Different versions of soft-photon theorems exemplified at leading and next-to-leading terms for pion-pion and pion-proton scattering

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We investigate the photon emission in pion-pion and pion-proton scattering in the soft-photon limit where the photon energy $\omega \to 0$. The expansions of the $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ and the $\pi^\pm p \to \pi^\pm p\gamma$ amplitudes, satisfying the energy-momentum relations, to the orders ω^{-1} and ω^0 are derived. We show that these terms can be expressed completely in terms of the on-shell amplitudes for $\pi^-\pi^0 \to \pi^-\pi^0$ and $\pi^\pm p \to \pi^\pm p$, respectively, and their partial derivatives with respect to s and t. The structure term which is nonsingular for $\omega \to 0$ is determined to the order ω^0 from the gauge-invariance constraint using the generalized Ward identities for pions and the proton. For the reaction $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ we discuss in detail the soft-photon theorems in the versions of both Low and Weinberg. We show that these two versions are different and must not be confounded. Weinberg's version gives the pole term of a Laurent expansion in ω of the amplitude for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ around the phase-space point of zero radiation. Low's version gives an approximate expression for the above amplitude at a fixed phase-space point, corresponding to nonzero radiation. Clearly, the leading and next-to-leading terms in theses two approaches must be, and are indeed, different. We show their relation. We also discuss the expansions of differential cross sections for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ with respect to ω for $\omega \to 0$.

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I. INTRODUCTION

In this article we shall discuss the production of soft photons in $\pi\pi$ and πp scattering. In particular, we shall study the following reactions:

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1) + \pi^0(p_2), \tag{1.1}$$

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon), \tag{1.2}$$

and

$$\pi^\pm(p_a) + p(p_b, \lambda_b) \to \pi^\pm(p_1) + p(p_2, \lambda_2), \tag{1.3}$$

$$\pi^\pm(p_a) + p(p_b, \lambda_b) \to \pi^\pm(p_1') + p(p_2', \lambda_2') + \gamma(k, \varepsilon). \tag{1.4}$$

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Here $p_a, p_b, p_1, p_2, p_1', p_2'$ and k are the momenta of the particles, $\lambda_b, \lambda_2, \lambda_2'$ are the spin indices of the protons, and ε is the polarization vector of the photon. Let $\omega = k^0$ be the photon energy in the overall c.m. system. We are interested in soft-photon production, $\omega \to 0$.

In a seminal paper Low [1] derived the theorem that the leading term for $\omega \to 0$ in the soft-photon-production amplitudes comes from the emission of photons from the external particles of the reaction. In [1] this was shown explicitly for the scattering of a charged scalar on an uncharged scalar particle, that is, for a reaction like (1.2), and for the scattering of a charged spin $1/2$ fermion on a neutral scalar boson. In [2,3] a soft-photon theorem was derived for general reactions with an arbitrary number of external particles. In the following soft-photon production was studied by many authors; see, e.g., [4–19].

In [1] also an expression for a next-to-leading term, of order ω^0, is given for the scattering of scalars. In our study of soft-photon production in $\pi^-\pi^0$ scattering we recalculated the next-to-leading term and found a different result [20]. In the present paper, we reconsider the leading and next-to-leading terms in the soft-photon expansion of (1.2). We shall show that Low's version [1] and Weinberg's version [3] of the soft-photon theorem are different and should not be confused. We also shall show how they are related. In this way, we shall also give the clear reason why there must be a difference between the formulas for the next-to-leading terms in Low's paper [1] and in our paper [20] and how the two results are related. These results have already been presented in a short form at a conference in October 2023; see [21]. There, also our misunderstanding of the results of [1] which led us to an error in [20] was corrected. This will be discussed in detail below.

Our paper is organized as follows. In Sec. II we discuss the phase space and the kinematics for the reactions (1.1) and (1.2). Section III deals with the reactions (1.1) and (1.2) from a general point of view. In Sec. IV we recall the expansion of the amplitude for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ around the phase-space point $p_1, p_2, k = 0$ as presented in [20], where the leading term is precisely given by the soft-photon theorem of [3]. Section V deals with Low's version of the soft-photon theorem [1] and its relation to the results of [3,20]. In Sec. VI we discuss cross sections for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$. In Sec. VII we give an outline of the calculation for the leading and next-to-leading terms of the $\pi^\pm p \to \pi^\pm p\gamma$ reactions (1.4). Section VIII contains a summary and our conclusions. Some details of our analysis are given in Appendixes A and B.

In our paper we use the following theoretical framework for the calculations. We consider the reactions (1.1)–(1.4) in QCD plus leading order in electromagnetism. We use only exact quantum field theory (QFT) methods in this framework:

(i) energy-momentum conservation,
(ii) gauge invariance,
(iii) parity (P), charge conjugation (C), and time-reversal (T) invariance,
(iv) the generalized Ward identity for the pion and the proton fields, which in QCD are composite local fields,
(v) analyticity properties of amplitudes, in particular the Landau conditions.

It turned out that the evaluation of the ω^0 term for the amplitude of (1.4) involved a lengthy and complex analysis. Therefore, we present in this paper only the basic ingredients of the calculation and the results. All details can be found in Ref. [22].

II. KINEMATICS AND PHASE SPACE FOR $\pi\pi \to \pi\pi$ AND $\pi\pi \to \pi\pi\gamma$

Let us start with the elastic reaction

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1) + \pi^0(p_2),$$
$$p_a + p_b = p_1 + p_2. \tag{2.1}$$

We set as usual

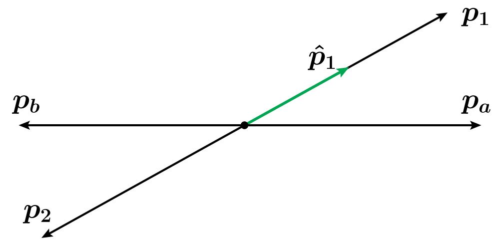


FIG. 1. The reaction $\pi^-\pi^0 \to \pi^-\pi^0$ (2.1) in the c.m. system.

$$s = (p_a + p_b)^2 = (p_1 + p_2)^2,$$
$$t = (p_a - p_1)^2 = (p_b - p_2)^2. \tag{2.2}$$

Let us look at the reaction (2.1) in the c.m. system and consider a given value of the c.m. energy squared s. Then the energies and absolute values of the momenta are fixed,

$$p_a^0 = p_b^0 = p_1^0 = p_2^0 = \frac{\sqrt{s}}{2},$$
$$|\boldsymbol{p_a}| = |\boldsymbol{p_b}| = |\boldsymbol{p_1}| = |\boldsymbol{p_2}| = \sqrt{\frac{s}{4} - m_\pi^2}. \tag{2.3}$$

For a given initial configuration we can vary only $\hat{\boldsymbol{p}}_{\boldsymbol{1}} = \boldsymbol{p_1}/|\boldsymbol{p_1}|$, the unit vector in direction of $\boldsymbol{p_1}$; see Fig. 1. The phase space is the unit sphere.

Now we go to the reaction with photon radiation,

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1') + \pi^0(p_2') + \gamma(k, \varepsilon),$$
$$p_a + p_b = p_1' + p_2' + k. \tag{2.4}$$

Here we set

$$s = (p_a + p_b)^2 = (p_1' + p_2' + k)^2,$$
$$t_1 = (p_a - p_1')^2 = (p_b - p_2' - k)^2,$$
$$t_2 = (p_b - p_2')^2 = (p_a - p_1' - k)^2. \tag{2.5}$$

We shall consider real and virtual photon emission and require

$$k^2 \geq 0, \qquad k^0 \geq 0, \tag{2.6}$$

and k small, say $|k^\mu| \ll \sqrt{s - 4m_\pi^2}$ $(\mu = 0, \ldots, 3)$ in the c.m. system. We consider a given value of s and ask what are the free parameters of the reaction (2.4). In the c.m. system (Fig. 2) a convenient set of such parameters is given by the four-vector k plus the unit vector $\hat{\boldsymbol{p}}_{\boldsymbol{1}}' = \boldsymbol{p}_{\boldsymbol{1}}'/|\boldsymbol{p}_{\boldsymbol{1}}'|$:

$$\text{Phase space of } (2.4) = \{(k, \hat{\boldsymbol{p}}_{\boldsymbol{1}}'), \quad k \in \text{part of } \mathrm{R}_4, |\hat{\boldsymbol{p}}_{\boldsymbol{1}}'| = 1\}. \tag{2.7}$$

We can easily see this by considering the reaction (2.4) for given k in the rest system of the four-vector

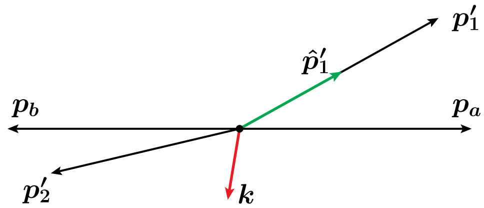


FIG. 2. The reaction $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ (2.4) in the c.m. system.

$p_a + p_b - k$. In this system $\boldsymbol{p}'_1$ and $\boldsymbol{p}'_2$ are back to back with fixed $|\boldsymbol{p}'_1|$ and $|\boldsymbol{p}'_2|$. The only freedom left is to vary $\boldsymbol{p}'_1$ in any direction. The same is then also true in the c.m. system if k is small enough.

III. GENERAL ANALYSIS OF $\pi\pi \to \pi\pi$ AND $\pi\pi \to \pi\pi\gamma$

We consider first the reaction

$$\pi^-(\tilde{p}_a) + \pi^0(\tilde{p}_b) \to \pi^-(\tilde{p}_1) + \pi^0(\tilde{p}_2) \tag{3.1}$$

off shell and on shell. We have always energy-momentum conservation

$$\tilde{p}_a + \tilde{p}_b = \tilde{p}_1 + \tilde{p}_2. \tag{3.2}$$

In relations which hold off shell and on shell we denote momenta with a tilde. The diagram for (3.1) is shown in Fig. 3.

As kinematic variables we have the masses of the, in general off-shell, pions, an energy variable, and a t variable:

$$\begin{aligned} &\tilde{\nu} = \tilde{p}_a \cdot \tilde{p}_b + \tilde{p}_1 \cdot \tilde{p}_2, \\ &\tilde{t} = (\tilde{p}_a - \tilde{p}_1)^2 = (\tilde{p}_b - \tilde{p}_2)^2, \\ &m_a^2 = \tilde{p}_a^2, \quad m_b^2 = \tilde{p}_b^2, \quad m_1^2 = \tilde{p}_1^2, \quad m_2^2 = \tilde{p}_2^2. \end{aligned} \tag{3.3}$$

We use here, following Low, $\tilde{\nu}$ as energy variable. For the Mandelstam $\tilde{s}$ variable we get

$$\tilde{s} = (\tilde{p}_a + \tilde{p}_b)^2 = \tilde{\nu} + \frac{1}{2}(m_a^2 + m_b^2 + m_1^2 + m_2^2). \tag{3.4}$$

The scattering amplitude for (3.1) can only depend on the variables (3.3):

$$\mathcal{T}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_a, \tilde{p}_b) = \mathcal{M}(\tilde{\nu}, \tilde{t}, m_a^2, m_b^2, m_1^2, m_2^2). \tag{3.5}$$

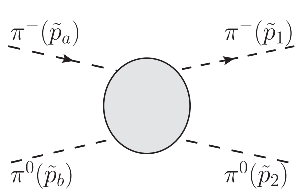


FIG. 3. Diagram for the off-shell and on-shell reaction (3.1).

The on-shell amplitude is obtained setting

$$\begin{aligned} &\tilde{p}_a \to p_a, \quad \tilde{p}_b \to p_b, \quad \tilde{p}_1 \to p_1, \quad \tilde{p}_2 \to p_2, \\ &m_a^2 = m_b^2 = m_1^2 = m_2^2 = m_\pi^2, \\ &\tilde{\nu} \to \nu, \quad \tilde{t} \to t, \end{aligned} \tag{3.6}$$

and we get

$$\begin{aligned} \mathcal{T}(p_1, p_2, p_a, p_b)|_{\text{on shell}} &= \mathcal{M}(\nu, t, m_\pi^2, m_\pi^2, m_\pi^2, m_\pi^2) \\ &\equiv \mathcal{M}^{(\text{on})}(\nu, t). \end{aligned} \tag{3.7}$$

Next we come to the photon-emission reaction (1.2):

$$\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p'_1) + \pi^0(p'_2) + \gamma(k, \varepsilon), \tag{3.8}$$

where energy-momentum conservation reads

$$p_a + p_b = p'_1 + p'_2 + k. \tag{3.9}$$

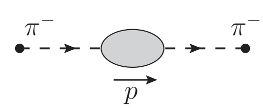
Note that for four-vector $k \neq 0$ we *must have* $p'_1 \neq p_1$, $p'_2 \neq p_2$ with p_1, p_2 from (2.1). The amplitude for (3.8) is

$$\begin{aligned} &\langle \pi^-(p'_1), \pi^0(p'_2), \gamma(k, \varepsilon)|\mathcal{T}|\pi^-(p_a), \pi^0(p_b)\rangle \\ &\quad = (\varepsilon^\lambda)^* \mathcal{M}_\lambda(p'_1, p'_2, k, p_a, p_b). \end{aligned} \tag{3.10}$$

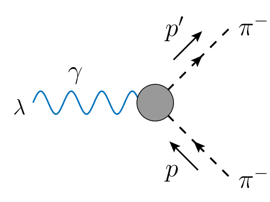
In the following we consider $\mathcal{M}_\lambda$ for real and also for virtual, timelike, photons, that is, for (2.6).

