_{K_1^0}^2 = \tilde{m}_1^2 + \tilde{\delta}_s - \frac{1}{2}\delta_3 + \frac{h_1}{2}(\phi_N^2 + \phi_3^2) + \frac{1}{4}(\phi_N - \phi_3)^2(h_2 + g_1^2) + \frac{\phi_S^2}{2}(h_1 + h_2 + g_1^2) \quad \text{(A4d)}$$

$$- \frac{1}{\sqrt{2}}(\phi_N - \phi_3)\phi_S(h_3 - g_1^2), \quad \text{(A4e)}$$

$$m_{f_{1N}}^2 = \tilde{m}_1^2 + \frac{1}{2}(\phi_N^2 + \phi_3^2)(h_1 + h_2 - h_3 + 2g_1^2) + \frac{h_1}{2}\phi_S^2 + \delta m_A^2, \quad \text{(A4f)}$$

$$m_{a_1^0}^2 = \tilde{m}_1^2 + \frac{1}{2}(\phi_N^2 + \phi_3^2)(h_1 + h_2 - h_3 + 2g_1^2) + \frac{h_1}{2}\phi_S^2 - \delta m_A^2, \quad \text{(A4g)}$$

$$m_{f_{1N}a_1^0}^2 = (h_2 - h_3 + 2g_1^2)\phi_N\phi_3 + \delta_3, \quad \text{(A4h)}$$

$$m_{f_{1S}}^2 = \tilde{m}_1^2 + 2\tilde{\delta}_s + \frac{h_1}{2}(\phi_N^2 + \phi_3^2) + \phi_S^2\left(\frac{h_1}{2} + h_2 - h_3 + 2g_1^2\right), \quad \text{(A4i)}$$

$$m_{f_{1N}f_{1S}}^2 = m_{a_1^0 f_{1S}}^2 = 0, \quad \text{(A4j)}$$

where

$$\tilde{m}_1^2 = m_1^2 + \delta_u + \delta_d, \;\; \tilde{\delta}_s = \delta_s - (\delta_u + \delta_d)/2, \;\; \delta_3 = \delta_u - \delta_d. \quad \text{(A4k)}$$

APPENDIX B: THE f_{π^0} WEAK-DECAY CONSTANT

We start from the PCAC hypotheses for the neutral pion,

$$\langle 0|\mathcal{J}^{\mu}_{\pi^0}(0)|\pi^0\rangle = ip^{\mu}f_{\pi^0}, \tag{B1}$$

where $\mathcal{J}^{\mu}_{\pi^0}$ is the Noether current pertaining to axial transformations. We need to calculate the current only in the $N-3-S$ sector, since π_3 only mixes with η_N and η_S to form the physical π^0,

$$\mathcal{J}^{\mu}_i\omega^A_i = \frac{\delta\mathcal{L}_y}{\delta(\partial_\mu y^{\mathrm{ph}}_k)}\delta y^{\mathrm{ph}}_k, \qquad i,k\in(N,3,S), \tag{B2}$$

where ω^A_i are the arbitrary infinitesimal parameters of the axial transformations and $y^{\mathrm{ph}}_k = (\pi^0,\eta,\eta')$ are the physical isoscalar fields in the pseudoscalar sector [see Eq. (39)], while δy^{ph}_k are the infinitesimal axial transformations of the considered fields that also need to be determined. The relevant part of the Lagrangian based on Eqs. (35a) and (39) is

$$\mathcal{L}_y = \frac{1}{2}\partial^\mu y^{\mathrm{ph}}_k\partial_\mu y^{\mathrm{ph}}_k - \frac{1}{2}y^{\mathrm{ph}}_k\mathrm{diag}(\tilde{\mathbb{M}}^2_P)_{kl}y^{\mathrm{ph}}_l, \tag{B3}$$

where $\mathrm{diag}(\tilde{\mathbb{M}}^2_P)$ is given by Eq. (36a). According to Eq. (37), $y^{\mathrm{ph}}_k = (\mathbb{O}_M\mathbb{Z}_P^{-1}\mathbb{O}_D)_{kl}x_l \equiv (\mathbb{O}_P^{-1})_{kl}x_l$ with $x_l = (\tilde{\eta}_N,\tilde{\pi}_0,\tilde{\eta}_S)$. Thus,

$$\mathcal{J}^{\mu}_i\omega^A_i = \partial^\mu y^{\mathrm{ph}}_k(\mathbb{O}_P^{-1})_{kl}\delta x_l, \qquad k,l\in(N,3,S). \tag{B4}$$

While δx_l can be determined as follows, let $\Phi = \sum_{l\in N,3,S}\phi_l T_l$, $\phi_l = \sigma_l + ix_l$ be the scalar-pseudoscalar $N,3,S$ fields and $T_l = \lambda_l/2$ the generators of U(3) in the $N,3,S$ directions. The effect of an infinitesimal axial transformation on Φ can be written as

$$\begin{aligned}\delta\Phi = T_l\delta\phi_l &= (\mathbb{1}+i\omega^A_iT_i)T_k\phi_k(\mathbb{1}+i\omega^A_iT_i) - T_k\phi_k\\ &= i\omega^A_i\{T_i,T_k\}\phi_k \equiv i\omega^A_i d_{ikl}\phi_kT_l,\end{aligned} \tag{B5}$$

where d_{ikl} are the totally symmetric structure constants of U(3). Since the i and k indices are in $(N,3,S)$ it can be shown that d_{ikl} is nonzero only if $l\in(N,3,S)$. Calculating the d_{ikl} structure constants in the $N-3-S$ basis, the nonzero elements are $d_{NNN}=1$, $d_{N33}=d_{3N3}=d_{33N}=1$, and $d_{SSS}=\sqrt{2}$. Thus, Eq. (B5) can be written as

$$\begin{aligned}\frac{1}{2}(i\delta x_l+\delta\sigma_l)\lambda_l = \frac{1}{2}[&(i(\omega^A_N\sigma_N+\omega^A_3\sigma_3)-(\omega^A_N\tilde{\eta}_N+\omega^A_3\pi_3))\lambda_N\\ &+(i(\omega^A_N\sigma_3+\omega^A_3\sigma_N)-(\omega^A_N\pi_3+\omega^A_3\tilde{\eta}_N))\lambda_3\\ &+(i\omega^A_S\sigma_S-\omega^A_S\tilde{\eta}_S)\lambda_S],\end{aligned} \tag{B6}$$

where $\sigma_3\equiv a^0_0$, $\pi_3=\tilde{\pi}^0$, and the δx_l can be read off as

$$\delta x_1 \equiv \delta\tilde{\eta}_N = \omega^A_N\sigma_N+\omega^A_3\sigma_3, \tag{B7a}$$

$$\delta x_2 \equiv \delta\tilde{\pi}^0 = \omega^A_N\sigma_3+\omega^A_3\sigma_N, \tag{B7b}$$

$$\delta x_3 \equiv \delta\tilde{\eta}_S = \omega^A_S\sigma_S. \tag{B7c}$$

Since the ω^A_i's are independent, we can set $\omega^A_N\neq 0$, $\omega^A_3=\omega^A_S=0$, which—after substitution into Eq. (B4) and considering only the physical π^0—leads to

$$\mathcal{J}^{\mu}_N\omega^A_N = \partial^\mu\pi^0(\mathbb{O}_P^{-1}{}_{11}\sigma_N+\mathbb{O}_P^{-1}{}_{12}\sigma_3)\omega^A_N. \tag{B8}$$

Similarly, $\omega^A_3\neq 0, \omega^A_{N,S}=0$ and $\omega^A_S\neq 0, \omega^A_{N,3}=0$ gives J^μ_3 and J^μ_S, respectively. Thus, the relevant parts of the currents are

$$\mathcal{J}^{\mu}_N = \partial^\mu\pi^0(\mathbb{O}_P^{-1}{}_{11}\sigma_N+\mathbb{O}_P^{-1}{}_{12}\sigma_3), \tag{B9a}$$

$$\mathcal{J}^{\mu}_3 = \partial^\mu\pi^0(\mathbb{O}_P^{-1}{}_{11}\sigma_3+\mathbb{O}_P^{-1}{}_{12}\sigma_N), \tag{B9b}$$

$$\mathcal{J}^{\mu}_S = \partial^\mu\pi^0(\mathbb{O}_P^{-1}{}_{13}\sigma_S), \tag{B9c}$$

while the total current is

$$\begin{aligned}\mathcal{J}^{\mu}_{\mathrm{tot}} &= \mathcal{J}^{\mu}_N+\mathcal{J}^{\mu}_3+\mathcal{J}^{\mu}_S\\ &= \partial^\mu\pi^0[(\mathbb{O}_P^{-1}{}_{11}+\mathbb{O}_P^{-1}{}_{12})(\sigma_N+\sigma_3)+\mathbb{O}_P^{-1}{}_{13}\sigma_S].\end{aligned} \tag{B10}$$

In the PCAC relation (B1) one can only get a nonzero contribution if the scalar fields assume nonvanishing expectation values [Eq. (C7)]. Thus, f_{π^0} is given by

$$f_{\pi^0} = (\mathbb{O}_P^{-1}{}_{11}+\mathbb{O}_P^{-1}{}_{12})(\phi_N+\phi_3)+\mathbb{O}_P^{-1}{}_{13}\phi_S. \tag{B11}$$

It should be noted that due to the $\mathbb{O}_P$ transformation this quantity can be negative. However, according to Ref. [42] (part 72), the measurements of leptonic decays of charged pseudoscalar mesons (like π^+, K^+, D^+_s, B^+) can only determine the combination $|f_P||V_{q_1,q_2}|$, where $|V_{q_1,q_2}|$ is a CKM matrix element. Thus, we can consider simply $|f_{\pi^0}|$, which is a prediction in our case since only the charged decay widths are measured.

APPENDIX C: EXPLICIT FORMS OF THE TREE-LEVEL DECAY WIDTHS

In this appendix, we explicitly list all tree-level expressions for the decay widths, grouped as vector-meson, axial-vector-meson, and scalar-meson decays.

1. Vector-meson decays

a. $\rho \to \pi\pi$

The relevant part of the Lagrangian reads

$$\mathcal{L}_{\rho\pi\pi} = iB_1^\rho \rho_\mu^0 \pi^- \partial^\mu \pi^+ + iC_1^\rho (\partial_\mu \rho_\nu^0)(\partial^\mu \pi^-)\partial^\nu \pi^+ + i\rho_\mu^+ [D_1^\rho \pi^0 \partial^\mu \pi^- + D_2^\rho \pi^- \partial^\mu \pi^0] + i(\partial_\mu \rho_\nu^+)[F_1^\rho (\partial^\mu \pi^0)\partial^\nu \pi^- + F_2^\rho (\partial^\mu \pi^-)\partial^\nu \pi^0] + \text{H.c.}, \tag{C1}$$

where

$$B_1^\rho = Z_{\pi^\pm}^2 \left\{ \left[g_1 + \phi_N (h_3 - g_1^2) w_{a_1^\pm} \right] \cos\vartheta_V - \frac{2}{3}\phi_3 (h_2 + h_3) w_{a_1^\pm} \sin\vartheta_V \right\}, \tag{C2a}$$

$$C_1^\rho = -g_2 Z_{\pi^\pm}^2 w_{a_1^\pm}^2 \cos\vartheta_V, \tag{C2b}$$

$$D_1^\rho = Z_{\pi^\pm} \left\{ \left[g_1 + \phi_N (h_3 - g_1^2) w_{a_1^\pm} \right] \mathbb{O}_{P21} - \phi_3 (h_3 - g_1^2) w_{a_1^\pm} \mathbb{O}_{P11} \right\}, \tag{C2c}$$

$$D_2^\rho = -Z_{\pi^\pm} [g_1 \mathbb{O}_{P21} + \phi_N (h_3 - g_1^2)(w_\eta^a \mathbb{O}_{P11} + w_\pi^a \mathbb{O}_{P21}) - \phi_3 (h_2 - h_3 + 2g_1^2)(w_\eta^f \mathbb{O}_{P11} + w_\pi^f \mathbb{O}_{P21})], \tag{C2d}$$

$$F_1^\rho = -g_2 Z_{\pi^\pm} w_{a_1^\pm} (w_\eta^a \mathbb{O}_{P11} + w_\pi^a \mathbb{O}_{P21}), \qquad F_2^\rho = -F_1^\rho. \tag{C2e}$$

The explicit forms of the tree-level decay widths for the neutral and charged ρ read

$$\Gamma_{\rho^0 \to \pi^+\pi^-} = \frac{k_{\rho^0}^3}{6\pi M_{\rho^0}^2} \left| B_1^\rho + \frac{1}{2} C_1^\rho M_{\rho^0}^2 \right|^2, \tag{C3a}$$

$$\Gamma_{\rho^\pm \to \pi^\pm\pi^0} = \frac{k_{\rho^\pm}^3}{24\pi M_{\rho^\pm}^2} |D_1^\rho - D_2^\rho + F_1^\rho M_{\rho^\pm}^2|^2, \tag{C3b}$$

where

$$k_{\rho^0} = \frac{1}{2}\sqrt{M_{\rho^0}^2 - 4M_{\pi^\pm}^2}, \tag{C4a}$$

$$k_{\rho^\pm} = \frac{\sqrt{(M_{\rho^\pm}^2 - M_{\pi^\pm}^2 - M_{\pi^0}^2)^2 - 4M_{\pi^\pm}^2 M_{\pi^0}^2}}{2M_{\rho^\pm}}. \tag{C4b}$$

b. $\rho \to \pi\eta$

This G parity-violating process will be a prediction. The relevant part of the Lagrangian reads

$$\mathcal{L}_{\rho\pi\eta} = i\rho_\mu^+ [G_1^\rho \eta \partial^\mu \pi^- + G_2^\rho \pi^- \partial^\mu \eta] + i(\partial_\mu \rho_\nu^+)[H_1^\rho (\partial^\mu \eta)\partial^\nu \pi^- + H_2^\rho (\partial^\mu \pi^-)\partial^\nu \eta] + \text{H.c.}, \tag{C5}$$

where

$$G_1^\rho = Z_{\pi^\pm} \left\{ \left[g_1 + \phi_N (h_3 - g_1^2) w_{a_1^\pm} \right] \mathbb{O}_{P22} - \phi_3 (h_3 - g_1^2) w_{a_1^\pm} \mathbb{O}_{P12} \right\}, \tag{C6a}$$

$$G_2^\rho = -Z_{\pi^\pm} [g_1 \mathbb{O}_{P22} + \phi_N (h_3 - g_1^2)(w_\eta^a \mathbb{O}_{P12} + w_\pi^a \mathbb{O}_{P22}) - \phi_3 (h_2 - h_3 + 2g_1^2)(w_\eta^f \mathbb{O}_{P12} + w_\pi^f \mathbb{O}_{P22})], \tag{C6b}$$

$$H_1^\rho = -g_2 Z_{\pi^\pm} w_{a_1^\pm} (w_\eta^a \mathbb{O}_{P12} + w_\pi^a \mathbb{O}_{P22}), \qquad H_2^\rho = -H_1^\rho, \tag{C6c}$$

and the decay width of the process is

$$\Gamma_{\rho^\pm\to\pi^\pm\eta} = \frac{k^3_{\rho^\pm}}{24\pi M^2_{\rho^\pm}}|G^\rho_1 - G^\rho_2 + H^\rho_1 M^2_{\rho^\pm}|^2, \tag{C7}$$

where

$$k_{\rho^\pm} = \frac{1}{2M_{\rho^\pm}}\sqrt{(M^2_{\rho^\pm} - M^2_{\pi^\pm} - M^2_\eta)^2 - 4M^2_{\pi^\pm}M^2_\eta}. \tag{C8}$$

c. $\omega \to \pi\pi$

The relevant part of the Lagrangian reads

$$\mathcal{L}_{\omega\pi\pi} = iB^\omega_1 \omega_\mu \pi^- \partial^\mu \pi^+ + \text{H.c.}, \tag{C9}$$

where

$$B^\omega_1 = Z^2_{\pi^\pm} w_{a_1^\pm} \phi_3 (h_2 + h_3)\cos\vartheta_V, \tag{C10}$$

