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Editors' Suggestion

Five-parton scattering in QCD at two loops

Bakul Agarwal,[1,*] Federico Buccioni,[2,†] Federica Devoto,[3,‡] Giulio Gambuti,[3,4,§] Andreas von Manteuffel,[5,6,∥] and Lorenzo Tancredi[2,¶]

[1]*Institute for Theoretical Physics, Karlsruhe Institute of Technology (KIT), D-76131 Karlsruhe, Germany*
[2]*Physik Department, James-Franck-Straße 1, Technische Universität München, D–85748 Garching, Germany*
[3]*Rudolf Peierls Centre for Theoretical Physics, University of Oxford, Clarendon Laboratory, Parks Road, Oxford OX1 3PU, United Kingdom*
[4]*New College, University of Oxford, Holywell Street, Oxford OX1 3BN, United Kingdom*
[5]*Institut für Theoretische Physik, Universität Regensburg, 93040 Regensburg, Germany*
[6]*Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824, USA*

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We compute all helicity amplitudes for the scattering of five partons in two-loop QCD in all the relevant flavor configurations, retaining all contributing color structures. We employ tensor projection to obtain helicity amplitudes in the 't Hooft-Veltman scheme starting from a set of primitive amplitudes. Our analytic results are expressed in terms of massless pentagon functions, and are easy to evaluate numerically. These amplitudes provide important input to investigations of soft-collinear factorization and to studies of the high-energy limit.

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I. INTRODUCTION

The calculation of scattering amplitudes for n partons in perturbative quantum chromodynamics (QCD) has attracted much attention since the discovery of the ubiquitous role of Yang-Mills theories in the description of particle interactions. These amplitudes constitute the building blocks for the calculation of cross sections for processes involving jets at hadron colliders, which have played a crucial role in providing direct experimental access to fundamental parameters of QCD such as the number of colors and the strong coupling constant. At the LHC, the measurement of multijet cross sections at large transverse momenta constitutes a unique opportunity to explore QCD dynamics in extreme regimes [1–4]. Such high-precision analyses need to be matched by accurate theoretical predictions which as of today have been carried out to second order in perturbative QCD [5–9]. In addition, analytic calculations of n-parton scattering amplitudes provide important insights into the fundamental properties of Yang-Mills theories, such as their high-energy (Regge) limit (see, e.g., [10]) or the universal structure of infrared divergences and factorization in soft and collinear limits (see, e.g., [11]).

Feynman diagram based calculations for multiparton scattering amplitudes become challenging at higher orders in perturbation theory, and the relative simplicity of their results is often obscured by the complexity of the intermediate expressions. More recently, new methods have been developed to exploit this simplicity and render higher order calculations manageable. Standard techniques based on integration-by-parts identities (IBPs) [12–14] and differential equations [15–17] have been augmented by finite-field methods [18,19] and the use of canonical bases [20] to partly bypass heavy use of computer algebra and enable new ways to calculate loop integrals and amplitudes. Moreover, a better understanding of the mathematical properties of special functions [21–24] defined as iterated integrals [25] has made it possible to devise efficient techniques for analytic and numerical evaluation of the ensuing integrals.

Thanks to these developments, a large number of previously unthinkable calculations have become possible, opening the way to entire new opportunities to test perturbative QCD. In particular, QCD form factors have now been computed to an astonishing four loops [26,27], scattering amplitudes for four strongly interacting partons to three loops [28–30], and five-parton scattering up to two

*bakul.agarwal@kit.edu
†federico.buccioni@tum.de
‡federica.devoto@physics.ox.ac.uk
§giulio.gambuti@physics.ox.ac.uk
∥manteuffel@ur.de
¶lorenzo.tancredi@tum.de

loops, largely in the leading-color approximation [31–37]. For the latter, while all the ingredients have been available for some time, the corresponding sub-leading-color contributions have remained elusive given the algebraic complexity introduced by the nonplanar diagrams. More recently, full-color calculations have been completed for five-particle scattering processes involving fewer colored particles [38–41].

In this paper, we remove the last remaining roadblock and address the calculation of complete two-loop corrections for the scattering of five partons in QCD. We consider all relevant partonic channels, i.e., the scattering of five gluons, of two quarks and three gluons, and of four quarks and one gluon, both for identical and different quark flavors. Employing a combination of sophisticated computational methods, we derive analytic results expressed in terms of relatively simple rational functions and so-called pentagon functions for massless particles [42].

II. KINEMATICS AND COLOR DECOMPOSITION

We consider the processes

$$\begin{aligned} &0 \to g(p_1) + g(p_2) + g(p_3) + g(p_4) + g(p_5), \\ &0 \to \bar{q}(p_1) + q(p_2) + g(p_3) + g(p_4) + g(p_5), \\ &0 \to \bar{q}(p_1) + q(p_2) + \bar{q}'(p_3) + q'(p_4) + g(p_5), \end{aligned} \tag{1}$$

where all momenta are outgoing and massless,

$$p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu + p_5^\mu = 0, \qquad p_i^2 = 0, \tag{2}$$

and q and q' correspond to strictly different flavors of quarks. All other channels, including those involving four identical quarks, can be reconstructed from these by suitable permutations of the external momenta, as described below.

