Holographic dual of a quantum spin chain as a Chern-Simons-scalar theory

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We construct a holographic dual theory of one-dimensional anisotropic Heisenberg spin chain, which includes two Chern-Simons gauge fields and a charged scalar field. Thermodynamic quantities of the spin chain at low temperatures, which are exactly calculated from the integrability, are completely reproduced by the dual theory on three-dimensional black hole backgrounds and the exact matching of the parameters between the dual theory and the spin chain is obtained. The holographic dual theory provides a new theoretical framework to analyze the quantum spin chain and one-dimensional quantum many-body systems.

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I. INTRODUCTION

The quantum spin chain has been serving a role as an important touchstone for the study on dynamics of quantum many-body systems, such as quantum entanglement [1] and quantum phase transition [2]. In the analysis of the quantum spin chain, several theoretical frameworks have been developed: the Bethe ansatz, bosonization, and (conformal) field theory techniques [3–5]. Thanks to these techniques, the quantum spin chain has been established as a quantum integrable model, and allowed to have some exact results from the analytical calculations.

Recently, for the nonperturbative analysis of quantum many-body systems, yet another framework has been emerged: the holographic duality [6,7]. The holographic duality, which is also known as the anti–de Sitter/conformal field theory (AdS/CFT) correspondence [8], is a correspondence between quantum many-body systems (or quantum field theories) in d-dimensional space and gravitational theories in $(d+2)$-dimensional curved spacetime. So far, the holographic duality has been applied to various condensed matter systems, such as superconductors [9–11] and magnetic materials [12–17]. Although those analyses based on the holographic duality qualitatively reproduce the critical behavior of temperature and external field dependences, few examples can successfully reproduce the physical quantities at the quantitative level in the applications to condensed matters. The quantum spin chain has the exact results based on various integrable techniques, and thus we can compare the physical quantities calculated from the holographic dual theory with such exact results quantitatively including numerical factors.[1]

In this paper, we construct the holographic dual theory of one-dimensional antiferromagnetic Heisenberg spin chain ($S=1/2$) with anisotropy, which is three-dimensional Abelian Chern-Simons gauge theory coupled with a charged scalar field, and show that the dual theory on three-dimensional black holes can analytically and quantitatively reproduce the physical quantities of the spin chain in the low-energy regions. In the course of this study, the matching between the parameters in the both theories are determined including numerical factors. From the three-dimensional perspectives, the dual gauge theory can lead to a new theoretical framework to analyze the quantum spin chain and also one-dimensional quantum many-body systems.

II. SPIN CHAIN AT LOW TEMPERATURES AS CHERN-SIMONS THEORY

The antiferromagnetic quantum spin chain ($S=1/2$) with the anisotropy along the S^z direction (XXZ spin chain) is defined by the Hamiltonian

$$\mathcal{H} = J\sum_{i=1}^{N}(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \delta S_i^z S_{i+1}^z), \qquad (1)$$

where $J>0$ is the exchange coupling constant, and $\delta = -\cos(\pi\beta^2)$ with $0 \le \beta^2 \le 1$ is the anisotropy parameter. Using the Jordan-Wigner transformation and the bosonization technique, the dynamics of the XXZ spin chain can be described by the Gaussian model of a compactified boson

[1]Integrability has been also discussed in the AdS/CFT correspondence from the string theory perspectives. See Ref. [18] for a review and references therein.

Φ with the radius $R(\beta) = \beta/\sqrt{2\pi}$, which is a $(1+1)$-dimensional conformal field theory (CFT) with the central charge $c = 1$, in the continuum limit at low temperatures [4,5]. The Gaussian model possesses the chiral $U(1)_L \times U(1)_R$ symmetry whose currents are given by $J_z^{(L)} \propto \partial_z \Phi$ and $J_{\bar{z}}^{(R)} \propto \partial_{\bar{z}} \Phi$ respectively. The both chiral currents suffer from the chiral anomaly, and only the vectorlike combination of the currents satisfies the conservation law, which corresponds to the total spin conservation, $S^z = \sum_{i=1}^N S_i^z$. From the calculation based on the CFT technique and the Bethe ansatz method, the susceptibility for the spin density ($s^z = S^z/N$) in the low temperature limit ($T \to 0$) has been exactly obtained to be

$$\chi_0 = \frac{\partial \langle s^z \rangle}{\partial H} = \frac{1}{2\pi\beta^2 v_s}, \tag{2}$$

where H is the external magnetic field and v_s is the spin-wave velocity [19]. In the following discussion, we take the unit of $v_s = 1$ for simplicity.