while the decay width reads

$$\Gamma_{\omega\to\pi^+\pi^-} = \frac{k^3_\omega}{6\pi M^2_\omega}|B^\omega_1|^2, \quad k_\omega = \frac{1}{2}\sqrt{M^2_\omega - 4M^2_{\pi^\pm}}. \tag{C11}$$

d. $K^\star \to K\pi$

The relevant part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{K^\star K\pi} = {} & iK^{\star 0}_\mu\Big[B^{K^\star}_1 \pi^0\partial^\mu\overline{K^0} + B^{K^\star}_2 \overline{K^0}\partial^\mu\pi^0 + B^{K^\star}_3 \pi^+\partial^\mu K^- + B^{K^\star}_4 K^-\partial^\mu\pi^+\Big] \\ & + i(\partial_\mu K^{\star 0}_\nu)\Big[C^{K^\star}_1(\partial^\mu\pi^0)\partial^\nu\overline{K^0} + C^{K^\star}_2(\partial^\mu\overline{K^0})\partial^\nu\pi^0 + C^{K^\star}_3(\partial^\mu\pi^+)\partial^\nu K^- + C^{K^\star}_4(\partial^\mu K^-)\partial^\nu\pi^+\Big] \\ & + iK^{\star +}_\mu\Big[D^{K^\star}_1\pi^0\partial^\mu K^- + D^{K^\star}_2 K^-\partial^\mu\pi^0 + D^{K^\star}_3\pi^-\partial^\mu\overline{K^0} + D^{K^\star}_4\overline{K^0}\partial^\mu\pi^-\Big] \\ & + i(\partial_\mu K^{\star +}_\nu)\Big[F^{K^\star}_1(\partial^\mu\pi^0)\partial^\nu K^- + F^{K^\star}_2(\partial^\mu K^-)\partial^\nu\pi^0 + F^{K^\star}_3(\partial^\mu\pi^-)\partial^\nu\overline{K^0} + F^{K^\star}_4(\partial^\mu\overline{K^0})\partial^\nu\pi^-\Big] + \text{H.c.},\end{aligned} \tag{C12}$$

where

$$B^{K^\star}_1 = Z_{K^0}\left\{\frac{g_1}{2}\left(\mathbb{O}_{P11} - \mathbb{O}_{P21} - \sqrt{2}\mathbb{O}_{P31}\right) + \frac{h_3 - g_1^2}{\sqrt{2}} w_{K_1^0}[(\phi_3 - \phi_N)\mathbb{O}_{P31} + \phi_S(\mathbb{O}_{P11} - \mathbb{O}_{P21})]\right\}, \tag{C13a}$$

$$\begin{aligned}B^{K^\star}_2 = {} & -Z_{K^0}\left\{\frac{g_1}{2}\left(\mathbb{O}_{P11} - \mathbb{O}_{P21} - \sqrt{2}\mathbb{O}_{P31}\right) + \frac{\phi_N - \phi_3}{4}\Big[W_m(h_2 - 2h_3 + 3g_1^2) - \sqrt{2}W_S(h_2 + g_1^2)\Big]\right. \\ & \left. - \frac{\sqrt{2}\phi_S}{4}\Big[W_m(h_2 + g_1^2) - \sqrt{2}W_S(h_2 - 2h_3 + 3g_1^2)\Big]\right\},\end{aligned} \tag{C13b}$$

$$B^{K^\star}_3 = \frac{1}{\sqrt{2}}Z_{\pi^\pm}Z_{K^\pm}\left\{g_1 + w_{K_1^\pm}\Big[-\phi_3(h_2 + g_1^2) + \sqrt{2}\phi_S(h_3 - g_1^2)\Big]\right\}, \tag{C13c}$$

$$B^{K^\star}_4 = \frac{1}{\sqrt{2}}Z_{\pi^\pm}Z_{K^\pm}\left\{-g_1 + \frac{1}{2}w_{a_1^\pm}\Big[\phi_N(h_2 - 2h_3 + 3g_1^2) + \phi_3(h_2 + 2h_3 - g_1^2) - \sqrt{2}\phi_S(h_2 + g_1^2)\Big]\right\}, \tag{C13d}$$

$$C^{K^\star}_1 = \frac{g_2}{2}Z_{K^0}w_{K_1^0}\left(W_m + \sqrt{2}W_S\right), \qquad C^{K^\star}_2 = -C^{K^\star}_1, \tag{C13e}$$

$$C^{K^\star}_3 = -\frac{g_2}{\sqrt{2}}Z_{\pi^\pm}Z_{K^\pm}w_{a_1^\pm}w_{K_1^\pm}, \qquad C^{K^\star}_4 = -C^{K^\star}_3, \tag{C13f}$$

$$D^{K^\star}_1 = Z_{K^\pm}\left\{\frac{g_1}{2}\left(\mathbb{O}_{P11} + \mathbb{O}_{P21} - \sqrt{2}\mathbb{O}_{P31}\right) + \frac{h_3 - g_1^2}{\sqrt{2}}w_{K_1^\pm}[-(\phi_N + \phi_3)\mathbb{O}_{P31} + \phi_S(\mathbb{O}_{P11} + \mathbb{O}_{P21})]\right\}, \tag{C13g}$$

$$D_2^{K^\star} = -Z_{K^\pm}\Big\{\frac{g_1}{2}\Big(\mathbb{O}_{P11} + \mathbb{O}_{P21} - \sqrt{2}\mathbb{O}_{P31}\Big) - \frac{\phi_N + \phi_3}{4}\Big[W_m(h_2 - 2h_3 + 3g_1^2) + \sqrt{2}W_S(h_2 + g_1^2)\Big] + \frac{\sqrt{2}\phi_S}{4}\Big[W_m(h_2 + g_1^2) + \sqrt{2}W_S(h_2 - 2h_3 + 3g_1^2)\Big]\Big\}, \tag{C13h}$$

$$D_3^{K^\star} = \frac{1}{\sqrt{2}}Z_{\pi^\pm}Z_{K^0}\Big\{g_1 + w_{K_1^0}\Big[\phi_3(h_2 + g_1^2) + \sqrt{2}\phi_S(h_3 - g_1^2)\Big]\Big\}, \tag{C13i}$$

$$D_4^{K^\star} = \frac{1}{\sqrt{2}}Z_{\pi^\pm}Z_{K^0}\Big\{-g_1 + \frac{1}{2}w_{a_1^\pm}\Big[\phi_N(h_2 - 2h_3 + 3g_1^2) - \phi_3(h_2 + 2h_3 - g_1^2) - \sqrt{2}\phi_S(h_2 + g_1^2)\Big]\Big\}, \tag{C13j}$$

$$F_1^{K^\star} = -\frac{g_2}{2}Z_{K^\pm}w_{K_1^\pm}\Big(W_p - \sqrt{2}W_S\Big), \qquad F_2^{K^\star} = -F_1^{K^\star}, \tag{C13k}$$

$$F_3^{K^\star} = -\frac{g_2}{\sqrt{2}}Z_{\pi^\pm}Z_{K^0}w_{a_1^\pm}w_{K_1^0}, \qquad F_4^{K^\star} = -F_3^{K^\star}, \tag{C13l}$$

and

$$W_m = (w_\eta^a - w_\eta^f)\mathbb{O}_{P11} + (w_\pi^a - w_\pi^f)\mathbb{O}_{P21}, \tag{C14a}$$

$$W_p = (w_\eta^a + w_\eta^f)\mathbb{O}_{P11} + (w_\pi^a + w_\pi^f)\mathbb{O}_{P21}, \tag{C14b}$$

$$W_S = w_{f_{1S}}\mathbb{O}_{P31}. \tag{C14c}$$

The explicit forms of the tree-level decay widths for the neutral and charged $K^\star$—both of them containing two subchannels—read

$$\Gamma_{\bar{K}^{\star 0}\to\pi^{0,+}\bar{K}^{0,-}} = \Gamma_{\bar{K}^{\star 0}\to\pi^0\bar{K}^0} + \Gamma_{\bar{K}^{\star 0}\to\pi^+K^-} = \frac{1}{24\pi M_{K^{\star 0}}^2}\Big\{k_{\bar{K}^{\star 0}\to\pi^0\bar{K}^0}^3\Big|B_1^{K^\star} - B_2^{K^\star} + C_1^{K^\star}M_{K^{\star 0}}^2\Big|^2 + k_{\bar{K}^{\star 0}\to\pi^+K^-}^3\Big|B_3^{K^\star} - B_4^{K^\star} + C_3^{K^\star}M_{K^{\star 0}}^2\Big|^2\Big\}, \tag{C15a}$$

$$\Gamma_{K^{\star -}\to\pi^{0,-}K^{-,\bar{0}}} = \Gamma_{K^{\star -}\to\pi^0K^-} + \Gamma_{K^{\star -}\to\pi^-\bar{K}^0} = \frac{1}{24\pi M_{K^{\star\pm}}^2}\Big\{k_{K^{\star -}\to\pi^0K^-}^3\Big|D_1^{K^\star} - D_2^{K^\star} + F_1^{K^\star}M_{K^{\star\pm}}^2\Big|^2 + k_{K^{\star -}\to\pi^-\bar{K}^0}^3\Big|D_3^{K^\star} - D_4^{K^\star} + F_3^{K^\star}M_{K^{\star 0}}^2\Big|^2\Big\}, \tag{C15b}$$

where

$$k_{A\to BC} = \frac{1}{2M_A}\sqrt{(M_A^2 - M_B^2 - M_C^2)^2 - 4M_B^2M_C^2}. \tag{C16}$$

e. $\Phi \to KK$

The relevant part of the Lagrangian reads

$$\mathcal{L}_{\Phi KK} = i\Phi_\mu[B_1^\Phi K^0\partial^\mu\overline{K^0} + B_2^\Phi K^+\partial^\mu K^-] + i(\partial_\mu\Phi_\nu)[C_1^\Phi(\partial^\mu K^0)\partial^\nu\overline{K^0} + C_2^\Phi(\partial^\mu K^+)\partial^\nu K^-] + \text{H.c.}, \tag{C17}$$

where

$$B_1^\Phi = \frac{1}{\sqrt{2}}Z_{K^0}^2\Big\{g_1 - \frac{1}{2}w_{K_1^0}\Big[(\phi_N - \phi_3)(h_2 + g_1^2) - \sqrt{2}\phi_S(h_2 + 2h_3 - g_1^2)\Big]\Big\}, \tag{C18a}$$

$$B_2^\Phi = \frac{1}{\sqrt{2}}Z_{K^\pm}^2\Big\{g_1 - \frac{1}{2}w_{K_1^\pm}\Big[(\phi_N + \phi_3)(h_2 + g_1^2) - \sqrt{2}\phi_S(h_2 + 2h_3 - g_1^2)\Big]\Big\}, \tag{C18b}$$

$$C_1^{\Phi} = -\frac{g_2}{\sqrt{2}} Z_{K^0}^2 w_{K_1^0}^2, \tag{C18c}$$

$$C_2^{\Phi} = -\frac{g_2}{\sqrt{2}} Z_{K^\pm}^2 w_{K_1^\pm}^2. \tag{C18d}$$

The explicit forms of the tree-level decay widths for the neutral and charged part of the $\Phi \to KK$ process read

$$\Gamma_{\Phi\to K^0\bar{K}^0} = \frac{k^3_{\Phi\to K^0\bar{K}^0}}{6\pi M_\Phi^2}\left| B_1^{\Phi} + \frac{1}{2} C_1^{\Phi} M_\Phi^2 \right|^2, \tag{C19a}$$

$$\Gamma_{\Phi\to K^+K^-} = \frac{k^3_{\Phi\to K^+K^-}}{6\pi M_\Phi^2}\left| B_2^{\Phi} + \frac{1}{2} C_2^{\Phi} M_\Phi^2 \right|^2, \tag{C19b}$$

where

$$k_{\Phi\to K^0\bar{K}^0} = \frac{1}{2}\sqrt{M_\Phi^2 - 4M_{K^0}^2}, \tag{C20a}$$

$$k_{\Phi\to K^+K^-} = \frac{1}{2}\sqrt{M_\Phi^2 - 4M_{K^\pm}^2}. \tag{C20b}$$

2. Axial-vector-meson decays

a. $a_1 \to \rho\pi$

The relevant part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{a_1\rho\pi} = {} & iB_1^{a_1} a^0_{1\mu}\rho^{+\mu}\pi^- + iC_1^{a_1} a^0_{1\mu}(\partial^\mu\rho^{+\nu})\partial_\nu\pi^- - iC_1^{a_1} a^0_{1\nu}(\partial^\mu\rho^{+\nu})\partial_\mu\pi^- - iC_1^{a_1}(\partial_\mu a^0_{1\nu})[\rho^{+\mu}\partial^\nu\pi^- - \rho^{+\nu}\partial^\mu\pi^-] \\ & + ia^+_{1\,\mu}\left[D_1^{a_1}\rho^{-\mu}\pi^0 + D_2^{a_1}\rho^{0\mu}\pi^-\right] + ia^+_{1\,\mu}\left[E_1^{a_1}(\partial^\mu\rho^{-\nu})\partial_\nu\pi^0 + E_2^{a_1}(\partial^\mu\rho^{0\nu})\partial_\nu\pi^-\right] \\ & - ia^+_{1\,\nu}\left[E_1^{a_1}(\partial^\mu\rho^{-\nu})\partial_\mu\pi^0 + E_2^{a_1}(\partial^\mu\rho^{0\nu})\partial_\mu\pi^-\right] - i(\partial_\mu a^+_{1\,\nu})\left[E_1^{a_1}(\rho^{-\mu}\partial^\nu\pi^0 - \rho^{-\nu}\partial^\mu\pi^0) + E_2^{a_1}(\rho^{0\mu}\partial^\nu\pi^- - \rho^{0\nu}\partial^\mu\pi^-)\right] + \text{H.c.},\end{aligned} \tag{C21}$$

where

$$B_1^{a_1} = -Z_{\pi^\pm}\phi_N(h_3 - g_1^2)\cos\vartheta_A, \tag{C22a}$$

$$C_1^{a_1} = -g_2 Z_{\pi^\pm} w_{a_1^\pm}\cos\vartheta_A, \tag{C22b}$$

$$D_1^{a_1} = -(h_3 - g_1^2)(\phi_N \mathbb{O}_{P21} - \phi_3 \mathbb{O}_{P11}), \tag{C22c}$$

$$D_2^{a_1} = Z_{\pi^\pm}[\phi_N(h_3 - g_1^2)\cos\vartheta_V - \phi_3(h_2 + h_3)\sin\vartheta_V], \tag{C22d}$$

$$E_1^{a_1} = -g_2(w^a_\eta \mathbb{O}_{P11} + w^a_\pi \mathbb{O}_{P21}), \tag{C22e}$$

$$E_2^{a_1} = g_2 Z_{\pi^\pm} w_{a_1^\pm}\cos\vartheta_V. \tag{C22f}$$

The decay width for the neutral a_1^0 reads

$$\Gamma_{a_1^0\to\rho^\pm\pi^\mp} = \frac{k_{a_1^0}}{12M^2_{a_1^0}\pi}\left(|V^1_{\mu\nu}|^2 - \frac{|V^1_{\mu\nu}k^\nu_{a_1^0}|^2}{M^2_{a_1^0}} - \frac{|V^1_{\mu\nu}k^\mu_{\rho^\pm}|^2}{M^2_{\rho^\pm}} + \frac{|V^1_{\mu\nu}k^\mu_{a_1^0}k^\nu_{\rho^\pm}|^2}{M^2_{a_1^0}M^2_{\rho^\pm}}\right), \tag{C23}$$

with (omitting the a_1 superscript from B_1, C_1)

$$V^1_{\mu\nu} = i\left\{B_1 g_{\mu\nu} + C_1\left[k_{\pi^\mp}\cdot(k_{a_1^0} + k_{\rho^\pm})g_{\mu\nu} - k_{\rho^\pm\,\mu}k_{\pi^\mp\,\nu} - k_{\pi^\mp\,\mu}k_{a^0_{1\,\nu}}\right]\right\}, \tag{C24a}$$