The five-point kinematics can be parametrized by five independent Mandelstam invariants $s_{ij} \equiv (p_i + p_j)^2$. We choose the cyclic set

$$s_{12}, s_{23}, s_{34}, s_{45}, s_{51}, \tag{3}$$

and identify the *physical region* with

$$s_{12}, s_{34}, s_{45} > 0, \qquad s_{23}, s_{51} < 0, \tag{4}$$

which corresponds to the $12 \to 345$ scattering process.

In order to describe all relevant helicity configurations we also employ the quantity

$$\mathrm{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma} p_1^\mu p_2^\nu p_3^\rho p_4^\sigma, \tag{5}$$

which has nontrivial transformation properties under parity, inherited from its definition through the Levi-Civita tensor as given in (5).

We employ *dimensional regularization* to regulate both ultraviolet (UV) and infrared (IR) divergences. Specifically, we employ the ’t Hooft-Veltman scheme (tHV) [43], which treats loop momenta in $d = 4 - 2\epsilon$ dimensions, while retaining momenta and polarizations for external particles in four dimensions.

The bare amplitudes for each process in (1) can be decomposed onto a set of color tensors $\mathcal{C}_c$ as

$$A = (4\pi\alpha_{s,b})^{\frac{3}{2}} \boldsymbol{\mathcal{A}} \cdot \boldsymbol{\mathcal{C}} = (4\pi\alpha_{s,b})^{\frac{3}{2}} \mathcal{A}_c \mathcal{C}_c, \tag{6}$$

where $\alpha_{s,b}$ is the bare strong coupling. In (6) $\mathcal{A}_c$ are color-ordered *partial amplitudes*, $\mathcal{C}_c$ the corresponding elements of the color tensor basis, and a summation over the index c is implied. Let us introduce the shorthand notation

$$\begin{aligned} t^{mn} &= \mathrm{Tr}[T^{a_m}T^{a_n}], \qquad t^{mnl}_{i_1 i_2} = (T^{a_m}T^{a_n}T^{a_l})_{i_1 i_2}, \\ t^{mn\ldots k} &= \mathrm{Tr}[T^{a_m}T^{a_n}\ldots T^{a_k}] - \mathrm{Tr}[T^{a_k}\ldots T^{a_n}T^{a_m}], \end{aligned} \tag{7}$$

where the $SU(N_c)$ adjoint index a_n is associated with the n-th external gluon, and the (anti)fundamental index i_n with the corresponding (anti)quark state. The matrices T^a_{ij} are generators of $SU(N_c)$ in the fundamental representation, and they obey the normalization condition $\mathrm{Tr}[T^aT^b] = \delta^{ab}/2$. With these definitions, the color basis for $ggggg$ reads as

$$\begin{aligned} \{\mathcal{C}_c\}_{c=1,\ldots,12} = \{&t^{12345}, t^{12354}, t^{12435}, t^{12453}, t^{12534}, \\ &t^{12543}, t^{13245}, t^{13254}, t^{13425}, t^{13524}, t^{14235}, t^{14325}\}, \\ \{\mathcal{C}_c\}_{c=13,\ldots,22} = \{&t^{12}t^{345}, t^{45}t^{123}, t^{35}t^{124}, t^{34}t^{125}, \\ &t^{13}t^{245}, t^{25}t^{134}, t^{24}t^{135}, t^{14}t^{235}, t^{23}t^{145}, t^{15}t^{234}\}, \end{aligned} \tag{8}$$

for $\bar{q}qggg$

$$\begin{aligned} \{\mathcal{C}_c\}_{c=1,\ldots,6} &= \{t^{345}_{i_1 i_2}, t^{453}_{i_1 i_2}, t^{534}_{i_1 i_2}, t^{354}_{i_1 i_2}, t^{543}_{i_1 i_2}, t^{435}_{i_1 i_2}\}, \\ \{\mathcal{C}_c\}_{c=7,8,9} &= \{T^{a_3}_{i_1 i_2}t^{45}, T^{a_4}_{i_1 i_2}t^{53}, T^{a_5}_{i_1 i_2}t^{34}\}, \\ \mathcal{C}_{10,11} &= \delta_{i_1 i_2}(\mathrm{Tr}[T^{a_3}T^{a_4}T^{a_5}] \mp \mathrm{Tr}[T^{a_5}T^{a_4}T^{a_3}]), \end{aligned} \tag{9}$$

and finally for $\bar{q}q\bar{q}'q'g$

$$\begin{aligned} \mathcal{C}_1 &= \delta_{i_2 i_3}T^{a_5}_{i_1 i_4}, \qquad \mathcal{C}_2 = \delta_{i_1 i_2}T^{a_5}_{i_3 i_4}, \\ \mathcal{C}_3 &= \delta_{i_3 i_4}T^{a_5}_{i_1 i_2}, \qquad \mathcal{C}_4 = \delta_{i_1 i_4}T^{a_5}_{i_3 i_2}. \end{aligned} \tag{10}$$