There are three diagrams for the reaction (3.8) as shown in Fig. 4, two one-particle reducible ones, (a) and (b), and one which is one-particle irreducible (c).

To calculate the diagrams (a) and (b) we need the off-shell $\pi\pi \to \pi\pi$ amplitude which we have already introduced, the pion propagator $\Delta_F(p^2)$, and the pion-photon vertex function $\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p', p)$:



$$i\Delta_F(p^2)\,, \tag{3.11}$$



$$ie\widehat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p', p)\,. \tag{3.12}$$

We denote by $e = \sqrt{4\pi\alpha_{\text{em}}} > 0$ the π^+ charge.

We get

$$\mathcal{M}_\lambda = \mathcal{M}_\lambda^{(a)} + \mathcal{M}_\lambda^{(b)} + \mathcal{M}_\lambda^{(c)}, \tag{3.13}$$

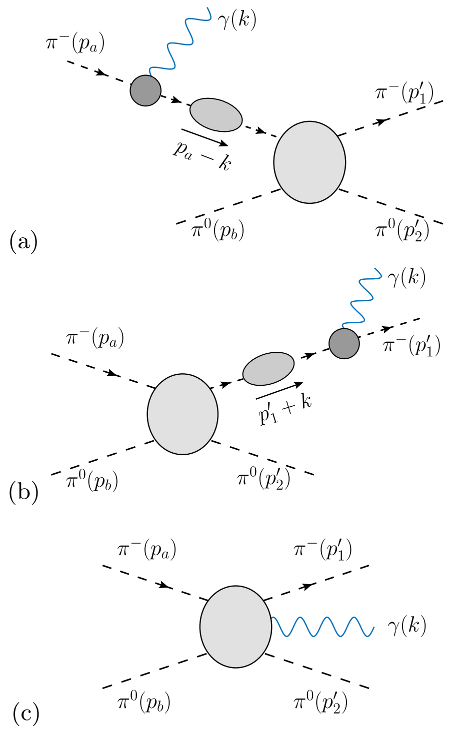


FIG. 4. Diagrams for the reaction $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ (3.8). The diagrams (a) and (b) describe the photon emission from the external charged lines, the diagram (c) corresponds to the photon emission from internal lines, the structure term. The blobs in (a) and (b) stand for the full pion propagator, the full $\gamma\pi\pi$ vertex function, and the off-shell $\pi\pi$ scattering amplitude.

where

$$\begin{aligned}
\mathcal{M}_\lambda^{(a)} &= -e\mathcal{M}^{(a)}\Delta_F[(p_a-k)^2]\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p_a-k,p_a),\\
\mathcal{M}^{(a)} &= \mathcal{T}(p_1',p_2',p_a-k,p_b)|_{\text{off shell}}\\
&= \mathcal{M}[(p_a-k,p_b)+p_1'\cdot p_2',\\
&\quad (p_b-p_2')^2,(p_a-k)^2,m_\pi^2,m_\pi^2,m_\pi^2],
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\mathcal{M}_\lambda^{(b)} &= -e\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p_1',p_1'+k)\Delta_F[(p_1'+k)^2]\mathcal{M}^{(b)},\\
\mathcal{M}^{(b)} &= \mathcal{T}(p_1'+k,p_2',p_a,p_b)|_{\text{off shell}}\\
&= \mathcal{M}[(p_a\cdot p_b)+(p_1'+k,p_2'),\\
&\quad (p_b-p_2')^2,m_\pi^2,m_\pi^2,(p_1'+k)^2,m_\pi^2].
\end{aligned} \tag{3.15}$$

We shall now use one of the best tools from QFT which we have: *gauge invariance*. This gives us the generalized Ward identity [23,24]

$$(p'-p)^\lambda\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p',p) = \Delta_F^{-1}(p'^2) - \Delta_F^{-1}(p^2) \tag{3.16}$$

and the condition

$$k^\lambda\mathcal{M}_\lambda = k^\lambda(\mathcal{M}_\lambda^{(a)}+\mathcal{M}_\lambda^{(b)}+\mathcal{M}_\lambda^{(c)}) = 0. \tag{3.17}$$

From these two conditions we get an exact relation between $\mathcal{M}_\lambda^{(c)}$, and $\mathcal{M}^{(a)}$, $\mathcal{M}^{(b)}$:

$$k^\lambda\mathcal{M}_\lambda^{(c)} = -e\mathcal{M}^{(a)} + e\mathcal{M}^{(b)}; \tag{3.18}$$

cf. (2.22) of [20]. These are the QFT relations which we shall use in the following.

IV. SOFT PHOTON THEOREM I

In this section we shall give the expansion of the amplitude $\mathcal{M}_\lambda$ around the phase-space point $(k=0, \hat{\boldsymbol{p}}_1' = \hat{\boldsymbol{p}}_1)$. In a small neighborhood of this phase-space point we set, assuming $|\boldsymbol{l}_{1\perp}| = \mathcal{O}(\omega)$,

$$\hat{\boldsymbol{p}}_1' = \hat{\boldsymbol{p}}_1 - \frac{\boldsymbol{l}_{1\perp}}{|\boldsymbol{p}_1|}, \qquad \boldsymbol{l}_{1\perp}\cdot\hat{\boldsymbol{p}}_1 = 0 + \mathcal{O}(\omega^2). \tag{4.1}$$

This neighborhood has six dimensions, schematically we represent it as shown in Fig. 5.

For $(k=0, \boldsymbol{l}_{1\perp}=0)$ we have the kinematics of $\pi^-\pi^0 \to \pi^-\pi^0$, the reaction without radiation (2.1). For $(k, \boldsymbol{l}_{1\perp})$ we have

$$p_1' = p_1 - l_1, \qquad p_2' = p_2 - l_2, \tag{4.2}$$

where in the c.m. frame we get after a simple calculation, up to order ω, the following result. We have in the overall c.m. system of (2.1) and (2.4) with $\hat{\boldsymbol{p}}_1 = \boldsymbol{p}_1/|\boldsymbol{p}_1|$,

$$\begin{aligned}
(p_1^\mu) &= \begin{pmatrix} p_1^0 \\ |\boldsymbol{p}_1|\hat{\boldsymbol{p}}_1 \end{pmatrix}, \qquad (p_2^\mu) = \begin{pmatrix} p_2^0 \\ -|\boldsymbol{p}_2|\hat{\boldsymbol{p}}_1 \end{pmatrix},\\
(k^\mu) &= \begin{pmatrix} k^0 \\ k_\parallel\hat{\boldsymbol{p}}_1 + \boldsymbol{k}_\perp \end{pmatrix}, \qquad \boldsymbol{k}_\perp\cdot\hat{\boldsymbol{p}}_1 = 0,
\end{aligned} \tag{4.3}$$

where $k^0 = \omega$, and $p_{1,2}^0$, $|\boldsymbol{p}_{1,2}|$ are as in (2.3). We find then [see (3.21) of [20]]

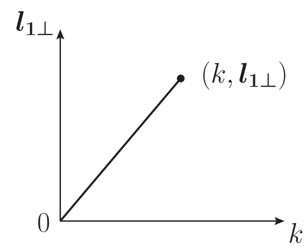


FIG. 5. Schematic representation of the six-dimensional neighborhood of the phase-space point $(k=0, \hat{\boldsymbol{p}}_1' = \hat{\boldsymbol{p}}_1)$; see (4.1).

$$(l_1^\mu) = \begin{pmatrix} \frac{1}{\sqrt{s}}(p_2 \cdot k) \\ \frac{p_1^0}{|\boldsymbol{p_1}|\sqrt{s}}(p_2 \cdot k)\hat{\boldsymbol{p}}_1 + \boldsymbol{l}_{1\perp} \end{pmatrix},$$

$$(l_2^\mu) = \begin{pmatrix} \frac{1}{\sqrt{s}}(p_1 \cdot k) \\ \boldsymbol{k} - \frac{p_1^0}{|\boldsymbol{p_1}|\sqrt{s}}(p_2 \cdot k)\hat{\boldsymbol{p}}_1 - \boldsymbol{l}_{1\perp} \end{pmatrix}. \tag{4.4}$$

We have written (4.4) in such a way that it holds, of course, for $\pi\pi$ scattering, inserting for s, $p_{1,2}$, $|\boldsymbol{p_1}|$, and $\hat{\boldsymbol{p}}_1 = \boldsymbol{p_1}/|\boldsymbol{p_1}|$ the expressions from (2.2), (2.3), and (4.3). We shall see in Sec. VII that for the πp scattering case the analogous quantities $l_{1,2}$ are again given by (4.4) with the appropriate expressions for s, $p_{1,2}$, $|\boldsymbol{p_1}|$, and $\hat{\boldsymbol{p}}_1$ inserted; see Eq. (7.11) and the discussion following it.

We illustrate the situation in the overall c.m. system in Fig. 6.

We have the important relations (up to order ω)

$$l_1 + l_2 = k,$$
$$p_1 \cdot l_1 = p_2 \cdot l_2 = 0. \tag{4.5}$$

Now we come to the expansion of the amplitude $\mathcal{M}_\lambda$ for $\omega \to 0$. To be precise we set, in the c.m. system,

$$(k^\mu) = \omega \begin{pmatrix} 1 \\ \tilde{\boldsymbol{k}} \end{pmatrix}, \qquad \omega \geq 0, \qquad \tilde{\boldsymbol{k}}^2 \leq 1. \tag{4.6}$$

In this way we have always

$$k^2 = \omega^2(1 - \tilde{\boldsymbol{k}}^2) \geq 0. \tag{4.7}$$

Furthermore we set

$$\boldsymbol{l}_{1\perp} = \omega \tilde{\boldsymbol{l}}_{1\perp}, \qquad |\tilde{\boldsymbol{l}}_{1\perp}| = \mathcal{O}(1). \tag{4.8}$$

We keep $\tilde{\boldsymbol{k}}$ and $\tilde{\boldsymbol{l}}_{1\perp}$ fixed and consider the expansion of the radiative amplitude for $\omega \to 0$. That is, we consider $\mathcal{M}_\lambda$ on a line starting from the origin in the phase space shown schematically in Fig. 5. Of course, this will be a *Laurent expansion*.

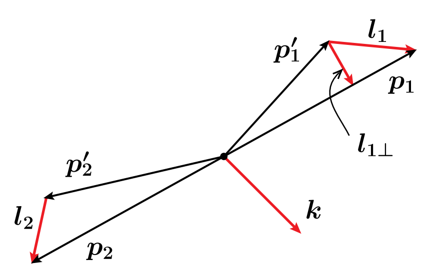


FIG. 6. The final states of (2.1) and (2.4) for $|\boldsymbol{l}_{1\perp}| = \mathcal{O}(\omega)$ in the c.m. system.

Note that when constructing this expansion we have to count $k^2 = \mathcal{O}(\omega^2)$ due to (4.7). In theoretical considerations we certainly can consider k^μ (4.6) and k^2 (4.7) for fixed $\tilde{\boldsymbol{k}}$ and $\omega \to 0$. If we want to realize $k^2 > 0$ in nature by virtual photon γ^* production with $\gamma^* \to e^+e^-$ we have, of course, the limit

$$\omega \geq 2m_e,$$
$$k^2 = \omega^2(1 - \tilde{\boldsymbol{k}}^2) \geq 4m_e^2, \tag{4.9}$$

where m_e is the electron mass. But the electron mass is very small on a hadronic scale. Thus, in γ^* production with the decay $\gamma^* \to e^+e^-$ we can reach very low values of ω and k^2 where the soft-photon expansion has a good chance to be valid.