$$|V^1_{\mu\nu}|^2 = \left\{4B_1^2 + C_1^2\left[\frac{5}{2}(M^2_{a_1^0} - M^2_{\rho^\pm})^2 + \frac{1}{2}M^2_{\pi^\pm}(2M^2_{a_1^0} + 2M^2_{\rho^\pm} - M^2_{\pi^\pm})\right] + 6B_1C_1(M^2_{a_1^0} - M^2_{\rho^\pm})\right\}, \tag{C24b}$$

$$|V^1_{\mu\nu}k^\nu_{a_1^0}|^2 = B_1^2 M^2_{a_1^0} - k^2_{a_1^0} C_1 M^2_{a_1^0}\Big[C_1 M^2_{\rho^\pm} - 2B_1\Big], \tag{C24c}$$

$$|V^1_{\mu\nu}k^\mu_{\rho^\pm}|^2 = B_1^2 M^2_{\rho^\pm} - k^2_{a_1^0} C_1 M^2_{a_1^0}\Big[C_1 M^2_{a_1^0} + 2B_1\Big], \tag{C24d}$$

$$|V^1_{\mu\nu}k^\mu_{a_1^0}k^\nu_{\rho^\pm}|^2 = B_1^2 M^2_{a_1^0}\Big(k^2_{a_1^0} + M^2_{\rho^\pm}\Big), \tag{C24e}$$

where $k_{a_1^0} \equiv k_{a_1^0\to\rho^\pm\pi^\mp}$ is given in Eq. (C16). The decay width of the charged $a_1^\pm$ consists of two subchannels and is given by

$$\begin{aligned}\Gamma_{a_1^\pm\to\rho^{\pm,0}\pi^{0,\pm}} &= \Gamma_{a_1^\pm\to\rho^\pm\pi^0} + \Gamma_{a_1^\pm\to\rho^0\pi^\pm}\\ &= \frac{k_{a_1^\pm\to\rho^\pm\pi^0}}{24M^2_{a_1^\pm}\pi}\left(|V^2_{\mu\nu}|^2 - \frac{|V^2_{\mu\nu}k^\nu_{a_1^\pm}|^2}{M^2_{a_1^\pm}} - \frac{|V^2_{\mu\nu}k^\mu_{\rho^\pm}|^2}{M^2_{\rho^\pm}} + \frac{|V^2_{\mu\nu}k^\mu_{a_1^\pm}k^\nu_{\rho^\pm}|^2}{M^2_{a_1^\pm}M^2_{\rho^\pm}}\right)\\ &\quad + \frac{k_{a_1^\pm\to\rho^0\pi^\pm}}{24M^2_{a_1^\pm}\pi}\left(|V^3_{\mu\nu}|^2 - \frac{|V^3_{\mu\nu}k^\nu_{a_1^\pm}|^2}{M^2_{a_1^\pm}} - \frac{|V^3_{\mu\nu}k^\mu_{\rho^0}|^2}{M^2_{\rho^0}} + \frac{|V^3_{\mu\nu}k^\mu_{a_1^\pm}k^\nu_{\rho^0}|^2}{M^2_{a_1^\pm}M^2_{\rho^0}}\right),\end{aligned} \tag{C25}$$

with (omitting the a_1 superscript from D_1, E_1, D_2, E_2)

$$V^2_{\mu\nu} = i\Big\{D_1 g_{\mu\nu} + E_1\Big[k_{\pi^0}\cdot(k_{a_1^\pm} + k_{\rho^\pm})g_{\mu\nu} - k_{\rho^\pm\mu}k_{\pi^0\nu} - k_{\pi^0\mu}k_{a_1^\pm\nu}\Big]\Big\}, \tag{C26a}$$

$$|V^2_{\mu\nu}|^2 = 4D_1^2 + E_1^2\left[\frac{5}{2}(M^2_{a_1^\pm} - M^2_{\rho^\pm})^2 + \frac{1}{2}M^2_{\pi^0}(2M^2_{a_1^\pm} + 2M^2_{\rho^\pm} - M^2_{\pi^0})\right] + 6D_1E_1(M^2_{a_1^\pm} - M^2_{\rho^\pm}), \tag{C26b}$$

$$|V^2_{\mu\nu}k^\nu_{a_1^\pm}|^2 = D_1^2 M^2_{a_1^\pm} - k^2_{a_1^\pm\to\rho^\pm\pi^0}E_1 M^2_{a_1^\pm}\Big(E_1 M^2_{\rho^\pm} - 2D_1\Big), \tag{C26c}$$

$$|V^2_{\mu\nu}k^\mu_{\rho^\pm}|^2 = D_1^2 M^2_{\rho^\pm} - k^2_{a_1^\pm\to\rho^\pm\pi^0}E_1 M^2_{a_1^\pm}\Big(E_1 M^2_{a_1^\pm} + 2D_1\Big), \tag{C26d}$$

$$|V^2_{\mu\nu}k^\mu_{a_1^\pm}k^\nu_{\rho^\pm}|^2 = D_1^2 M^2_{a_1^\pm}\Big(k^2_{a_1^\pm\to\rho^\pm\pi^0} + M^2_{\rho^\pm}\Big), \tag{C26e}$$

$$V^3_{\mu\nu} = i\Big\{D_2 g_{\mu\nu} + E_2\Big[k_{\pi^\pm}\cdot(k_{a_1^\pm} + k_{\rho^0})g_{\mu\nu} - k_{\rho^0\mu}k_{\pi^\pm\nu} - k_{\pi^\pm\mu}k_{a_1^\pm\nu}\Big]\Big\}, \tag{C26f}$$

$$|V^3_{\mu\nu}|^2 = 4D_2^2 + E_2^2\left[\frac{5}{2}(M^2_{a_1^\pm} - M^2_{\rho^0})^2 + \frac{1}{2}M^2_{\pi^\pm}(2M^2_{a_1^\pm} + 2M^2_{\rho^0} - M^2_{\pi^\pm})\right] + 6D_2E_2(M^2_{a_1^\pm} - M^2_{\rho^0}), \tag{C26g}$$

$$|V^3_{\mu\nu}k^\nu_{a_1^\pm}|^2 = D_2^2 M^2_{a_1^\pm} - k^2_{a_1^\pm\to\rho^0\pi^\pm}E_2 M^2_{a_1^\pm}(E_2 M^2_{\rho^0} - 2D_2), \tag{C26h}$$

$$|V^3_{\mu\nu}k^\mu_{\rho^0}|^2 = D_2^2 M^2_{\rho^0} - k^2_{a_1^\pm\to\rho^0\pi^\pm}E_2 M^2_{a_1^\pm}\Big(E_2 M^2_{a_1^\pm} + 2D_2\Big), \tag{C26i}$$

$$|V^3_{\mu\nu}k^\mu_{a_1^\pm}k^\nu_{\rho^0}|^2 = D_2^2 M^2_{a_1^\pm}\Big(k^2_{a_1^\pm\to\rho^0\pi^\pm} + M^2_{\rho^0}\Big), \tag{C26j}$$

where $k_{a_1^\pm\to\rho^\pm\pi^0}$ and $k_{a_1^\pm\to\rho^0\pi^\pm}$ are given in Eq. (C16).

b. $a_1 \to \pi\gamma$

The relevant part of the Lagrangian reads

$$\mathcal{L}_{a_1\pi\gamma} = iB^\gamma a^+_{1\,\mu}\pi^- A^\mu_e + iC^\gamma\left(\partial_\mu a^+_{1\,\nu} - \partial_\nu a^+_{1\,\mu}\right) \times (\partial^\nu \pi^-)A^\mu_e + \text{H.c.}, \tag{C27}$$

where

$$B^\gamma = -eg_1\phi_N Z_{\pi^\pm}, \qquad C^\gamma = -eZ_{\pi^\pm} w_{a_1^\pm}. \tag{C28}$$

The decay width for the charged $a_1^\pm$ has the simple form

$$\Gamma_{a_1^\pm\to\pi^\pm\gamma} = \frac{e^2 g_1^2 \phi_N^2}{96\pi M_{a_1^\pm}} Z^2_{\pi^\pm}\left[1 - \left(\frac{M_{\pi^\pm}}{M_{a_1^\pm}}\right)^2\right]^3. \tag{C29}$$

c. $f_1^H \to K^\star K$

The relevant part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{f_1^H K^\star K} = {} & if^H_{1\,\mu}\left[B_1^{f_1} K^{\star+\mu}K^- + B_2^{f_1} K^{\star 0\mu}\bar{K}^0\right] + if^H_{1\,\mu}\left[C_1^{f_1}(\partial^\mu K^{\star+\nu})\partial_\nu K^- + C_2^{f_1}(\partial^\mu K^{\star 0\nu})\partial_\nu \bar{K}^0\right] \\ & - if^H_{1\,\nu}\left[C_1^{f_1}(\partial^\mu K^{\star+\nu})\partial_\mu K^- + C_2^{f_1}(\partial^\mu K^{\star 0\nu})\partial_\mu \bar{K}^0\right] \\ & - i(\partial_\mu f^H_{1\,\nu})\left[C_1^{f_1}(K^{\star+\mu}\partial^\nu K^- - K^{\star+\nu}\partial^\mu K^-) + C_2^{f_1}(K^{\star 0\mu}\partial^\nu \bar{K}^0 - K^{\star 0\nu}\partial^\mu \bar{K}^0)\right] + \text{H.c.},\end{aligned} \tag{C30}$$

where

$$B_1^{f_1} = \frac{\sqrt{2}}{4} Z_{K^\pm}\left[(\phi_N + \phi_3)(h_2 + g_1^2) - \sqrt{2}\phi_S(h_2 - 2h_3 + 3g_1^2)\right], \tag{C31a}$$

$$B_2^{f_1} = \frac{\sqrt{2}}{4} Z_{K^0}\left[(\phi_N - \phi_3)(h_2 + g_1^2) - \sqrt{2}\phi_S(h_2 - 2h_3 + 3g_1^2)\right], \tag{C31b}$$

$$C_1^{f_1} = \frac{g_2}{\sqrt{2}} Z_{K^\pm} w_{K_1^\pm}, \tag{C31c}$$

$$C_2^{f_1} = \frac{g_2}{\sqrt{2}} Z_{K^0} w_{K_1^0}. \tag{C31d}$$

Similar to as in the case of $a_1^\pm$ there are two subchannels, and the decay width is given by

$$\begin{aligned}\Gamma_{f_1^H\to K^{\star\pm,0,\bar{0}}K^{\mp,\bar{0},0}} &= \Gamma_{f_1^H\to K^{\star\pm}K^\mp} + \Gamma_{f_1^H\to K^{0,\bar{0}}K^{\bar{0},0}} \\ &= \frac{k_{f_1^H\to K^{\star\pm}K^\mp}}{12M^2_{f_1^H}\pi}\left(|V^4_{\mu\nu}|^2 - \frac{|V^4_{\mu\nu}k^\nu_{f_1^H}|^2}{M^2_{f_1^H}} - \frac{|V^4_{\mu\nu}k^\mu_{K^{\star\pm}}|^2}{M^2_{K^{\star\pm}}} + \frac{|V^4_{\mu\nu}k^\mu_{f_1^H}k^\nu_{K^{\star\pm}}|^2}{M^2_{f_1^H}M^2_{K^{\star\pm}}}\right) \\ &\quad + \frac{k_{f_1^H\to K^{\star 0}\bar{K}^0}}{12M^2_{f_1^H}\pi}\left(|V^5_{\mu\nu}|^2 - \frac{|V^5_{\mu\nu}k^\nu_{f_1^H}|^2}{M^2_{f_1^H}} - \frac{|V^5_{\mu\nu}k^\mu_{K^{\star 0}}|^2}{M^2_{K^{\star 0}}} + \frac{|V^5_{\mu\nu}k^\mu_{f_1^H}k^\nu_{K^{\star 0}}|^2}{M^2_{f_1^H}M^2_{K^{\star 0}}}\right),\end{aligned} \tag{C32}$$

with (omitting the f_1 superscript from B_1, C_1, B_2, C_2)

$$V^4_{\mu\nu} = i\left\{B_1 g_{\mu\nu} + C_1\left[k_{K^\mp}\cdot(k_{f_1^H} + k_{K^{\star\pm}})g_{\mu\nu} - k_{K^{\star\pm}\mu}k_{K^\mp\nu} - k_{K^\mp\mu}k_{f^H_{1}\nu}\right]\right\}, \tag{C33a}$$

$$|V^4_{\mu\nu}|^2 = 4B_1^2 + C_1^2\left[\frac{5}{2}\left(M^2_{f_1^H} - M^2_{K^{\star\pm}}\right)^2 + \frac{1}{2}M^2_{K^\pm}\left(2M^2_{f_1^H} + 2M^2_{K^{\star\pm}} - M^2_{K^\pm}\right)\right] + 6B_1C_1(M^2_{f_1^H} - M^2_{K^{\star\pm}}), \tag{C33b}$$

$$|V^4_{\mu\nu}k^\nu_{f_1^H}|^2 = B_1^2 M^2_{f_1^H} - k^2_{f_1^H\to K^{\star\pm}K^\mp} C_1 M^2_{f_1^H}(C_1 M^2_{K^{\star\pm}} - 2B_1), \tag{C33c}$$

$$|V^4_{\mu\nu}k^\mu_{K^{\star\pm}}|^2 = B_1^2 M^2_{K^{\star\pm}} - k^2_{f_1^H\to K^{\star\pm}K^\mp} C_1 M^2_{f_1^H}\left(C_1 M^2_{f_1^H} + 2B_1\right), \tag{C33d}$$

$$|V^4_{\mu\nu}k^\mu_{f_1^H}k^\nu_{K^{\star\pm}}|^2 = B_1^2 M^2_{f_1^H}\left(k^2_{f_1^H\to K^{\star\pm}K^\mp} + M^2_{K^{\star\pm}}\right), \tag{C33e}$$

$$V^5_{\mu\nu} = i\left\{B_2 g_{\mu\nu} + C_2\left[k_{\bar{K}^0}\cdot(k_{f_1^H} + k_{K^{\star 0}})g_{\mu\nu} - k_{K^{\star 0}\mu}k_{\bar{K}^0\nu} - k_{\bar{K}^0\mu}k_{f_1^H\nu}\right]\right\}, \tag{C33f}$$

$$|V^5_{\mu\nu}|^2 = 4B_2^2 + C_2^2\left[\frac{5}{2}\left(M^2_{f_1^H} - M^2_{K^{\star 0}}\right)^2 + \frac{1}{2}M^2_{K^0}\left(2M^2_{f_1^H} + 2M^2_{K^{\star 0}} - M^2_{K^0}\right)\right] + 6B_2C_2\left(M^2_{f_1^H} - M^2_{K^{\star 0}}\right), \tag{C33g}$$

$$|V^5_{\mu\nu}k^\nu_{f_1^H}|^2 = B_2^2 M^2_{f_1^H} - k^2_{f_1^H\to K^{\star 0}\bar{K}^0} C_2 M^2_{f_1^H}(C_2 M^2_{K^{\star 0}} - 2B_2), \tag{C33h}$$

$$|V^5_{\mu\nu}k^\mu_{K^{\star 0}}|^2 = B_2^2 M^2_{K^{\star 0}} - k^2_{f_1^H\to K^{\star 0}\bar{K}^0} C_2 M^2_{f_1^H}\left(C_2 M^2_{f_1^H} + 2B_2\right), \tag{C33i}$$

$$|V^5_{\mu\nu}k^\mu_{f_1^H}k^\nu_{K^{\star 0}}|^2 = B_2^2 M^2_{f_1^H}\left(k^2_{f_1^H\to K^{\star 0}\bar{K}^0} + M^2_{K^{\star 0}}\right). \tag{C33j}$$

