Penrose limit of T dual of Maldacena-Nastase solution and dual orbifold field theory

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Here we study the Penrose limit of the T dual of the Maldacena-Nastase solution and its field theory dual, in order to better understand the effect of T duality in this case. We find a matching of string pp wave oscillators and their masses to the field theory modes, that are rearranged after T duality. The effect of T duality on the long "annulon-type" operators is found as a symmetry of the $(2+1)$-dimensional confining theory with spontaneous supersymmetry breaking.

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I. INTRODUCTION

Most of the interest in the AdS/CFT correspondence [1] and its gauge/gravity generalizations (see the books [2,3] for reviews) comes from the description it gives of nonperturbative quantum field theories via perturbative string theory in gravitational backgrounds. However, in the cases of most interest, which are closer to the real world, and to QCD in particular, the holographic map is less understood, so it is worth exploring ways to understand it better. One way is by using the Penrose limit, leading to the pp wave correspondence, originally defined in [4] for the $AdS_5 \times S^5$ case, leading to a maximally supersymmetric type IIB pp wave [5,6] on the string side (see the book [7] for more details about the pp wave correspondence). In the Penrose limit one field theory side, one restricts to a sector of long operators, of a large global (R-)charge, corresponding to long discretized strings in the pp wave.

Cases of more interest are confining theories like the ones of Klebanov-Strassler [8] and Maldacena-Núñez [9] (also Polchinski-Strassler [10]) in $3+1$ dimensions. The Penrose limits of these theories were first analyzed in [11] where in the IR the Klebanov-Strassler case gave a theory of heavy hadrons, described by long gauge invariant operators and dubbed "annulons" due to the ring structure of the resulting hadrons. The Maldacena-Núñez case was argued to be qualitatively similar, though more difficult to analyze.

In $2+1$ dimensions, the $\mathcal{N}=1$ supersymmetric confining case similar to the above is the case of the Maldacena-Nastase (MNa) model, for NS5-branes wrapped on S^3 with a twist, a case that also has spontaneous supersymmetry breaking. The analysis of the Penrose limit and the resulting annulonlike long operators for hadrons was started in [12,13], based on the ideas in [11] but was not completed until our previous paper [14].

A very puzzling issue in holography has been the understanding of T duality. In the case of Abelian T duality of $AdS_5 \times S^5$, the interpretation has been in terms of a circular quiver field theory coming from NS5-branes and D4-branes [15–17], but the rules of the T-duality map action on the field theory dual were not very clear. After the inclusion of RR-charged fields in non-Abelian T duality [18], in [19,20] and many subsequent papers, it was shown that the field theory corresponds to an infinite linear quiver, but again, the T-duality map in field theory was not very clear. On the other hand, the role of T duality in the Penrose limit was pioneered in [21,22]. In [23], the Penrose limit method was applied in order to understand better the Abelian and non-Abelian T duals of $AdS_5 \times S^5$, and to see what their effect is in field theory.

In this paper, we consider the application of the Penrose limit method on the T dual of the MNa model, where the T duality is applied in one of the directions of the S^3 on which the 5-branes are wrapped and twisted. The goal is to understand better the effect of this T duality on the effective $(2+1)$-dimensional confining $\mathcal{N}=1$ supersymmetry field theory and its hadronic states.

The paper is organized as follows. In Sec. II we review the MNa solution, present its T dual, and analyze the resulting supersymmetry. In Sec. III we consider the Penrose limit of the T dual of the MNa gravity solution and quantize strings in the background. In Sec. IV we explain the orbifold field theory dual to the MNa T dual, and construct the "spin chain" for the annulonlike hadrons in $2+1$ dimensions. In Sec. V we conclude.

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II. THE MNa SOLUTION AND ITS T DUAL

A. MNa solution and generalization

The Maldacena-Nastase [1] gravity background of 10-dimensional type IIB supergravity, dual to NS5-branes wrapped on an S^3, with a twist that preserves $\mathcal{N}=1$ supersymmetry, is given by (writing only the Neveu-Schwarz (NS)-NS fields: metric, B field, and dilaton)

$$\begin{aligned}
ds^2_{10,\text{string}} &= d\vec{x}^2_{2,1} + \alpha' N\left[d\rho^2 + R^2(\rho)d\Omega_3^2 + \frac{1}{4}(\tilde{w}^a_L - A^a)^2\right]\\
H = dB &= N\left[-\frac{1}{4}\frac{1}{6}\epsilon_{abc}(\tilde{w}^a_L - A^a)(\tilde{w}^b_L - A^b)(\tilde{w}^c_L - A^c) + \frac{1}{4}F^a(\tilde{w}^a_L - A^a)\right] + h\\
h &= N[w^3(\rho) - 3w(\rho) + 2]\frac{1}{16}\frac{1}{6}\epsilon_{abc}w^a_L w^b_L w^c_L\\
A &= \frac{w(\rho)+1}{2}w^a_L\\
\Phi &= \Phi(\rho).
\end{aligned} \tag{2.1}$$

Here w^a_L and w^a_R are the left- and right-invariant one-forms on the S^3 on which the NS5-branes are wrapped, respectively. Its metric is $d\Omega_3^2 = w^a_L w^a_L = w^a_R w^a_R$. $\tilde{w}^a_L$ and $\tilde{w}^a_R$ are the corresponding forms on the sphere transverse to the NS5-branes, i.e., at infinity, S^3_∞. The one-forms are parametrized by angles ψ, θ, ϕ as

$$\begin{aligned}
&w^1_L = \sin\psi d\theta - \sin\theta\cos\psi d\phi, \qquad w^2_L = \cos\psi d\theta + \sin\theta\sin\psi d\phi, \qquad w^3_L = d\psi + \cos\theta d\phi\\
&w^1_R = -\sin\phi d\theta + \sin\theta\cos\phi d\psi, \qquad w^2_R = \cos\phi d\theta + \sin\theta\sin\phi d\psi, \qquad w^3_R = d\phi + \cos\theta d\psi.
\end{aligned} \tag{2.2}$$

The functions $w(\rho), R^2(\rho)$, and $\Phi(\rho)$ are found perturbatively or numerically, subject to boundary conditions in the UV. Note that one can also S-dualize the solution to a solution describing D5-branes, using ($\Phi_D = -\Phi$ is the S-dual dilaton)

$$ds^2_{10,\text{D,string}} = e^{\Phi_D(\rho)}ds^2_{10,\text{string}}, \quad H^{(D)} = e^{\Phi_D(\rho)}H. \tag{2.3}$$

The generalization by Canoura *et al.* [24] has metric

$$\begin{aligned}
ds^2_{st} &= e^{\Phi}(dx^2_{1,2} + ds^2_7)\\
ds^2_7 &= N\left[e^{2g}d\rho^2 + \frac{e^{2h}}{4}(w^i_L)^2 + \frac{e^{2g}}{4}\left(\tilde{w}^i_L - \frac{1}{2}(1+w)w^i_L\right)^2\right],
\end{aligned} \tag{2.4}$$

and the Ramond-Ramond (RR) three-form field is

$$\begin{aligned}
F_3 = \frac{N}{4}\Big\{&(w^1_L\wedge w^2_L\wedge w^3_L - \tilde{w}^1_L\wedge\tilde{w}^2_L\wedge\tilde{w}^3_L)\\
&-\frac{\gamma'}{2}d\rho\wedge\tilde{w}^i_L\wedge w^i_L\\
&-\frac{(1+\gamma)}{4}\epsilon_{ijk}[w^i_L\wedge w^j_L\wedge\tilde{w}^k_L - \tilde{w}^i_L\wedge\tilde{w}^j_L\wedge w^k_L]\Big\}.
\end{aligned} \tag{2.5}$$