The holographic dual theory of the XXZ quantum spin chain in the low temperature limit is given by the $U(1)_L \times U(1)_R$ Chern-Simons (CS) gauge theory on a three-dimensional manifold M,[2] whose action is given by

$$S_{\mathrm{CS}}[A, B] = -\frac{k}{4\pi} \int_M d^3x \epsilon^{\mu\nu\rho} (A_\mu \partial_\nu A_\rho - B_\mu \partial_\nu B_\rho). \tag{3}$$

Here, A_μ and B_μ are the gauge fields of $U(1)_L$ and $U(1)_R$ respectively, k is a positive constant, and $\epsilon^{\mu\nu\rho}$ is the (normalized) third-rank totally antisymmetric tensor. Note that this action does not depend on the metric of three-dimensional base space, and thus the CS theory is topologically invariant.

The holographic duality assumes the negatively curved spacetime with the (conformal) boundary, so-called asymptotically anti–de Sitter (AdS) spacetime, as the background geometry. Here, for concreteness, we consider the base space as the Euclidean three-dimensional AdS spacetime (AdS_3), whose metric is given by the Poincaré metric:

$$ds^2 = \frac{\ell^2}{y^2}(dy^2 + dz d\bar{z}), \tag{4}$$

where $z = x + it$ and $\bar{z} = x - it$, and we take $\epsilon^{yz\bar{z}} = 1$. Note that the background geometries are given by the solutions of Einstein equation with a negative cosmological constant, $\Lambda = -1/\ell^2$, and we consider the probe approximation ignoring further dynamics of gravity in the following discussion.

The CS action (3) is gauge invariant, and we take $A_y = B_y = 0$ as the gauge fixing condition. Using this gauge fixing and the parametrization (4), the bulk equations of motion are given by the flatness conditions:

$$\partial_y A_z = \partial_y B_z = 0, \qquad F_{z\bar{z}}^{(A)} = F_{z\bar{z}}^{(B)} = 0, \tag{5}$$

with $F^{(A)} = dA$ and $F^{(B)} = dB$. Here, in order to cancel the boundary terms, we include the boundary action as a counterterm, following Ref. [23]:

$$S_{\mathrm{bd}} = -\frac{k}{4\pi} \int_{y=0} d^2z (A_z A_{\bar{z}} + B_{\bar{z}} B_z - 2A_{\bar{z}} B_z). \tag{6}$$

with the Dirichlet boundary condition, $\delta A_{\bar{z}} = \delta B_z = 0$ at $y = 0$. In the context of the holographic duality, the boundary action and the boundary condition imply that $A_{\bar{z}}$ and B_z correspond to the external sources for the chiral current operators $J_z^{(L)}$ (dual to A_z) and $J_{\bar{z}}^{(R)}$ (dual to $B_{\bar{z}}$) in the boundary CFT.

In the holographic duality, the Gubser-Klebanov-Polyakov-Witten (GKPW) relation [24,25] claims the equivalence of the partition functions of the quantum many-body system and the holographic dual theory: $Z_{\mathrm{qms}} = Z_{\mathrm{hol}} \simeq e^{-S_{\mathrm{hol}}}$, where the last relation is the semi-classical approximation in the dual theory. From this relation, we can obtain the expectation values of the current operators in the boundary CFT:

$$\begin{aligned} \langle J_z^{(L)} \rangle_{A_{\bar{z}}} &= -\frac{\delta(S_{\mathrm{CS}} + S_{\mathrm{bd}})}{\delta A_{\bar{z}}} = \frac{k}{2\pi}[A_z - B_z]_{y=0}, \\ \langle J_{\bar{z}}^{(R)} \rangle_{B_z} &= -\frac{\delta(S_{\mathrm{CS}} + S_{\mathrm{bd}})}{\delta B_z} = \frac{k}{2\pi}[B_{\bar{z}} - A_{\bar{z}}]_{y=0}. \end{aligned} \tag{7}$$