3. Scalar-meson decays

a. $K_0^\star \to K\pi$

The relevant part of the Lagrangian reads

$$\begin{aligned}\mathcal{L}_{K_0^\star K\pi} = {} & K_0^{\star 0}\left[B_1^{K_0^\star}\pi^0\overline{K^0} + B_2^{K_0^\star}(\partial_\mu\pi^0)\partial^\mu\overline{K^0} + B_3^{K_0^\star}\pi^+K^- + B_4^{K_0^\star}(\partial_\mu\pi^+)\partial^\mu K^-\right] \\ & + (\partial_\mu K_0^{\star 0})\left[C_1^{K_0^\star}\pi^0\partial^\mu\overline{K^0} + C_2^{K_0^\star}(\partial^\mu\pi^0)\overline{K^0} + C_3^{K_0^\star}\pi^+\partial^\mu K^- + C_4^{K_0^\star}(\partial^\mu\pi^+)K^-\right] \\ & + K_0^{\star +}\left[D_1^{K_0^\star}\pi^0K^- + D_2^{K_0^\star}(\partial_\mu\pi^0)\partial^\mu K^- + D_3^{K_0^\star}\pi^-\overline{K^0} + D_4^{K_0^\star}(\partial_\mu\pi^-)\partial^\mu\overline{K^0}\right] \\ & + (\partial_\mu K_0^{\star +})\left[F_1^{K_0^\star}\pi^0\partial^\mu K^- + F_2^{K_0^\star}(\partial^\mu\pi^0)K^- + F_3^{K_0^\star}\pi^-\partial^\mu\overline{K^0} + F_4^{K_0^\star}(\partial^\mu\pi^-)\overline{K^0}\right] + \text{H.c.},\end{aligned} \tag{C34}$$

where

$$B_1^{K_0^\star} = \frac{1}{\sqrt{2}}Z_{K_0^{\star 0}}Z_{K^0}\mathbb{O}_{P21}\phi_S[\lambda_2 - c_1\phi_3(\phi_N + \phi_3)], \tag{C35a}$$

$$B_2^{K_0^\star} = \frac{1}{4}Z_{K_0^{\star 0}}Z_{K^0}\left\{2g_1(W_a + w_{K_1^0}\mathbb{O}_{P21}) - W_a w_{K_1^0}\left[(\phi_N - \phi_3)(h_2 - 2h_3 + 3g_1^2) + \sqrt{2}\phi_S(h_2 + g_1^2)\right]\right\}, \tag{C35b}$$

$$B_3^{K_0^\star} = -Z_{K_0^{\star 0}}Z_{K^\pm}Z_{\pi^\pm}\left(\phi_S - \sqrt{2}\phi_3\right)\lambda_2, \tag{C35c}$$

$$B_4^{K_0^\star} = -\frac{1}{2\sqrt{2}}Z_{K_0^{\star 0}}Z_{K^\pm}Z_{\pi^\pm}\left\{2g_1(w_{a_1^\pm} + w_{K_1^\pm}) - w_{a_1^\pm}w_{K_1^\pm}[\phi_N(h_2 - 2h_3 + 3g_1^2) - \phi_3(h_2 + 2h_3 - g_1^2) + \sqrt{2}\phi_S(h_2 + g_1^2)]\right\}, \tag{C35d}$$

$$C_1^{K_0^\star} = -\frac{1}{2}Z_{K_0^{\star 0}}Z_{K^0}\mathbb{O}_{P21}\left[g_1(w_{K_1^0} + \bar{w}_{K^{\star 0}}) - w_{K_1^0}\bar{w}_{K^{\star 0}}\sqrt{2}\phi_S(g_1^2 - h_3)\right], \tag{C35e}$$

$$C_2^{K_0^\star} = -\frac{1}{4} Z_{K_0^{\star 0}} Z_{K^0} \Big\{ 2g_1 (W_a - \bar{w}_{K^{\star 0}} \mathbb{O}_{P21}) + W_a \bar{w}_{K^{\star 0}} \Big[(\phi_N - \phi_3)(h_2 - 2h_3 + 3g_1^2) - \sqrt{2}\phi_S (h_2 + g_1^2) \Big] \Big\}, \tag{C35f}$$

$$C_3^{K_0^\star} = \frac{1}{\sqrt{2}} Z_{K_0^{\star 0}} Z_{K^\pm} Z_{\pi^\pm} \Big\{ g_1 (w_{K_1^\pm} + \bar{w}_{K^{\star 0}}) - w_{K_1^\pm} \bar{w}_{K^{\star 0}} \Big[\phi_3 (h_2 + g_1^2) + \sqrt{2}\phi_S (g_1^2 - h_3) \Big] \Big\}, \tag{C35g}$$

$$C_4^{K_0^\star} = \frac{1}{2\sqrt{2}} Z_{K_0^{\star 0}} Z_{K^\pm} Z_{\pi^\pm} \Big\{ 2g_1 (w_{a_1^\pm} - \bar{w}_{K^{\star 0}}) + w_{a_1^\pm} \bar{w}_{K^{\star 0}} [(\phi_N (h_2 - 2h_3 + 3g_1^2) + \phi_3 (h_2 + 2h_3 - 3g_1^2) - \sqrt{2}\phi_S (h_2 + g_1^2)] \Big\}, \tag{C35h}$$

$$D_1^{K_0^\star} = -\frac{1}{\sqrt{2}} Z_{K_0^{\star \pm}} Z_{K^\pm} \mathbb{O}_{P21} \phi_S [\lambda_2 - c_1 \phi_3 (\phi_N - \phi_3)], \tag{C35i}$$

$$D_2^{K_0^\star} = -\frac{1}{4} Z_{K_0^{\star \pm}} Z_{K^\pm} \Big\{ 2g_1 (W_a + w_{K_1^\pm} \mathbb{O}_{P21}) - W_a w_{K_1^\pm} \Big[(\phi_N + \phi_3)(h_2 - 2h_3 + 3g_1^2) + \sqrt{2}\phi_S (h_2 + g_1^2) \Big] \Big\}, \tag{C35j}$$

$$D_3^{K_0^\star} = -Z_{K_0^{\star \pm}} Z_{K^0} Z_{\pi^\pm} \Big(\phi_S + \sqrt{2}\phi_3 \Big) \lambda_2, \tag{C35k}$$

$$D_4^{K_0^\star} = -\frac{1}{2\sqrt{2}} Z_{K_0^{\star \pm}} Z_{K^0} Z_{\pi^\pm} \Big\{ 2g_1 (w_{a_1^\pm} + w_{K_1^0}) - w_{a_1^\pm} w_{K_1^0} [\phi_N (h_2 - 2h_3 + 3g_1^2) + \phi_3 (h_2 + 2h_3 - g_1^2) + \sqrt{2}\phi_S (h_2 + g_1^2)] \Big\}, \tag{C35l}$$

$$F_1^{K_0^\star} = \frac{1}{2} Z_{K_0^{\star \pm}} Z_{K^\pm} \mathbb{O}_{P21} \Big[g_1 (w_{K_1^\pm} + \bar{w}_{K^{\star +}}) - w_{K_1^\pm} \bar{w}_{K^{\star +}} \sqrt{2}\phi_S (g_1^2 - h_3) \Big], \tag{C35m}$$

$$F_2^{K_0^\star} = \frac{1}{4} Z_{K_0^{\star \pm}} Z_{K^\pm} \Big\{ 2g_1 (W_a - \bar{w}_{K^{\star +}} \mathbb{O}_{P21}) + W_a \bar{w}_{K^{\star +}} \Big[(\phi_N + \phi_3)(h_2 - 2h_3 + 3g_1^2) - \sqrt{2}\phi_S (h_2 + g_1^2) \Big] \Big\}, \tag{C35n}$$

$$F_3^{K_0^\star} = \frac{1}{\sqrt{2}} Z_{K_0^{\star \pm}} Z_{K^0} Z_{\pi^\pm} \Big\{ g_1 (w_{K_1^0} + \bar{w}_{K^{\star +}}) - w_{K_1^0} \bar{w}_{K^{\star +}} \Big[-\phi_3 (h_2 + g_1^2) + \sqrt{2}\phi_S (g_1^2 - h_3) \Big] \Big\}, \tag{C35o}$$

$$F_4^{K_0^\star} = \frac{1}{2\sqrt{2}} Z_{K_0^{\star \pm}} Z_{K^0} Z_{\pi^\pm} \Big\{ 2g_1 (w_{a_1^\pm} - \bar{w}_{K^{\star +}}) + w_{a_1^\pm} \bar{w}_{K^{\star +}} [\phi_N (h_2 - 2h_3 + 3g_1^2) - \phi_3 (h_2 + 2h_3 - 3g_1^2) - \sqrt{2}\phi_S (h_2 + g_1^2)] \Big\}, \tag{C35p}$$

and

$$W_a = w_\eta^a \mathbb{O}_{P11} + w_\pi^a \mathbb{O}_{P21}, \tag{C36a}$$

$$\bar{w}_{K^{\star 0}} \equiv i w_{K^{\star 0}} = -\frac{g_1 (\phi_N - \phi_3 - \sqrt{2}\phi_S)}{2 m_{K^{\star 0}}^2}, \tag{C36b}$$

$$\bar{w}_{K^{\star +}} \equiv i w_{K^{\star +}} = -\frac{g_1 (\phi_N + \phi_3 - \sqrt{2}\phi_S)}{2 m_{K^{\star \pm}}^2}. \tag{C36c}$$

The tree-level decay widths for the neutral and charged $K_0^\star$ are given by

$$\Gamma_{\bar{K}_0^{\star 0}\to\pi^{0,+}\bar{K}^{0,-}} = \Gamma_{\bar{K}_0^{\star 0}\to\pi^0\bar{K}^0} + \Gamma_{\bar{K}_0^{\star 0}\to\pi^+K^-}$$
$$= \frac{1}{8\pi M_{K_0^{\star 0}}^2}\left\{k_{\bar{K}_0^{\star 0}\to\pi^0\bar{K}^0}\left|B_1^{K_0^\star} + \frac{1}{2}\left(C_1^{K_0^\star} + C_2^{K_0^\star} - B_2^{K_0^\star}\right)(M_{K_0^{\star 0}}^2 - M_{K^0}^2 - M_{\pi^0}^2) + C_1^{K_0^\star}M_{K^0}^2 + C_2^{K_0^\star}M_{\pi^0}^2\right|^2\right.$$
$$\left. + k_{\bar{K}_0^{\star 0}\to\pi^+K^-}\left|B_3^{K_0^\star} + \frac{1}{2}\left(C_3^{K_0^\star} + C_4^{K_0^\star} - B_4^{K_0^\star}\right)(M_{K_0^{\star 0}}^2 - M_{K^\pm}^2 - M_{\pi^\pm}^2) + C_3^{K_0^\star}M_{K^\pm}^2 + C_4^{K_0^\star}M_{\pi^\pm}^2\right|^2\right\}, \quad \text{(C37)}$$

$$\Gamma_{K_0^{\star -}\to\pi^{0,-}K^{-,\bar{0}}} = \Gamma_{K_0^{\star -}\to\pi^0K^-} + \Gamma_{K_0^{\star -}\to\pi^-\bar{K}^0}$$
$$= \frac{1}{8\pi M_{K_0^{\star\pm}}^2}\left\{k_{K_0^{\star -}\to\pi^0K^-}\left|D_1^{K_0^\star} + \frac{1}{2}\left(F_1^{K_0^\star} + F_2^{K_0^\star} - D_2^{K_0^\star}\right)(M_{K_0^{\star\pm}}^2 - M_{K^\pm}^2 - M_{\pi^0}^2) + F_1^{K_0^\star}M_{K^\pm}^2 + F_2^{K_0^\star}M_{\pi^0}^2\right|^2\right.$$
$$\left. + k_{K_0^{\star -}\to\pi^-\bar{K}^0}\left|D_3^{K_0^\star} + \frac{1}{2}\left(F_3^{K_0^\star} + F_4^{K_0^\star} - D_4^{K_0^\star}\right)(M_{K_0^{\star\pm}}^2 - M_{K^0}^2 - M_{\pi^\pm}^2) + F_3^{K_0^\star}M_{K^0}^2 + F_4^{K_0^\star}M_{\pi^\pm}^2\right|^2\right\}, \quad \text{(C38)}$$

where $k_{A\to BC}$ is defined in Eq. (C16).

b. $a_0 \to KK$

The relevant part of the Lagrangian reads

$$\mathcal{L}_{a_0KK} = a_0^0[B_1^{a_0}K^0\overline{K^0} + B_2^{a_0}(\partial_\mu K^0)\partial^\mu\overline{K^0} + B_3^{a_0}K^+K^- + B_4^{a_0}(\partial_\mu K^+)\partial^\mu K^-]$$
$$+ (\partial_\mu a_0^0)[C_1^{a_0}K^0\partial^\mu\overline{K^0} + C_2^{a_0}(\partial^\mu K^0)\overline{K^0} + C_3^{a_0}K^+\partial^\mu K^- + C_4^{a_0}(\partial^\mu K^+)K^-]$$
$$+ a_0^+[D_1^{a_0}K^0K^- + D_2^{a_0}(\partial_\mu K^0)\partial^\mu K^-] + (\partial_\mu a_0^+)[F_1^{a_0}K^0\partial^\mu K^- + F_2^{a_0}(\partial^\mu K^0)K^-] + \text{H.c.}, \quad \text{(C39)}$$

where we introduce

$$\mathbb{O}_{S2}^{a_0} = \mathbb{O}_{S22}, \quad \text{(C40a)}$$

$$\mathbb{O}_{SNS0}^{a_0} = \sqrt{2}\mathbb{O}_{S21} + \mathbb{O}_{S23}, \quad \text{(C40b)}$$

$$\mathbb{O}_{SNS8}^{a_0} = \mathbb{O}_{S21} - \sqrt{2}\mathbb{O}_{S23}, \quad \text{(C40c)}$$

$$B_1^{a_0} = \frac{1}{2}Z_{K^0}^2\mathbb{O}_{S2}^{a_0}\left[\lambda_2(2\phi_N - \sqrt{2}\phi_S) - 2\phi_3(2\lambda_1 + \lambda_2)\right] - \frac{\sqrt{2}}{6}Z_{K^0}^2\mathbb{O}_{SNS0}^{a_0}\left[\phi_N(4\lambda_1 + \lambda_2) - \phi_3\lambda_2 + \sqrt{2}\phi_S(2\lambda_1 + \lambda_2)\right]$$
$$- \frac{1}{6}Z_{K^0}^2\mathbb{O}_{SNS8}^{a_0}\left[4\phi_N(\lambda_1 + \lambda_2) - 4\phi_3\lambda_2 - \sqrt{2}\phi_S(4\lambda_1 + 5\lambda_2)\right], \quad \text{(C41a)}$$

$$B_2^{a_0} = \frac{1}{2}Z_{K^0}^2\mathbb{O}_{S2}^{a_0}w_{K_1^0}\left\{2g_1 - w_{K_1^0}\left[\phi_N(h_2 + g_1^2) - \phi_3(2h_1 + h_2 + g_1^2) + \sqrt{2}\phi_S(g_1^2 - h_3)\right]\right\}$$
$$- \frac{\sqrt{2}}{6}Z_{K^0}^2\mathbb{O}_{SNS0}^{a_0}w_{K_1^0}\left\{4g_1 - w_{K_1^0}[\phi_N(2h_1 + h_2 - h_3 + 2g_1^2) - \phi_3(h_2 - h_3 + 2g_1^2) + \sqrt{2}\phi_S(h_1 + h_2 - h_3 + 2g_1^2)]\right\}$$
$$+ \frac{1}{6}Z_{K^0}^2\mathbb{O}_{SNS8}^{a_0}w_{K_1^0}\left\{2g_1 + w_{K_1^0}\left[\phi_N(2h_1 + h_2 + 2h_3 - g_1^2) - \phi_3(h_2 + 2h_3 - g_1^2) - \sqrt{2}\phi_S(2h_1 + 2h_2 + h_3 + g_1^2)\right]\right\}, \quad \text{(C41b)}$$