The functions e^{2g}, e^{2h} [generalizing $R^2(\rho)$] and $w(\rho), \Phi(\rho)$, and $\gamma(\rho)$ are in the IR (at small ρ)

$$\begin{aligned}
e^{2g}(\rho) &= g_0 + \frac{(g_0-1)(9g_0+5)}{12g_0}\rho^2 + \ldots,\\
e^{2h}(\rho) &= g_0\rho^2 - \frac{3g_0^2 - 4g_0 + 4}{18g_0}\rho^4 + \ldots,\\
w(\rho) &= 1 - \frac{3g_0 - 2}{3g_0}\rho^2 + \ldots,\\
\gamma(\rho) &= 1 - \frac{1}{3}\rho^2 + \ldots,\\
\Phi(\rho) &= \Phi_0 + \frac{7}{24g_0^2}\rho^2,
\end{aligned} \tag{2.6}$$

and for $g_0 = 1$, i.e., for $g = 0$, we find the MNa original solution, S-dualized to the D5-brane solution.

B. T dual of (generalization of) MNa

The T duality of the above background was performed in [25], by use of Buscher's T-duality rules,

$$\begin{aligned}
&e^{2\tilde{\Phi}} = \frac{e^{2\Phi}}{|G_{99}|}, \qquad \tilde{G}_{99} = \frac{1}{G_{99}},\\
&\tilde{G}_{MN} = G_{MN} - \frac{G_{9M}G_{9N} - B_{9M}B_{9N}}{G_{99}}, \qquad \tilde{G}_{9M} = \frac{1}{G_{99}}B_{9M},\\
&\tilde{B}_{MN} = B_{MN} - 2\frac{B_{9[M}G_{N]9}}{G_{99}}, \qquad \tilde{B}_{M9} = -\frac{G_{M9}}{G_{99}},
\end{aligned} \tag{2.7}$$

and the corresponding rules for the RR form fields,

$$C^{(2n+1)}_{M_1\ldots M_{2n+1}} = C^{(2n+2)}_{M_1\ldots M_{2n+1}\tilde{\phi}} + (2n+1)B_{[M_1|\tilde{\phi}|}C^{(2n)}_{M_2\ldots M_{2n+1}]} + 2n(2n+1)B_{[M_1|\tilde{\phi}|}g_{M_2|\tilde{\phi}|}C^{(2n)}_{M_3\ldots M_{2n+1}]\tilde{\phi}}/g_{\tilde{\phi}\tilde{\phi}}, \tag{2.8a}$$

$$C^{(2n+1)}_{M_1\ldots M_{2n}\tilde{\phi}} = C^{(2n)}_{M_1\ldots M_{2n}} - 2ng_{[M_1|\tilde{\phi}|}C^{(2n)}_{M_2\ldots M_{2n}]\tilde{\phi}}/g_{\tilde{\phi}\tilde{\phi}}. \tag{2.8b}$$

We define

$$\begin{aligned}
\Delta &= e^{\Phi}Ne^{2g},\\
\Sigma &= \frac{e^{\Phi}}{4}N\left(e^{2h}+\frac{e^{2g}}{4}(1+w)^2\right) \equiv e^{\Phi}\tilde{\Sigma} \equiv e^{2\tilde{\Phi}}\tilde{\Sigma}^2,\\
\Omega &= \frac{e^{\Phi}}{4}Ne^{2g} \equiv \frac{\Delta}{4},\\
\Xi &= -\frac{e^{\Phi}}{8}(1+w)Ne^{2g} \equiv -\frac{\Omega}{2}(1+w),
\end{aligned} \tag{2.9}$$

though in the following we will restrict ourselves to the strict MNa case $g_0 = 1$ [so in the IR $e^{2g} = 1 + \mathcal{O}(\rho^4)$, $e^{2h} = \rho^2 + \mathcal{O}(\rho^4)$, $w = 1 - \rho^2/3 + \mathcal{O}(\rho^4)$, $\Phi = \Phi_0 + 7\rho^2/24$], corresponding to $b = 1/3$ in the notation used in [14]. After the T duality on ϕ_1, one finds the metric

$$\begin{aligned}
d\tilde{s}^2_{st} = {}& e^{2\tilde{\Phi}}\frac{N}{4}\left(e^{2h}+\frac{e^{2g}}{4}(1+w)^2\right)d\vec{x}^2_{1,2} + \Delta d\rho^2 + \frac{1}{\Sigma}d\phi_1^2 + \Sigma(d\theta_1^2 + \sin^2\theta_1 d\psi_1^2)\\
&+ 2\Xi[\cos(\psi_1-\psi_2)d\theta_1 d\theta_2 - \sin(\psi_1-\psi_2)\sin\theta_2 d\theta_1 d\phi_2\\
&- \sin(\psi_1-\psi_2)\sin\theta_1\cos\theta_1 d\psi_1 d\theta_2\\
&+ (\cos\theta_2\sin^2\theta_1 - \cos\theta_1\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2))d\psi_1 d\phi_2 + \sin^2\theta_1 d\psi_1 d\psi_2]\\
&+ \left(\Omega - \frac{\Xi^2}{\Sigma}\sin^2(\psi_1-\psi_2)\sin^2\theta_1\right)d\theta_2^2\\
&+ \left(\Omega - \frac{\Xi^2}{\Sigma}[\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2) + \cos\theta_1\cos\theta_2]^2\right)d\phi_2^2\\
&+ \left(\Omega - \frac{\Xi^2}{\Sigma}\cos^2\theta_1\right)d\psi_2^2\\
&+ 2\left(\Omega\cos\theta_2 - \frac{\Xi^2}{\Sigma}\cos\theta_1[\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2) + \cos\theta_1\cos\theta_2]\right)d\phi_2 d\psi_2\\
&- 2\frac{\Xi^2}{\Sigma}[\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2) + \cos\theta_1\cos\theta_2]\sin(\psi_1-\psi_2)\sin\theta_1 d\theta_2 d\phi_2\\
&- 2\frac{\Xi^2}{\Sigma}\sin(\psi_1-\psi_2)\sin\theta_1\cos\theta_1 d\theta_2 d\psi_2,
\end{aligned} \tag{2.10}$$

where $e^{2\tilde{\Phi}} = e^{2\Phi}/\Sigma$ is the T-dual dilaton, the coefficient of $dx^2_{1,2}$ equals e^{Φ} (the original dilaton), the B field is

$$\begin{aligned}
\mathcal{B} = -\Big\{&\cos\theta_1 d\psi_1\wedge d\phi_1 + \frac{\Xi}{\Sigma}\sin(\psi_1-\psi_2)\sin\theta_1 d\theta_2\wedge d\phi_1\\
&+ \frac{\Xi}{\Sigma}[\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2) + \cos\theta_1\cos\theta_2]d\phi_2\wedge d\phi_1 + \frac{\Xi}{\Sigma}\cos\theta_1 d\psi_2\wedge d\phi_1\Big\},
\end{aligned} \tag{2.11}$$

and the angles with the index "1" parametrize the internal (on which the branes are wrapped) S^3 and the angles with index "2" parametrize the S^3_∞. There are also RR fields $F_2 \neq 0$, $F_4 \neq 0$ but, since we will not be considering fermions in the string worldsheet action, we will not write them here.