where the equations of motion (5) have been used for the on-shell evaluation. From the conformal invariance, the left-current $J_z^{(L)}$ should be a holomorphic function and the right-current $J_{\bar{z}}^{(R)}$ should be an antiholomorphic function and thus these should satisfy the conditions, $\partial_{\bar{z}} J_z^{(L)} = 0$ and $\partial_z J_{\bar{z}}^{(R)} = 0$ on the boundary. However, the holographic expectation values (7) implies

$$\begin{aligned} \partial_{\bar{z}} \langle J_z^{(L)} \rangle &= \frac{k}{2\pi}(\partial_z A_{\bar{z}} - \partial_{\bar{z}} B_z), \\ \partial_z \langle J_{\bar{z}}^{(R)} \rangle &= \frac{k}{2\pi}(\partial_{\bar{z}} B_z - \partial_z A_{\bar{z}}), \end{aligned} \tag{8}$$

where the flatness conditions of the gauge fields have been used. This relation corresponds to the chiral anomaly in the boundary CFT (the Gaussian model), which spoils the $U(1)_L \times U(1)_R$ invariance under the nontrivial background gauge fields, $A_{\bar{z}}$ and B_z [20–22]. However, it is important that we have the conserved vectorlike Noether current, $J = J_z^{(L)} dz + J_{\bar{z}}^{(R)} d\bar{z}$, on the boundary[3]:

[2]This type of the CS gauge theory has been also discussed in the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence with the string theory constructions [20–22].

[3]Our convention for the two-dimensional current is $J_z = \frac{1}{2}(J_x - iJ_t)$, $J_{\bar{z}} = \frac{1}{2}(J_x + iJ_t)$.

$$d\langle *J\rangle = \partial_{\bar{z}}\langle J_z^{(L)}\rangle + \partial_z\langle J_{\bar{z}}^{(R)}\rangle = 0. \tag{9}$$

This is consistent with the total spin conservation in the XXZ spin chain. Note that the conservation is attributed to the last term in the boundary action (6), which vanishes under the variation with the boundary condition [23].

The conserved charge density, which corresponds to the spin density s^z, is given by $(J_z^{(L)} - J_{\bar{z}}^{(R)})/2 \propto J_t/2$ and, accordingly, the conjugate external source for the spin density corresponds to the combination of gauge fields, $H = A_{\bar{z}} - B_z$. For the regularity of general bulk geometries, we assume the vanishing temporal components of gauge fields ($A_t = B_t = 0$) and only consider the nonvanishing spatial components of gauge fields [20,21].[4] Then, the external source for the spin density, i.e., the external magnetic field, can be identified as $H = \frac{1}{2}(A_x - B_x)$. With this identification, the spin density can be expressed as $\langle s^z\rangle = \frac{1}{2}(\langle J_z^{(L)}\rangle - \langle J_{\bar{z}}^{(R)}\rangle) = \frac{k}{2\pi}H$, and the spin susceptibility in the low temperature limit is given by

$$\chi_0 = \frac{\partial\langle s^z\rangle}{\partial H} = \frac{k}{2\pi}, \tag{10}$$

and the susceptibility implies that the free energy density of the XXZ spin chain is

$$\frac{F_{XXZ}^{(0)}}{\ell} = \frac{k}{4\pi}H^2. \tag{11}$$

Comparing the holographic result with the exact one (2), we have the parameter matching between the dual CS gauge theory and the quantum spin chain:

$$k = \frac{1}{\beta^2}. \tag{12}$$

A. Specific heat from Chern-Simons theory on BTZ black holes

Next, we calculate another thermodynamic quantity, the specific heat in the low temperature limit without the external field. In the holographic duality, finite-temperature effects of quantum many-body systems can be included by introducing the black hole background in the holographic dual theory. In the three-dimensional gravity with a negative cosmological constant, there exist the black hole solutions, so-called Banados-Teitelboim-Zanelli (BTZ) black holes [26], which have the asymptotically AdS geometry at $r \sim \infty$. In what follows, for simplicity, we consider the nonrotating (Euclidean) BTZ black holes, whose metric is given by

$$ds^2 = \left(\frac{r^2 - r_+^2}{\ell^2}\right)dt^2 + \left(\frac{\ell^2}{r^2 - r_+^2}\right)dr^2 + r^2 d\theta^2. \tag{13}$$