$$B_3^{a_0} = -\frac{1}{2} Z_{K^\pm}^2 \mathbb{O}_{S2}^{a_0} \Big[\lambda_2(2\phi_N - \sqrt{2}\phi_S) + 2\phi_3(2\lambda_1 + \lambda_2)\Big] - \frac{\sqrt{2}}{6} Z_{K^\pm}^2 \mathbb{O}_{SNS0}^{a_0} \Big[\phi_N(4\lambda_1 + \lambda_2) + \phi_3\lambda_2 + \sqrt{2}\phi_S(2\lambda_1 + \lambda_2)\Big] - \frac{1}{6} Z_{K^\pm}^2 \mathbb{O}_{SNS8}^{a_0} \Big[4\phi_N(\lambda_1 + \lambda_2) + 4\phi_3\lambda_2 - \sqrt{2}\phi_S(4\lambda_1 + 5\lambda_2)\Big], \tag{C41c}$$

$$B_4^{a_0} = -\frac{1}{2} Z_{K^\pm}^2 \mathbb{O}_{S2}^{a_0} w_{K_1^\pm} \Big\{2g_1 - w_{K_1^\pm}\Big[\phi_N(h_2 + g_1^2) + \phi_3(2h_1 + h_2 + g_1^2) + \sqrt{2}\phi_S(g_1^2 - h_3)\Big]\Big\} - \frac{\sqrt{2}}{6} Z_{K^\pm}^2 \mathbb{O}_{SNS0}^{a_0} w_{K_1^\pm} \Big\{4g_1 - w_{K_1^\pm}\Big[\phi_N(2h_1 + h_2 - h_3 + 2g_1^2) + \phi_3(h_2 - h_3 + 2g_1^2) + \sqrt{2}\phi_S(h_1 + h_2 - h_3 + 2g_1^2)\Big]\Big\} + \frac{1}{6} Z_{K^\pm}^2 \mathbb{O}_{SNS8}^{a_0} w_{K_1^\pm} \Big\{2g_1 + w_{K_1^\pm}\Big[\phi_N(2h_1 + h_2 + 2h_3 - g_1^2) + \phi_3(h_2 + 2h_3 - g_1^2) - \sqrt{2}\phi_S(2h_1 + 2h_2 + h_3 + g_1^2)\Big]\Big\}, \tag{C41d}$$

$$C_1^{a_0} = -\frac{g_1}{2} Z_{K^0}^2 \mathbb{O}_{S2}^{a_0} w_{K_1^0} + g_1 \frac{\sqrt{2}}{3} Z_{K^0}^2 \mathbb{O}_{SNS0}^{a_0} w_{K_1^0} - \frac{g_1}{6} Z_{K^0}^2 \mathbb{O}_{SNS8}^{a_0} w_{K_1^0}, \qquad C_2^{a_0} = C_1^{a_0}, \tag{C41e}$$

$$C_3^{a_0} = \frac{g_1}{2} Z_{K^\pm}^2 \mathbb{O}_{S2}^{a_0} w_{K_1^\pm} + g_1 \frac{\sqrt{2}}{3} Z_{K^\pm}^2 \mathbb{O}_{SNS0}^{a_0} w_{K_1^\pm} - \frac{g_1}{6} Z_{K^\pm}^2 \mathbb{O}_{SNS8}^{a_0} w_{K_1^\pm}, \qquad C_4^{a_0} = C_3^{a_0}, \tag{C41f}$$

$$D_1^{a_0} = -\frac{1}{\sqrt{2}} Z_{K^0} Z_{K^\pm} Z_{a_0^\pm} \lambda_2(2\phi_N - \sqrt{2}\phi_S), \tag{C41g}$$

$$D_2^{a_0} = -\frac{1}{\sqrt{2}} Z_{K^0} Z_{K^\pm} Z_{a_0^\pm} \Big\{g_1(w_{K_1^0} + w_{K_1^\pm}) - w_{K_1^0} w_{K_1^\pm}\Big[\phi_N(h_2 + g_1^2) + \sqrt{2}\phi_S(g_1^2 - h_3)\Big]\Big\}, \tag{C41h}$$

$$F_1^{a_0} = \frac{1}{2\sqrt{2}} Z_{K^0} Z_{K^\pm} Z_{a_0^\pm} \Big\{2g_1(w_{K_1^\pm} - \bar{w}_{\rho^+}) - w_{K_1^\pm}\bar{w}_{\rho^+}\Big[\phi_N(h_2 + 2h_3 - g_1^2) - \phi_3(h_2 - 2h_3 + 3g_1^2) - \sqrt{2}\phi_S(g_1^2 + h_2)\Big]\Big\}, \tag{C41i}$$

$$F_2^{a_0} = \frac{1}{2\sqrt{2}} Z_{K^0} Z_{K^\pm} Z_{a_0^\pm} \Big\{2g_1(w_{K_1^0} + \bar{w}_{\rho^+}) + w_{K_1^0}\bar{w}_{\rho^+}\Big[\phi_N(h_2 + 2h_3 - g_1^2) + \phi_3(h_2 - 2h_3 + 3g_1^2) - \sqrt{2}\phi_S(g_1^2 + h_2)\Big]\Big\}, \tag{C41j}$$

and

$$\bar{w}_{\rho^+} \equiv i w_{\rho^+} = -\frac{g_1\phi_3}{m_{\rho^\pm}^2}. \tag{C42}$$

The tree-level decay widths for the neutral and charged a_0 are given by

$$\Gamma_{a_0^0\to K^{0,+}\bar{K}^{0,-}} = \Gamma_{a_0^0\to K^0\bar{K}^0} + \Gamma_{a_0^0\to K^+K^-} = \frac{1}{8\pi M_{a_0^0}^2}\Bigg\{k_{a_0^0\to K^0\bar{K}^0}\left|B_1^{a_0} + \frac{1}{2}(C_1^{a_0} + C_2^{a_0} - B_2^{a_0})M_{a_0^0}^2 + B_2^{a_0}M_{K^0}^2\right|^2 + k_{a_0^0\to K^+K^-}\left|B_3^{a_0} + \frac{1}{2}(C_3^{a_0} + C_4^{a_0} - B_4^{a_0})M_{a_0^0}^2 + B_4^{a_0}M_{K^\pm}^2\right|^2\Bigg\}, \tag{C43a}$$

$$\Gamma_{a_0^-\to K^0K^-} = \frac{1}{8\pi M_{a_0^\pm}^2}\Bigg\{k_{a_0^-\to K^0K^-}\left|D_1^{a_0} + \frac{1}{2}(F_1^{a_0} + F_2^{a_0} - D_2^{a_0})(M_{a_0^\pm}^2 - M_{K^\pm}^2 - M_{K^0}^2) + F_1^{a_0}M_{K^\pm}^2 + F_2^{a_0}M_{K^0}^2\right|^2\Bigg\}, \tag{C43b}$$

where $k_{A\to BC}$ is defined in Eq. (C16).

c. $a_0 \to \pi\eta,\pi\eta'$

We start from the following part of the Lagrangian:

$$\mathcal{L}_{a_0\pi\eta_N/\eta_S} = a_0^0[\mathbf{x}^T\mathbb{B}_1^\eta\mathbf{x} + (\partial_\mu\mathbf{x})^T\mathbb{B}_2^\eta\partial^\mu\mathbf{x}] + (\partial_\mu a_0^0)[(\partial^\mu\mathbf{x})^T\mathbb{C}^\eta\mathbf{x}] + a_0^+\Big[\pi^-\mathbf{D}_1^{\eta T}\mathbf{x} + (\partial_\mu\pi^-)\mathbf{D}_2^{\eta T}\partial^\mu\mathbf{x}\Big] + (\partial_\mu a_0^+)\Big[\pi^-\mathbf{F}_1^{\eta T}\partial^\mu\mathbf{x} + (\partial^\mu\pi^-)\mathbf{F}_2^{\eta T}\mathbf{x}\Big] + \text{H.c.}, \tag{C44}$$

where $\mathbf{x}^T = (\tilde\eta_N, \tilde\pi^0, \tilde\eta_S)$ and the coefficient matrices and vectors are

$$\mathbb{B}^\eta_{1\,11} = -\frac{1}{6}\mathbb{O}^{a_0}_{SNS0}\Big[2c_1\phi_N\phi_S(\phi_N+\sqrt{2}\phi_S) + \sqrt{2}\phi_N(2\lambda_1+\lambda_2) + 2\lambda_1\phi_S\Big] - \frac{1}{2}\mathbb{O}^{a_0}_{S2}(2\lambda_1+\lambda_2)\phi_3 + \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\Big[2c_1\phi_N\phi_S(\sqrt{2}\phi_N-\phi_S) - \phi_N(2\lambda_1+\lambda_2) + 2\sqrt{2}\lambda_1\phi_S\Big], \tag{C45a}$$

$$\mathbb{B}^\eta_{1\,12} = \frac{1}{6}\mathbb{O}^{a_0}_{SNS0}\Big[c_1\phi_S(2\phi_N+\sqrt{2}\phi_S) - \sqrt{2}\lambda_2\Big]\phi_3 + \frac{1}{2}\mathbb{O}^{a_0}_{S2}(c_1\phi_S^2-\lambda_2)\phi_N - \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\Big[c_1\phi_S(2\sqrt{2}\phi_N-\phi_S)+\lambda_2\Big]\phi_3, \tag{C45b}$$

$$\mathbb{B}^\eta_{1\,13} = -\frac{1}{12}\mathbb{O}^{a_0}_{SNS0}c_1\Big[\phi_N^2(\phi_N+3\sqrt{2}\phi_S) - \phi_3^2(\phi_N+\sqrt{2}\phi_S)\Big] + \frac{1}{2}\mathbb{O}^{a_0}_{S2}c_1\phi_N\phi_S\phi_3 + \frac{1}{12}\mathbb{O}^{a_0}_{SNS8}c_1\Big[\phi_N^2(\sqrt{2}\phi_N-3\phi_S) - \phi_3^2(\sqrt{2}\phi_N-\phi_S)\Big], \tag{C45c}$$

$$\mathbb{B}^\eta_{1\,21} = \mathbb{B}^\eta_{1\,12}, \tag{C45d}$$

$$\mathbb{B}^\eta_{1\,22} = -\frac{1}{6}\mathbb{O}^{a_0}_{SNS0}\Big[2c_1\phi_3^2\phi_S + \sqrt{2}\phi_N(2\lambda_1+\lambda_2) + 2\lambda_1\phi_S\Big] - \frac{1}{2}\mathbb{O}^{a_0}_{S2}(2\lambda_1+\lambda_2+2c_1\phi_S^2)\phi_3 + \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\Big[2\sqrt{2}c_1\phi_3^2\phi_S - \phi_N(2\lambda_1+\lambda_2) + 2\sqrt{2}\lambda_1\phi_S\Big], \tag{C45e}$$

$$\mathbb{B}^\eta_{1\,23} = \frac{1}{12}\mathbb{O}^{a_0}_{SNS0}c_1\phi_3\left(\phi_N^2 - \phi_3^2 + 2\sqrt{2}\phi_N\phi_S\right) + \frac{1}{4}\mathbb{O}^{a_0}_{S2}c_1\phi_S(\phi_N^2-3\phi_3^2) - \frac{1}{12}\mathbb{O}^{a_0}_{SNS8}c_1\phi_3\Big[\sqrt{2}(\phi_N^2-\phi_3^2) - 2\phi_N\phi_S\Big], \tag{C45f}$$

$$\mathbb{B}^\eta_{1\,31} = \mathbb{B}^\eta_{1\,13}, \tag{C45g}$$

$$\mathbb{B}^\eta_{1\,32} = \mathbb{B}^\eta_{1\,23}, \tag{C45h}$$

$$\mathbb{B}^\eta_{1\,33} = -\frac{1}{6}\mathbb{O}^{a_0}_{SNS0}\Big[\sqrt{2}c_1\phi_N(\phi_N^2-\phi_3^2) + 2\sqrt{2}\lambda_1\phi_N + 2(\lambda_1+\lambda_2)\phi_S\Big] + \frac{1}{2}\mathbb{O}^{a_0}_{S2}[-2\lambda_1 + c_1(\phi_N^2-\phi_3^2)]\phi_3 + \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\Big[-c_1\phi_N(\phi_N^2-\phi_3^2) - 2\lambda_1\phi_N + 2\sqrt{2}(\lambda_1+\lambda_2)\phi_S\Big], \tag{C45i}$$

$$\mathbb{B}^\eta_{2\,11} = \frac{\sqrt{2}}{6}\mathbb{O}^{a_0}_{SNS0}\left\{-2g_1w^f_\eta + [(w^a_\eta)^2+(w^f_\eta)^2](2g_1^2+h_1+h_2-h_3)\phi_N + 2w^a_\eta w^f_\eta(2g_1^2+h_2-h_3)\phi_3 + \frac{1}{\sqrt{2}}((w^a_\eta)^2+(w^f_\eta)^2)h_1\phi_S\right\} + \mathbb{O}^{a_0}_{S2}\left\{-g_1w^a_\eta + w^a_\eta w^f_\eta(2g_1^2+h_2-h_3)\phi_N + \frac{1}{2}[(w^a_\eta)^2+(w^f_\eta)^2](2g_1^2+h_1+h_2-h_3)\phi_3\right\} + \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\left\{-2g_1w^f_\eta + [(w^a_\eta)^2+(w^f_\eta)^2](2g_1^2+h_1+h_2-h_3)\phi_N + 2w^a_\eta w^f_\eta(2g_1^2+h_2-h_3)\phi_3 - \sqrt{2}((w^a_\eta)^2+(w^f_\eta)^2)h_1\phi_S\right\}, \tag{C45j}$$

$$\mathbb{B}^{\eta}_{2\ 12} = \frac{\sqrt{2}}{12}\mathbb{O}^{a_0}_{SNS0}\left\{-2g_1(w^a_\eta + w^f_\pi) + 2(w^a_\pi w^a_\eta + w^f_\pi w^f_\eta)\left[(2g_1^2 + h_2 - h_3)(\phi_N + \phi_3) + h_1\left(\phi_N + \frac{1}{\sqrt{2}}\phi_S\right)\right]\right\}$$
$$+ \mathbb{O}^{a_0}_{S2}\left[-\frac{g_1}{2}(w^a_\pi + w^f_\eta) + \frac{1}{2}(w^a_\eta w^f_\pi + w^f_\eta w^a_\pi)(2g_1^2 + h_2 - h_3)\phi_N\right.$$
$$\left.+ \frac{1}{2}(w^a_\eta w^a_\pi + w^f_\eta w^f_\pi)(2g_1^2 + h_1 + h_2 - h_3)\phi_3\right]$$
$$+ \frac{1}{12}\mathbb{O}^{a_0}_{SNS8}\left\{-2g_1(w^a_\eta + w^f_\pi) + 2(w^a_\pi w^a_\eta + w^f_\pi w^f_\eta)\left[(2g_1^2 + h_2 - h_3)(\phi_N + \phi_3) + h_1(\phi_N - \sqrt{2}\phi_S)\right]\right\}, \tag{C45k}$$

$$\mathbb{B}^{\eta}_{2\ 13} = 0, \tag{C45l}$$

$$\mathbb{B}^{\eta}_{2\ 21} = \mathbb{B}^{\eta}_{2 12}, \tag{C45m}$$

$$\mathbb{B}^{\eta}_{2\ 22} = \frac{\sqrt{2}}{6}\mathbb{O}^{a_0}_{SNS0}\left\{-2g_1 w^a_\pi + [(w^a_\pi)^2 + (w^f_\pi)^2](2g_1^2 + h_1 + h_2 - h_3)\phi_N + 2w^a_\pi w^f_\pi(2g_1^2 + h_2 - h_3)\phi_3 + \frac{1}{\sqrt{2}}[(w^a_\pi)^2 + (w^f_\pi)^2]h_1\phi_S\right\}$$
$$+ \mathbb{O}^{a_0}_{S2}\left\{-g_1 w^f_\pi + w^a_\pi w^f_\pi(2g_1^2 + h_2 - h_3)\phi_N + \frac{1}{2}[(w^a_\pi)^2 + (w^f_\pi)^2](2g_1^2 + h_1 + h_2 - h_3)\phi_3\right\}$$
$$+ \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\left\{-2g_1 w^a_\pi + [(w^a_\pi)^2 + (w^f_\pi)^2](2g_1^2 + h_1 + h_2 - h_3)\phi_N + 2w^a_\pi w^f_\pi(2g_1^2 + h_2 - h_3)\phi_3 - \sqrt{2}[(w^a_\pi)^2 + (w^f_\pi)^2]h_1\phi_S\right\}, \tag{C45n}$$