III. HELICITY AMPLITUDES

We consider the scattering amplitudes for fixed helicity configurations of the external particles. We work in the spinor-helicity formalism and define the polarization vectors for gluons as

$$\epsilon^{\mu}_{i,-} = \frac{[i+1|\gamma^{\mu}|i\rangle}{\sqrt{2}[i|i+1]}, \qquad \epsilon^{\mu}_{i,+} = \frac{[i|\gamma^{\mu}|i+1\rangle}{\sqrt{2}\langle i+1|i\rangle}, \tag{11}$$

and the spinors for (anti)quarks as

$$\bar{u}_{i,-} = \langle i|, \quad \bar{u}_{i,+} = [i|, \quad u_{i,-} = |i\rangle, \quad u_{i,+} = |i]. \tag{12}$$

Helicities are given in the all-outgoing convention.

We define spinor-stripped helicity amplitudes $\mathcal{H}_c(\boldsymbol{\lambda})$ as

$$\mathcal{A}_c(\boldsymbol{\lambda}) = \sqrt{2}\Phi_c(\boldsymbol{\lambda})\mathcal{H}_c(\boldsymbol{\lambda}), \tag{13}$$

where $\Phi_c(\boldsymbol{\lambda})$ is a color and helicity dependent spinor factor that fully accounts for the little-group scaling of the corresponding amplitude. The scalar quantities $\mathcal{H}_c(\boldsymbol{\lambda})$ can be further split into their parity even and odd parts,

$$\mathcal{H}_c(\boldsymbol{\lambda}) = \mathcal{H}^E_c(\boldsymbol{\lambda}) + \mathrm{tr}_5\mathcal{H}^O_c(\boldsymbol{\lambda}) \tag{14}$$

that are individually gauge invariant and can be computed independently. We will only consider a minimal set of helicity and color configurations needed to reconstruct the whole amplitude via crossings of the external states. These are listed in Table I along with the corresponding spinor factors.

The spinor-stripped helicity amplitudes contain both ultraviolet (UV) and infrared (IR) divergences, which manifest as poles in the dimensional regulator ϵ. UV divergences can be removed by expressing the amplitudes in terms of the $\overline{\mathrm{MS}}$ renormalized strong coupling $\alpha_s(\mu)$

$$\alpha_{s,b}\mu_0^{2\epsilon}\, S_\epsilon = \alpha_s(\mu)\,\mu^{2\epsilon}\, Z[\alpha_s(\mu)], \tag{15}$$

where μ_0 and μ are the regularization and renormalization scales respectively, and $S_\epsilon = (4\pi)^\epsilon e^{-\epsilon\gamma_E}$, with $\gamma_E \approx 0.5772$ the Euler constant. Up to two loops, the renormalization factor Z reads as

$$Z[\alpha_s] = 1 - \frac{\alpha_s}{4\pi}\frac{\beta_0}{\epsilon} + \left(\frac{\alpha_s}{4\pi}\right)^2\left(\frac{\beta_0^2}{\epsilon^2} - \frac{\beta_1}{2\epsilon}\right), \tag{16}$$

with the β-function coefficients

$$\beta_0 = \frac{11}{3}N_c - \frac{2}{3}N_f,$$
$$\beta_1 = \frac{34}{3}N_c^2 - \frac{10}{3}N_cN_f - \frac{N_c^2-1}{N_c}N_f, \tag{17}$$

where N_c is the number of colors and N_f the number of light fermions. The renormalized helicity amplitudes can be expanded as a perturbative series in the renormalized strong coupling

TABLE I. Definition of the minimal set of helicity and color configurations for the different processes. The list of helicity configuration is given in the left-most column of each table. The middle columns contain our definition of the spinor factors $\Phi_c(\boldsymbol{\lambda})$ for the different color factors. The right columns give the list of partial amplitudes we computed analytically for the corresponding helicity configuration. Spinor factors are chosen to set the relative tree level (when nonvanishing) to 1. Because of this, two spinor factor choices are needed for the $\bar{q}q\bar{q}'q'g$ channel.

$ggggg$	$\Phi(\boldsymbol{\lambda})$	$\mathcal{A}_c$
$+++++$	$\frac{2s_{12}^2/3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	1, 13
$-++++$	$\frac{[21]\langle 12\rangle^4\langle 13\rangle^3}{\langle 15\rangle^2\langle 23\rangle^5\langle 14\rangle^2}$	1, 13
$--+++$	$\frac{4\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	1, 13
$-+-++$	$\frac{4\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	1, 13
$++++-$	$\frac{[51]\langle 15\rangle^4\langle 25\rangle^3}{\langle 54\rangle^2\langle 12\rangle^5\langle 53\rangle^2}$	13
$+++--$	$\frac{4\langle 45\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	13

$\bar{q}qggg$	$\Phi(\boldsymbol{\lambda})$	$\mathcal{A}_c$
$+-++-$	$\frac{2\langle 15\rangle\langle 52\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	1, 7, 10, 11
$+-+-+$	$\frac{2\langle 14\rangle\langle 42\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	1
$+--++$	$\frac{2\langle 13\rangle\langle 32\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}$	1
$+----$	$\frac{2\langle 23\rangle[31]}{[34][45][53]}$	1, 7, 10, 11
$+-+--$	$\frac{2[23][31]^3}{[12][23][34][45][51]}$	7, 10, 11