Here, r_+ is the radius of the horizon given by $r_+ = \sqrt{8G_N M}\ell$, where G_N is three-dimensional Newton constant, and M is mass of the black holes. Note that the BTZ black holes have locally same geometry as AdS_3 and the metric (13) can be transformed to the AdS_3 metric (4) by the coordinate transformations,

$$y = \left(\frac{r_+}{r}\right)e^{\frac{r_+}{\ell}\theta}, \qquad z = \sqrt{1 - \frac{r_+^2}{r^2}}e^{\frac{r_+}{\ell}(\theta + i\frac{t}{\ell})}. \tag{14}$$

The Hawking temperature of the BTZ black hole is given by $T_{\mathrm{BH}} = (\frac{1}{2\pi\ell})(\frac{r_+}{\ell})$, which is identified as the temperature of the quantum many-body system. For convenience, we introduce another coordinate, $u = \frac{\ell^2}{r}$ (and $u_+ = \frac{\ell^2}{r_+}$), which leads to

$$ds^2 = \ell^2\left(\frac{f(u)dt^2 + f(u)^{-1}dy^2 + \ell^2 d\theta^2}{u^2}\right) \tag{15}$$

with $f(u) = 1 - \frac{u^2}{u_+^2}$.

In the CS gauge theory (3) with the boundary term (6), the classical on-shell action vanishes without the external field ($A_x = B_x = 0$) and the classical partition function is trivial. In the literature [27–29], the quantum one-loop (and exact) partition function on the BTZ black hole (13) is obtained by the explicit path-integral calculation of the CS gauge theory:

$$Z_{\mathrm{CS}}(\tau) = \frac{1}{|\eta(\tau)|^2} = |q|^{-1/12}\prod_{n=1}^{\infty}\frac{1}{|(1-q^n)|^2}, \tag{16}$$

where $q = e^{2\pi i\tau}$ and $\eta(\tau)$ is the Dedekind's η-function. τ is the moduli parameter of the boundary torus at $r \to \infty$, which is given by $\tau = i\ell T_{\mathrm{BH}}$ for the BTZ black holes [30–32]. At low (but finite) temperatures, the partition function can be well approximated by $Z_{\mathrm{CS}} \simeq \exp(\pi\ell T_{\mathrm{BH}}/6)$ for $T_{\mathrm{BH}} \gtrsim 1/\ell$, and thus the free energy density is given by[5]

$$\frac{F_{\mathrm{CS}}}{\ell} = -\frac{T_{\mathrm{BH}}}{\ell}\log Z_{\mathrm{CS}} \simeq -\frac{\pi}{6}T_{\mathrm{BH}}^2. \tag{17}$$

Using the GKPW relation, the free energy F_{CS} can be identified with that of the spin chain and we obtain the specific heat $c_v \simeq \frac{\pi}{3}T_{\mathrm{BH}}$, which is consistent with the exact result based on the finite-size scaling method in CFT and the Bethe ansatz method [33,34]. In the following discussion, we set the curvature radius of AdS_3 to be unity, $\ell = 1$, for simplicity.

[4]As discussed below, the backgrounds of AdS_3 and BTZ black holes correspond to the boundary CFT at zero and finite temperature, respectively. The bulk gauge fields need to be consistent in the both backgrounds so that the thermal effects can be continuously incorporated in the boundary CFT.

[5]$T_{\mathrm{BH}} \gtrsim 1/\ell$ indicates $r_+ \gtrsim 2\pi\ell$, which guarantees that the BTZ black holes are large black holes in AdS_3 and stable with respect to the Hawking-Page phase transition.

III. FINITE TEMPERATURE CORRECTION FROM BULK SCALAR FIELD

We have seen, so far, the conformally invariant effective description of the XXZ spin chain in the low temperature limit. Here, we consider an irrelevant scalar perturbation of the CFT of the XXZ spin chain within the holographic dual theory. In this paper, we consider the scalar perturbation of the sine-Gordon (SG) type,

$$S_{\mathrm{SG}} = \frac{1}{2}\int d^2x(\lambda(x)\mathcal{O}(x) + \mathrm{H.c.}) = \lambda\int d^2x \cos(\Phi(x)/\beta), \tag{18}$$

where a real constant external source (i.e., coupling constant) λ is assumed, and $\Phi(x)$ is the compactified boson in the CFT description of the spin chain [35]. The operator $\mathcal{O}(x) = e^{i\Phi(x)/\beta}$ has the scaling dimension $d = 2/\beta^2$ and the charge $(q_L, q_R) = (1/\beta, -1/\beta)$ under the $U(1)_L \times U(1)_R$ chiral symmetry. The correlation function is normalized to be $\langle\mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \langle\mathcal{O}^\dagger(x)\mathcal{O}(y)\rangle = 1/|x-y|^{2d}$. The SG perturbation actually originates from the term proportional to anisotropy ($\sim S_i^z S_{i+1}^z$) in the spin-chain Hamiltonian (1).