$$\mathbb{B}^{\eta}_{2\ 23} = 0, \tag{C45o}$$

$$\mathbb{B}^{\eta}_{2\ 31} = 0, \tag{C45p}$$

$$\mathbb{B}^{\eta}_{2\ 32} = 0, \tag{C45q}$$

$$\mathbb{B}^{\eta}_{2\ 33} = \frac{\sqrt{2}}{6}\mathbb{O}^{a_0}_{SNS0}\left\{-2g_1 w_{f_{1S}} + (w_{f_{1S}})^2\left[h_1\phi_N + \frac{1}{\sqrt{2}}(4g_1^2 + h_1 + 2h_2 - 2h_3)\phi_S\right]\right\} + \frac{1}{2}\mathbb{O}^{a_0}_{S2}h_1\phi_3 w^2_{f_{1S}}$$
$$+ \frac{1}{6}\mathbb{O}^{a_0}_{SNS8}\left\{4g_1 w_{f_{1S}} + (w_{f_{1S}})^2\left[h_1\phi_N - \sqrt{2}(4g_1^2 + h_1 + 2h_2 - 2h_3)\phi_S\right]\right\}, \tag{C45r}$$

$$\mathbb{C}^{\eta} = g_1\begin{pmatrix} \mathbb{O}_{S21}w^f_\eta + \mathbb{O}_{S22}w^a_\eta & \mathbb{O}_{S21}w^a_\eta + \mathbb{O}_{S22}w^f_\eta & 0 \\ \mathbb{O}_{S21}w^f_\pi + \mathbb{O}_{S22}w^a_\pi & \mathbb{O}_{S21}w^a_\pi + \mathbb{O}_{S22}w^f_\pi & 0 \\ 0 & 0 & \sqrt{2}\mathbb{O}_{S23}w_{f_{1S}} \end{pmatrix}, \tag{C45s}$$

$$\mathbf{D}^{\eta}_1 = Z_{a_0^\pm}Z_{\pi^\pm}\begin{pmatrix} (c_1\phi_S^2 - \lambda_2)\phi_N \\ -(c_1\phi_S^2 - \lambda_2)\phi_3 \\ \frac{1}{2}c_1\phi_S(\phi_N^2 - \phi_3^2) \end{pmatrix}, \tag{C45t}$$

$$\mathbf{D}^{\eta}_2 = Z_{a_0^\pm}Z_{\pi^\pm}\begin{pmatrix} -g_1 w^f_\eta + w_{a_1^\pm}[w^f_\eta(2g_1^2 + h_2 - h_3)\phi_N + w^a_\eta(g_1^2 - h_3)\phi_3 - g_1] \\ -g_1 w^f_\pi + w_{a_1^\pm}[w^f_\pi(2g_1^2 + h_2 - h_3)\phi_N + w^a_\pi(g_1^2 - h_3)\phi_3] \\ 0 \end{pmatrix}, \tag{C45u}$$

$$\mathbf{F}_1^\eta = Z_{a_0^\pm} Z_{\pi^\pm} \begin{pmatrix} g_1 w_\eta^f + \bar{w}_{\rho^+}[w_\eta^f (2g_1^2 + h_2 - h_3)\phi_3 + w_\eta^a (g_1^2 - h_3)\phi_N] \\ g_1 w_\pi^f + \bar{w}_{\rho^+}[w_\pi^f (2g_1^2 + h_2 - h_3)\phi_3 + w_\pi^a (g_1^2 - h_3)\phi_N] \\ 0 \end{pmatrix}, \tag{C45v}$$

$$\mathbf{F}_2^\eta = Z_{a_0^\pm} Z_{\pi^\pm} \begin{pmatrix} w_{a_1^\pm}[g_1 + \bar{w}_{\rho^+}(g_1^2 - h_3)\phi_3] \\ \bar{w}_{\rho^+}[g_1 - w_{a_1^\pm}(g_1^2 - h_3)\phi_N] \\ 0 \end{pmatrix}. \tag{C45w}$$

Then we apply for $\mathbf{x}$ the transformation in Eq. (37) to obtain

$$\mathcal{L}_{a_0\pi\eta/\eta'} = a_0^0\left[\mathbf{y}^{\mathrm{ph}T}\tilde{\mathbb{B}}_1^\eta\mathbf{y}^{\mathrm{ph}} + (\partial_\mu\mathbf{y}^{\mathrm{ph}})^T\tilde{\mathbb{B}}_2^\eta\partial^\mu\mathbf{y}^{\mathrm{ph}}\right] + (\partial_\mu a_0^0)\left[(\partial^\mu\mathbf{y}^{\mathrm{ph}})^T\tilde{\mathbb{C}}^\eta\mathbf{y}^{\mathrm{ph}}\right]$$
$$+ a_0^+\left[\pi^-\tilde{\mathbf{D}}_1^{\eta T}\mathbf{y}^{\mathrm{ph}} + (\partial_\mu\pi^-)\tilde{\mathbf{D}}_2^{\eta T}\partial^\mu\mathbf{y}^{\mathrm{ph}}\right] + (\partial_\mu a_0^+)\left[\pi^-\tilde{\mathbf{F}}_1^{\eta T}\partial^\mu\mathbf{y}^{\mathrm{ph}} + (\partial^\mu\pi^-)\tilde{\mathbf{F}}_2^{\eta T}\mathbf{y}^{\mathrm{ph}}\right] + \mathrm{H.c.}, \tag{C46}$$

where

$$\tilde{\mathbb{B}}_1^\eta \equiv \mathbb{O}_P^T\mathbb{B}_1^\eta\mathbb{O}_P, \qquad \tilde{\mathbb{B}}_2^\eta \equiv \mathbb{O}_P^T\mathbb{B}_2^\eta\mathbb{O}_P, \qquad \tilde{\mathbb{C}}^\eta \equiv \mathbb{O}_P^T\mathbb{C}^\eta\mathbb{O}_P, \tag{C47a}$$

$$\tilde{\mathbf{D}}_1^{\eta T} \equiv \mathbf{D}_1^{\eta T}\mathbb{O}_P, \qquad \tilde{\mathbf{D}}_2^{\eta T} \equiv \mathbf{D}_2^{\eta T}\mathbb{O}_P, \qquad \tilde{\mathbf{F}}_1^{\eta T} \equiv \mathbf{F}_1^{\eta T}\mathbb{O}_P, \qquad \tilde{\mathbf{F}}_2^{\eta T} \equiv \mathbf{F}_2^{\eta T}\mathbb{O}_P, \tag{C47b}$$

which finally leads to

$$\mathcal{L}_{a_0\pi\eta/\eta'} = a_0^0\left[(\tilde{\mathbb{B}}^\eta_{1\ 12} + \tilde{\mathbb{B}}^\eta_{1\ 21})\pi^0\eta + (\tilde{\mathbb{B}}^\eta_{2\ 12} + \tilde{\mathbb{B}}^\eta_{2\ 21})(\partial_\mu\pi^0)(\partial^\mu\eta)\right] + (\partial_\mu a_0^0)\left[\tilde{\mathbb{C}}^\eta{}_{21}\pi^0\partial^\mu\eta + \tilde{\mathbb{C}}^\eta{}_{12}(\partial^\mu\pi^0)\eta\right]$$
$$+ a_0^0\left[(\tilde{\mathbb{B}}^\eta_{1\ 13} + \tilde{\mathbb{B}}^\eta_{1\ 31})\pi^0\eta' + (\tilde{\mathbb{B}}^\eta_{2\ 13} + \tilde{\mathbb{B}}^\eta_{2\ 31})(\partial_\mu\pi^0)(\partial^\mu\eta')\right]$$
$$+ a_0^+\left[\left(\tilde{\mathbf{D}}_1^{\eta T}\right)_2\pi^-\eta + \left(\tilde{\mathbf{D}}_2^{\eta T}\right)_2(\partial_\mu\pi^-)(\partial^\mu\eta)\right] + (\partial_\mu a_0^+)\left[\left(\tilde{\mathbf{F}}_1^{\eta T}\right)_2\pi^-\partial^\mu\eta + \left(\tilde{\mathbf{F}}_2^{\eta T}\right)_2(\partial^\mu\pi^-)\eta\right]$$
$$+ a_0^+\left(\tilde{\mathbf{D}}_1^{\eta T}\right)_3\pi^-\eta' + \mathrm{H.c.} \tag{C48}$$

The tree-level decay widths for the neutral and charged a_0 are given by

$$\Gamma_{a_0^0\to\pi^0\eta} = \frac{k_{a_0^0\to\pi^0\eta}}{8\pi M_{a_0^0}^2}\left|\tilde{\mathbb{B}}^\eta_{1\ 12} + \tilde{\mathbb{B}}^\eta_{1\ 21} + \frac{1}{2}\left(\tilde{\mathbb{C}}^\eta{}_{12} + \tilde{\mathbb{C}}^\eta{}_{21} - \tilde{\mathbb{B}}^\eta_{2\ 12} - \tilde{\mathbb{B}}^\eta_{2\ 21}\right)\left(M_{a_0^0}^2 - M_\eta^2 - M_{\pi^0}^2\right) + \tilde{\mathbb{C}}^\eta{}_{21}M_\eta^2 + \tilde{\mathbb{C}}^\eta{}_{12}M_{\pi^0}^2\right|^2, \tag{C49a}$$

$$\Gamma_{a_0^0\to\pi^0\eta'} = \frac{k_{a_0^0\to\pi^0\eta'}}{8\pi M_{a_0^0}^2}\left|\tilde{\mathbb{B}}^\eta_{1\ 13} + \tilde{\mathbb{B}}^\eta_{1\ 31} + \frac{1}{2}\left(\tilde{\mathbb{C}}^\eta{}_{13} + \tilde{\mathbb{C}}^\eta{}_{31} - \tilde{\mathbb{B}}^\eta_{2\ 13} - \tilde{\mathbb{B}}^\eta_{2\ 31}\right)\left(M_{a_0^0}^2 - M_{\eta'}^2 - M_{\pi^0}^2\right) + \tilde{\mathbb{C}}^\eta{}_{31}M_{\eta'}^2 + \tilde{\mathbb{C}}^\eta{}_{13}M_{\pi^0}^2\right|^2, \tag{C49b}$$

$$\Gamma_{a_0^-\to\pi^-\eta} = \frac{k_{a_0^-\to\pi^-\eta}}{8\pi M_{a_0^\pm}^2}\left|\left(\tilde{\mathbf{D}}_1^{\eta T}\right)_2 + \frac{1}{2}\left[\left(\tilde{\mathbf{F}}_1^{\eta T}\right)_2 + \left(\tilde{\mathbf{F}}_2^{\eta T}\right)_2 - \left(\tilde{\mathbf{D}}_2^{\eta T}\right)_2\right]\left(M_{a_0^\pm}^2 - M_\eta^2 - M_{\pi^\pm}^2\right) + \left(\tilde{\mathbf{F}}_1^{\eta T}\right)_2 M_\eta^2 + \left(\tilde{\mathbf{F}}_2^{\eta T}\right)_2 M_{\pi^\pm}^2\right|^2, \tag{C49c}$$

$$\Gamma_{a_0^-\to\pi^-\eta'} = \frac{k_{a_0^-\to\pi^-\eta'}}{8\pi M_{a_0^\pm}^2}\left|\left(\tilde{\mathbf{D}}_1^{\eta T}\right)_3 + \frac{1}{2}\left[\left(\tilde{\mathbf{F}}_1^{\eta T}\right)_3 + \left(\tilde{\mathbf{F}}_2^{\eta T}\right)_3 - \left(\tilde{\mathbf{D}}_2^{\eta T}\right)_3\right]\left(M_{a_0^\pm}^2 - M_{\eta'}^2 - M_{\pi^\pm}^2\right) + \left(\tilde{\mathbf{F}}_1^{\eta T}\right)_3 M_{\eta'}^2 + \left(\tilde{\mathbf{F}}_2^{\eta T}\right)_3 M_{\pi^\pm}^2\right|^2, \tag{C49d}$$

where $k_{A\to BC}$ is defined in Eq. (C16).

d. $f_0^{L/H} \to KK$

The relevant part of the Lagrangian reads

$$\mathcal{L}_{f_0KK} = f_0^{L/H}\Big[B_1^{L/H} K^0\overline{K^0} + B_2^{L/H}(\partial_\mu K^0)\partial^\mu\overline{K^0} + B_3^{L/H}K^+K^- + B_4^{L/H}(\partial_\mu K^+)\partial^\mu K^-\Big] + (\partial_\mu f_0^{L/H})\Big[C_1^{L/H}K^0\partial^\mu\overline{K^0} + C_2^{L/H}(\partial^\mu K^0)\overline{K^0} + C_3^{L/H}K^+\partial^\mu K^- + C_4^{L/H}(\partial^\mu K^+)K^-\Big], \tag{C50}$$

where we introduced

$$\mathbb{O}_{S2}{}^{L} = \mathbb{O}_{S12}, \qquad \mathbb{O}_{S2}{}^{H} = \mathbb{O}_{S32}, \tag{C51a}$$

$$\mathbb{O}_{SNS0}^{L} = \sqrt{2}\mathbb{O}_{S11} + \mathbb{O}_{S13}, \qquad \mathbb{O}_{SNS0}^{H} = \sqrt{2}\mathbb{O}_{S31} + \mathbb{O}_{S33}, \tag{C51b}$$

$$\mathbb{O}_{SNS8}^{L} = \mathbb{O}_{S11} - \sqrt{2}\mathbb{O}_{S13}, \qquad \mathbb{O}_{SNS8}^{H} = \mathbb{O}_{S31} - \sqrt{2}\mathbb{O}_{S33}, \tag{C51c}$$

$$B_1^{L/H} = \frac{1}{2}Z_{K^0}^2\mathbb{O}_{S2}{}^{L/H}\Big[\lambda_2(2\phi_N - \sqrt{2}\phi_S) - 2\phi_3(2\lambda_1+\lambda_2)\Big] - \frac{\sqrt{2}}{6}Z_{K^0}^2\mathbb{O}_{SNS0}^{L/H}\Big[\phi_N(4\lambda_1+\lambda_2) - \phi_3\lambda_2 + \sqrt{2}\phi_S(2\lambda_1+\lambda_2)\Big] - \frac{1}{6}Z_{K^0}^2\mathbb{O}_{SNS8}^{L/H}\Big[4\phi_N(\lambda_1+\lambda_2) - 4\phi_3\lambda_2 - \sqrt{2}\phi_S(4\lambda_1+5\lambda_2)\Big], \tag{C52a}$$

$$B_2^{L/H} = \frac{1}{2}Z_{K^0}^2\mathbb{O}_{S2}{}^{L/H}w_{K_1^0}\Big\{2g_1 - w_{K_1^0}\Big[\phi_N(h_2+g_1^2) - \phi_3(2h_1+h_2+g_1^2) + \sqrt{2}\phi_S(g_1^2-h_3)\Big]\Big\} - \frac{\sqrt{2}}{6}Z_{K^0}^2\mathbb{O}_{SNS0}^{L/H}w_{K_1^0}\Big\{4g_1 - w_{K_1^0}\Big[\phi_N(2h_1+h_2-h_3+2g_1^2) - \phi_3(h_2-h_3+2g_1^2) + \sqrt{2}\phi_S(h_1+h_2-h_3+2g_1^2)\Big]\Big\} + \frac{1}{6}Z_{K^0}^2\mathbb{O}_{SNS8}^{L/H}w_{K_1^0}\Big\{2g_1 + w_{K_1^0}\Big[\phi_N(2h_1+h_2+2h_3-g_1^2) - \phi_3(h_2+2h_3-g_1^2) - \sqrt{2}\phi_S(2h_1+2h_2+h_3+g_1^2)\Big]\Big\}, \tag{C52b}$$