Now we illustrate the construction of the Laurent expansion for the term $\mathcal{M}_\lambda^{(a)}$; see Fig. 4(a) and (3.14). We have

$$\mathcal{M}_\lambda^{(a)} = -e\mathcal{T}(p_1', p_2', p_a - k, p_b)|_{\text{off shell}} \times \Delta_F[(p_a - k)^2]\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p_a - k, p_a). \tag{4.10}$$

Note that our off-shell amplitude satisfies energy-momentum conservation:

$$p_a - k + p_b = p_1' + p_2'. \tag{4.11}$$

From the generalized Ward identity (3.16) we find for $\omega \to 0$ [see (3.13) of [20]]

$$\Delta_F[(p_a - k)^2]\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}(p_a - k, p_a) = \frac{(2p_a - k)_\lambda}{-2p_a \cdot k + k^2 + i\varepsilon} + \mathcal{O}(\omega). \tag{4.12}$$

This shows that in order to get the expansion for $\mathcal{M}_\lambda^{(a)}$ to the orders ω^{-1} and ω^0 we have to calculate the expansion of

$$\begin{aligned}\mathcal{T}(p_1', p_2', p_a - k, p_b)|_{\text{off shell}} &= \mathcal{T}(p_1 - l_1, p_2 - l_2, p_a - k, p_b)|_{\text{off shell}} \\ &= \mathcal{M}[(p_a - k, p_b) + (p_1 - l_1, p_2 - l_2), \\ &\quad (p_b - p_2 + l_2)^2, (p_a - k)^2, m_\pi^2, m_\pi^2, m_\pi^2]\end{aligned} \tag{4.13}$$

to the orders ω^0 and ω. Note that, of course, l_1 and l_2 have to be taken into account. Treating in this way $\mathcal{M}_\lambda^{(a)}$ and $\mathcal{M}_\lambda^{(b)}$ and determining $\mathcal{M}_\lambda^{(c)}$ to the required order in ω from the gauge invariance condition (3.18) we get for the case of real photon emission, $k^2 = 0$, the following [see (3.27) of [20]], where we neglect gauge terms $\propto k_\lambda$:

$$\begin{aligned}\mathcal{M}_\lambda(p_1', p_2', k, p_a, p_b) = e\bigg\{&\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p'_{1\lambda}}{p'_1\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu,t) - 2\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu,t)\\ &-2\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\left[(p_a-p_1,k) - p_a\cdot l_1\right]\frac{\partial}{\partial t}\mathcal{M}^{(\mathrm{on})}(\nu,t)\bigg\} + \mathcal{O}(\omega),\\ \nu = s - 2m_\pi^2,\qquad & t = (p_a-p_1)^2 = (p_b-p_2)^2. \end{aligned} \tag{4.14}$$

Here $\mathcal{M}^{(\mathrm{on})}(\nu, t)$ is the on-shell $\pi^-\pi^0 \to \pi^-\pi^0$ amplitude (3.7). We can, for consistency, still expand (cf. Appendix A)

$$\begin{aligned}\frac{p'_{1\lambda}}{p'_1\cdot k} &= \frac{(p_1-l_1)_\lambda}{(p_1-l_1,k)}\\ &= \frac{p_{1\lambda}}{p_1\cdot k} + \frac{1}{(p_1\cdot k)^2}\left[p_{1\lambda}(l_1\cdot k) - l_{1\lambda}(p_1\cdot k)\right] + \mathcal{O}(\omega).\end{aligned} \tag{4.15}$$

In this way we get [see (A1) of [20]]

$$\begin{aligned}\mathcal{M}_\lambda(p_1', p_2', k, p_a, p_b) = e\bigg\{&\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu,t) - \frac{1}{(p_1\cdot k)^2}\left[p_{1\lambda}(l_1\cdot k) - l_{1\lambda}(p_1\cdot k)\right]\mathcal{M}^{(\mathrm{on})}(\nu,t)\\ &-2\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu,t) - 2\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\left[(p_a-p_1,k) - p_a\cdot l_1\right]\\ &\times\frac{\partial}{\partial t}\mathcal{M}^{(\mathrm{on})}(\nu,t)\bigg\} + \mathcal{O}(\omega).\end{aligned} \tag{4.16}$$

We leave it to the readers to insert in (4.16) l_1, l_2 and k from (4.4)–(4.8) and to convince themselves that in this way we have given the terms of order ω^{-1} and ω^0 in the Laurent expansion of the amplitude $\mathcal{M}_\lambda$ for the reaction (3.8) for $\omega \to 0$.

The first term on the rhs of Eq. (4.16) is the pole term $\propto \omega^{-1}$ and this is exactly Weinberg's soft-photon term. He writes in [3]: "Hence the effect of attaching one soft-photon line to an arbitrary diagram is simply to supply an extra factor,

$$\sum_n e_n\eta_n\frac{p_n^\mu}{p_n\cdot q - i\eta_n\varepsilon}, \tag{4.17}$$

the sum running over all external lines in the original diagram." Here q is the photon four-momentum and $\eta_n = +1$ for an outgoing charged particle, $\eta_n = -1$ for an incoming charged particle. In our work, [20] and (4.16) here, we have given the *next-to-leading term to Weinberg's pole term* for the reaction (3.8).

V. SOFT PHOTON THEOREM II

Now we want to discuss Low's version of soft-photon theorem [1]. Of course, as a starting point he considers again the diagrams (a), (b), and (c) of Fig. 4 for $\mathcal{M}_\lambda$. He also uses the generalized Ward identity which gave us (4.12) for $\Delta_F\hat{\Gamma}_\lambda^{(\gamma\pi\pi)}$. Considering only real photon emission we have then for the term $\mathcal{M}_\lambda^{(a)}$ [see Eq. (2.11) of [1] but note that a different metric convention is used there]

$$\begin{aligned}\mathcal{M}_\lambda^{(a)}(p_1', p_2', k, p_a, p_b) = e\mathcal{M}[&(p_a-k,p_b) + p_1'\cdot p_2',\\ &(p_b-p_2')^2, m_a^2 = (p_a-k)^2, m_\pi^2, m_\pi^2, m_\pi^2]\frac{p_{a\lambda}}{p_a\cdot k}.\end{aligned} \tag{5.1}$$

Now Low expands $\mathcal{M}$ with respect to k keeping p_1' and p_2' fixed. That is, he only expands with respect to k which is *explicit* in the parametrization chosen. In this way we get

$$\begin{aligned}\mathcal{M}_\lambda^{(a)}(p_1', p_2', k, p_a, p_b) = e\frac{p_{a\lambda}}{p_a\cdot k}\bigg\{&\mathcal{M}^{(\mathrm{on})}[p_a\cdot p_b + p_1'\cdot p_2', (p_b-p_2')^2]\\ &-(p_b\cdot k)\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}[p_a\cdot p_b + p_1'\cdot p_2', (p_b-p_2')^2]\\ &-2(p_a\cdot k)\frac{\partial}{\partial m_a^2}\mathcal{M}[p_a\cdot p_b + p_1'\cdot p_2', (p_b-p_2')^2, m_a^2, m_\pi^2, m_\pi^2, m_\pi^2]\big|_{m_a^2=m_\pi^2}\bigg\} + \mathcal{O}(k).\end{aligned} \tag{5.2}$$

Note an important point: Whereas the expansion of the scalar function $\mathcal{M}$ on the rhs of (5.1) with respect to k, keeping p_1' and p_2' fixed, is completely standard, this is *not the case* for $\mathcal{M}_\lambda^{(a)}$. Keeping in $\mathcal{M}_\lambda^{(a)}(p_1', p_2', k, p_a, p_b) p_a$, p_b, p_1', and p_2' fixed and expanding in k, in the usual sense with varying k, we go *outside* the physical region where we must have $p_a + p_b = p_1' + p_2' + k$ (3.9). Thus, the rhs of (5.2) should, in our opinion, be considered as giving an approximate expression for $\mathcal{M}_\lambda^{(a)}$ making sense *only* for the physical value of k satisfying (3.9).

Now we can treat $\mathcal{M}_\lambda^{(b)}$ in a similar way and then determine $\mathcal{M}_\lambda^{(c)}$ approximately from the gauge-invariance condition (3.18). The result is Low's formula [see (1.7) of [1]]

$$\begin{aligned}\mathcal{M}_\lambda(p_1', p_2', k, p_a, p_b) = e\bigg\{&\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}'}{p_1'\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu_L, t_2)\\ &-\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} + p_{1\lambda}'\frac{p_2'\cdot k}{p_1'\cdot k}\right.\\ &\left.- p_{b\lambda} - p_{2\lambda}'\right]\frac{\partial}{\partial\nu_L}\mathcal{M}^{(\mathrm{on})}(\nu_L, t_2)\bigg\}\\ &+\mathcal{O}(k), \end{aligned} \tag{5.3}$$

where

$$\begin{aligned}\nu_L &= p_a\cdot p_b + p_1'\cdot p_2' = \nu - (p_a + p_b, k),\\ t_2 &= (p_b - p_2')^2 = (p_a - p_1' - k)^2. \end{aligned}\tag{5.4}$$

Note that the amplitude $\mathcal{M}^{(\mathrm{on})}$ is evaluated at values of the momenta appropriate to the radiative process.

Again we emphasize that (5.3) is *not* the expansion of $\mathcal{M}_\lambda$ *around some phase-space point*. The rhs of (5.3) gives an approximate expression for $\mathcal{M}_\lambda$ at a *given phase-space point* p_1', p_2', k. Also, the leading approximation in (5.3) does *not* give what is frequently called Low's theorem, but really is Weinberg's version of the soft-photon theorem; see (4.17). We see this best by considering the reactions

$$\pi^-(p_a) + \pi^+(p_b) \to \pi^-(p_1) + \pi^+(p_2), \tag{5.5}$$

and

$$\pi^-(p_a) + \pi^+(p_b) \to \pi^-(p_1') + \pi^+(p_2') + \gamma(k, \varepsilon). \tag{5.6}$$

The leading term according to Low for (5.6) is

$$\begin{aligned}&\mathcal{M}_\lambda(p_1', p_2', k, p_a, p_b)\\ &= e\bigg\{\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}'}{p_1'\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu_L, t_2)\\ &\quad+\left[-\frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}'}{p_2'\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu_L, t_1)\bigg\} + \mathcal{O}(\omega^0), \end{aligned}\tag{5.7}$$

where t_1 and t_2 are defined in (2.5). According to Weinberg, see (4.17), we have, on the other hand,

$$\begin{aligned}&\mathcal{M}_\lambda(p_1', p_2', k, p_a, p_b)\\ &= e\bigg\{\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k} - \frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}}{p_2\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu, t)\bigg\}\\ &\quad+\mathcal{O}(\omega^0). \end{aligned}\tag{5.8}$$

In (5.7) we have an approximate expression for $\mathcal{M}_\lambda$ valid at the given phase-space point p_1', p_2', k. In (5.8) we have the pole term of the Laurent expansion of $\mathcal{M}_\lambda$ around the phase-space point $p_1' = p_1$, $p_2' = p_2$, $k = 0$.

Let us go back to the $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ case. In (5.3) we have Low's formula which gives us an approximate expression for $\mathcal{M}_\lambda$ at a given phase-space point. We can construct, as we did in Sec. IV, the corresponding expansion of this approximate expression around the phase-space point $(k = 0, \hat{\boldsymbol{p}}_1)$. Inserting in (5.3) $p_1' = p_1 - l_1$, $p_2' = p_2 - l_2$ from (4.2) we get

$$\begin{aligned}\mathcal{M}^{(\mathrm{on})}(\nu_L, t_2) &= \mathcal{M}^{(\mathrm{on})}[\nu - (p_a + p_b, k),\\ &\qquad t - 2((p_a - p_1, k) - p_a\cdot l_1)] + \mathcal{O}(\omega^2)\\ &= \mathcal{M}^{(\mathrm{on})}(\nu, t) - (p_a + p_b, k)\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu, t)\\ &\quad - 2((p_a - p_1, k) - p_a\cdot l_1)\frac{\partial}{\partial t}\mathcal{M}^{(\mathrm{on})}(\nu, t)\\ &\quad + \mathcal{O}(\omega^2). \end{aligned}\tag{5.9}$$

Furthermore, we find

$$\begin{aligned}&-\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}'}{p_1'\cdot k}\right](p_a + p_b, k)\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu, t)\\ &\quad-\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} + p_{1\lambda}'\frac{p_2'\cdot k}{p_1'\cdot k} - p_{b\lambda} - p_{2\lambda}'\right]\frac{\partial}{\partial\nu_L}\mathcal{M}^{(\mathrm{on})}(\nu_L, t_2)\\ &= -2\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu, t) + \mathcal{O}(\omega). \end{aligned}\tag{5.10}$$

From (5.3), (5.9), and (5.10) we get the result for $\mathcal{M}_\lambda$ identical to our result (4.14). We can go on to (4.16) using (4.15). In this way we have given the relation between Low's theorem (5.3) and the Laurent series (4.16) where the pole term $\propto \omega^{-1}$ is given by Weinberg's soft-photon theorem and the next-to-leading term $\propto \omega^0$ by our calculation, (3.27) and (A1) of [20].