When T dualizing, we have the consistency condition [26]

$$\int d\phi_1 \wedge d\tilde{\phi}_1 = (2\pi)^2, \tag{2.12}$$

where we denote the original coordinate as ϕ_1 and the T dual as $\tilde{\phi}_1$.

If the ϕ_1 direction is orbifolded by $\mathbb{Z}_{N_1}$, giving a periodicity $2\pi/N_1$ for ϕ_1, the above condition implies that $\tilde{\phi}_1 \in [0, 2\pi N_1]$. This is the case that is relevant for us, since otherwise it is hard to make sense of the T duality in the field theory dual [23].

C. Supersymmetry

To understand the amount of supersymmetry, we need to either construct the Killing spinor equations, or use symmetry arguments.

If we did not have any orbifolding, we would be back in the MNa case, analyzed in [14]. As it is, we need to understand whether, when taking the orbifold of $S^3/\mathbb{Z}_{N_1}$, where $\mathbb{Z}_{N_1}$ acts on an S^1 fiber inside S^3, any supersymmetry survives. In general, for a quotient manifold X/G, the unbroken supersymmetry from X is the one that is invariant under G.

To see that, we will analyze the action of $U(1)_{S^1}$ symmetry (reduced by $\mathbb{Z}_{N_1}$) on the supercharges, reduced down to $2+1$ dimensions.

The $(5+1)$-dimensional theory on the NS5-branes has $\mathcal{N} = (1,1)$ supersymmetries. The 10-dimensional Majorana-Weyl spinor decomposes under the decomposition $SO(9,1) \to SO(5,1) \times SO(4)$ as

$$\mathbf{16} \to (\mathbf{4}, \mathbf{1}, \mathbf{2}) \oplus (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}), \tag{2.13}$$

where we used the fact that $SO(5,1) \simeq SU(4)$ and $SO(4) \simeq SU(2)_A \times SU(2)_B$. Compactifying on the (internal) S^3 means the decomposition

$$\begin{aligned} &SO(5,1) \times SO(4) \to SO(2,1) \times SU(2)_T \times SU(2)_A \times SU(2)_B \\ &\quad \Rightarrow (\mathbf{4}, \mathbf{1}, \mathbf{2}) \oplus (\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1}) \to (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}). \end{aligned} \tag{2.14}$$

Finally, twisting means considering only the diagonal subgroup, $(SU(2)_T \times SU(2)_A)_{\rm diag}$ (embedding the gauge group in the spin connection). This gives the decomposition

$$\begin{aligned} &SO(2,1) \times SU(2)_T \times SU(2)_A \times SU(2)_B \to SO(2,1) \times (SU(2)_T \times SU(2)_A)_{\rm diag} \times SU(2)_B \\ &\quad \Rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}) \to (\mathbf{2}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{3}, \mathbf{1}). \end{aligned} \tag{2.15}$$

Since there is only one $SO(2,1)$ spinor that is invariant ($\mathbf{1}$) under the twisted connection (diagonal group), we have $\mathcal{N} = 1$ supersymmetry.

In our case, $U(1)_{S^1} \subset SU(2)_{\rm diag}$, which means that the remaining supercharge is invariant under $\mathbb{Z}_{N_1}$, and survives the orbifolding.

This is as we want, since the orbifolding we consider should not interfere with the symmetries of the solution.

III. STRINGS IN PENROSE LIMIT OF T DUAL OF MNa

A. Penrose limit of T dual of MNa

We want to consider the Penrose limit in order to better understand the effect of T duality on gravity dual pairs. But then, the most useful Penrose limit is in the T-duality direction, ϕ_1. As in previous cases, however (for instance, [14] and [23,27]), that is not consistent, and we must consider also motion in another spatial coordinate (and of course, in time t). Based on what we expect from the field theory, we want the motion to combine with one of the other two obvious isometries of the metric, ϕ_2 and ψ_+, where

$$\psi_\pm \equiv \psi_1 \pm \psi_2. \tag{3.1}$$

We choose to mix the motion on ϕ_1 with motion on ϕ_2.

In order to find a null geodesic, we also impose the usual conditions, which on our metric reduce to

$$\partial^\mu g_{\phi_2\phi_2} = 0, \qquad \partial^\mu g_{\phi_1\phi_1} = 0. \tag{3.2}$$

The first observation is that, for $\mu = \rho$, since at small ρ the functions in the metric depend on ρ^2, so ∂^ρ is proportional to ρ (and the multiplying functions do not have anything special), the solution would be $\rho = 0$. However, we can easily check that for $\rho = 0$ we get singularities in the metric (metric coefficients vanish for the case of the other angles being on their solutions). That means that we will need to keep $\rho \neq 0$, though very small (so it will be almost a solution to the geodesic equation, up to vanishingly small corrections, $\rho \to 0$). But that is fine for the Penrose limit, since ρ will be transverse to the geodesic, so will scale with $1/R \to 0$.

For the angles (that are not isometries), we get the conditions

$$\begin{aligned}
&\sin\psi_-\cos\psi_-\sin^2\theta_1=0,\qquad \sin^2\psi_-\sin\theta_1\cos\theta_1=0,\\
&[\sin\theta_1\sin\theta_2\cos\psi_-+\cos\theta_1\cos\theta_2]\sin\theta_1\sin\theta_2\sin\psi_-=0,\\
&[\sin\theta_1\sin\theta_2\cos\psi_-+\cos\theta_1\cos\theta_2](\cos\theta_1\sin\theta_2\cos\psi_--\sin\theta_1\cos\theta_2)=0,\\
&[\sin\theta_1\sin\theta_2\cos\psi_-+\cos\theta_1\cos\theta_2](\sin\theta_1\cos\theta_2\cos\psi_--\cos\theta_1\sin\theta_2)=0,
\end{aligned}\tag{3.3}$$

which we see are all satisfied by

$$\theta_1=\pi/2,\qquad \psi_-=\pi/2,\qquad \theta_2=\text{arbitrary}.\tag{3.4}$$

(There are other solutions, but they lead to even more singular coefficients for the metric, so we will ignore them.)

We then consider the null geodesic defined by $\rho=0$, $\theta_1=\pi/2$, $\psi_-=\pi/2$, with θ_2 and ψ_+ arbitrary *angles* (so not interacting with the geodesic). Then they will not be rescaled by $1/R$ in the Penrose limit. We note then that the coefficient of $d\theta_2^2$ in the metric is singular (vanishes) at the strict geodesic point, but in the Penrose limit that is fine, since θ_2 is not rescaled. The same comment also applies for the coefficient of $d\psi_+^2$ in the metric. Since, of course, there is no true singularity in the metric, the apparent singularity *must be* of the type of the one near the center in polar coordinates, and this is indeed what we will find.