$\bar{q}q\bar{q}'q'g$	$\Phi(\boldsymbol{\lambda})$	$\mathcal{A}_c$	$\Phi(\boldsymbol{\lambda})$	$\mathcal{A}_c$
$-+-+-$	$\frac{[14][42]^2}{[12][43][51][54]}$	1	$\frac{[42]^2}{[12][53][54]}$	2
$-+-++$	$\frac{\langle 41\rangle\langle 13\rangle^2}{\langle 15\rangle\langle 21\rangle\langle 34\rangle\langle 45\rangle}$	1	$\frac{\langle 13\rangle^2}{\langle 21\rangle\langle 35\rangle\langle 45\rangle}$	2
$-++--$	$\frac{[32]^2[41]}{[12][43][51][54]}$	1	$\frac{[32]^2}{[21][53][54]}$	2
$-++-+$	$\frac{\langle 14\rangle^3}{\langle 15\rangle\langle 21\rangle\langle 34\rangle\langle 45\rangle}$	1	$\frac{\langle 14\rangle^2}{\langle 12\rangle\langle 35\rangle\langle 45\rangle}$	2

$$\boldsymbol{\mathcal{A}}(\boldsymbol{\lambda}) = \sum_{\ell=0}^{2}\left(\frac{\alpha_s}{4\pi}\right)^{\ell}\boldsymbol{\mathcal{A}}^{(\ell)}_{\boldsymbol{\lambda},\mathrm{ren}} + \mathcal{O}(\alpha_s^3), \tag{18}$$

where $\boldsymbol{\mathcal{A}}^{(\ell)}_{\boldsymbol{\lambda},\mathrm{ren}}$ is the renormalized ℓ-loop contribution. These still contain IR singularities, which can be subtracted by defining their finite remainders at each loop order as

$$\boldsymbol{\mathcal{A}}^{(0)}_{\boldsymbol{\lambda},\mathrm{fin}} = \boldsymbol{\mathcal{A}}^{(0)}_{\boldsymbol{\lambda}}, \qquad \boldsymbol{\mathcal{A}}^{(1)}_{\boldsymbol{\lambda},\mathrm{fin}} = \boldsymbol{\mathcal{A}}^{(1)}_{\boldsymbol{\lambda},\mathrm{ren}} - \boldsymbol{\mathcal{I}}_1(\epsilon)\boldsymbol{\mathcal{A}}^{(0)}_{\boldsymbol{\lambda},\mathrm{ren}},$$
$$\boldsymbol{\mathcal{A}}^{(2)}_{\boldsymbol{\lambda},\mathrm{fin}} = \boldsymbol{\mathcal{A}}^{(2)}_{\boldsymbol{\lambda},\mathrm{ren}} - \boldsymbol{\mathcal{I}}_2(\epsilon)\boldsymbol{\mathcal{A}}^{(0)}_{\boldsymbol{\lambda},\mathrm{ren}} - \boldsymbol{\mathcal{I}}_1(\epsilon)\boldsymbol{\mathcal{A}}^{(1)}_{\boldsymbol{\lambda},\mathrm{ren}}. \tag{19}$$

The color operators $\boldsymbol{\mathcal{I}}_1$ and $\boldsymbol{\mathcal{I}}_2$ were first defined in Ref. [44] and then in [45,46]. A precise definition of the operators $\boldsymbol{\mathcal{I}}_{1,2}$ is provided in the Appendix.

IV. DETAILS OF THE CALCULATION

To compute the helicity amplitudes up to two loops, we proceed as follows. We generate all Feynman diagrams for (1) using QGRAF [47]. At two loops there are a total of 28020, 9136, and 2129 diagrams for the three channels respectively. However, not all of them contribute to the independent color structures described in Eqs. (8)–(10). The relevant diagrams can be selected by repeated application of the color identities

$$\mathrm{Tr}T^a = 0, \qquad T^a_{ij}T^a_{kh} = \frac{1}{2}\left(\delta_{ih}\delta_{kj} - \frac{1}{N_c}\delta_{ij}\delta_{kh}\right),$$
$$f^{abc} = -2i\mathrm{Tr}(T^a[T^b, T^c]), \tag{20}$$

which reduce their color structure to a linear combination of the basis elements. From here, one can easily read off the contribution of the corresponding diagram to each element of the vector $\boldsymbol{\mathcal{A}}$. We then compute the contribution of each diagram to the helicity amplitudes using projectors in the tHV scheme as in Refs. [48,49]. For each process we define helicity projectors $P_{\lambda,c}$ as

$$P_{\lambda,c} \circ \mathcal{A}_c = \mathcal{H}_c(\boldsymbol{\lambda}), \tag{21}$$

where the operation "∘" stands for summation over polarizations. The $P_{\lambda,c}$ can be identified as follows. Each partial amplitude can be decomposed as