In the context of the holographic duality, the irrelevant scalar perturbations can be described by introducing massive bulk scalar fields in the dual theory. Thus, we introduce a complex scalar field with the mass m in the dual theory, whose action is given by

$$S_{\mathrm{sc}} = \int d^3x\sqrt{g}[g^{\mu\nu}(D_\mu\phi^*)(D_\nu\phi) + m^2\phi^*\phi], \tag{19}$$

where the complex scalar has the charge $+q$ for A_μ and $-q$ for B_μ respectively, and the gauge covariant derivative is defined as $D_\mu\phi = \partial_\mu\phi - iq(A_\mu - B_\mu)\phi$. The resulting equation of motion of ϕ from the action (19) becomes

$$\frac{1}{\sqrt{g}}D_\mu(\sqrt{g}g^{\mu\nu}D_\nu\phi) - m^2\phi = 0. \tag{20}$$

In the holographic dictionary, the bulk scalar field with the mass m corresponds to the scalar operator in the boundary CFT with the scaling dimension, $d = \dim\mathcal{O} = 1 + \sqrt{1+m^2}$. The solutions to the equation of motion have the asymptotic expansion near the boundary ($u \sim 0$):

$$\phi_{\mathrm{sol}} \simeq \phi_0(x)u^{1-d} + \psi_0(x)u^d + \cdots, \tag{21}$$

where $x = (t, \theta)$ is boundary coordinates and the dots represent the higher order terms. By means of the GKPW relation, the coefficient ϕ_0 gives the external source with the renormalization factor, $\lambda/(d-1)$, and ψ_0 gives the expectation value, $\langle\mathcal{O}(x)\rangle$, under the perturbation (18) [36,37]. Along with the bulk scalar field, we consider finite-temperature corrections to the free energy of the XXZ spin chain from the scalar perturbation (18) in the holographic dual theory.

In the perturbed case, the GKPW relation leads to the correspondence between the on-shell classical action S_{hol} of the CS theory with the scalar field and the free energy F_{XXZ} of the XXZ spin chain:

$$Z_{\mathrm{hol}}[A, B, \phi] = Z_{\mathrm{qms}}(T, H)$$
$$\Rightarrow S_{\mathrm{hol}}[A_{\mathrm{sol}}, B_{\mathrm{sol}}, \phi_{\mathrm{sol}}] \simeq F_{\mathrm{qms}}(T, H)/T. \tag{22}$$

Here, the on-shell action is evaluated based upon the solutions of (classical) equations of motion of the bulk fields. In order to obtain the free energy of the XXZ spin chain including finite temperature corrections, we should have the solution to the equation of motion of the complex scalar field on the BTZ black hole (15).

We assume the product form of the solution, $\phi(u,x) = \phi(u)\phi_0(x)$, and further restrict ϕ_0 to a real constant which corresponds to the coupling constant λ of the SG perturbation.[6] Using the correspondence between the external magnetic field and the CS gauge fields, $H = \frac{1}{2}(A_\theta - B_\theta)$, the radial equation of motion of the scalar field $\phi(u)$ is explicitly given by

$$u^2 f(u)\partial_u^2\phi - u\left(1 + \frac{u^2}{u_+^2}\right)\partial_u\phi - (4h^2u^2 + m^2)\phi = 0, \tag{23}$$

where $h = qH$ with the charge q of the scalar field and the uniform magnetic field ($H = \mathrm{const}$) has been assumed for simplicity.