We define the coefficient of $d\phi_2^2$ in the metric as the function

$$f\equiv\Omega-\frac{\Xi^2}{\Sigma}[\sin\theta_1\sin\theta_2\cos(\psi_1-\psi_2)+\cos\theta_1\cos\theta_2]^2.\tag{3.5}$$

Then, at the geodesic point we have the values

$$f\to f_0=\Delta_0=\Sigma_0=\Omega_0=\frac{\Xi_0^2}{\Sigma_0}=-\Xi_0=\frac{e^{\Phi_0}N}{4}.\tag{3.6}$$

For later use, we write the full metric at the geodesic location,

$$\begin{aligned}
ds^2=e^{\Phi_0}d\vec{x}_{1,2}^2&+\frac{1}{f_0}d\phi_1^2+f_0[4d\rho^2+d\theta_1^2+0\cdot d\theta_2^2\\
&+0\cdot d\psi_+^2+d\psi_-^2+d\phi_2^2-2\cos\theta_2 d\phi_2 d\psi_-\\
&+2\sin\theta_2 d\phi_2 d\theta_1].
\end{aligned}\tag{3.7}$$

The Lagrangian for a particle moving on this geodesic is (considering the values of the functions *at the geodesic*)

$$\begin{aligned}
L&=-e^{2\tilde{\Phi}_0}\tilde{\Sigma}_0\frac{\dot{t}^2}{2}+\frac{\dot{\phi}_1^2}{2\Sigma_0}+f_0\frac{\dot{\phi}_2^2}{2}\\
&=-e^{\Phi_0}\frac{\dot{t}^2}{2}+\frac{\dot{\phi}_1^2}{2f_0}+f_0\frac{\dot{\phi}_2^2}{2},
\end{aligned}\tag{3.8}$$

and the null condition for the geodesic means $L=0$.

The cyclic coordinates are t, ϕ_1, and ϕ_2 and the conserved momenta are

$$\frac{\partial L}{\partial\dot{t}}=-e^{\Phi_0}\dot{t}=-E,\quad \frac{\partial L}{\partial\dot{\phi}_1}=\frac{\dot{\phi}_1}{f_0}=-J_1,\quad \frac{\partial L}{\partial\dot{\phi}_2}=f_0\dot{\phi}_2=J_2.\tag{3.9}$$

The $L=0$ (geodesic being null) constraint then gives

$$\frac{J_2^2}{f_0}=\frac{E^2}{e^{\Phi_0}}-\Sigma_0 J_1^2.\tag{3.10}$$

Considering u as the affine parameter on the null geodesic, we make the change of coordinates

$$\begin{aligned}
dt&=\dot{t}du=\frac{E}{e^{\Phi_0}}du,\\
d\phi_1&=\dot{\phi}_1du+dw=J_1f_0du+dw,\\
d\phi_2&=\dot{\phi}_2du+dv=\frac{J_2}{f_0}du+dv,
\end{aligned}\tag{3.11}$$

after which the metric becomes

$$\begin{aligned}
ds^2=du^2&\left[-E^2e^{\Phi-2\Phi_0}+J_1^2\frac{\Sigma_0^2}{\Sigma}+\frac{f}{f_0^2}J_2^2\right]+\frac{dw^2}{\Sigma}+fdv^2\\
&+e^{\Phi}d\vec{x}_{1,2}^2+\Delta d\rho^2+g_{lm}dy^ldy^m\\
&+2du\left[J_1dw\frac{\Sigma_0}{\Sigma}+J_2dv\frac{f}{f_0}+g_{\phi_2 l}dy^l\right],
\end{aligned}\tag{3.12}$$

where $y^l=(\theta_1,\psi_-,\theta_2,\psi_+)$.

Now we finally make the coordinate change

$$\begin{aligned}
f_0dV&\equiv J_1dw+\frac{f}{f_0}J_2dv+g_{\phi_2 l}dy^l\Rightarrow dv\\
&=\frac{f_0}{fJ_2}\left[f_0dV-J_1\frac{\Sigma_0}{\Sigma}dw-g_{\phi_2 l}dy^l\right],
\end{aligned}\tag{3.13}$$

which gets rid of the mixing term with du, and puts the metric in a form appropriate for taking the rescaling and the Penrose limit,

$$
\begin{aligned}
ds^2 = du^2\left[-E^2 e^{\Phi-2\Phi_0} + J_1^2\frac{\Sigma_0^2}{\Sigma} + \frac{f}{f_0^2}J_2^2\right] + \frac{dw^2}{\Sigma} + e^{\Phi}d\vec{x}_{1,2}^2 \\
+ \Delta d\rho^2 + g_{lm}dy^l dy^m + 2du dV + \frac{f_0^2}{fJ_2^2}\left[J_1^2\frac{\Sigma_0^2}{\Sigma^2}dw^2\right. \\
+ 2J_1\frac{\Sigma_0}{\Sigma}g_{\phi_2 l}dw dy_l + g_{\phi_2 l}dy^l g_{\phi_2 m}dy^m \\
\left. + \text{terms with } dV\right]. \qquad (3.14)
\end{aligned}
$$

We have not written the terms with dV in the brackets ($f_0^2 dV^2 - 2f_0 dV J_1\frac{\Sigma_0}{\Sigma}dw - 2f_0 dV g_{\phi_2 l}dy^l$), since they will be scaled away in the Penrose limit, as we can easily check. Note that in $g_{lm}dy^l dy^m$, *in the neighbourhood of the geodesic* we have the metric (3.7), where there are no ψ_+, θ_2 components, so there we have a *de facto* reduction to $y^l = (\psi_-, \theta_1)$ only. This is good, since otherwise the term with $dV g_{\phi_2\psi_+}d\psi_+$ and $dV g_{\phi_2\theta_2}d\theta_2$ would contribute. Then there we also have

$$
g_{\phi_2 l}:\; g_{\phi_2\psi_-} = -\cos\theta_2 f_0, \qquad g_{\phi_2\theta_1} = \sin\theta_2 f_0. \qquad (3.15)
$$

Now we see that indeed the Penrose limit rescaling needs to be, as we advertised,

$$
\begin{aligned}
u = U, \qquad V = \frac{V'}{R^2}, \qquad \theta_2 = \theta_2', \qquad \psi_+ = \psi_+', \\
w = \frac{w'}{R}, \qquad \rho = \frac{\rho'}{R}, \qquad x_i = \frac{x_i'}{R}, \\
\theta_1 - \frac{\pi}{2} = \frac{\theta_1'}{R}, \qquad \psi_- - \frac{\pi}{2} = \frac{\psi_-'}{R}, \qquad (3.16)
\end{aligned}
$$

and for simplicity of notation we remove the primes after the procedure. Moreover, we can, as usual, identify the overall scale of the metric, $f_0 = e^{\Phi_0}N/4 = g_s N/4$, with R^2.

We need to consider the coefficients of $d\theta_2^2$ and $d\psi_+^2$ at subleading order in ρ, θ_1, ψ_-, and the same for the coefficient of du^2 in the metric above, since these will all contribute to the Penrose limit. We first obtain

$$
\begin{aligned}
\Sigma &\simeq \frac{e^{\Phi}N}{4}\left(1+\frac{2\rho^2}{3}\right) \simeq f_0\left(1+\frac{23\rho^2}{24}\right), \qquad e^{\Phi} \simeq e^{\Phi_0}\left(1+\frac{7\rho^2}{24}\right), \\
\Xi &\simeq -\frac{e^{\Phi}N}{4}\left(1-\frac{\rho^2}{6}\right) \simeq -f_0\left(1+\frac{\rho^2}{8}\right), \\
\Omega &\simeq \frac{e^{\Phi}N}{4} \simeq f_0\left(1+\frac{7\rho^2}{4}\right), \\
\Omega - \frac{\Xi^2}{\Sigma}\sin^2\psi_-\sin^2\theta_1 &\simeq \frac{e^{\Phi}N}{4}(\rho^2+\delta\psi_-^2+\delta\theta_1^2) \simeq f_0(\rho^2+\delta\psi_-^2+\delta\theta_1^2) \\
f &\simeq \frac{e^{\Phi}N}{4}[1-(\sin\theta_2\delta\psi_- + \cos\theta_2\delta\theta_1)^2] \\
&\simeq f_0\left[1+\frac{7\rho^2}{24} - (\sin\theta_2\delta\psi_- + \cos\theta_2\delta\theta_1)^2\right]. \qquad (3.17)
\end{aligned}
$$

With the proposed Penrose scaling in (3.16), we obtain that the metric in the ψ_1, ψ_2, and θ_2 directions is, at leading order,

$$
f_0(\rho^2+\delta\psi_-^2+\delta\theta_1^2) + f_0\delta\psi_-^2 + f_0\frac{\rho^2}{4}\delta\psi_+^2, \qquad (3.18)
$$

where $\delta\psi_- = \psi_- - \pi/2$, $\delta\theta_1 = \theta_1 - \pi/2$.