$$B_3^{L/H} = -\frac{1}{2}Z_{K^\pm}^2\mathbb{O}_{S2}{}^{L/H}\Big[\lambda_2(2\phi_N - \sqrt{2}\phi_S) + 2\phi_3(2\lambda_1+\lambda_2)\Big] - \frac{\sqrt{2}}{6}Z_{K^\pm}^2\mathbb{O}_{SNS0}^{L/H}\Big[\phi_N(4\lambda_1+\lambda_2) + \phi_3\lambda_2 + \sqrt{2}\phi_S(2\lambda_1+\lambda_2)\Big] - \frac{1}{6}Z_{K^\pm}^2\mathbb{O}_{SNS8}^{L/H}\Big[4\phi_N(\lambda_1+\lambda_2) + 4\phi_3\lambda_2 - \sqrt{2}\phi_S(4\lambda_1+5\lambda_2)\Big], \tag{C52c}$$

$$B_4^{L/H} = -\frac{1}{2}Z_{K^\pm}^2\mathbb{O}_{S2}{}^{L/H}w_{K_1^\pm}\Big\{2g_1 - w_{K_1^\pm}\Big[\phi_N(h_2+g_1^2) + \phi_3(2h_1+h_2+g_1^2) + \sqrt{2}\phi_S(g_1^2-h_3)\Big]\Big\} - \frac{\sqrt{2}}{6}Z_{K^\pm}^2\mathbb{O}_{SNS0}^{L/H}w_{K_1^\pm}\Big\{4g_1 - w_{K_1^\pm}\Big[\phi_N(2h_1+h_2-h_3+2g_1^2) + \phi_3(h_2-h_3+2g_1^2) + \sqrt{2}\phi_S(h_1+h_2-h_3+2g_1^2)\Big]\Big\} + \frac{1}{6}Z_{K^\pm}^2\mathbb{O}_{SNS8}^{L/H}w_{K_1^\pm}\Big\{2g_1 + w_{K_1^\pm}\Big[\phi_N(2h_1+h_2+2h_3-g_1^2) + \phi_3(h_2+2h_3-g_1^2) - \sqrt{2}\phi_S(2h_1+2h_2+h_3+g_1^2)\Big]\Big\}, \tag{C52d}$$

$$C_1^{L/H} = -\frac{g_1}{2}Z_{K^0}^2\mathbb{O}_{S2}{}^{L/H}w_{K_1^0} + g_1\frac{\sqrt{2}}{3}Z_{K^0}^2\mathbb{O}_{SNS0}^{L/H}w_{K_1^0} - \frac{g_1}{6}Z_{K^0}^2\mathbb{O}_{SNS8}^{L/H}w_{K_1^0}, \qquad C_2^{L/H} = C_1^{L/H}, \tag{C52e}$$

$$C_3^{L/H} = \frac{g_1}{2}Z_{K^\pm}^2\mathbb{O}_{S2}{}^{L/H}w_{K_1^\pm} + g_1\frac{\sqrt{2}}{3}Z_{K^\pm}^2\mathbb{O}_{SNS0}^{L/H}w_{K_1^\pm} - \frac{g_1}{6}Z_{K^\pm}^2\mathbb{O}_{SNS8}^{L/H}w_{K_1^\pm}, \qquad C_4^{L/H} = C_3^{L/H}. \tag{C52f}$$

The tree-level decay widths for $f_0^{L/H}$ are given by

$$\begin{aligned}\Gamma_{f_0^{L/H}\to K^{0,+}\bar{K}^{0,-}} &= \Gamma_{f_0^{L/H}\to K^0\bar{K}^0} + \Gamma_{f_0^{L/H}\to K^+K^-} \\ &= \frac{1}{8\pi M^2_{f_0^{L/H}}}\left\{k_{f_0^{L/H}\to K^0\bar{K}^0}\left|B_1^{L/H} + \frac{1}{2}\left(2C_1^{L/H} - B_2^{L/H}\right)M^2_{f_0^{L/H}} + B_2^{L/H}M^2_{K^0}\right|^2\right. \\ &\quad \left. + k_{f_0^{L/H}\to K^+K^-}\left|B_3^{L/H} + \frac{1}{2}\left(2C_3^{L/H} - B_4^{L/H}\right)M^2_{f_0^{L/H}} + B_4^{L/H}M^2_{K^\pm}\right|^2\right\},\end{aligned} \tag{C53}$$

where $k_{A\to BC}$ is defined in Eq. (C16).

e. $f_0^{L/H} \to \pi\pi$

We start from the following part of the Lagrangian:

$$\begin{aligned}\mathcal{L}_{f_0\pi\pi} &= f_0^{L/H}\left[\mathbf{x}^T\mathbb{B}_1^{L/H}\mathbf{x} + (\partial_\mu\mathbf{x})^T\mathbb{B}_2^{L/H}\partial^\mu\mathbf{x}\right] + (\partial_\mu f_0^{L/H})\left[(\partial^\mu\mathbf{x})^T\mathbb{C}^{L/H}\mathbf{x}\right] \\ &\quad + f_0^{L/H}\left[\pi^+D_1^{L/H}\pi^- + (\partial_\mu\pi^+)D_2^{L/H}\partial^\mu\pi^-\right] + (\partial_\mu f_0^{L/H})\left[\pi^+F_1^{L/H}\partial^\mu\pi^- + (\partial^\mu\pi^+)F_2^{L/H}\pi^-\right],\end{aligned} \tag{C54}$$

where $\mathbf{x}^T = (\tilde{\eta}_N, \tilde{\pi}^0, \tilde{\eta}_S)$ and the coefficient matrices and vectors are

$$\begin{aligned}\mathbb{B}_1^{L/H}{}_{11} &= -\frac{1}{6}\mathbb{O}_{SNS0}^{L/H}\left[2c_1\phi_N\phi_S(\phi_N + \sqrt{2}\phi_S) + \sqrt{2}\phi_N(2\lambda_1 + \lambda_2) + 2\lambda_1\phi_S\right] - \frac{1}{2}\mathbb{O}_{S2}{}^{L/H}(2\lambda_1 + \lambda_2)\phi_3 \\ &\quad + \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left[2c_1\phi_N\phi_S(\sqrt{2}\phi_N - \phi_S) - \phi_N(2\lambda_1 + \lambda_2) + 2\sqrt{2}\lambda_1\phi_S\right],\end{aligned} \tag{C55a}$$

$$\begin{aligned}\mathbb{B}_1^{L/H}{}_{12} &= \frac{1}{6}\mathbb{O}_{SNS0}^{L/H}\left[c_1\phi_S(2\phi_N + \sqrt{2}\phi_S) - \sqrt{2}\lambda_2\right]\phi_3 + \frac{1}{2}\mathbb{O}_{S2}{}^{L/H}(c_1\phi_S^2 - \lambda_2)\phi_N \\ &\quad - \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left[c_1\phi_S(2\sqrt{2}\phi_N - \phi_S) + \lambda_2\right]\phi_3,\end{aligned} \tag{C55b}$$

$$\begin{aligned}\mathbb{B}_1^{L/H}{}_{13} &= -\frac{1}{12}\mathbb{O}_{SNS0}^{L/H}c_1\left[\phi_N^2(\phi_N + 3\sqrt{2}\phi_S) - \phi_3^2(\phi_N + \sqrt{2}\phi_S)\right] + \frac{1}{2}\mathbb{O}_{S2}{}^{L/H}c_1\phi_N\phi_S\phi_3 \\ &\quad + \frac{1}{12}\mathbb{O}_{SNS8}^{L/H}c_1\left[\phi_N^2(\sqrt{2}\phi_N - 3\phi_S) - \phi_3^2(\sqrt{2}\phi_N - \phi_S)\right],\end{aligned} \tag{C55c}$$

$$\mathbb{B}_1^{L/H}{}_{21} = \mathbb{B}_1^{L/H}{}_{12}, \tag{C55d}$$

$$\begin{aligned}\mathbb{B}_1^{L/H}{}_{22} &= -\frac{1}{6}\mathbb{O}_{SNS0}^{L/H}\left[2c_1\phi_3^2\phi_S + \sqrt{2}\phi_N(2\lambda_1 + \lambda_2) + 2\lambda_1\phi_S\right] - \frac{1}{2}\mathbb{O}_{S2}{}^{L/H}(2\lambda_1 + \lambda_2 + 2c_1\phi_S^2)\phi_3 \\ &\quad + \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left[2\sqrt{2}c_1\phi_3^2\phi_S - \phi_N(2\lambda_1 + \lambda_2) + 2\sqrt{2}\lambda_1\phi_S\right],\end{aligned} \tag{C55e}$$

$$\mathbb{B}_1^{L/H}{}_{23} = \frac{1}{12}\mathbb{O}_{SNS0}^{L/H}c_1\phi_3\left[\phi_N^2 - \phi_3^2 + 2\sqrt{2}\phi_N\phi_S\right] + \frac{1}{4}\mathbb{O}_{S2}{}^{L/H}c_1\phi_S(\phi_N^2 - 3\phi_3^2) - \frac{1}{12}\mathbb{O}_{SNS8}^{L/H}c_1\phi_3\left[\sqrt{2}(\phi_N^2 - \phi_3^2) - 2\phi_N\phi_S\right], \tag{C55f}$$

$$\mathbb{B}_1^{L/H}{}_{31} = \mathbb{B}_1^{L/H}{}_{13}, \tag{C55g}$$

$$\mathbb{B}_1^{L/H}{}_{32} = \mathbb{B}_1^{L/H}{}_{23}, \tag{C55h}$$

$$\mathbb{B}_1^{L/H}{}_{33} = -\frac{1}{6}\mathbb{O}_{SNS0}^{L/H}\left[\sqrt{2}c_1\phi_N(\phi_N^2-\phi_3^2)+2\sqrt{2}\lambda_1\phi_N+2(\lambda_1+\lambda_2)\phi_S\right]+\frac{1}{2}\mathbb{O}_{S2}{}^{L/H}(-2\lambda_1+c_1(\phi_N^2-\phi_3^2))\phi_3 + \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left[-c_1\phi_N(\phi_N^2-\phi_3^2)-2\lambda_1\phi_N+2\sqrt{2}(\lambda_1+\lambda_2)\phi_S\right], \tag{C55i}$$

$$\mathbb{B}_2^{L/H}{}_{11} = \frac{\sqrt{2}}{6}\mathbb{O}_{SNS0}^{L/H}\left\{-2g_1w_\eta^f+[(w_\eta^a)^2+(w_\eta^f)^2]e(2g_1^2+h_1+h_2-h_3)\phi_N+2w_\eta^aw_\eta^f(2g_1^2+h_2-h_3)\phi_3 + \frac{1}{\sqrt{2}}[(w_\eta^a)^2+(w_\eta^f)^2]h_1\phi_S\right\} + \mathbb{O}_{S2}{}^{L/H}\left\{-g_1w_\eta^a+w_\eta^aw_\eta^f(2g_1^2+h_2-h_3)\phi_N+\frac{1}{2}[(w_\eta^a)^2+(w_\eta^f)^2](2g_1^2+h_1+h_2-h_3)\phi_3\right\} + \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left[-2g_1w_\eta^f+((w_\eta^a)^2+(w_\eta^f)^2)(2g_1^2+h_1+h_2-h_3)\phi_N+2w_\eta^aw_\eta^f(2g_1^2+h_2-h_3)\phi_3 - \sqrt{2}((w_\eta^a)^2+(w_\eta^f)^2)h_1\phi_S\right], \tag{C55j}$$

$$\mathbb{B}_2^{L/H}{}_{12} = \frac{\sqrt{2}}{12}\mathbb{O}_{SNS0}^{L/H}\left\{-2g_1(w_\eta^a+w_\pi^f)+2(w_\pi^aw_\eta^a+w_\pi^fw_\eta^f)\left[(2g_1^2+h_2-h_3)(\phi_N+\phi_3)+h_1\left(\phi_N+\frac{1}{\sqrt{2}}\phi_S\right)\right]\right\} \tag{C55k}$$

$$+\mathbb{O}_{S2}{}^{L/H}\left[-\frac{g_1}{2}(w_\pi^a+w_\eta^f)+\frac{1}{2}(w_\eta^aw_\pi^f+w_\eta^fw_\pi^a)(2g_1^2+h_2-h_3)\phi_N+\frac{1}{2}(w_\eta^aw_\pi^a+w_\eta^fw_\pi^f)(2g_1^2+h_1+h_2-h_3)\phi_3\right] + \frac{1}{12}\mathbb{O}_{SNS8}^{L/H}\left\{-2g_1(w_\eta^a+w_\pi^f)+2(w_\pi^aw_\eta^a+w_\pi^fw_\eta^f)\left[(2g_1^2+h_2-h_3)(\phi_N+\phi_3)+h_1(\phi_N-\sqrt{2}\phi_S)\right]\right\}, \tag{C55l}$$

$$\mathbb{B}_2^{L/H}{}_{13} = 0, \tag{C55m}$$

$$\mathbb{B}_2^{L/H}{}_{21} = \mathbb{B}_2^{L/H}{}_{12}, \tag{C55n}$$

$$\mathbb{B}_2^{L/H}{}_{22} = \frac{\sqrt{2}}{6}\mathbb{O}_{SNS0}^{L/H}\left\{-2g_1w_\pi^a+[(w_\pi^a)^2+(w_\pi^f)^2](2g_1^2+h_1+h_2-h_3)\phi_N+2w_\pi^aw_\pi^f(2g_1^2+h_2-h_3)\phi_3 + \frac{1}{\sqrt{2}}[(w_\pi^a)^2+(w_\pi^f)^2]h_1\phi_S\right\} + \mathbb{O}_{S2}{}^{L/H}\left[-g_1w_\pi^f+w_\pi^aw_\pi^f(2g_1^2+h_2-h_3)\phi_N+\frac{1}{2}[(w_\pi^a)^2+(w_\pi^f)^2](2g_1^2+h_1+h_2-h_3)\phi_3\right] + \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left\{-2g_1w_\pi^a+[(w_\pi^a)^2+(w_\pi^f)^2](2g_1^2+h_1+h_2-h_3)\phi_N+2w_\pi^aw_\pi^f(2g_1^2+h_2-h_3)\phi_3 - \sqrt{2}[(w_\pi^a)^2+(w_\pi^f)^2]h_1\phi_S\right\}, \tag{C55o}$$

$$\mathbb{B}_2^{L/H}{}_{23} = 0, \tag{C55p}$$

$$\mathbb{B}_2^{L/H}{}_{31} = 0, \tag{C55q}$$

$$\mathbb{B}_2^{L/H}{}_{32} = 0, \tag{C55r}$$

$$\mathbb{B}_2^{L/H}{}_{33} = \frac{\sqrt{2}}{6}\mathbb{O}_{SNS0}^{L/H}\left\{-2g_1w_{f_{1S}}+(w_{f_{1S}})^2\left[h_1\phi_N+\frac{1}{\sqrt{2}}(4g_1^2+h_1+2h_2-2h_3)\phi_S\right]\right\}+\frac{1}{2}\mathbb{O}_{S2}{}^{L/H}h_1\phi_3w_{f_{1S}}^2 + \frac{1}{6}\mathbb{O}_{SNS8}^{L/H}\left\{4g_1w_{f_{1S}}+(w_{f_{1S}})^2\left[h_1\phi_N-\sqrt{2}(4g_1^2+h_1+2h_2-2h_3)\phi_S\right]\right\}, \tag{C55s}$$

$$\mathbb{C}^{L/H} = \begin{cases} g_1 \begin{pmatrix} \mathbb{O}_{S11} w_\eta^f + \mathbb{O}_{S12} w_\eta^a & \mathbb{O}_{S11} w_\eta^a + \mathbb{O}_{S12} w_\eta^f & 0 \\ \mathbb{O}_{S11} w_\pi^f + \mathbb{O}_{S12} w_\pi^a & \mathbb{O}_{S11} w_\pi^a + \mathbb{O}_{S12} w_\pi^f & 0 \\ 0 & 0 & \sqrt{2}\mathbb{O}_{S13} w_{f_{1S}} \end{pmatrix}, & \text{for } L \\ g_1 \begin{pmatrix} \mathbb{O}_{S31} w_\eta^f + \mathbb{O}_{S32} w_\eta^a & \mathbb{O}_{S31} w_\eta^a + \mathbb{O}_{S32} w_\eta^f & 0 \\ \mathbb{O}_{S31} w_\pi^f + \mathbb{O}_{S32} w_\pi^a & \mathbb{O}_{S31} w_\pi^a + \mathbb{O}_{S32} w_\pi^f & 0 \\ 0 & 0 & \sqrt{2}\mathbb{O}_{S33} w_{f_{1S}} \end{pmatrix}, & \text{for } H \end{cases} \tag{C55t}$$