The change of variable, $\zeta = 1 - \frac{u^2}{u_+^2}$, and the ansatz for the solution, $\phi(\zeta) = (1-\zeta)^\alpha\varphi(\zeta)$, lead to the Gauss' hypergeometric differential equation [38]:

$$\zeta(1-\zeta)\frac{d^2\varphi(\zeta)}{d\zeta^2} + [c - (1+a+b)\zeta]\frac{d\varphi(\zeta)}{d\zeta} - ab\varphi(\zeta) = 0, \tag{24}$$

where $a = \alpha + i\tilde{h}$, $b = \alpha - i\tilde{h}$, and $c = 1$ with $\tilde{h} = hu_+$. The parameter α is determined by the mass parameter of the scalar field: $\alpha = \Delta_\pm = \frac{1\pm\sqrt{1+m^2}}{2}$. We thus obtain the solution using the Gauss' hypergeometric function $F(a, b, c; \zeta)$ [39]:

$$\phi_{\mathrm{sol}}(\zeta) = C(1-\zeta)^\Delta F(\Delta + i\tilde{h}, \Delta - i\tilde{h}, 1; \zeta) \tag{25}$$

where C is a normalization constant and we choose $\alpha = \Delta_- \equiv \Delta$ without loss of generality.[7] Actually, we have

[6] We also assume a sufficiently small value of ϕ_0 which guarantees the probe approximation ignoring the gravitational backreaction.

[7] Using the identity of the Gauss' hypergeometric function, we can show $(1-\zeta)^{\Delta_+}F(\Delta_+ + i\tilde{h}, \Delta_+ - i\tilde{h}, 1; \zeta) = (1-\zeta)^{\Delta_-}F(\Delta_- + i\tilde{h}, \Delta_- - i\tilde{h}, 1; \zeta)$ using the relation $\Delta_+ = 1 - \Delta_-$.

another linearly independent solution which is logarithmically divergent at the horizon ($\zeta = 0$), and thus we discard this solution for the regularity inside the bulk. We should notice that, for $\Delta < 0$ (or $m^2 > 0$), this solution diverges at $u = 0$ and the position of the boundary should be slightly shifted to $u = \varepsilon$ for well-defined calculations, where ε is a cutoff parameter ($0 < \varepsilon \ll 1$). Following the holographic scheme [36,37], we take the radial solution normalized at the shifted boundary at $u = \varepsilon$,

$$\phi_{\mathrm{sol}}(u) = \left(\frac{u^2}{\varepsilon^2}\right)^{\Delta} \frac{F(\Delta + i\tilde{h}, \Delta - i\tilde{h}, 1; 1 - u^2/u_+^2)}{F(\Delta + i\tilde{h}, \Delta - i\tilde{h}, 1; 1 - \varepsilon^2/u_+^2)}, \quad (26)$$

and fix the normalization constant,

$$C \simeq \left(\frac{u_+^2}{\varepsilon^2}\right)^{\Delta} \left(\frac{\Gamma(1 - \Delta - i\tilde{h})\Gamma(1 - \Delta + i\tilde{h})}{\Gamma(1 - 2\Delta)}\right), \quad (27)$$

where we used the asymptotic form of the hypergeometric function near $\zeta = 1$.

In order to obtain the on-shell action, $S_{\mathrm{hol}}[\phi_{\mathrm{sol}}]$, the partial integration is performed in the kinetic term in the scalar action (19):

$$S_{\mathrm{sc}} = -\int d^3x \sqrt{g} \phi^* \left[\frac{1}{\sqrt{g}} D_\mu(\sqrt{g} g^{\mu\nu} D_\nu \phi) - m^2 \phi\right] + \int_{u=\varepsilon} d^2x [\sqrt{g} g^{uu} \phi^* \partial_u \phi]. \quad (28)$$

The bulk term vanishes upon the solution of the scalar equation of motion (20), and thus we only need to evaluate the boundary term at $u = \varepsilon$.[8] Using the analytic continuation formula for the hypergeometric function [39], we have the asymptotic expansion of $\phi_{\mathrm{sol}}(u)$ near $u = 0$:

$$\phi_{\mathrm{sol}}(u) \simeq \gamma \left[\left(\frac{u^{2\Delta}}{u_+^{2\Delta}}\right) + \left(\frac{\Delta^2 + \tilde{h}^2}{2\Delta}\right)\left(\frac{u^{2\Delta+2}}{u_+^{2\Delta+2}}\right)\right] + \xi \left[\left(\frac{u^{2-2\Delta}}{u_+^{2-2\Delta}}\right) + \left(\frac{(1-\Delta)^2 + \tilde{h}^2}{2(1-\Delta)}\right)\left(\frac{u^{4-2\Delta}}{u_+^{4-2\Delta}}\right)\right]. \quad (29)$$