VI. CROSS SECTION FOR $\pi^-\pi^0 \to \pi^-\pi^0\gamma$

Here we consider the cross section for $\pi^-\pi^0$ scattering with real photon emission and summed over the photon polarizations. We get

$$\begin{aligned}
&d\sigma(\pi^-(p_a)+\pi^0(p_b)\to\pi^-(p'_1)+\pi^0(p'_2)+\gamma(k))\\
&=\frac{1}{2\sqrt{s(s-4m_\pi^2)}}\frac{d^3k}{(2\pi)^3 2k^0}\frac{d^3p'_1}{(2\pi)^3 2p'^0_1}\frac{d^3p'_2}{(2\pi)^3 2p'^0_2}\\
&\quad\times(2\pi)^4\delta^{(4)}(p'_1+p'_2+k-p_a-p_b)\\
&\quad\times(-1)\mathcal{M}_\lambda(p'_1,p'_2,k,p_a,p_b)\mathcal{M}^{\lambda*}(p'_1,p'_2,k,p_a,p_b)
\end{aligned} \tag{6.1}$$

with $\mathcal{M}_\lambda$ from (3.10). We are interested here in small ω, where we found that the phase space can be parametrized by $(k,\hat{\boldsymbol{p}}'_1)$; see (2.7). Here we have for real photons, of course,

$$(k^\mu)=\omega\begin{pmatrix}1\\ \hat{\boldsymbol{k}}\end{pmatrix}, \qquad |\hat{\boldsymbol{k}}|=1. \tag{6.2}$$

We can, as well, choose $(k,\hat{\boldsymbol{p}}'_2)$ as phase-space variables. Below we shall discuss the following cross sections:

$$\sigma_1=\frac{\omega d\sigma}{d\omega d\Omega_{\hat{k}}d\Omega_{\hat{p}'_1}}, \qquad \sigma_2=\frac{\omega d\sigma}{d\omega d\Omega_{\hat{k}}d\Omega_{\hat{p}'_2}}, \qquad \sigma_3=\frac{\omega d\sigma}{d\omega}, \tag{6.3}$$

where $d\Omega_{\hat{k}}$, $d\Omega_{\hat{p}'_1}$, and $d\Omega_{\hat{p}'_2}$ are the solid-angle elements to $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{p}}'_1$, and $\hat{\boldsymbol{p}}'_2$ in the overall c.m. system, respectively. For calculating the expansions in ω of the cross sections (6.3) it is important to choose the appropriate expansion of $\mathcal{M}_\lambda(p'_1,p'_2,k,p_a,p_b)$. For calculating the cross section with respect to $(\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}'_1)$, we shall use the expansion around $(\hat{\boldsymbol{k}}=0,\hat{\boldsymbol{p}}'_1)$ keeping $\hat{\boldsymbol{p}}'_1$ constant. Similarly, for the cross section with respect to $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}'_2$ it will be convenient to use the expansion where $\hat{\boldsymbol{p}}'_2$ is kept constant. We illustrate this in Fig. 7. We set

$$\hat{\boldsymbol{p}}''_1=-\hat{\boldsymbol{p}}'_2 \tag{6.4}$$

and get, after a simple calculation,

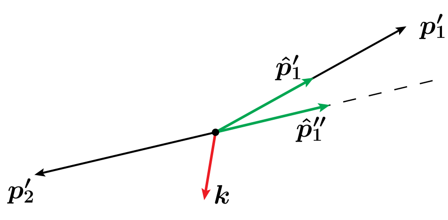


FIG. 7. Sketch of the momentum configuration for the final state of $\pi^-(p_a)\pi^0(p_b)\to\pi^-(p'_1)\pi^0(p'_2)\gamma(k)$ with the definition of the unit vectors $\hat{\boldsymbol{p}}'_1$ and $\hat{\boldsymbol{p}}''_1=-\hat{\boldsymbol{p}}'_2$.

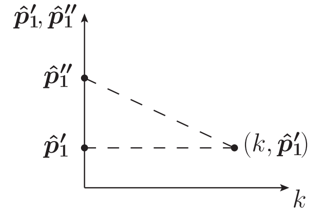


FIG. 8. Sketch for the two expansions (6.6) of $\mathcal{M}_\lambda(p'_1,p'_2,k,p_a,p_b)$ in the phase space $(k,\hat{\boldsymbol{p}}'_1)$.

$$\hat{\boldsymbol{p}}'_1=\hat{\boldsymbol{p}}''_1-\frac{\omega}{|\boldsymbol{p}_1|}\left[\hat{\boldsymbol{k}}(\hat{\boldsymbol{p}}''_1)^2-(\hat{\boldsymbol{p}}''_1\cdot\hat{\boldsymbol{k}})\hat{\boldsymbol{p}}''_1\right]+\mathcal{O}(\omega^2). \tag{6.5}$$

The next point to realize is, that in expanding $\mathcal{M}_\lambda(p'_1,p'_2,k,p_a,p_b)$ we have the freedom to choose the starting point of the expansion appropriately, of course, always staying with (p_1,p_2) close to (p'_1,p'_2) as sketched in Fig. 6. Referring always to the phase-space variables $(k,\hat{\boldsymbol{p}}'_1)$ we choose as a starting point for expanding σ_1 from (6.3) the point $(k=0,\hat{\boldsymbol{p}}'_1)$, for σ_2 the point $(k=0,\hat{\boldsymbol{p}}''_1)$. Then, σ_3 from (6.3) should be independent of these two choices and this is indeed what we shall see below. We can, therefore, write the following expansions:

$$\begin{aligned}
&\omega\mathcal{M}_\lambda(p'_1,p'_2,k,p_a,p_b)\\
&=\hat{\mathcal{M}}^{(0)}_\lambda(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})+\omega\hat{\mathcal{M}}^{(1)}_\lambda(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}'_1)+\mathcal{O}(\omega^2)\\
&=\hat{\mathcal{M}}^{(0)}_\lambda(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}''_1,\hat{\boldsymbol{k}})+\omega\hat{\mathcal{M}}^{(1)}_\lambda(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}''_1)+\mathcal{O}(\omega^2).
\end{aligned} \tag{6.6}$$

Here we indicate by the last variable in $\hat{\mathcal{M}}^{(1)}_\lambda$ the starting point of the expansion, $(k=0,\hat{\boldsymbol{p}}'_1)$ or $(k=0,\hat{\boldsymbol{p}}''_1)$, respectively; see Fig. 8. The precise definitions of $\hat{\mathcal{M}}^{(0)}_\lambda$ and $\hat{\mathcal{M}}^{(1)}_\lambda$ following from (4.16) are given below.

Now we come to the calculation of the expansion of σ_1 (6.3).

The cross section with respect to the phase-space variables ω, $\hat{\boldsymbol{k}}$, $\hat{\boldsymbol{p}}'_1$ reads (see Appendix B of [20])

$$\begin{aligned}
d\sigma(\pi^-\pi^0\to\pi^-\pi^0\gamma)=&\frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}\\
&\times\omega d\omega d\Omega_{\hat{k}}d\Omega_{\hat{p}'_1}J(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})\\
&\times(-\mathcal{M}_\lambda\mathcal{M}^{\lambda*}).
\end{aligned} \tag{6.7}$$

Here J is a kinematic function given in (B7) of [20] and we consider $\mathcal{M}_\lambda$ as a function of the independent initial variables s, $\hat{\boldsymbol{p}}_a$ and the phase-space variables (2.7)

$$\mathcal{M}_\lambda(p'_1,p'_2,k,p_a,p_b)\equiv\mathcal{M}_\lambda(s,\hat{\boldsymbol{p}}_a,\omega,\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}'_1). \tag{6.8}$$

We are interested in the cross section (6.7) for $\omega \to 0$. The expansion of the phase-space factor J is easily obtained from (B3)–(B8) of [20]

$$J(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) = J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) + \mathcal{O}(\omega^2),$$
$$J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) = \frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}} - \frac{\omega}{\sqrt{s}}\left(\frac{2m_\pi^2}{\sqrt{s(s-4m_\pi^2)}} + \hat{\boldsymbol{p}}'_1\cdot\hat{\boldsymbol{k}}\right). \tag{6.9}$$

The expansion in ω of $\mathcal{M}_\lambda$ (6.8) was the topic of Sec. IV. Now we consider given values for $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}'_1$ and vary ω. Therefore, in the schematic diagrams of Figs. 5 and 8, we move for fixed $\hat{\boldsymbol{p}}'_1 = \hat{\boldsymbol{p}}_1$, that is, for fixed $\boldsymbol{l}_{1\perp} = 0$ [see (4.1)] along the line $k = \omega(1,\hat{\boldsymbol{k}})^\top$. The expansion of $\mathcal{M}_\lambda$ (6.8) on this line is given in (4.16), of course, inserting l_1 and l_2 from (4.4) with $\boldsymbol{l}_{1\perp} = 0$. We denote the corresponding values of l_i by l'_i ($i = 1, 2$) in the following. In this way we obtain with $\hat{\boldsymbol{p}}'_1 = \hat{\boldsymbol{p}}_1$ [see (B12)–(B14) of [20]] for the first line on the rhs of (6.6)

$$\mathcal{M}_\lambda(s,\hat{\boldsymbol{p}}_a,\omega,\hat{\boldsymbol{k}},\hat{\boldsymbol{p}}'_1) = \frac{1}{\omega}\hat{\mathcal{M}}_\lambda^{(0,1)} + \hat{\mathcal{M}}_\lambda^{(1,1)} + \mathcal{O}(\omega), \tag{6.10}$$

where

$$\hat{\mathcal{M}}_\lambda^{(0,1)} = \hat{\mathcal{M}}_\lambda^{(0)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) = e\omega\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu,t), \tag{6.11}$$

$$\begin{aligned}\hat{\mathcal{M}}_\lambda^{(1,1)} &= \hat{\mathcal{M}}_\lambda^{(1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}'_1)\\ &= e\left\{-\frac{1}{(p_1\cdot k)^2}\left[p_{1\lambda}(l'_1\cdot k) - l'_{1\lambda}(p_1\cdot k)\right]\mathcal{M}^{(\mathrm{on})}(\nu,t) - 2\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k} - p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu,t)\right.\\ &\quad\left. - 2\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right]\left[(p_a - p_1, k) - (p_a\cdot l'_1)\right]\frac{\partial}{\partial t}\mathcal{M}^{(\mathrm{on})}(\nu,t)\right\}.\end{aligned} \tag{6.12}$$

Note that both $\hat{\mathcal{M}}_\lambda^{(0,1)}$ and $\hat{\mathcal{M}}_\lambda^{(1,1)}$ are independent of ω and they are unambiguously defined inserting l'_1 and l'_2 which are the values of l_1 and l_2 from (4.4) with $\boldsymbol{l}_{1\perp} = 0$ and $\nu = s - 2m_\pi^2$, $t = (p_a - p_1)^2$ from (3.4)–(3.6). Note also that the expansion (4.15) which is used here is, for our case $\boldsymbol{l}_{1\perp} = 0$, alright for $\omega \ll |\boldsymbol{p}_1|$ and all values of $\hat{\boldsymbol{p}}_1$ and $\hat{\boldsymbol{k}}$; see (A3).

Now we define

$$\mathcal{A}^{(0,1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) = \frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}\frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}}(-\hat{\mathcal{M}}_\lambda^{(0,1)}\hat{\mathcal{M}}^{(0,1)\lambda*}), \tag{6.13}$$

$$\begin{aligned}\mathcal{A}^{(1,1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}'_1) = \frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}&\left[\frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}}\left(-\hat{\mathcal{M}}_\lambda^{(1,1)}\hat{\mathcal{M}}^{(0,1)\lambda*} - \hat{\mathcal{M}}_\lambda^{(0,1)}\hat{\mathcal{M}}^{(1,1)\lambda*}\right)\right.\\ &\left.-\frac{1}{\sqrt{s}}\left(\frac{2m_\pi^2}{\sqrt{s(s-4m_\pi^2)}} + \hat{\boldsymbol{p}}'_1\cdot\hat{\boldsymbol{k}}\right)\left(-\hat{\mathcal{M}}_\lambda^{(0,1)}\hat{\mathcal{M}}^{(0,1)\lambda*}\right)\right].\end{aligned} \tag{6.14}$$

Inserting (6.9)–(6.12) in (6.7) and using (6.13) and (6.14) we get

$$\begin{aligned}\frac{\omega\mathrm{d}\sigma(\pi^-\pi^0\to\pi^-\pi^0\gamma)}{\mathrm{d}\omega\mathrm{d}\Omega_{\hat{k}}\mathrm{d}\Omega_{\hat{p}'_1}} &= \frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})\\ &\quad\times(-1)\left(\hat{\mathcal{M}}_\lambda^{(0,1)} + \omega\hat{\mathcal{M}}_\lambda^{(1,1)}\right)\left(\hat{\mathcal{M}}^{(0,1)\lambda} + \omega\hat{\mathcal{M}}^{(1,1)\lambda}\right)^* + \mathcal{O}(\omega^2)\\ &= \mathcal{A}^{(0,1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) + \omega\mathcal{A}^{(1,1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}'_1) + \mathcal{O}(\omega^2).\end{aligned} \tag{6.15}$$

Integrating (6.15) over the solid angles of $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}'_1$, we find the expansion of the cross section $\omega\mathrm{d}\sigma/\mathrm{d}\omega$ for $\omega \to 0$,

$$\begin{aligned}\omega\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}(\pi^-\pi^0\to\pi^-\pi^0\gamma) &= \frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}\int \mathrm{d}\Omega_{\hat{k}}\mathrm{d}\Omega_{\hat{p}'_1}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})\\ &\quad\times(-1)\Big(\hat{\mathcal{M}}^{(0,1)}_\lambda+\omega\hat{\mathcal{M}}^{(1,1)}_\lambda\Big)\Big(\hat{\mathcal{M}}^{(0,1)\lambda}+\omega\hat{\mathcal{M}}^{(1,1)\lambda}\Big)^*+\mathcal{O}(\omega^2)\\ &=\int \mathrm{d}\Omega_{\hat{k}}\mathrm{d}\Omega_{\hat{p}'_1}\mathcal{A}^{(0,1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})+\omega\int \mathrm{d}\Omega_{\hat{k}}\mathrm{d}\Omega_{\hat{p}'_1}\mathcal{A}^{(1,1)}(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}'_1)+\mathcal{O}(\omega^2).\end{aligned}\tag{6.16}$$

Note that in the expansions (6.15) and (6.16) all terms are unambiguously defined. The expansion coefficients are independent of ω, as it should be.