$$\mathcal{A}_c(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \mathcal{F}_{i,c} T_i(\boldsymbol{\lambda}), \tag{22}$$

where the T_i are a set of tensor structures containing all polarization vectors, and the $F_{i,c}$ are scalar *form factors* for the different color structures. While the form of the decomposition is loop independent, the form factors themselves have a perturbative expansion and, at a given perturbative order, are linear combinations of scalar Feynman integrals. In our scheme, the number of tensor structures N always equals the number of helicity amplitudes [48,49]. These are 32, 16, and 8 for the processes listed in (1), respectively. In practice this means that after numbering the helicity configurations $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots$ for each process, we can change to a new basis of tensors $\bar{T}_{i,c}(\boldsymbol{\lambda})$, which satisfy

$$\bar{T}_{i,c}(\boldsymbol{\lambda} = \boldsymbol{\lambda_j}) = \delta_{ij}\Phi_c(\boldsymbol{\lambda}_j). \tag{23}$$

This allows us to define the helicity projectors $P_{\lambda,c}$ by requiring

$$P_{\lambda,c} \circ \bar{T}_{i,c}(\boldsymbol{\lambda}') = \delta_{\lambda,\lambda'}. \tag{24}$$

The Lorentz and color algebra required to isolate the partial amplitudes and to apply the projectors are performed with FORM [50,51].

As a result, the spinor-stripped helicity amplitudes $\mathcal{H}(\boldsymbol{\lambda})$ in (13) are expressed as linear combination of scalar Feynman integrals. All L-loop integrals required for the evaluation of these amplitudes are of the usual form,

$$\mathcal{I}^{\mathrm{fam}}_{n_1,\ldots,n_N} = \mu_0^{2L\epsilon} e^{L\epsilon\gamma_E} \int \prod_{i=1}^{L} \left(\frac{\mathrm{d}^d k_i}{i\pi^{\frac{d}{2}}}\right) \frac{1}{D_1^{n_1}\ldots D_N^{n_N}}, \tag{25}$$

where the loop momenta are labeled by k_i. The label "fam" represents one of the two integral families $\{A, B\}$ as well as their crossed versions, which differ by a permutation of the external particles. Every integral family specifies the list of inverse propagators $D_e = q_e^2 + i\varepsilon$ in Eq. (25), where ε implements the Feynman prescription. The definitions of these integral families is immaterial to the discussion, but we report them in the Appendix for completeness. Collectively, across the various primitive amplitudes, we are left with $\sim\mathcal{O}(10^6)$ different scalar integrals.

It is well known that Feynman integrals satisfy many linear relations which can be obtained via symmetry relations and IBPs. These can be used to express all Feynman integrals for a given family in terms of a set of basis integrals, referred to as *master integrals*. For the processes considered in this paper, the set of master integrals was computed in Refs. [52–57] via the method of differential equations, and expressed as a Laurent series in the dimensional regulator ϵ. We use the uniform representation from Ref. [42] in terms of massless *pentagon functions*.

In practice, we first derive shift relations and sector symmetries using Reduze 2 [58,59]. This allows us to reduce the number of integrals appearing in the unreduced amplitudes by 2 orders of magnitude. We then perform IBP reduction using the public code Kira [60,61] and, specifically for the nonplanar topologies, the code Finred. The latter is an in-house implementation of Laporta's algorithm which employs finite field techniques [18,19,62,63] enhanced with denominator guessing from a rational sample [64] (see [31] for an alternative approach) for an efficient reconstruction of multivariate rational functions and syzygy algorithms [65–70].

Although the number of integrals to reduce is substantially smaller compared to other challenging multiloop QCD calculations with fewer scales, see, e.g., [27,29,30,71–73], the multiscale kinematics of five-point amplitudes is responsible for a large swell in intermediate expressions, in particular in the nonplanar sectors. In fact in the course of this calculation one encounters individual IBP identities with disk sizes of up to 3 GB [38,74,75]. A possible way around this consists in avoiding the reconstruction of IBP identities from finite-field samples for individual integrals and attempting, instead, to directly obtain the analytic expressions for the IR subtracted finite remainders of the various helicity amplitudes. As the finite remainders are expected to be simpler, this strategy has been very successful in many state-of-the-art five-point calculations.

In this work, we follow a different approach [38,75], based on the observation that partial fraction decomposition can reduce expression size for single IBPs of various orders of magnitude. We perform IBP reduction for a minimal subset of integrals and decompose the resulting integral coefficients into multivariate partial fractions. Reduction identities for all integrals contributing to the amplitudes are then obtained using crossings of external invariants. To perform the partial-fraction decomposition, we utilize MultivariateApart [64] augmented by Singular [76]. This step requires first defining an appropriate ordering among the denominator factors that appear in the individual color factors of each helicity amplitudes. As expected, the complexity of the final expressions depends greatly on this choice of monomial ordering. We find it beneficial, in particular, to choose an ordering that avoids as much as possible the appearance of spurious singularities in the denominators; for more details see Refs. [38,75]. Once the simplified reduction identities are inserted into the amplitude, we perform an additional partial-fraction decomposition to simplify the resulting expressions further; in addition to the IBP reduction, this proves to be the step with the highest computational cost in our framework. Once the bare amplitudes have been computed with a consistent choice of monomial ordering, UV renormalization and IR subtraction can be performed to yield relatively compact expressions for the finite remainders.