$$D_1^{L/H} = -\frac{\sqrt{2}}{3} Z_{\pi^\pm}^2 \mathbb{O}_{SNS0}^{L/H}\Big[\phi_N(2\lambda_1+\lambda_2) + \sqrt{2}\lambda_1\phi_S\Big] - Z_{\pi^\pm}^2 \mathbb{O}_{S2}^{L/H}(2\lambda_1+3\lambda_2)\phi_3 - \frac{1}{3} Z_{\pi^\pm}^2 \mathbb{O}_{SNS8}^{L/H}\Big[\phi_N(2\lambda_1+\lambda_2) - 2\sqrt{2}\lambda_1\phi_S\Big], \tag{C55u}$$

$$\begin{aligned} D_2^{L/H} = {}& \frac{\sqrt{2}}{3} Z_{\pi^\pm}^2 \mathbb{O}_{SNS0}^{L/H}\Big\{-2g_1 w_{a_1^\pm} + w_{a_1^\pm}^2\Big[(2g_1^2+h_1+h_2-h_3)\phi_N + \frac{1}{\sqrt{2}}h_1\phi_S\Big]\Big\} + Z_{\pi^\pm}^2 \mathbb{O}_{S2}^{L/H} w_{a_1^\pm}^2 (h_1+h_2+h_3)\phi_3 \\ &+ \frac{1}{3} Z_{\pi^\pm}^2 \mathbb{O}_{SNS8}^{L/H}\Big\{-2g_1 w_{a_1^\pm} + w_{a_1^\pm}^2\Big[(2g_1^2+h_1+h_2-h_3)\phi_N - \sqrt{2}h_1\phi_S\Big]\Big\}, \end{aligned} \tag{C55v}$$

$$F_1^{L/H} = Z_{\pi^\pm}^2 g_1 w_{a_1^\pm} \begin{cases} \mathbb{O}_{S11}, & \text{for } L \\ \mathbb{O}_{S13}, & \text{for } H \end{cases} \tag{C55w}$$

$$F_2^{L/H} = F_1^{L/H}. \tag{C55x}$$

Then we apply for $\mathbf{x}$ the transformation in Eq. (37) to obtain

$$\begin{aligned} \mathcal{L}_{f_0\pi\pi} = {}& f_0^{L/H}\Big[\mathbf{y}^{\mathrm{ph}T}\tilde{\mathbb{B}}_1^{L/H}\mathbf{y}^{\mathrm{ph}} + (\partial_\mu \mathbf{y}^{\mathrm{ph}})^T \tilde{\mathbb{B}}_2^{L/H}\partial^\mu \mathbf{y}^{\mathrm{ph}}\Big] + (\partial_\mu f_0^{L/H})\Big[(\partial^\mu \mathbf{y}^{\mathrm{ph}})^T \tilde{\mathbb{C}}^{L/H}\mathbf{y}^{\mathrm{ph}}\Big] \\ &+ f_0^{L/H}\Big[\pi^+ D_1^{L/H}\pi^- + (\partial_\mu\pi^+)D_2^{L/H}\partial^\mu\pi^-\Big] + (\partial_\mu f_0^{L/H})\Big[\pi^+ F_1^{L/H}\partial^\mu\pi^- + (\partial^\mu\pi^+)F_2^{L/H}\pi^-\Big], \end{aligned} \tag{C56}$$

where

$$\tilde{\mathbb{B}}_1^{L/H} \equiv \mathbb{O}_P^T \mathbb{B}_1^{L/H}\mathbb{O}_P, \qquad \tilde{\mathbb{B}}_2^{L/H} \equiv \mathbb{O}_P^T \mathbb{B}_2^{L/H}\mathbb{O}_P, \qquad \tilde{\mathbb{C}}^{L/H} \equiv \mathbb{O}_P^T \mathbb{C}^{L/H}\mathbb{O}_P, \tag{C57}$$

which finally leads to

$$\begin{aligned} \mathcal{L}_{f_0\pi\pi} = {}& f_0^{L/H}\Big[\Big(\tilde{\mathbb{B}}_1^{L/H}\Big)_{11}\pi^0\pi^0 + \Big(\tilde{\mathbb{B}}_2^{L/H}\Big)_{11}(\partial_\mu\pi^0)(\partial^\mu\pi^0)\Big] + (\partial_\mu f_0^{L/H})\Big[\tilde{\mathbb{C}}^{L/H}{}_{11}\pi^0\partial^\mu\pi^0\Big] \\ &+ f_0^{L/H}\Big[\pi^+ D_1^{L/H}\pi^- + (\partial_\mu\pi^+)D_2^{L/H}\partial^\mu\pi^-\Big] + (\partial_\mu f_0^{L/H})\Big[\pi^+ F_1^{L/H}\partial^\mu\pi^- + (\partial^\mu\pi^+)F_2^{L/H}\pi^-\Big]. \end{aligned} \tag{C58}$$

The tree-level decay widths for $f_0^{L/H}$ are given by

$$\begin{aligned} \Gamma_{f_0^{L/H}\to\pi^{0,+}\pi^{0,-}} &= \Gamma_{f_0^{L/H}\to\pi^0\pi^0} + \Gamma_{f_0^{L/H}\to\pi^+\pi^-} \\ &= \frac{1}{8\pi M_{f_0^{L/H}}^2}\Big\{2k_{f_0^{L/H}\to\pi^0\pi^0}\Big|\Big(\tilde{\mathbb{B}}_1^{L/H}\Big)_{11} + \frac{1}{2}\Big(\tilde{\mathbb{C}}^{L/H}{}_{11} - \Big(\tilde{\mathbb{B}}_2^{L/H}\Big)_{11}\Big)M_{f_0^{L/H}}^2 + \Big(\tilde{\mathbb{B}}_2^{L/H}\Big)_{11}M_{\pi^0}^2\Big|^2 \\ &\quad + k_{f_0^{L/H}\to\pi^\pm\pi^\mp}\Big|D_1^{L/H} + \frac{1}{2}\Big(F_1^{L/H} + F_2^{L/H} - D_2^{L/H}\Big)M_{f_0^{L/H}}^2 + D_2^{L/H}M_{\pi^\pm}^2\Big|^2\Big\}, \end{aligned} \tag{C59}$$

where $k_{A\to BC}$ is defined in Eq. (C16).

APPENDIX D: DETAILED FIT RESULTS AND PARAMETER VALUES

In this appendix, the best-fit results with and without the ω decay are presented (Table II) for the DS and for the DVS-I (see Sec. III for more details). For comparison, we have also listed the experimental values, which we have already discussed in detail in Sec. V. Note that the decays of the scalar-isoscalar f_0's are not fitted; the values in the tables are just calculated with the parameters obtained. The parameter sets for the fits are given in Table III.

TABLE II. Detailed fit results. Second column contains the experimental values (Expt. val), third, fifth, seventh, and ninth columns hold our fit results in the DS and in the DVS-I, and cases where the $\omega \to \pi\pi$ decay is fitted (ω) and where it is not (no-ω), while in the fourth, sixth, eighth, and tenth columns the χ^2 values for the given quantities are listed.

Observable	Expt. val. (MeV)	$\text{Fit}_{\text{DS},\omega}$ (MeV)	χ^2	$\text{Fit}_{\text{DS,no}-\omega}$ (MeV)	χ^2	$\text{Fit}_{\text{DVS-I},\omega}$ (MeV)	χ^2	$\text{Fit}_{\text{DVS-I,no}-\omega}$ (MeV)	χ^2
f_{π^+}	92.06 ± 4.60	96.78	1.1	96.72	1.0	96.13	0.8	96.61	1.0
f_{K^+}	110.10 ± 5.51	109.20	0.0	110.45	0.0	108.73	0.1	109.54	0.0
$\bar{M}_\pi$	138.04 ± 6.90	140.56	0.1	140.20	0.1	140.32	0.1	140.61	0.1
ΔM_π	-4.59 ± 0.92	−4.54	0.0	−4.56	0.0	−4.57	0.0	−4.55	0.0
M_η	547.86 ± 27.39	550.38	0.0	547.39	0.0	546.90	0.0	548.07	0.0
$M_{\eta'}$	957.78 ± 47.89	949.69	0.0	952.44	0.0	957.45	0.0	958.13	0.0
$\bar{M}_K$	495.64 ± 24.78	476.73	0.6	482.47	0.3	484.04	0.2	480.23	0.4
ΔM_K	3.93 ± 0.79	3.89	0.0	3.93	0.0	3.92	0.0	3.90	0.0
$\bar{M}_\rho$	775.16 ± 38.76	762.00	0.1	761.61	0.1	744.48	0.6	762.31	0.1
ΔM_ρ	0.15 ± 0.57	0.13	0.0	0.13	0.0	0.14	0.0	0.10	0.0
M_ω	782.66 ± 39.13	762.13	0.3	761.86	0.3	755.64	0.5	763.22	0.2
M_ϕ	1019.46 ± 50.97	979.66	0.6	986.41	0.4	998.68	0.2	982.42	0.5
$\bar{M}_{K^\star}$	895.50 ± 44.78	878.64	0.1	882.52	0.1	882.97	0.1	880.57	0.1
$\Delta M_{K^\star}$	0.08 ± 0.90	0.14	0.0	0.19	0.0	0.15	0.0	0.22	0.0
$\bar{M}_{a_1}$	1230.00 ± 246.00	1109.54	0.2	1115.71	0.2	1050.84	0.5	1115.80	0.2
$M_{f_1^L}$	1281.90 ± 256.38	1246.50	0.0	1222.40	0.1	1334.34	0.0	1233.96	0.0
$M_{f_1^H}$	1426.30 ± 285.26	1357.51	0.1	1367.72	0.0	1366.92	0.0	1363.18	0.0
$\bar{M}_{K_1}$	1253.00 ± 250.60	1256.58	0.0	1260.84	0.0	1256.35	0.0	1260.05	0.0
$\bar{M}_{a_0}$	1474.00 ± 294.80	1251.40	0.5	1140.38	1.0	1308.90	0.3	1187.00	0.8
$\bar{M}_{K_0^\star}$	1425.00 ± 285.00	1321.76	0.1	1237.23	0.4	1411.77	0.0	1282.16	0.2
$M_{f_0^L}$	1350.00 ± 675.00	1229.39	0.0	1136.72	0.1	1295.92	0.0	1187.92	0.0
$M_{f_0^H}$	1733.00 ± 866.50	1515.87	0.1	1326.58	0.2	1499.63	0.1	1375.34	0.1
$\bar{\Gamma}_{\rho\to\pi\pi}$	148.53 ± 7.43	154.81	0.7	154.85	0.7	153.63	0.5	155.85	1.0
$\Delta\Gamma_{\rho\to\pi\pi}$	-1.70 ± 1.60	−1.90	0.0	−1.89	0.0	−1.87	0.0	−1.78	0.0
$\bar{\Gamma}_{\omega\to\pi\pi}$	0.13 ± 0.03	0.00	25.0	0.0	…	0.00	25.0	0.0	…
$\bar{\Gamma}_{\phi\to\bar{K}K}$	1.76 ± 0.09	1.44	0.4	1.10	0.3	1.87	0.3	1.12	0.4
$\Delta\Gamma_{\phi\to\bar{K}K}$	-0.65 ± 0.13	−0.63	0.1	−0.58	0.2	−0.70	0.1	−0.58	0.1
$\bar{\Gamma}_{K^\star\to K\pi}$	46.75 ± 2.34	46.27	0.0	46.20	0.0	46.63	0.0	46.02	0.1
$\Delta\Gamma_{K^\star\to K\pi}$	1.10 ± 1.80	0.66	0.1	0.40	0.2	1.09	0.0	0.31	0.2
$\bar{\Gamma}_{a_1\to\rho\pi}$	425.00 ± 175.00	533.23	0.4	428.34	0.0	566.85	0.7	489.59	0.1
$\Gamma_{a_1\to\pi\gamma}$	0.64 ± 0.25	0.62	0.0	0.68	0.0	0.50	0.3	0.65	0.0
$\Gamma_{f_1^H\to K^\star K}$	43.60 ± 8.72	43.53	0.0	43.84	0.0	43.42	0.0	43.72	0.0
Γ_{a_0}	265.00 ± 53.00	238.25	0.3	239.03	0.2	263.73	0.0	259.45	0.0
$\Gamma_{K_0^\star\to K\pi}$	270.00 ± 80.00	336.80	0.7	333.96	0.6	256.33	0.0	308.70	0.2
$\Gamma_{f_0^L\to\pi\pi}$ (no fit)	250.00 ± 125.00	0.004	…	0.96	…	152.68	…	1.74	…
$\Gamma_{f_0^L\to KK}$ (no fit)	150.00 ± 100.00	114.17	…	86.74	…	0.40	…	95.34	…
$\Gamma_{f_0^H\to\pi\pi}$ (no fit)	20.20 ± 10.10	1000.0	…	443.19	…	720.07	…	493.93	…
$\Gamma_{f_0^H\to KK}$ (no fit)	87.70 ± 43.85	594.43	…	984.60	…	60.15	…	999.64	…

TABLE III. Parameter sets for four different cases. From left to right, DS with fitting the ω decay, DS without fitting the ω decay, DVS-I with fitting the ω decay, and DVS-I without fitting the ω decay.

Parameter	DS, ω	DS, no-ω	DVS-I, ω	DVS-I, no-ω
ϕ_N (MeV)	163.95	163.93	153.49	166.37
ϕ_S (MeV)	127.65	133.40	119.63	130.17
ϕ_3 (MeV)	2.62×10^{-2}	-4.72×10^{-3}	-3.25×10^{-3}	-1.50×10^{-2}
m_0^2 (MeV^2)	-9.91×10^5	-6.39×10^5	-8.13×10^5	-7.04×10^5
$\tilde{m}_1^2$ (MeV^2)	8.00×10^5	8.00×10^5	8.00×10^5	8.00×10^5
λ_1	5.73	0.09	−0.82	0.23
λ_2	55.91	44.79	72.55	50.68
h_1	−72.31	26.24	−30.98	26.23
h_2	16.76	23.82	−1.55	18.01
h_3	4.64	5.41	2.58	5.05
g_1	5.63	5.53	5.75	5.60
g_2	2.42	3.01	0.38	2.72
c_1 (MeV^{-2})	2.98×10^{-4}	2.68×10^{-4}	-2.76×10^{-5}	2.95×10^{-4}
$\tilde{\delta}_S$ (MeV^2)	1.59×10^5	1.46×10^5	2.16×10^5	1.56×10^5
δ_3 (MeV^2)	91.47	3.75	4.025×10^3	10.02
δm_V^2 (MeV^2)	30.77	-1.26×10^2	7.13×10^3	-6.41×10^2
δm_A^2 (MeV^2)	-2.45×10^5	-1.91×10^5	-5.01×10^5	-2.11×10^5
$m_{\mathrm{em},S}^2$ (MeV^2)	-9.95×10^3	9.93×10^3	-9.96×10^3	-9.65×10^3
$m_{\mathrm{em},P}^2$ (MeV^2)	-4.30×10^3	-3.69×10^3	-6.94×10^3	-3.89×10^3
$m_{\mathrm{em},V}^2$ (MeV^2)	-2.33×10^2	-3.20×10^2	-8.51×10^3	-7.95×10^2
$m_{\mathrm{em},A}^2$ (MeV^2)	9.70×10^3	9.91×10^3	8.42×10^3	9.59×10^3
$m_{\mathrm{em},K}^2$ (MeV^2)	⋯	⋯	3.99×10^3	4.54×10^2

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