Next we consider the cross section σ_2 from (6.3), that is, the cross section with respect to ω, $\hat{\boldsymbol{k}}$, and $\hat{\boldsymbol{p}}'_2$. For this we define here [see (6.4)–(6.6)] the pion momenta p''_1 and p''_2 of the now appropriate nonradiative starting point of the expansion (see Figs. 7 and 8). We have

$$\begin{aligned}\boldsymbol{p}''_1 &= -\boldsymbol{p}''_2=\hat{\boldsymbol{p}}''_1|\boldsymbol{p}_1|=\hat{\boldsymbol{p}}''_1\sqrt{\frac{s}{4}-m_\pi^2},\\ p''^0_1 &= p''^0_2=\frac{\sqrt{s}}{2},\\ t'' &= (p_a-p''_1)^2.\end{aligned}\tag{6.17}$$

We get then the following matrix elements for the second line on the rhs of (6.6) using (4.16):

$$\begin{aligned}\hat{\mathcal{M}}^{(0,2)}_\lambda &= \hat{\mathcal{M}}^{(0)}_\lambda(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}''_1,\hat{\boldsymbol{k}})\\ &= e\omega\left[\frac{p_{a\lambda}}{p_a\cdot k}-\frac{p''_{1\lambda}}{p''_1\cdot k}\right]\mathcal{M}^{(\mathrm{on})}(\nu,t''),\end{aligned}\tag{6.18}$$

$$\begin{aligned}\hat{\mathcal{M}}^{(1,2)}_\lambda &= \hat{\mathcal{M}}^{(1)}_\lambda(s,\hat{\boldsymbol{p}}_a,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}};\hat{\boldsymbol{p}}''_1)\\ &= e\Big\{-\frac{1}{(p''_1\cdot k)^2}\Big[p''_{1\lambda}(l''_1\cdot k)-l''_{1\lambda}(p''_1\cdot k)\Big]\mathcal{M}^{(\mathrm{on})}(\nu,t'')\\ &\quad-2\left[p_{a\lambda}\frac{p_b\cdot k}{p_a\cdot k}-p_{b\lambda}\right]\frac{\partial}{\partial\nu}\mathcal{M}^{(\mathrm{on})}(\nu,t'')\\ &\quad-2\left[\frac{p_{a\lambda}}{p_a\cdot k}-\frac{p''_{1\lambda}}{p''_1\cdot k}\right]\Big[(p_a-p''_1,k)-(p_a\cdot l''_1)\Big]\frac{\partial}{\partial t''}\mathcal{M}^{(\mathrm{on})}(\nu,t'')\Big\}.\end{aligned}\tag{6.19}$$

Here we have to set in (4.16) $l_1=l''_1$ and $l_2=l''_2$ according to (4.4) but with the replacements

$$\begin{aligned}&p_1\to p''_1,\qquad \hat{\boldsymbol{p}}_1\to\hat{\boldsymbol{p}}''_1,\qquad p_2\to p''_2,\\ &\boldsymbol{l}_{1\perp}\to\boldsymbol{l}''_{1\perp}=\boldsymbol{k}(\hat{\boldsymbol{p}}'_1)^2-(\boldsymbol{k}\cdot\hat{\boldsymbol{p}}''_1)\hat{\boldsymbol{p}}''_1.\end{aligned}\tag{6.20}$$

We get then $\boldsymbol{l}''_{2\perp}=0$, that is, $\hat{\boldsymbol{p}}'_2=\hat{\boldsymbol{p}}''_2$, as we require here.

For the cross section σ_2 of (6.3) we obtain

$$\begin{aligned}\frac{\omega\mathrm{d}\sigma(\pi^-\pi^0\to\pi^-\pi^0\gamma)}{\mathrm{d}\omega\mathrm{d}\Omega_{\hat{k}}\mathrm{d}\Omega_{\hat{p}'_2}} &= \frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_2,\hat{\boldsymbol{k}})\\ &\quad\times(-1)\Big(\hat{\mathcal{M}}^{(0,2)}_\lambda+\omega\hat{\mathcal{M}}^{(1,2)}_\lambda\Big)\Big(\hat{\mathcal{M}}^{(0,2)\lambda}+\omega\hat{\mathcal{M}}^{(1,2)\lambda}\Big)^*+\mathcal{O}(\omega^2)\\ &= \frac{1}{\sqrt{s(s-4m_\pi^2)}}\frac{1}{2^4(2\pi)^5}\left[\frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}}-\frac{\omega}{\sqrt{s}}\left(\frac{2m_\pi^2}{\sqrt{s(s-4m_\pi^2)}}+\hat{\boldsymbol{p}}'_2\cdot\hat{\boldsymbol{k}}\right)\right]\\ &\quad\times(-1)\Big(\hat{\mathcal{M}}^{(0,2)}_\lambda+\omega\hat{\mathcal{M}}^{(1,2)}_\lambda\Big)\Big(\hat{\mathcal{M}}^{(0,2)\lambda}+\omega\hat{\mathcal{M}}^{(1,2)\lambda}\Big)^*+\mathcal{O}(\omega^2),\end{aligned}\tag{6.21}$$

and finally

$$\frac{\omega \mathrm{d}\sigma(\pi^-\pi^0 \to \pi^-\pi^0\gamma)}{\mathrm{d}\omega \mathrm{d}\Omega_{\hat{k}} \mathrm{d}\Omega_{\hat{p}'_2}} = \frac{1}{\sqrt{s(s-4m_\pi^2)}} \frac{1}{2^4(2\pi)^5} \left\{ \frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}} \left(-\hat{\mathcal{M}}_\lambda^{(0,2)} \hat{\mathcal{M}}^{(0,2)\lambda *}\right) \right.$$
$$+ \omega \left[\frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}} \left(-\hat{\mathcal{M}}_\lambda^{(1,2)} \hat{\mathcal{M}}^{(0,2)\lambda *} - \hat{\mathcal{M}}_\lambda^{(0,2)} \hat{\mathcal{M}}^{(1,2)\lambda *}\right) \right.$$
$$\left.\left. - \frac{1}{\sqrt{s}} \left(\frac{2m_\pi^2}{\sqrt{s(s-4m_\pi^2)}} + \hat{\boldsymbol{p}}'_2 \cdot \hat{\boldsymbol{k}} \right) \left(-\hat{\mathcal{M}}_\lambda^{(0,2)} \hat{\mathcal{M}}^{(0,2)\lambda *} \right) \right] \right\} + \mathcal{O}(\omega^2). \tag{6.22}$$

Now we turn to σ_3 in (6.3). In (6.16) we have obtained $\omega \mathrm{d}\sigma/\mathrm{d}\omega$ by integrating over the solid angles of $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}'_1$. We can, however, also integrate (6.21) over the solid angles of $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{p}}'_2$. Will we get the same result up to order ω? From (6.21) we get

$$\omega \frac{\mathrm{d}\sigma}{\mathrm{d}\omega}(\pi^-\pi^0 \to \pi^-\pi^0\gamma) = \frac{1}{\sqrt{s(s-4m_\pi^2)}} \frac{1}{2^4(2\pi)^5} \int \mathrm{d}\Omega_{\hat{k}} \mathrm{d}\Omega_{\hat{p}'_2} J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_2,\hat{\boldsymbol{k}})$$
$$\times (-1)(\hat{\mathcal{M}}_\lambda^{(0,2)} + \omega \hat{\mathcal{M}}_\lambda^{(1,2)})(\hat{\mathcal{M}}^{(0,2)\lambda} + \omega \hat{\mathcal{M}}^{(1,2)\lambda})^* + \mathcal{O}(\omega^2). \tag{6.23}$$

In Appendix B we give the result for the change of measure under the variable transformation $\hat{\boldsymbol{p}}'_1 \to \hat{\boldsymbol{p}}''_1 = -\hat{\boldsymbol{p}}'_2$ from (6.5) for fixed $\hat{\boldsymbol{k}}$. We find from (B9)

$$\begin{aligned} &\mathrm{d}\Omega_{\hat{p}'_1} J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) \\ &= \mathrm{d}\Omega_{\hat{p}''_1} J^{(1)}(s,\omega,-\hat{\boldsymbol{p}}''_1,\hat{\boldsymbol{k}}) + \mathcal{O}(\omega^2) \\ &= \mathrm{d}\Omega_{\hat{p}'_2} J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_2,\hat{\boldsymbol{k}}) + \mathcal{O}(\omega^2). \end{aligned} \tag{6.24}$$

Inserting (6.24) in (6.23) and using (6.6), (6.10)–(6.12), (6.18), and (6.19), we find, indeed, that the expansions of the cross sections $\omega \mathrm{d}\sigma/\mathrm{d}\omega$ from (6.16) and (6.23) are the same up to order ω which is the order up to which we calculate here.

To conclude this chapter, we emphasize that for the discussions of cross sections in (6.3) it was essential to have at our disposal the general expansion of the radiative amplitude $\mathcal{M}_\lambda$ around a phase-space point $(k=0,\hat{\boldsymbol{p}}_1)$, respectively $(k=0,\boldsymbol{l}_{1\perp}=0)$; see Fig. 5. The expansion parameters were $(k,\boldsymbol{l}_{1\perp})$. We found that in calculating σ_1 and σ_2 of (6.3) we had to use the general expansion (4.16) of $\mathcal{M}_\lambda$ but with *different* starting points and *different* values of $\boldsymbol{l}_{1\perp}$, respectively; see Fig. 8.

VII. OUTLINE OF THE CALCULATION FOR $\pi p \to \pi p \gamma$

We use the framework of QCD and treat electromagnetism to lowest relevant order. In QCD we have the symmetries: parity (P), charge conjugation (C), and time reversal (T). These give us restrictions for the propagators, vertices, and amplitudes. Furthermore, we use the generalized Ward identities for pions and the proton [23,24] and the Landau conditions for determining the singularities in amplitudes [25,26]. All our results are derived using only these rigorous methods.