A further level of simplification can be achieved by expressing the finite remainders in terms of a minimal set of independent rational functions. Following Refs. [38,75], we start from a partial-fractioned form of the rational functions based on a fixed monomial ordering. We exploit the uniqueness of this representation in order to search for linear relations among the rational functions, which we solve using row reductions. As a measure of complexity we use the byte size of the rational functions, and we reduce the system of linear relations to express more "complicated" functions in terms of simpler ones; see Ref. [38] for more details. This final step allows us to obtain an improvement of around 1 order of magnitude in the disk size of independent color-ordered primitive helicity amplitudes.

V. FINAL RESULTS AND CONCLUSIONS

The color and helicity configurations listed in Eqs. (8)–(10), and Table I for the processes in Eq. (1) are sufficient to reconstruct all possible partonic channels. In particular, using parity and charge conjugation, as well as permutations of the external particles, one can obtain all amplitudes for the processes

$$
\begin{aligned}
&gg\to ggg, \quad q\bar{q}\to ggg, \quad qg\to qgg, \quad gg\to q\bar{q}g, \\
&q\bar{q}\to \bar{q}'q'g, \quad q\bar{q}'\to \bar{q}'qg, \quad \bar{q}'\bar{q}\to \bar{q}'\bar{q}g, \quad g\bar{q}\to \bar{q}'q'\bar{q}.
\end{aligned} \tag{26}
$$

The same-flavor quark amplitudes

$$
q\bar{q}\to \bar{q}qg, \qquad qq\to qqg, \qquad qg\to qq\bar{q} \tag{27}
$$

can then be obtained as linear combinations of appropriate components of the different-flavor quark amplitudes. More specifically, in the notation of Eq. (13) we can write

$$
\begin{aligned}
\boldsymbol{\mathcal{A}}_{q\bar{q}\to q\bar{q}g}(\boldsymbol{\lambda}) &= \boldsymbol{\mathcal{A}}_{q\bar{q}'\to q\bar{q}'g}(\boldsymbol{\lambda}) - \boldsymbol{\mathcal{A}}_{q\bar{q}\to q'\bar{q}'g}(\boldsymbol{\lambda}), \\
\boldsymbol{\mathcal{A}}_{qq\to qqg}(\boldsymbol{\lambda}) &= \boldsymbol{\mathcal{A}}_{qq'\to qq'g}(\boldsymbol{\lambda}) - \boldsymbol{\mathcal{A}}_{qq'\to q'qg}(\boldsymbol{\lambda}), \\
\boldsymbol{\mathcal{A}}_{qg\to qq\bar{q}}(\boldsymbol{\lambda}) &= \boldsymbol{\mathcal{A}}_{qg\to qq'\bar{q}'}(\boldsymbol{\lambda}) - \boldsymbol{\mathcal{A}}_{qg\to q'q\bar{q}'}(\boldsymbol{\lambda}),
\end{aligned} \tag{28}
$$

where all channels on the rhs of these identities are either in Eq. (26) or can be obtained from them by crossings which do not require analytic continuation.

Crossings of rational functions and spinor factors amount to a simple renumbering of momenta and their helicities (including flipping of helicities from incoming to outgoing states). Crossing of the transcendental functions, on the other hand, may require a dedicated analytic continuation.

We note that the analytic continuation of each pentagon function individually is nontrivial but also not necessary. In fact, the information needed to perform the required continuation is available implicitly in the results provided by Ref. [42], where all master integrals are evaluated for all 120 permutations of the external invariants in terms of a minimal set of pentagon functions. In practice, we took every uncrossed master integral and applied the required crossing on its analytic expression by formally crossing all appearing pentagon functions. We then equated these formal expressions to the crossed master integrals available in Ref. [42], which are written in terms of uncrossed pentagon functions. Repeating this for all master integrals for a given crossing, we obtained a linear system of equations, which we solved using FiniteFlow [19] to express the crossed pentagon functions in terms of the uncrossed ones. Typically, the system is underdetermined, and some crossed pentagon functions remain unsolved for. Nevertheless, since the results in Ref. [42] are, by construction, sufficient to represent any crossing of the amplitudes, all remaining crossed pentagon functions must cancel upon inserting these relations in the amplitude. We verified this cancellation explicitly for each crossing required to obtain the helicity and color-ordered amplitudes for all partonic subchannels. This provided a strong check of the consistency of our procedure.

We performed numerous checks on our results. First, we observed full cancellation of UV and IR poles of the bare amplitudes after UV renormalization and IR subtraction. For the five-gluon channel, we also verified the one- and two-loop $U(1)$ decoupling identities as well as the generalized color-trace identities described in [77]. For the same channel, we also computed a redundant set of single-trace partial amplitudes; the crossing relations among them allowed us to verify the consistency of our calculation at the level of finite remainders. We also compared our

TABLE II. Benchmark results for the interference of the tree level with the one- and two-loop finite remainders (first and third columns) and for the squared one-loop finite remainder (second column). $\overline{\sum}$ refers to summation over color and helicity states, and normalization over the corresponding leading order term.