Finally taking the Penrose rescaling and limit, and dropping the primes on the rescaled variables, we obtain the metric [after using (3.10)]

$$
\begin{aligned}
R^2 ds^2 = 2f_0 dU dV - f_0\left[\frac{5}{4}J_1^2\rho^2 + \left(\frac{E^2}{e^{\Phi_0}f_0} - J_1^2\right)\left(\sin\theta_2\psi_- + \cos\theta_2\theta_1\right)^2\right]dU^2 + e^{\Phi_0}d\vec{x}_2^2 \\
+ f_0\left[4d\rho^2 + \frac{\rho^2}{4}d\psi_+^2 + (\rho^2+\psi_-^2+\theta_1^2)d\theta_2^2 + d\psi_-^2 + d\theta_1^2\right. \\
\left. + \frac{J_1^2}{J_2^2}dw^2 + 2\frac{J_1}{J_2}dw\frac{(\sin\theta_2 d\theta_1 - \cos\theta_2 d\psi_-)}{J_2} + \frac{(\sin\theta_2 d\theta_1 - \cos\theta_2 d\psi_-)^2}{J_2^2}\right], \qquad (3.19)
\end{aligned}
$$

where note that $E^2/(e^{\Phi_0}f_0) - J_1^2 = J_2^2/f_0^2 \geq 0$, so the coefficient of dU^2 is negative definite, as it should be, and in the last line we have kept J_2 as it is, to make the formulas clearer. Since J_2 is a derived quantity, we define $J_2/f_0 \equiv \tilde{J}_2$ for convenience. Moreover, we finally identify $R^2 = f_0$ and they will drop from the metric.

Making the rotation $\sin\theta_2 d\psi_- + \cos\theta_2 d\theta_1 \equiv d\tilde{\psi}_-$, $-\cos\theta_2 d\psi_- + \sin\theta_2 d\theta_1 \equiv d\tilde{\theta}_1$, rescaling x_i by $N/4$, $\rho = \tilde{\rho}/4$ and $(J_1/J_2)dw \equiv d\tilde{w}$, we get

$$ds^2 = 2dUdV - \left[\frac{5}{16}J_1^2\tilde{\rho}^2 + \left(\frac{E^2}{e^{\Phi_0 f_0}} - J_1^2\right)\tilde{\psi}_-^2\right]dU^2 + d\vec{x}_2^2 + d\tilde{\rho}^2 + \tilde{\rho}^2 d\left(\frac{\psi_+}{4}\right)^2 + \left(\frac{\tilde{\rho}^2}{4} + \tilde{\psi}_-^2 + \tilde{\theta}_1^2\right)d\theta_2^2 + d\tilde{\psi}_-^2 + d\tilde{\theta}_1^2 + d\left(\tilde{w} + \frac{d\tilde{\theta}_1}{f_0\tilde{J}_2}\right)^2. \tag{3.20}$$

We note that, since we are in the limit $R^2 = f_0 \to \infty$, the term with $d\tilde{\psi}_-/(f_0\tilde{J}_2) \to 0$ drops out. Then, defining $d\tilde{\rho}^2 + \tilde{\rho}^2 d(\psi_+/4)^2 \equiv d\vec{y}_2^2$, $\tilde{\psi}_- \equiv z_1, \tilde{\theta}_1 = z_2$, we finally get the metric

$$ds^2 = 2dUdV - \left[\frac{5}{16}J_1^2\vec{y}^2 + \left(\frac{E^2}{e^{\Phi_0}f_0} - J_1^2\right)z_1^2\right]dU^2 + d\vec{x}_2^2 + d\vec{z}_2^2 + d\vec{y}_2^2 + \left(\frac{1}{4}\vec{y}^2 + \vec{z}^2\right)d\theta_2^2 + d\tilde{w}^2. \tag{3.21}$$

We can define $d\theta_2\sqrt{\vec{y}^2/4 + \vec{z}^2} \equiv d\eta$, with η a Cartesian coordinate transverse to $\vec{y}$ and $\vec{z}$, such that it has $d\eta^2$ in the metric.

The B field becomes then

$$B = f_0\left(J_1 dU + \frac{dw}{f_0}\right) \wedge \left\{\delta\theta_1 d\delta\psi_- - \left(1 - \frac{\delta\psi_-^2}{2} - \frac{\delta\theta_1^2}{2} - \frac{5}{6}\rho^2\right)d\theta_2^2 - (\sin\theta_2\delta\psi_- + \cos\theta_2\delta\theta_1)\left(\tilde{J}_2 dU + \frac{dV}{\tilde{J}_2} - \frac{J_1 dw}{f_0\tilde{J}_2} - \frac{-\cos\theta_2 d\psi_- + \sin\theta_2 d\theta_1}{\tilde{J}_2}\right)\right\}. \tag{3.22}$$

There is a term of order f_0 that under the Penrose limit becomes of order $R^2 \to \infty$, but it can be removed by making a gauge transformation $\delta B = d\lambda$, with

$$\Lambda = -f_0\left(J_1 U + \frac{w}{f_0}\right)d\theta_2. \tag{3.23}$$

Then under the Penrose limit there is a term of order $f_0/R = R$, but that is $\propto dU \wedge dU$, so vanishes. Finally, after the Penrose limit, dropping the primes on the rescaled coordinates, as before, we get

$$\begin{aligned} B &= J_1 dU \wedge \left\{\theta_1 d\psi_- + \left(\psi_-^2 + \theta_1^2 + \frac{5}{24}\tilde{\rho}^2\right)d\theta_2^2 + \tilde{\psi}_-\left[d\tilde{w}\left(1 + \frac{\tilde{J}_2^2}{J_1^2}\right) + \frac{d\tilde{\theta}_1}{\tilde{J}_2}\right]\right\} \\ &= J_1 dU \wedge \left\{\theta_1 d\psi_- + \left(\vec{z}^2 + \frac{5}{24}\vec{y}^2\right)d\theta_2 + \tilde{\psi}_-\left[d\tilde{w}\left(1 + \frac{\tilde{J}_2^2}{J_1^2}\right) + \frac{dz_2}{\tilde{J}_2}\right]\right\}. \end{aligned} \tag{3.24}$$

Note that, if we define $(\psi_-^2 + \theta_1^2 + \frac{5}{24}\tilde{\rho}^2) \equiv T_1^2$ and $T_1 d\theta_2 \equiv dT_2$ (since T_1 is a radius and θ_2 an angle), we have

$$H = dB = J_1 dU \wedge \left\{d\theta_1 \wedge d\psi_- + dT_1 \wedge dT_2 + d\tilde{\psi}_- \wedge \left[d\tilde{w}\left(1 + \frac{\tilde{J}_2^2}{J_1^2}\right) + \frac{d\tilde{\theta}_1}{\tilde{J}_2}\right]\right\}, \tag{3.25}$$

so has constant components in some Cartesian coordinates: $H = dU \wedge h$ gives $h = h_{ij}dX^i \wedge dX^j$, with h_{ij} constant.