Consider now the reactions (1.3) where energy-momentum conservation reads

$$p_a + p_b = p_1 + p_2. \tag{7.1}$$

Thus, there are only three independent momenta which we choose as

$$\begin{aligned} p_s &= p_a + p_b = p_1 + p_2, \\ p_t &= p_a - p_1 = p_2 - p_b, \\ p_u &= p_a - p_2 = p_1 - p_b. \end{aligned} \tag{7.2}$$

We have

$$s = p_s^2, \qquad t = p_t^2, \qquad u = p_u^2. \tag{7.3}$$

The amplitude for (1.3) has the general structure

$$\begin{aligned} &\langle \pi^\pm(p_1), p(p_2,\lambda_2)|\mathcal{T}|\pi^\pm(p_a), p(p_b,\lambda_b)\rangle \\ &= \bar{u}(p_2,\lambda_2)\left[A^{(\mathrm{on})\pm}(s,t) + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\mathrm{on})\pm}(s,t)\right] \\ &\quad \times u(p_b,\lambda_b), \end{aligned} \tag{7.4}$$

with invariant functions $A^{(\mathrm{on})\pm}$ and $B^{(\mathrm{on})\pm}$; see, e.g., [26]. In the calculation of the amplitude for (1.4) we need, however, the off-shell amplitude for $\pi p \to \pi p$ which is much more complicated than (7.4). Writing for the on- or off-shell momenta of the general reaction (1.3) $\tilde{p}_a$, $\tilde{p}_b$, $\tilde{p}_1$, $\tilde{p}_2$ and defining $\tilde{p}_s$, $\tilde{p}_t$, $\tilde{p}_u$, $\tilde{s}$, $\tilde{t}$ in analogy to (7.2) and (7.3) we find for the off-shell amplitudes

$$\begin{aligned}\mathcal{M}^{(0)\pm}(\tilde{p}_1,\tilde{p}_2,\tilde{p}_a,\tilde{p}_b) = {}& \mathcal{M}_1^\pm + \tilde{\not{p}}_s\mathcal{M}_2^\pm + \tilde{\not{p}}_t\mathcal{M}_3^\pm + \tilde{\not{p}}_u\mathcal{M}_4^\pm \\ &+ i\sigma_{\mu\nu}\tilde{p}_s{}^\mu\tilde{p}_t{}^\nu\mathcal{M}_5^\pm + i\sigma_{\mu\nu}\tilde{p}_s{}^\mu\tilde{p}_u{}^\nu\mathcal{M}_6^\pm \\ &+ i\sigma_{\mu\nu}\tilde{p}_t{}^\mu\tilde{p}_u{}^\nu\mathcal{M}_7^\pm \\ &+ i\gamma_\mu\gamma_5\varepsilon^{\mu\nu\rho\sigma}\tilde{p}_{s\nu}\tilde{p}_{t\rho}\tilde{p}_{u\sigma}\mathcal{M}_8^\pm. \end{aligned} \quad (7.5)$$

We use the convention $\varepsilon_{0123} = 1$. Here the invariant amplitudes $\mathcal{M}_j^\pm$ $(j = 1, \ldots, 8)$ can only depend on $\tilde{s}$, $\tilde{t}$ and the invariant squared masses,

$$\begin{aligned}&\mathcal{M}_j^\pm = \mathcal{M}_j^\pm(\tilde{s},\tilde{t},m_1^2,m_2^2,m_a^2,m_b^2), \\ &m_a^2 = \tilde{p}_a^2, \quad m_b^2 = \tilde{p}_b^2, \quad m_1^2 = \tilde{p}_1^2, \quad m_2^2 = \tilde{p}_2^2. \end{aligned} \quad (7.6)$$

Specializing (7.5) for the on-shell case we get back (7.4) with $\tilde{s} \to s$, $\tilde{t} \to t$, $m_a^2 = m_1^2 = m_\pi^2$, $m_b^2 = m_2^2 = m_p^2$, and

$$\begin{aligned}A^{(\mathrm{on})\pm}(s,t) = {}& \mathcal{M}_1^{(\mathrm{on})\pm} + m_p\mathcal{M}_2^{(\mathrm{on})\pm} - m_p\mathcal{M}_4^{(\mathrm{on})\pm} \\ &+ (-s + m_p^2 + m_\pi^2)\mathcal{M}_5^{(\mathrm{on})\pm} \\ &+ (s + t - m_p^2 - m_\pi^2)\mathcal{M}_7^{(\mathrm{on})\pm} \\ &- m_p(2s + t - 2m_p^2 - 2m_\pi^2)\mathcal{M}_8^{(\mathrm{on})\pm}, \end{aligned} \quad (7.7)$$

$$\begin{aligned}B^{(\mathrm{on})\pm}(s,t) = {}& \mathcal{M}_2^{(\mathrm{on})\pm} + \mathcal{M}_4^{(\mathrm{on})\pm} \\ &+ 2m_p\mathcal{M}_5^{(\mathrm{on})\pm} - 2m_p\mathcal{M}_7^{(\mathrm{on})\pm} \\ &+ (4m_p^2 - t)\mathcal{M}_8^{(\mathrm{on})\pm}. \end{aligned} \quad (7.8)$$

On shell the amplitudes $\mathcal{M}_3^\pm$ and $\mathcal{M}_6^\pm$ are zero from C and T invariance.

Next we consider the reactions (1.4). We have five diagrams for $\pi^- p \to \pi^- p\gamma$ as shown in Fig. 9. For $\pi^+ p \to \pi^+ p\gamma$ the diagrams are analogous.

Let $\mathcal{M}_\lambda^\pm$ be the amplitude without spinors for (1.4). We define a matrix amplitude $\mathcal{N}_\lambda^\pm$ by

$$\begin{aligned}&\mathcal{N}_\lambda^\pm(p_1', p_2', k, p_a, p_b) \\ &= (\not{p}_2' + m_p)\mathcal{M}_\lambda^\pm(p_1', p_2', k, p_a, p_b)(\not{p}_b + m_p). \end{aligned} \quad (7.9)$$

The $\mathcal{T}$ matrix element for (1.4) is then

$$\begin{aligned}&\langle\pi^\pm(p_1'), p(p_2', \lambda_2'), \gamma(k, \varepsilon)|\mathcal{T}|\pi^\pm(p_a), p(p_b, \lambda_b)\rangle \\ &= (\varepsilon^\lambda)^* \frac{1}{(2m_p)^2}\bar{u}(p_2', \lambda_2')\mathcal{N}_\lambda^\pm(p_1', p_2', k, p_a, p_b)u(p_b, \lambda_b). \end{aligned} \quad (7.10)$$

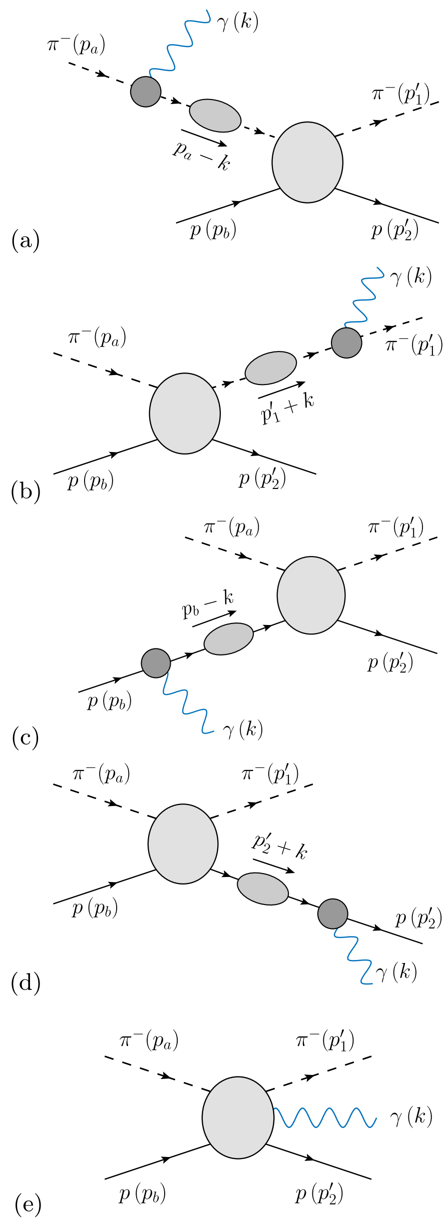


FIG. 9. Diagrams for the $\pi^- p \to \pi^- p\gamma$ reaction. Photon emission from external particles is shown in (a)–(d), the structure term (e) is nonsingular for $k \to 0$. The blobs in (a)–(d) represent the full propagators and vertices and the off-shell $\pi^- p \to \pi^- p$ amplitude.

The advantage of working with $\mathcal{N}_\lambda^\pm$ instead of $\mathcal{M}_\lambda^\pm$, sandwiched between spinors, is that we do not have to specify any particular spin basis for the protons.

For real photon emission we have $k^2 = 0$ in (7.10) and this is what we consider here. In [22] we treat the amplitude $\mathcal{N}_\lambda^\pm$ (7.9) also for virtual photons, that is, for $k^2 \neq 0$. The discussion of the kinematics of the reactions (1.3) and (1.4) is analogous to the one for (1.1) and (1.2) in Sec. II. We work again in the c.m. system where we have for (1.3)

$$p_a^0 = p_1^0 = \frac{1}{2\sqrt{s}}(s + m_\pi^2 - m_p^2),$$
$$p_b^0 = p_2^0 = \frac{1}{2\sqrt{s}}(s - m_\pi^2 + m_p^2),$$
$$|\boldsymbol{p_a}| = |\boldsymbol{p_b}| = |\boldsymbol{p_1}| = |\boldsymbol{p_2}| = \sqrt{(p_a^0)^2 - m_\pi^2} = \sqrt{(p_b^0)^2 - m_p^2}. \quad (7.11)$$

Given the initial state the phase space of the final state for (1.3) is parametrized by $\hat{\boldsymbol{p}}_1 = \boldsymbol{p}_1/|\boldsymbol{p}_1|$ and for (1.4) by $(k, \hat{\boldsymbol{p}}_1')$, where $\hat{\boldsymbol{p}}_1' = \boldsymbol{p}_1'/|\boldsymbol{p}_1'|$. For small ω ($\omega \ll |\boldsymbol{p}_1|$) $\hat{\boldsymbol{p}}_1'$ can vary over the whole unit sphere.

We consider again a neighborhood of a phase-space point $(k = 0, \hat{\boldsymbol{p}}_1)$ and set there $\hat{\boldsymbol{p}}_1' = \hat{\boldsymbol{p}}_1 - \boldsymbol{l}_{1\perp}/|\boldsymbol{p}_1|$ with $|\boldsymbol{l}_{1\perp}| = \mathcal{O}(\omega)$; see (4.1). This neighborhood is then parametrized by $(k, \boldsymbol{l}_{1\perp})$; see Fig. 5. For the momenta of the reaction (1.4) at this phase-space point we set again $p_1' = p_1 - l_1$ and $p_2' = p_2 - l_2$, where $l_{1,2}$ are determined up to order ω with the same result as in (4.4) but inserting for s, $p_{1,2}$, $|\boldsymbol{p}_1|$, and $\hat{\boldsymbol{p}}_1 = \boldsymbol{p}_1/|\boldsymbol{p}_1|$, the appropriate values for πp scattering; see (7.1)–(7.3) and (7.11).

Our aim is to derive the expansion of $\mathcal{N}_\lambda^\pm(p_1 - l_1, p_2 - l_2, k, p_a, p_b)$ from (7.9) for $\omega \to 0$ and to give the terms of order ω^{-1} and ω^0 explicitly. Note that l_1, l_2, and k are all of order ω; see (4.4). Thus, we have to expand $\mathcal{N}_\lambda^\pm$ with respect to *all* these momenta. Setting $l_1 = l_2 = 0$ and expanding then only in k makes no sense since this violates energy-momentum conservation and leads outside the physical region of the amplitude. In the following we shall give the analog of the Laurent expansion for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ as discussed in Sec. IV.

In Figs. 9(a)–9(d), the combinations of propagator times photon vertex occur for pion and proton. Using the generalized Ward identities we find for the pion (4.12), see Fig. 9(a), and for the proton in Fig. 9(c)

$$S_F(p_b - k)\hat{\Gamma}^{(\gamma pp)\mu}(p_b - k, p_b)(\not{p}_b + m_p) = \frac{\not{p}_b + m_p - \not{k}}{-2p_b \cdot k + k^2 + i\epsilon}\left[\gamma^\mu - \frac{i}{2m_p}\sigma^{\mu\nu}k_\nu F_2(0)\right] \times (\not{p}_b + m_p) + \mathcal{O}(\omega). \quad (7.12)$$

Here $F_2(0) = \mu_p/\mu_N - 1$ with μ_p the magnetic moment of the proton and μ_N the nuclear magneton. Expressions similar to (4.12) and (7.12) apply for the vertex times propagator terms in Figs. 9(b) and 9(d), respectively.