	$\overline{\sum} 2\mathrm{Re}[\boldsymbol{\mathcal{A}}^{0\dagger}\boldsymbol{\mathcal{A}}^{1}_{\mathrm{fin}}]$	$\lvert\boldsymbol{\mathcal{A}}^{1}_{\mathrm{fin}}\rvert^2$	$\overline{\sum} 2\mathrm{Re}[\boldsymbol{\mathcal{A}}^{0\dagger}\boldsymbol{\mathcal{A}}^{2}_{\mathrm{fin}}]$
$gg \to ggg$	−90.64321	3348.355	2856.837
$q\bar{q} \to ggg$	−115.3289	3939.841	3833.951
$qg \to qgg$	−74.31499	1917.467	1195.185
$gg \to q\bar{q}g$	−72.79952	3093.624	1503.403
$q\bar{q} \to \bar{q}'q'g$	−101.6531	3271.088	2511.430
$q\bar{q}' \to \bar{q}'qg$	−82.09317	4338.144	−768.0230
$\bar{q}'\bar{q} \to \bar{q}'\bar{q}g$	−47.41403	769.2739	82.75641
$g\bar{q} \to \bar{q}'q'\bar{q}$	−57.86782	1181.730	1341.638
$q\bar{q} \to \bar{q}qg$	−88.39101	3926.462	379.8467
$qq \to qqg$	−45.63443	767.2815	94.00947
$qg \to qq\bar{q}$	−71.33829	1686.104	1626.085

tree-level and one-loop results against existing analytic calculations [78,79] as well as OpenLoops 2 [80], by numerically evaluating all helicity configurations of the channels listed in (26) and (27). Finally, we found perfect agreement with the analytic two-loop full-color all-plus gluon amplitude of Ref. [36] and with the numerical benchmarks in the leading-color approximation provided in Ref. [37] for all channels in (26).

In Table II we present benchmark results for all the relevant partonic channels. We provide the finite remainder of the squared matrix elements summed over color and helicities, normalized by the corresponding leading order term. We choose the kinematic configuration

$$s_{12} = 10^6, \qquad s_{23} = -761244.13, \qquad s_{34} = 865719.14,$$
$$s_{45} = 126204.05, \qquad s_{51} = -29885.560, \qquad \mu^2 = 10^4, \qquad (29)$$

and fix $N_c = 3$ and $N_f = 5$. To evaluate the pentagon functions we use the PentagonMI package [42].

To conclude, in this Letter we have presented the calculation of the two-loop corrections to five-parton scattering in massless QCD, retaining full color dependence. Our calculation leveraged many state-of-the-art techniques in the evaluation of multiloop scattering amplitudes, including the helicity projector technique in the ’t Hooft-Veltman scheme, finite field and syzygy based reduction algorithms, and multivariate partial-fraction decomposition. We considered all relevant partonic channels and derived compact analytic results, that can be easily evaluated numerically for physical scattering kinematics. The amplitudes presented here constitute the last missing building block to obtain full-color Next-to-next-to Leading-Order (NNLO) predictions for three-jet observables at the LHC. Moreover, they can furnish important information to study multi-Regge kinematics in QCD and to investigate collinear factorization breaking [81,82].

Note added. During the final stages of completion of this project, we have become aware of another concurrent calculation of the gluonic [83] and quark processes [84]. While the two calculations have been performed in two different infrared subtraction schemes, we have verified that, after scheme change, the results agree numerically to high precision.

The supporting data for this paper are openly available from [85].

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APPENDIX

1. Integral families

We provide here the definition our two integral families. We recall that for each family fam $= \{A, B\}$ we define the integrals as

$$\mathcal{I}^{\mathrm{fam}}_{n_1,\ldots,n_{11}} = e^{2\epsilon\gamma_E} \int \prod_{i=1}^{2} \left(\frac{d^d k_i}{i\pi^{d/2}} \right) \frac{1}{D_1^{n_1} \ldots D_{11}^{n_{11}}}, \qquad (A1)$$

where $d = 4 - 2\epsilon$ is the space-time dimension, $\gamma_E \sim 0.5772$ is the Euler-Mascheroni constant, and the D_i are the propagators. For the two families, the propagators read (see, e.g., Refs. [38,53,75]) as

	Family A	Family B
D_1	k_1^2	k_1^2
D_2	$(k_1 + p_1)^2$	$(k_1 - p_1)^2$
D_3	$(k_1 + p_1 + p_2)^2$	$(k_1 - p_1 - p_2)^2$
D_4	$(k_1 + p_1 + p_2 + p_3)^2$	$(k_1 - p_1 - p_2 - p_3)^2$
D_5	k_2^2	k_2^2
D_6	$(k_2 + p_1 + p_2 + p_3)^2$	$(k_2 - p_1 - p_2 - p_3 - p_4)^2$
D_7	$(k_2 + p_1 + p_2 + p_3 + p_4)^2$	$(k_1 - k_2)^2$
D_8	$(k_1 - k_2)^2$	$(k_1 - k_2 + p_4)^2$
D_9	$(k_1 + p_1 + p_2 + p_3 + p_4)^2$	$(k_2 - p_1)^2$
D_{10}	$(k_2 + p_1)^2$	$(k_2 - p_1 - p_2)^2$
D_{11}	$(k_2 + p_1 + p_2)^2$	$(k_2 - p_1 - p_2 - p_3)^2$

where we suppressed the $+i\varepsilon$ from Feynman's prescription everywhere. All required crossed families are obtained from the two reference families above by permutations of the external momenta.