B. String quantization in the *pp* wave background

To quantize the string in the pp wave background, we write, as usual, the Polyakov action in the pp wave background, choose the conformal gauge $\sqrt{h}h^{\alpha\beta} = \eta^{\alpha\beta}$ (h is the worldsheet metric) and light-cone gauge $x^+ = p_-\tau$, where τ is the worldsheet time, obtaining

$$S = -\frac{1}{4\pi\alpha'}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \Bigg[\sum_{i=1}^{2}(\partial_a x_i)^2 + \sum_{i=1}^{2}(\partial_a y_i)^2 + \sum_{i=1}^{2}(\partial_a z_i)^2 + (\partial_a \tilde{w})^2 + \left(\frac{\vec{y}^2}{4} + \vec{z}^2\right)\sum_{a=1}^{2}(\partial_a \theta_2)^2 - \left(\frac{5}{16}J_1^2\vec{y}^2 + \tilde{J}_2^2 z_1^2\right)\Bigg] - \frac{J_1}{4\pi}\int d\tau \int_0^{2\pi\alpha' p^+} d\sigma \times \left\{\theta_1 \partial_\sigma \psi_- + \left(\vec{z}^2 + \frac{5}{24}\vec{y}^2\right)\partial_\sigma \theta_2 + z_1\left[\partial_\sigma \tilde{w}\left(1 + \frac{\tilde{J}_2^2}{J_1^2}\right) + \frac{\partial_\sigma z_2}{\tilde{J}_2}\right]\right\}. \tag{3.26}$$

For the solutions of the equations of motion we choose, as usual, free wave (Fourier mode) solutions of the type

$$\Phi_A = \Phi_{A,0}\exp\left[-i\omega\tau + ik_A\sigma\right], \tag{3.27}$$

where Φ_A stands for all the oscillators. Moreover, also as usual, with the rescaling by p^+ of the above, the quantization of momenta around the closed string circle in σ gives

$$k_{A,n} = \frac{n_A}{\alpha' p^+}. \tag{3.28}$$

Using this ansatz, the equations of motion for the oscillators become algebraic, although a bit complicated and unyielding to solve, so we will not attempt it.

The B field, being of the type $h_{ij}X_i dX^j$, with h_{ij} constant, as noted already, will then contribute to the equations of motion terms with $\sim ik_A = in_A/(\alpha' p^+)$. These will appear in skew-diagonal contributions (proportional to ϵ^{ij}) coupling the various oscillators. Like in the non-T-dualized MNa case in [14], then if we take all $n_A = n$ equal, we will find that the term with n in ω_n will be simply *added* to the mass term.

But in any case we are only interested in the mass terms, which are independent on the B field (since the B field is proportional to k, so to n). The masses are then

$$x_i,\quad i=1,2\colon\ m=0,\quad \tilde{w}\colon\ m=0,\quad \eta\colon\ m=0,$$
$$y_i,\quad i=1,2\colon\ m=\frac{\sqrt{5}J_1}{4},\quad z_i,\quad i=1,2\colon\ m=\tilde{J}_2. \tag{3.29}$$

As in [23], we will see that from the dual field theory we have only the relevant case when the two masses are equal, when $\sqrt{5}J_1/4 = \tilde{J}_2$, and it will not be clear how to obtain the nontrivial parameter J_1 to vary from the field theory point of view.

IV. FIELD THEORY AND SPIN CHAIN

A. Orbifold field theory

On the N 5-branes we start with a $U(N)$ theory. The action of the $U(1)_{S^1}$ on the $S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ defined as

$$|z_0|^2 + |z_1|^2 = 1 \tag{4.1}$$

is

$$z_0 \to z_0,\qquad z_1 \to e^{i\alpha} z_1. \tag{4.2}$$

As mentioned, $\mathbb{Z}_{N_1}$ acts by restricting the range of $U(1)$, so by $\alpha \to \alpha/N_1$.

Then, the gauge group of the orbifolded gauge theory on the wrapped 5-branes is $U(n_1)\times\cdots\times U(n_{N_1})$ (see, for instance, [28,29]), where

$$\sum_{i=1}^{M} n_i \dim(\mathbf{r}_i) = \sum_{i=1}^{M} n_i = N. \tag{4.3}$$

In the covering space, the group is $U(N_1 N)$. The bosonic spectrum is as follows: one gauge field, $(A_\mu^i)^I{}_J$, five real scalars, $(\Phi_\beta^i)^I{}_J$, transforming in the adjoint representation of the gauge group, one complex scalar (or two real ones), $(B^{I_i}{}_{J_{i+1}})^i{}_{i+1}$, transforming in the bifundamental representation of two consecutive $U(n)$'s in the gauge group, where μ, ν are Lorentz indices, I, J are fundamental group indices (I_i for the ith group), such that (IJ) is in the adjoint, i, j label the $U(n)$'s, and β refers to other internal indices.

The quiver diagram is circular, with nodes connected by the bifundamental fields.

Then the action for the gauge fields and the bifundamental scalars is [30]

$$S = \sum_i \frac{k_i}{8\pi} S_{CS_i}^{\mathcal{N}=1}(\Gamma_i^\alpha) - \int d^3x \int d^2\theta_1 \left[\sum_{W_{ij}} \mathrm{Tr}((D^\alpha + i\Gamma_j^\alpha)W_{ij}^\dagger (D_\alpha - i\Gamma_\alpha^i)W_{ij}) + \mathcal{W}^{\mathcal{N}=1}(W_{ij}, W_{ij}^*)\right], \tag{4.4}$$

where

$$\mathcal{W}^{\mathcal{N}=1}(Y_{ij},Y^*_{ij}) = \mathcal{W}(W_{ij}) + \mathcal{W}(W^*_{ij}) + \sum_{i,k_i\neq 0}\frac{k_i}{4\pi}R_i^2$$
$$\frac{k_i}{2\pi}R_i = \sum_j W_{ij}W^\dagger_{ij} - \sum_k W^\dagger_{ki}W_{ki}. \tag{4.5}$$

The Chern-Simons (CS) levels are as follows:

First, in the original theory, the H_3^{RR} field has a k_6 flux on S^3, leading to the same CS level, which is reduced to $k = k_6 - N/2$ by integrating out the fermions to have just a pure CS theory.

Second, in the orbifold theory, the H_3^{RR} flux is distributed between the nodes of the quiver, leading at each node to a CS level of $k_6 n_i/N$, reduced to $k_i = k_6 n_i/N - n_i/2$ by integrating out the fermions of the node. Note that then we have $\sum_i k_i = k$.