We see from (4.12) and (7.12) that for the determination of the amplitudes $\mathcal{N}_\lambda^\pm$ to the orders ω^{-1} and ω^0 we have to know the off-shell amplitudes for $\pi^\pm p \to \pi^\pm p$ in Figs. 9(a)–9(d) to the orders ω^0 and ω^1. Thus, we have to use (7.5) and make this expansion for the terms $\tilde{\not{p}}_s$, $\tilde{\not{p}}_t$, etc. as well as for the amplitudes $\mathcal{M}_1^\pm, \ldots, \mathcal{M}_8^\pm$. This expansion is *different* for the terms corresponding to the diagrams of Fig. 9(a)–9(d). Finally, the structure term corresponding to Fig. 9(e) can be determined from the gauge-invariance constraint

$$k^\lambda \mathcal{N}_\lambda = k^\lambda(\mathcal{N}_\lambda^{(a)} + \mathcal{N}_\lambda^{(b)} + \mathcal{N}_\lambda^{(c)} + \mathcal{N}_\lambda^{(d)} + \mathcal{N}_\lambda^{(e)}) = 0. \quad (7.13)$$

After a long and rather complicated calculation, we find that the result for the amplitudes $\mathcal{N}_\lambda^\pm$ (7.9) for $\pi^\pm p \to \pi^\pm p\gamma$ to the orders ω^{-1} and ω^0 can be expressed completely in terms of the on-shell amplitudes $A^{(\mathrm{on})\pm}(s,t)$ and $B^{(\mathrm{on})\pm}(s,t)$ for $\pi^\pm p \to \pi^\pm p$ and their partial derivatives with respect to s and t,

$$A_{,s}^{(\mathrm{on})\pm}(s,t) = \frac{\partial}{\partial s}A^{(\mathrm{on})\pm}(s,t),$$
$$A_{,t}^{(\mathrm{on})\pm}(s,t) = \frac{\partial}{\partial t}A^{(\mathrm{on})\pm}(s,t),$$
$$B_{,s}^{(\mathrm{on})\pm}(s,t) = \frac{\partial}{\partial s}B^{(\mathrm{on})\pm}(s,t),$$
$$B_{,t}^{(\mathrm{on})\pm}(s,t) = \frac{\partial}{\partial t}B^{(\mathrm{on})\pm}(s,t). \quad (7.14)$$

We find for real photon emission

$$\mathcal{N}_\lambda^\pm = \frac{1}{\omega}\hat{\mathcal{N}}_\lambda^{(0)\pm} + \hat{\mathcal{N}}_\lambda^{(1)\pm} + \mathcal{O}(\omega), \quad (7.15)$$

where with $j = 0, 1$

$$\hat{\mathcal{N}}_\lambda^{(j)\pm} = \hat{\mathcal{N}}_\lambda^{(a+b+e1)(j)\pm} + \hat{\mathcal{N}}_\lambda^{(c+d+e2)(j)\pm}, \quad (7.16)$$

$$\hat{\mathcal{N}}_\lambda^{(a+b+e1)(0)\pm} = \pm e(\not{p}_2 + m_p)\left[A^{(\mathrm{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\mathrm{on})\pm}\right](\not{p}_b + m_p)\omega\left[-\frac{p_{a\lambda}}{p_a \cdot k} + \frac{p_{1\lambda}}{p_1 \cdot k}\right], \quad (7.17)$$

$$\hat{\mathcal{N}}_\lambda^{(a+b+e1)(1)\pm} = \pm e(\not{p}_2 + m_p)\left[A^{(\mathrm{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\mathrm{on})\pm}\right](\not{p}_b + m_p)\frac{1}{(p_1 \cdot k)^2}\left[p_{1\lambda}(l_1 \cdot k) - l_{1\lambda}(p_1 \cdot k)\right]$$
$$\pm e(-\not{l}_2)\left[A^{(\mathrm{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\mathrm{on})\pm}\right](\not{p}_b + m_p)\left[-\frac{p_{a\lambda}}{p_a \cdot k} + \frac{p_{1\lambda}}{p_1 \cdot k}\right]$$

$$
\begin{aligned}
&\pm e(\not{p}_2 + m_p)\frac{1}{2}(-\not{l}_1)B^{(\text{on})\pm}(\not{p}_b + m_p)\left[-\frac{p_{a\lambda}}{p_a\cdot k} + \frac{p_{1\lambda}}{p_1\cdot k}\right] \\
&\pm e(\not{p}_2 + m_p)\left\{\left[A,_s^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B,_s^{(\text{on})\pm}\right]\left[2(p_s\cdot k)\frac{p_{a\lambda}}{p_a\cdot k} - 2p_{s\lambda}\right]\right. \\
&+ \left[A,_t^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B,_t^{(\text{on})\pm}\right]2(p_t\cdot l_2)\left[\frac{p_{a\lambda}}{p_a\cdot k} - \frac{p_{1\lambda}}{p_1\cdot k}\right] \\
&\left.+ B^{(\text{on})\pm}\left[\frac{1}{2}\not{k}\left(\frac{p_{a\lambda}}{p_a\cdot k} + \frac{p_{1\lambda}}{p_1\cdot k}\right) - \gamma_\lambda\right]\right\}(\not{p}_b + m_p),
\end{aligned}
\tag{7.18}
$$

$$
\hat{\mathcal{N}}_\lambda^{(c+d+e2)(0)\pm} = e(\not{p}_2 + m_p)\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right](\not{p}_b + m_p)\omega\left[-\frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}}{p_2\cdot k}\right],
\tag{7.19}
$$

$$
\begin{aligned}
\hat{\mathcal{N}}_\lambda^{(c+d+e2)(1)\pm} = {}& e(\not{p}_2 + m_p)\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right](\not{p}_b + m_p)\frac{1}{(p_2\cdot k)^2}\left[p_{2\lambda}(l_2\cdot k) - l_{2\lambda}(p_2\cdot k)\right] \\
&+ e(-\not{l}_2)\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right](\not{p}_b + m_p)\left[-\frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}}{p_2\cdot k}\right] \\
&+ e(\not{p}_2 + m_p)\frac{1}{2}(-\not{l}_1)B^{(\text{on})\pm}(\not{p}_b + m_p)\left[-\frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}}{p_2\cdot k}\right] \\
&+ e(\not{p}_2 + m_p)\left[A,_s^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B,_s^{(\text{on})\pm}\right](\not{p}_b + m_p)\left[2(p_s\cdot k)\frac{p_{b\lambda}}{p_b\cdot k} - 2p_{s\lambda}\right] \\
&+ e(\not{p}_2 + m_p)\left[A,_t^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B,_t^{(\text{on})\pm}\right](\not{p}_b + m_p)2(p_t\cdot l_1)\left[-\frac{p_{b\lambda}}{p_b\cdot k} + \frac{p_{2\lambda}}{p_2\cdot k}\right] \\
&+ e(\not{p}_2 + m_p)\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right](k_\lambda - \not{k}\gamma_\lambda)(\not{p}_b + m_p)\frac{1}{(-2p_b\cdot k)} \\
&- e\frac{1}{(2p_2\cdot k)}(\not{p}_2 + m_p)(k_\lambda - \gamma_\lambda\not{k})\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right](\not{p}_b + m_p) \\
&+ e(\not{p}_2 + m_p)\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right]\left[m_p(k_\lambda - \not{k}\gamma_\lambda) + (p_{b\lambda}\not{k} - (p_b\cdot k)\gamma_\lambda)\right](\not{p}_b + m_p) \\
&\times\frac{F_2(0)}{m_p}\frac{1}{(-2p_b\cdot k)} - e\frac{F_2(0)}{m_p}\frac{1}{(2p_2\cdot k)}(\not{p}_2 + m_p)\left[m_p(k_\lambda - \gamma_\lambda\not{k}) + \left(p_{2\lambda}\not{k} - (p_2\cdot k)\gamma_\lambda\right)\right] \\
&\times\left[A^{(\text{on})\pm} + \frac{1}{2}(\not{p}_a + \not{p}_1)B^{(\text{on})\pm}\right](\not{p}_b + m_p).
\end{aligned}
\tag{7.20}
$$

When our calculations for the amplitudes for $\pi^\pm p \to \pi^\pm p\gamma$ in the soft-photon limit were finished we learned about Refs. [7,8,27–29]. There, as in our work, it is emphasized that one has to make a consistent expansion of all terms in the amplitude for $\omega \to 0$. But our results rely on more general premisses. We consider, for instance, the most general off-shell $\pi p \to \pi p$ amplitudes which contain eight invariant amplitudes; see (7.5). In [28] only two invariant off-shell amplitudes are considered. A detailed comparison of our methods and results with those of [27–29] shall be given in an update of [22].

For $\pi^0 p \to \pi^0 p\gamma$ we are left with the diagrams (c), (d), and (e) of Fig. 9 with $\pi^- \to \pi^0$. Therefore, in our method, the amplitude $\mathcal{N}_\lambda^0$ for $\pi^0 p \to \pi^0 p\gamma$ to the orders ω^{-1} and ω^0 for real photon emission can be expressed as

$$
\mathcal{N}_\lambda^0 = \frac{1}{\omega}\hat{\mathcal{N}}_\lambda^{(0)0} + \hat{\mathcal{N}}_\lambda^{(1)0} + \mathcal{O}(\omega),
\tag{7.21}
$$

with

$$
\hat{\mathcal{N}}_\lambda^{(j)0} = \hat{\mathcal{N}}_\lambda^{(c+d+e2)(j)0}, \quad j = 0, 1,
\tag{7.22}
$$

using expressions analogous to (7.19) and (7.20) replacing, of course, the $\pi^\pm p$ by the $\pi^0 p$ amplitudes. Again we emphasize that Low's formula, (3.16) of [1], for the radiative amplitude for $\pi^0 p \to \pi^0 p\gamma$ gives an approximate result valid at the fixed phase-space point (p_1', p_2', k), while our result (7.22) corresponds to the Laurent expansion in ω around the phase-space point $(p_1, p_2, k = 0)$.

VIII. CONCLUSIONS

In this article we have first discussed $\pi^-\pi^0$ elastic scattering and soft-photon radiation in this reaction. We have shown that the soft-photon theorems of Low [1] and Weinberg [2,3] have a different meaning. In [1] an approximate expression for the radiative amplitude at a *given* phase-space point (p_1', p_2', k) is presented. In [3] the pole term of the radiative amplitude in a Laurent expansion in the photon energy ω around the phase-space point $(p_1, p_2, k = 0)$ is given, where p_1, p_2 are the momenta of the nonradiative reaction. We have recalled and discussed our calculation of [20] where we presented the ω^0 term in the above Laurent expansion. We have discussed in detail that the next-to-leading terms presented in [1] and [20], respectively, must be different, since their meaning is different. We have then constructed the Laurent expansion of Low's formula (5.3) around the phase-space point $(p_1, p_2, k = 0)$ and found complete agreement with our results of [20]. We emphasized that Low's formula (5.3) for the radiative amplitude $\mathcal{M}_\lambda$ is *only* valid for the *one physical value* of $k = p_a + p_b - p_1' - p_2'$. In contrast, in (4.16) we give a Laurent expansion of $\mathcal{M}_\lambda$ around the phase-space point $(p_1, p_2, k = 0)$ where, of course, we can vary the expansion parameters $(k, \boldsymbol{l}_{1\perp})$ which are defined at the beginning of Sec. IV.

In [20] we wrongly interpreted Low's formula as an expansion of $\mathcal{M}_\lambda$ around $(p_1, p_2, k = 0)$. With this premiss we came to the conclusion that Low's formula violates energy-momentum conservation. We have now discussed in detail that the above premiss does not hold. Thus, also our above conclusion does not hold. Therefore, the part of Sec. III of [20] after Eq. (3.29) is obsolete since it was based on a wrong premiss. This was already explained in [21] and will be subject of a forthcoming erratum to [20]. We were confused by the fact that in the literature frequently Weinberg's formula (4.17) is referred to as Low's formula. But, as we hope to have demonstrated in the present paper, these two formulas have a *different* meaning. This was the subject of our Secs. IV and V.

In Sec. VI we discussed the expansions of different cross sections of $\pi^-(p_a) + \pi^0(p_b) \to \pi^-(p_1') + \pi^0(p_2') + \gamma(k)$ for $\omega \to 0$. We found that for calculating these expansions for $\omega d\sigma/(d\omega d\Omega_{\hat{k}} d\Omega_{\hat{p}_1'})$ and $\omega d\sigma/(d\omega d\Omega_{\hat{k}} d\Omega_{\hat{p}_2'})$ it was necessary to have at our disposal the expansion of the radiative amplitude in *all* directions $(k, \hat{\boldsymbol{p}}_1')$ of the phase space around the nonradiative point $(k = 0, \hat{\boldsymbol{p}}_1)$. We calculated then the expansion in ω for $\omega \to 0$ of the cross section $\omega d\sigma/d\omega$ from the above two differential cross sections. As it must be, we found the same result using these two ways.

Finally, we have discussed in Sec. VII the soft-photon expansion of the amplitudes for the reactions $\pi^\pm p \to \pi^\pm p\gamma$. We have presented a strict theorem of QCD. Given the amplitudes for $\pi^\pm p \to \pi^\pm p$ scattering the amplitudes for soft-photon production, $\pi^\pm p \to \pi^\pm p\gamma$, have been calculated exactly to the orders ω^{-1} and ω^0. These two orders of the expansion of the $\pi^\pm p \to \pi^\pm p\gamma$ amplitudes are completely determined by the $\pi^\pm p \to \pi^\pm p$ on-shell amplitudes. For real photon emission the result is given in (7.15)–(7.20). The results for soft virtual photon emission and consequences for cross sections are given in [22].

The derivation of our results, especially for the $\pi^\pm p \to \pi^\pm p\gamma$ reaction, involved lengthy calculations. Thus, we found it convenient to check our general results in a model which satisfies the QFT constraints for these reactions as listed at the end of Sec. I. Such a model is the tensor-Pomeron model of [30] but improved for reactions involving photons as shown in Sec. IV A of [20] for $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ and in [31,32] for $pp \to pp\gamma$. Applying this model as done in [20] to $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ and expanding the resulting expressions [Eqs. (4.19), (4.22)–(4.24) from [20]] in ω for $\omega \to 0$ we get indeed the result expected from (3.27), (3.28), and (A1), of [20] which, for real photon emission, we reproduce in (4.14)–(4.16) of the present paper. We also checked that this improved tensor-Pomeron model gives amplitudes for $\pi^\pm p \to \pi^\pm p\gamma$ where the expansion in ω agrees with the general results (7.14)–(7.20) in our present paper. The details for this will be discussed elsewhere.