2. Infrared subtraction operators

In this section, we define the operators $\mathcal{I}_1$ and $\mathcal{I}_2$, which were used in Eq. (19) to construct the finite remainders of the amplitudes. We employ the expressions given in Ref. [46], that we report here for the completeness:

$$\mathcal{I}_1(\epsilon) = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_i\left(\frac{1}{\epsilon^2}-\frac{\gamma_0^i}{2\epsilon}\frac{1}{\mathbf{T}_i^2}\right)\sum_{i\neq j}\frac{\mathbf{T}_i\cdot\mathbf{T}_j}{2}\left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon},$$

$$\mathcal{I}_2(\epsilon) = \frac{e^{-\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)}\left(\frac{\gamma_1^{\mathrm{cusp}}}{8}+\frac{\beta_0}{2\epsilon}\right)\mathcal{I}_1(2\epsilon) - \frac{1}{2}\mathcal{I}_1(\epsilon)\left(\mathcal{I}_1(\epsilon)+\frac{\beta_0}{\epsilon}\right)+\frac{1}{\epsilon}\mathcal{H}_2. \tag{A2}$$

In the above equation, the quantities $\mathbf{T}_i$ are color operators related to the $\mathrm{SU}(N_c)$ generators associated with particle i. More specifically

$$\begin{aligned}
(\mathbf{T}_i)^a_{b_ic_i} &= -if^a_{b_ic_i} \quad \text{if } i \text{ is a gluon},\\
(\mathbf{T}_i)^a_{i_ij_i} &= +T^a_{i_ij_i} \quad \text{if } i \text{ is a final (initial) state quark (antiquark)},\\
(\mathbf{T}_i)^a_{i_ij_i} &= -T^a_{j_ii_i} \quad \text{if } i \text{ is an initial (final) state quark (antiquark)}.
\end{aligned} \tag{A3}$$

The square of the color charge operator yields the Casimir operator, i.e., $\mathbf{T}_i^2 = C_i$, where $C_g = C_A$ and $C_q = C_F$, with $C_A = N_c$ and $C_F = (N_c^2-1)/(2N_c)$. The operator $\mathcal{H}_2$ is given by

$$\begin{aligned}
\mathcal{H}_2 &= \frac{1}{16}\sum_i\left(\gamma_1^i - \frac{1}{4}\gamma_1^{\mathrm{cusp}}\gamma_0^i + \frac{\pi^2}{16}\beta_0\gamma_0^{\mathrm{cusp}}C_i\right) + \frac{if^{abc}}{24}\sum_{(i,j,k)}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\log\frac{-s_{ij}}{-s_{jk}}\log\frac{-s_{jk}}{-s_{ki}}\log\frac{-s_{ki}}{-s_{ij}}\\
&\quad - \frac{if^{abc}}{128}\gamma_0^{\mathrm{cusp}}\sum_{(i,j,k)}\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\left(\frac{\gamma_0^i}{C_i}-\frac{\gamma_0^j}{C_j}\right)\log\frac{-s_{ij}}{-s_{jk}}\log\frac{-s_{ki}}{-s_{ij}},
\end{aligned} \tag{A4}$$

where the cusp anomalous dimensions read as

$$\begin{aligned}
\gamma_0^{\mathrm{cusp}} &= 4,\\
\gamma_1^{\mathrm{cusp}} &= \left(\frac{268}{9}-\frac{4\pi^2}{3}\right)C_A - \frac{40}{9}N_f,
\end{aligned} \tag{A5}$$

the quark and gluon collinear anomalous dimensions are given by

$$\begin{aligned}
\gamma_0^q &= -3C_F,\\
\gamma_1^q &= C_F^2\left(-\frac{3}{2}+2\pi^2-24\zeta_3\right) + C_AC_F\left(-\frac{961}{54}-\frac{11}{6}\pi^2+26\zeta_3\right) + C_FN_f\left(\frac{65}{27}+\frac{\pi^2}{3}\right),\\
\gamma_0^g &= -\beta_0,\\
\gamma_1^g &= C_A^2\left(-\frac{692}{27}+\frac{11}{18}\pi^2+2\zeta_3\right) + C_AN_f\left(\frac{128}{27}-\frac{\pi^2}{9}\right) + 2C_FN_f,
\end{aligned} \tag{A6}$$

and β_0 is defined in Eq. (17).

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