B. Spin chain

In Sec. II C we have seen that under the twisted reduction on S^3, the symmetry group decomposes into $SO(2,1)\times (SU(2)_T\times SU(2)_A)_{\rm diag}\times SU(2)_B$ and, as we have shown in [14], the gauge fields result in the decomposition $(\mathbf{3},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{3},\mathbf{1})$ and the scalar in $(\mathbf{1},\mathbf{2},\mathbf{2})$. Moreover, the ϕ^M scalar modes in $(\mathbf{1},\mathbf{2},\mathbf{2})$ are written, under the $SO(4)\simeq SU(2)_L\times SU(2)_R$ bifundamental decomposition, as

$$\Phi^{\alpha\beta} = \frac{1}{\sqrt{2}}(\sigma_M)^{\alpha\beta}\phi^M = \frac{1}{\sqrt{2}}\begin{pmatrix} i\Phi^0+\Phi^3 & \Phi^1 - i\Phi^2\\ \Phi^i + i\Phi^2 & i\Phi^0-\Phi^3\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} W & Z^*\\ Z & -W^*\end{pmatrix}. \tag{4.6}$$

Under the orbifold action, the two real bifundamental scalars $(B^I{}_J)^i{}_{i+1}$ descend from the complex Z scalars, and the five real scalars $(\Phi^i_\beta)^I{}_J$ at a single node i descend from the three A_a scalars [in the $(\mathbf{1},\mathbf{3},\mathbf{1})$, from the gauge field decomposition], and the two scalars in the complex W.

As explained in [14], the spin chain before orbifolding was described in terms of a vacuum $|0,p^+\rangle\sim {\rm Tr}[Z^J]$, in which we inserted the eight bosonic oscillators $\Phi^Q = (D_\mu, A_a, W, \bar{W})$, $\mu = 0,1,2$; $a = 1,2,3$.

But as described in [23] in the case of T duals of AdS$_5\times S^5$, one must consider the quiver orbifold theory, based on the works [21,22], and then consider a kind of T dual in the quiver ($\mathbb{Z}_{N_1}$) direction.

As explained in [22,23] then, the vacuum of the original orbifold theory, with momentum $p=1$ and winding $m=0$ in the $\mathbb{Z}_{N_1}$ direction, is associated with a gauge-invariant state that has "winding" around the quiver. But under the T duality, p and m are interchanged, so we have a state of momentum $p=0$ and winding $m=1$, created by multiplying the bifundamental scalars (only the holomorphic ones in the complex notation, B and not $\bar{B}$),

$$|p=1,m=0\rangle_{\rm before\,Td} = |p=0,m=1\rangle_{\rm after\,Td} = \mathcal{O}_{N_1} = \frac{1}{\sqrt{\mathcal{N}}}{\rm Tr}[B^1{}_2B^2{}_3\ldots B^i{}_{i+1}\ldots B^{N_1}{}_1]. \tag{4.7}$$

The eight bosonic oscillators on these states with winding are provided by the following eight objects: (i) as usual, the three covariant derivatives D_μ, $\mu = 0,1,2$, (ii) the five real scalars in the adjoint, Φ^i_β, $\beta = 1,\ldots,5$, comprised of the three scalars from A_a [$(\mathbf{1},\mathbf{3},\mathbf{1})$] and the scalars in $W,\bar{W}$.

However, we will see shortly that the energies of these objects are not the same, as they will have different values of J. Then the insertions of D_μ and Φ^i_β at zero transverse momentum give the operators:

$$\mathcal{O}^a_{D,0} = a^{a;\dagger}_{D,0}|p=0,m=1\rangle_{\rm after\,Td} = \frac{1}{\sqrt{NN_1}}\frac{1}{\sqrt{\mathcal{N}}}\sum_{i=1}^{N_1}{\rm Tr}[B^1{}_2B^2{}_3\ldots B^{i-1}{}_i(D_aB^i{}_{i+1})\ldots B^{N_1}{}_1],$$
$$\mathcal{O}^\beta_{\Phi,0} = a^{\beta;\dagger}_{\Phi,0}|p=0,m=1\rangle_{\rm after\,Td} = \frac{1}{\sqrt{NN_1}}\frac{1}{\sqrt{\mathcal{N}}}\sum_{i=1}^{N_1}{\rm Tr}[B^1{}_2B^2{}_3\ldots B^{i-1}{}_i\Phi^i_\beta B^i{}_{i+1}\ldots B^{N_1}{}_1]. \tag{4.8}$$

The state of momentum p, distributed as a sum of mode numbers n_q, is (for instance for Φ^l_β insertions)

$$\mathcal{O}^\beta_{\Phi,p} = a^{\beta;\dagger}_{\Phi,n_q}|p;m=1\rangle_{\rm after\,Td} = \frac{1}{\sqrt{NN_1}}\frac{1}{\sqrt{\mathcal{N}}}\sum_{l=1}^{N_1}{\rm Tr}[B^1{}_2B^2{}_3\ldots B^{l-1}{}_l\Phi^l_\beta B^l{}_{l+1}\ldots B^{N_1}{}_1]e^{\frac{2\pi i l n_q}{N_1}}, \tag{4.9}$$

where we have only shown one mode number (momentum) n_q insertion for simplicity, and the total momentum p (what used to be the winding before the T duality, i.e., in the original orbifolded theory) is the sum,

$$p = \sum_q n_q. \tag{4.10}$$

To find the correspondence with the string oscillator states, we will consider the comparison with the case before T duality. The first observation that we have is that D_μ splits into D_{x_i} and D_t, as before T duality, but now also Φ_β split: A_a into (A_1, A_2) and A_3, and also W and $\bar{W}$.

The oscillators in this case are then reshuffled, as is their charge J. Indeed, now we are interested in the gravitational symmetry charge

$$J = J_1 + \tilde{J}_2 = J_{\phi_1} + J_{\phi_2}. \tag{4.11}$$

But J_{ϕ_1} corresponds now not to the J_ϕ from [14], which was a $U(1) \subset (SU(2)'_L \times SU(2)'_R)_{\text{diag}}$ of the S^3 (internal), but rather it is a $U(1)$ on which we T dualize, thus breaking $SO(4)' = SU(2)'_L \times SU(2)'_R$ of the S^3 to $U(1)_{\phi_1} \times SU(2)$. This has also the effect of breaking the three gauge fields A_a into (A_1, A_2) rotated by $U(1)_{\phi_1}$ and A_3, invariant. Further, J_{ϕ_2} is the same $J_{\tilde{\phi}}$ in [14], which is a $U(1) \subset SU(2)_L$ of the S^3_∞. But, because we do not have anymore the $J_{\tilde{\psi}}$ of [14], it will now be convenient to choose the normalization of the charges a factor of 2 larger, so J_{ϕ_1} of A_1, A_2 and D_t is now $+1$, and J_{ϕ_2} of Z, W is $-1/2$. The Hamiltonian is $H = \mu(\Delta - J - E_0)$, with $E_0 = 1$ as before T duality, and with $\mu = -1$.