To summarize, we hope to have clarified the meaning of the soft-photon expansions in the versions of Low [1] and Weinberg [2,3]. These expansions must not be confounded, as they have a *different* meaning. We have shown how these two expansions are related. We have discussed the Laurent expansions in the photon energy ω for $\omega \to 0$ for the reactions $\pi^-\pi^0 \to \pi^-\pi^0\gamma$ and $\pi^\pm p \to \pi^\pm p\gamma$ to the orders ω^{-1} (the pole term of [3]) and ω^0. Our results are strict consequences of QCD. We hope that they will be helpful also for experimentalists who are embarked to check soft-photon theorems. For $\pi^\pm p \to \pi^\pm p\gamma$ this could be done perhaps at COMPASS and for $\pi^\pm p \to \pi^\pm p\gamma^*(\to e^+e^-)$ in HADES at GSI [33]. For ALICE 3 [34] we would need the corresponding theoretical calculations for $pp \to pp\gamma$. Our methods are suited to calculate in a rigorous way the expansion of the amplitude to the orders ω^{-1} and ω^0. These calculations certainly will not be easy.

Note added. Recently, there appeared on the arXiv the paper [35] where our work of [20] was criticized. Some of this criticism is acceptable, but we have already discussed our misinterpretation of the soft-photon theorem of [1] in

Ref. [21]. In our present paper we again discuss extensively this misinterpretation. But, in this connection, a main topic of our paper is to show that the soft-photon theorems of [2,3] (to which there is no reference in [35]) and of [1] are different and should not be confounded. Of course, they are related, as we have discussed in Sec. V.

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APPENDIX A: THE EXPANSION OF $1/p_1' \cdot k$

In this Appendix we discuss the region of validity of the expansion (4.15). For this we have to consider $|l_1 \cdot k|/|p_1 \cdot k|$. Working in the c.m. system we have [see (2.3), (4.4) and (4.8)]

$$\begin{aligned} p_1 \cdot k &= p_1^0\omega - \boldsymbol{p_1}\cdot\boldsymbol{k} = \omega(p_1^0 - |\boldsymbol{p_1}|\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}) \\ &= \omega\left[\frac{m_\pi^2}{p_1^0+|\boldsymbol{p_1}|} + |\boldsymbol{p_1}|(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}})\right], \\ l_1\cdot k &= l_1^0\omega - \boldsymbol{l_1}\cdot\boldsymbol{k} = \omega(l_1^0 - \boldsymbol{l_1}\cdot\hat{\boldsymbol{k}}) \\ &= \omega\left[\frac{p_2\cdot k}{2}\left(\frac{1}{p_1^0} - \frac{1}{|\boldsymbol{p_1}|}\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}\right) - \boldsymbol{l}_{1\perp}\cdot\hat{\boldsymbol{k}}\right] \\ &= \omega\left[\frac{p_2\cdot k}{2}\frac{|\boldsymbol{p_1}|-p_1^0}{p_1^0|\boldsymbol{p_1}|} + \frac{p_2\cdot k}{2|\boldsymbol{p_1}|}(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}) - \boldsymbol{l}_{1\perp}\cdot\hat{\boldsymbol{k}}\right] \\ &= \omega^2\left[-\frac{m_\pi^2(p_1^0+\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}|\boldsymbol{p_1}|)}{2(p_1^0+|\boldsymbol{p_1}|)p_1^0|\boldsymbol{p_1}|} + \frac{p_1^0+\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}|\boldsymbol{p_1}|}{2|\boldsymbol{p_1}|}(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}) - \tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}\right]. \end{aligned} \tag{A1}$$

From (A1) we get

$$\frac{|l_1\cdot k|}{|p_1\cdot k|} \le \frac{\omega}{|\boldsymbol{p_1}|}\left[\frac{m_\pi^2}{p_1^0+|\boldsymbol{p_1}|}(2-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}) + |\boldsymbol{p_1}|(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}) + |\boldsymbol{p_1}||\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}|\right]\left[\frac{m_\pi^2}{p_1^0+|\boldsymbol{p_1}|} + |\boldsymbol{p_1}|(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}})\right]^{-1}. \tag{A2}$$

For $\tilde{\boldsymbol{l}}_{1\perp} = 0$ we have, therefore,

$$\frac{|l_1\cdot k|}{|p_1\cdot k|} = \mathcal{O}\left(\frac{\omega}{|\boldsymbol{p_1}|}\right) \tag{A3}$$

and the expansion (4.15) is valid for $\omega \ll |\boldsymbol{p_1}|$ and all $\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}$.

For $|\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}| \neq 0$ and $1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}} = \mathcal{O}(1)$ we still get (A3). But for $\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}} = 1$ we find from (A2) only

$$\frac{|l_1\cdot k|}{|p_1\cdot k|} \le \omega\left[\frac{1}{|\boldsymbol{p_1}|} + \frac{p_1^0+|\boldsymbol{p_1}|}{m_\pi^2}|\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}|\right]. \tag{A4}$$

In this case we have to require for the expansion (4.15) to be valid

$$\omega \le \frac{m_\pi^2|\boldsymbol{p_1}|}{m_\pi^2 + |\boldsymbol{p_1}|(p_1^0+|\boldsymbol{p_1}|)|\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}|}, \tag{A5}$$

which is very small for momenta $|\boldsymbol{p_1}| \gg m_\pi$.

To conclude, for $p_1' = p_1 - l_1$ the expansion

$$\frac{1}{(p_1-l_1,k)} = \frac{1}{p_1\cdot k}\left[1 + \frac{l_1\cdot k}{p_1\cdot k} + \mathcal{O}(\omega^2)\right] \tag{A6}$$

and thus (4.15) is alright under the following conditions.

(i) For $\hat{\boldsymbol{p}}_1' = \hat{\boldsymbol{p}}_1$ in the c.m. system we have to require from (A3)

$$\omega \ll |\boldsymbol{p_1}|. \tag{A7}$$

(ii) For $\hat{\boldsymbol{p}}_1' = \hat{\boldsymbol{p}}_1 - \boldsymbol{l}_{1\perp}/|\boldsymbol{p_1}|$ and $\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}} = 0$ the requirement is again (A7). For $\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}} \neq 0$ we get here from (A2) the condition

$$\omega \ll |\boldsymbol{p_1}|\left[\frac{m_\pi^2}{p_1^0+|\boldsymbol{p_1}|}+|\boldsymbol{p_1}|(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}})\right] \times\left[\frac{m_\pi^2}{p_1^0+|\boldsymbol{p_1}|}(2-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}})+|\boldsymbol{p_1}|(1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}) +|\boldsymbol{p_1}||\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}|\right]^{-1}. \tag{A8}$$

For $1-\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}\neq 0$ and of order 1 we get again (A7). But for $\hat{\boldsymbol{p}}_1\cdot\hat{\boldsymbol{k}}=1$ (A8) gives (A5) which is very small for $|\tilde{\boldsymbol{l}}_{1\perp}\cdot\hat{\boldsymbol{k}}|=\mathcal{O}(1)$ and $|\boldsymbol{p_1}|\gg m_\pi$.

APPENDIX B: VARIABLE TRANSFORMATION

We consider here the variable transformation (6.5) $\hat{\boldsymbol{p}}'_1\to\hat{\boldsymbol{p}}''_1$ for fixed $\hat{\boldsymbol{k}}$ and ω:

$$\hat{p}'_{1i}=\hat{p}''_{1i}-\frac{\omega}{|\boldsymbol{p_1}|}\left[\hat{k}_i\hat{p}''_{1j}\hat{p}''_{1j}-\hat{p}''_{1j}\hat{k}_j\hat{p}''_{1i}\right]. \tag{B1}$$

We have

$$(\hat{\boldsymbol{p}}'_1)^2=(\hat{\boldsymbol{p}}''_1)^2+\mathcal{O}(\omega^2), \tag{B2}$$

$$\frac{\partial\hat{p}'_{1i}}{\partial\hat{p}''_{1e}}=\delta_{ie}+\frac{\omega}{|\boldsymbol{p_1}|}\left[\hat{p}''_{1i}\hat{k}_e+\hat{\boldsymbol{p}}''_1\cdot\hat{\boldsymbol{k}}\delta_{ie}-2\hat{k}_i\hat{p}''_{1e}\right], \tag{B3}$$

$$\det\left(\frac{\partial\hat{p}'_{1i}}{\partial\hat{p}''_{1e}}\right)=1+\frac{2\omega}{|\boldsymbol{p_1}|}\hat{\boldsymbol{p}}''_1\cdot\hat{\boldsymbol{k}}+\mathcal{O}(\omega^2). \tag{B4}$$

Now we consider an arbitrary function $f(\hat{\boldsymbol{p}}'_1)$ setting

$$\tilde{f}(\hat{\boldsymbol{p}}''_1)=f(\hat{\boldsymbol{p}}'_1). \tag{B5}$$

We are interested in the following integral and its variable transformed:

$$\begin{aligned}\int \mathrm{d}\Omega_{\hat{p}'_1}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})f(\hat{\boldsymbol{p}}'_1) &= 2\int \mathrm{d}^3\hat{p}'_1\delta((\hat{\boldsymbol{p}}'_1)^2-1)J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})f(\hat{\boldsymbol{p}}'_1)\\ &= 2\int \mathrm{d}^3\hat{p}''_1\det\left(\frac{\partial\hat{p}'_{1i}}{\partial\hat{p}''_{1e}}\right)\delta((\hat{\boldsymbol{p}}''_1)^2-1)J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})\tilde{f}(\hat{\boldsymbol{p}}''_1)+\mathcal{O}(\omega^2)\\ &= 2\int \mathrm{d}^3\hat{p}''_1\delta((\hat{\boldsymbol{p}}''_1)^2-1)\left[1+\frac{2\omega}{|\boldsymbol{p_1}|}\hat{p}''_1\cdot\hat{\boldsymbol{k}}\right]J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})\tilde{f}(\hat{\boldsymbol{p}}''_1)+\mathcal{O}(\omega^2).\end{aligned} \tag{B6}$$

From (6.9) we find

$$\begin{aligned}\left[1+\frac{2\omega}{|\boldsymbol{p_1}|}\hat{\boldsymbol{p}}''_1\cdot\hat{\boldsymbol{k}}\right]J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) &= \frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}}-\frac{\omega}{\sqrt{s}}\left(\frac{2m_\pi^2}{\sqrt{s(s-4m_\pi^2)}}+\hat{\boldsymbol{p}}'_1\cdot\hat{\boldsymbol{k}}\right)+\frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}}\frac{2\omega}{|\boldsymbol{p_1}|}\hat{\boldsymbol{p}}''_1\cdot\hat{\boldsymbol{k}}+\mathcal{O}(\omega^2)\\ &= \frac{1}{2}\sqrt{1-\frac{4m_\pi^2}{s}}-\frac{\omega}{\sqrt{s}}\frac{2m_\pi^2}{\sqrt{s(s-4m_\pi^2)}}+\frac{\omega}{\sqrt{s}}\hat{\boldsymbol{p}}''_1\cdot\hat{\boldsymbol{k}}+\mathcal{O}(\omega^2)\\ &= J^{(1)}(s,\omega,-\hat{\boldsymbol{p}}''_1,\hat{\boldsymbol{k}})+\mathcal{O}(\omega^2)\\ &= J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_2,\hat{\boldsymbol{k}})+\mathcal{O}(\omega^2),\end{aligned} \tag{B7}$$

where we use $\hat{\boldsymbol{p}}'_2=-\hat{\boldsymbol{p}}''_1$; see (6.4).

Inserting (B7) in (B6) we obtain

$$\int \mathrm{d}\Omega_{\hat{p}'_1}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}})f(\hat{\boldsymbol{p}}'_1)=\int \mathrm{d}\Omega_{\hat{p}''_1}J^{(1)}(s,\omega,-\hat{\boldsymbol{p}}''_1,\hat{\boldsymbol{k}})\tilde{f}(\hat{\boldsymbol{p}}''_1)+\mathcal{O}(\omega^2). \tag{B8}$$

Since the function $f(\hat{\boldsymbol{p}}'_1)$ was arbitrary, we get the following transformation of the measures:

$$\begin{aligned}\mathrm{d}\Omega_{\hat{p}'_1}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_1,\hat{\boldsymbol{k}}) &= \mathrm{d}\Omega_{\hat{p}''_1}J^{(1)}(s,\omega,-\hat{\boldsymbol{p}}''_1,\hat{\boldsymbol{k}})+\mathcal{O}(\omega^2)\\ &= \mathrm{d}\Omega_{\hat{p}'_2}J^{(1)}(s,\omega,\hat{\boldsymbol{p}}'_2,\hat{\boldsymbol{k}})+\mathcal{O}(\omega^2).\end{aligned} \tag{B9}$$

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