Then we get the table (with respect to the fields before T duality)

Field	Z	W	$\bar{Z}$	$\bar{W}$	$A_{1,2}$	A_3	D_t	D_{x_i}
Δ	$1/2$	$1/2$	$1/2$	$1/2$	1	1	1	1
J_{ϕ_1}	0	0	0	0	1	0	1	0
J_{ϕ_2}	$-1/2$	$-1/2$	$+1/2$	$+1/2$	0	0	0	0
J	$-1/2$	$-1/2$	$+1/2$	$+1/2$	1	0	1	0
$\Delta - J$	1	1	0	0	1	0	1	0
$H/(-1) = \Delta - J - E_0$	0	0	1	1	1	0	1	0
Oscillator	$\cdots$	η	$\cdots$	y_1	z_1, z_2	$\tilde{w}$	y_2	x_1, x_2

We have listed the oscillators that correspond to the various insertions, as follows. D_{x_i} obviously correspond to x_i, and $A_{1,2}$ to $z_{1,2}$ (as they come from $\tilde{\theta}_1, \tilde{\psi}_-$); A_3 corresponds to $\tilde{w}$ (since this comes from a combination of ϕ_1, ϕ_2), and W corresponds to η, which is generated by the angle θ_2 in a transverse direction different than the one of Z (which is ϕ_2). Finally $\bar{W}$ to y_1 (which is the transverse direction $\tilde{\rho}$), and D_t to y_2, which is ψ_+, that includes ψ_1, that also charges the D_t direction, due to the twist.

We see that the masses of the pp wave oscillators indeed match the Hamiltonian, for the case of all masses equal ($\sqrt{5}J_1/4 = \tilde{J}_2$). As we already commented, it is unclear, just as in the T dual of $\text{AdS}_5 \times S^5$ case studied in [23], why the free parameter J_1 is not represented in the field theory.

V. DISCUSSION AND CONCLUSIONS

Here we have constructed the T dual of the MNa solution, and taken the Penrose limit, in order to have a simpler way to study the resulting field theory dual, now of orbifold type. We have found the spectrum of string theory oscillators, and matched it to the spectrum of insertions into the field theory "annulonlike" long gauge invariant operators. The effect of T duality on these operators (and thus on the hadrons associated with them) was described.

It is still not clear in general how to quantify the effect of the T duality [in a transverse direction to the $(2+1)$-dimensional field theory, namely on the S^3 that the 5-branes wrap] on general states in the orbifold field theory. We have described the action in the pp wave limit, which in the case of the spin chain for the annulons corresponds to the "dilute gas approximation" (or few "impurities"), but it generalizes to an action on a generic state of the (long) spin chain. It is also still not clear why the T-duality action on general short states is a symmetry, as is the case for the string theory on the gravity dual. The hadrons (annulons) are states of the confining theory in the IR, and it is not in general clear that such a theory should have a symmetry. Moreover, the effect (if any) on the spontaneous supersymmetry breaking needs to be understood.

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[1] J. M. Maldacena and H. S. Nastase, The supergravity dual of a theory with dynamical supersymmetry breaking, J. High Energy Phys. 09 (2001) 024.

[2] H. Nastase, *Introduction to the ADS/CFT Correspondence* (Cambridge University Press, Cambridge, England, 2015), p. 9.

[3] M. Ammon and J. Erdmenger, *Gauge/Gravity Duality: Foundations and Applications* (Cambridge University Press, Cambridge, England, 2015), p. 4.

[4] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, Strings in flat space and pp waves from $N = 4$ super-Yang-Mills, J. High Energy Phys. 04 (2002) 013.

[5] M. Blau, J. M. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, A new maximally supersymmetric background of IIB superstring theory, J. High Energy Phys. 01 (2002) 047.
[6] M. Blau, J. M. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, Penrose limits and maximal supersymmetry, Classical Quantum Gravity **19**, L87 (2002).
[7] H. Nastase, *String Theory Methods for Condensed Matter Physics* (Cambridge University Press, Cambridge, England, 2017), p. 9.
[8] I. R. Klebanov and M. J. Strassler, Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities, J. High Energy Phys. 08 (2000) 052.
[9] J. M. Maldacena and C. Nunez, Towards the large N limit of pure $N = 1$ super-Yang-Mills, Phys. Rev. Lett. **86**, 588 (2001).
[10] J. Polchinski and M. J. Strassler, The string dual of a confining four-dimensional gauge theory, arXiv:hep-th/0003136.
[11] E. G. Gimon, L. A. Pando Zayas, J. Sonnenschein, and M. J. Strassler, A soluble string theory of hadrons, J. High Energy Phys. 05 (2003) 039.
[12] G. Bertoldi, F. Bigazzi, A. L. Cotrone, C. Nunez, and L. A. Pando Zayas, On the universality class of certain string theory hadrons, Nucl. Phys. **B700**, 89 (2004).
[13] F. Bigazzi and A. L. Cotrone, PP waves and softly broken $N = 1$ SYM, Classical Quantum Gravity **21**, S1297 (2004).
[14] H. Nastase and M. R. Barbosa, Penrose limit of MNa solution and spin chains in three-dimensional field theories, J. High Energy Phys. 05 (2022) 181.
[15] E. Witten, Solutions of four-dimensional field theories via M theory, Nucl. Phys. **B500**, 3 (1997).
[16] A. Fayyazuddin and D. J. Smith, Localized intersections of M5-branes and four-dimensional superconformal field theories, J. High Energy Phys. 04 (1999) 030.
[17] M. Alishahiha and Y. Oz, AdS/CFT and BPS strings in four dimensions, Phys. Lett. B **465**, 136 (1999).
[18] K. Sfetsos and D. C. Thompson, On non-Abelian T-dual geometries with Ramond fluxes, Nucl. Phys. **B846**, 21 (2011).
[19] Y. Lozano and C. Núñez, Field theory aspects of non-Abelian T-duality and $\mathcal{N} = 2$ linear quivers, J. High Energy Phys. 05 (2016) 107.
[20] Y. Lozano, N. T. Macpherson, J. Montero, and C. Nunez, Three-dimensional $\mathcal{N} = 4$ linear quivers and non-Abelian T-duals, J. High Energy Phys. 11 (2016) 133.
[21] M. Alishahiha and M. M. Sheikh-Jabbari, The pp wave limits of orbifolded $AdS_5 \times S^5$, Phys. Lett. B **535**, 328 (2002).
[22] S. Mukhi, M. Rangamani, and E. P. Verlinde, Strings from quivers, membranes from moose, J. High Energy Phys. 05 (2002) 023.
[23] G. Itsios, H. Nastase, C. Núñez, K. Sfetsos, and S. Zacarías, Penrose limits of Abelian and non-Abelian T-duals of $AdS_5 \times S^5$ and their field theory duals, J. High Energy Phys. 01 (2018) 071.
[24] F. Canoura, P. Merlatti, and A. V. Ramallo, The supergravity dual of 3d supersymmetric gauge theories with unquenched flavors, J. High Energy Phys. 05 (2008) 011.
[25] T. R. Araujo and H. Nastase, Comments on the T-dual of the gravity dual of D5-branes on S^3, J. High Energy Phys. 04 (2015) 081.
[26] M. Rocek and E. P. Verlinde, Duality, quotients, and currents, Nucl. Phys. **B373**, 630 (1992).
[27] T. Araujo, G. Itsios, H. Nastase, and E. O. Colgáin, Penrose limits and spin chains in the GJV/CS-SYM duality, J. High Energy Phys. 12 (2017) 137.
[28] A. E. Lawrence, N. Nekrasov, and C. Vafa, On conformal field theories in four-dimensions, Nucl. Phys. **B533**, 199 (1998).
[29] M. Bershadsky and A. Johansen, Large N limit of orbifold field theories, Nucl. Phys. **B536**, 141 (1998).
[30] D. Forcella and A. Zaffaroni, N = 1 Chern-Simons theories, orientifolds and spin(7) cones, J. High Energy Phys. 05 (2010) 045.