with the coefficients $\gamma = C \frac{\Gamma(1-2\Delta)}{\Gamma(1-\Delta-i\tilde{h})\Gamma(1-\Delta+i\tilde{h})}$ and $\xi = C \frac{\Gamma(2\Delta-1)}{\Gamma(\Delta+i\tilde{h})\Gamma(\Delta-i\tilde{h})}$. From this asymptotic expansion, the leading terms in the boundary term of the on-shell action near the boundary is given by

$$S_{\mathrm{sc}} \simeq \phi_0^2 \int d^2x \left[2\gamma^2 \Delta \left(\frac{\varepsilon^{-2+4\Delta}}{u_+^{4\Delta}}\right) + \gamma^2 \left(2(\Delta^2 + \tilde{h}^2) + \frac{\tilde{h}^2 - \Delta^2}{\Delta}\right)\left(\frac{\varepsilon^{4\Delta}}{u_+^{2+4\Delta}}\right) + \left(\frac{2\gamma\xi}{u_+^2}\right)\right]. \quad (30)$$

Note that the boundary action has the divergent terms in the limit of $\varepsilon \to 0$, and thus we should subtract such terms using the holographic renormalization method [40,41]. Following the recipe of holographic renormalization, we add the local counter term at the boundary ($u = \varepsilon$) which cancels the divergent terms,

$$S_{\mathrm{ct}} = -\left[2\Delta + \left(\frac{\tilde{h}^2}{\Delta}\right)\left(\frac{\varepsilon^2}{u_+^2}\right)\right] \int_{u=\varepsilon} d^2x \sqrt{h} |\phi(\varepsilon, x)|^2, \quad (31)$$

and obtain the renormalized on-shell action of the bulk scalar field:

$$S_{\mathrm{ren}} = S_{\mathrm{sc}} + S_{\mathrm{ct}} = \left(\frac{2\phi_0^2 \gamma \xi (1 - 2\Delta)}{u_+^2}\right) + \mathcal{O}(\varepsilon^{2+4\Delta}). \quad (32)$$

where we have assumed $-1/2 < \Delta < 0$. Using the scaling dimension $d = 1 + \sqrt{1 + m^2} = 2(1 - \Delta)$, we obtain the free energy of the XXZ spin chain from $S_{\mathrm{ren}} = F_{\mathrm{xxz}}/T$:

$$F_{\mathrm{xxz}} = 4\phi_0^2 (d-1)^2 \varepsilon^{2d-4} (2\pi T_{\mathrm{BH}})^{2d-2} \sin \pi d \left(\frac{\Gamma(\frac{d}{2} - i\tilde{h})\Gamma(\frac{d}{2} + i\tilde{h})}{\Gamma(1 - \frac{d}{2} + i\tilde{h})\Gamma(1 - \frac{d}{2} - i\tilde{h})}\right) \Gamma(1 - d)^2, \quad (33)$$

where the explicit forms of constants, γ, ξ, and C, have been recovered and the expression of the Hawking temperature, $u_+ = \frac{1}{2\pi T_{\mathrm{BH}}}$, has been used. Under the parameter matching between the holographic dual theory and the XXZ spin chain (with the lattice spacing a) summarized in Table I, the resulting free energy (33), together with the CS results (11) and (17), completely reproduces the free energy of the XXZ spin chain (with $\frac{2}{3} < \beta^2 < 1$), which is calculated by the field theory and integrability techniques [35], including the numerical factors.[9] Here, the leading term in the cutoff expansion is compared to

[8] Other two coordinates (t, θ) are periodic and have no boundary.

[9] Note that the definition of external magnetic field is different between ours and the Ref. [35] by the factor of 2.

TABLE I. Parameter matching: XXZ spin chain anisotropy is $\delta = -\cos(\pi\beta^2)$. Irrelevant coupling constant $\lambda(\beta)$ is given by Eq. (2.24) in [35].

Parameter in dual theory		Matching with spin chain
Cutoff	ε	a
CS coupling	k	$1/\beta^2$
Charge	q	$1/\beta^2$
Mass	m^2	$4(1-\beta^2)/\beta^4$
SG coupling	ϕ_0	$\lambda(\beta)/4(2\beta^{-2}-1)$

the spin chain, but using this parameter matching, it is also possible to calculate and compare the subleading terms perturbatively in the holographic renormalization [40,41].

It should be noticed that, in the massless limit of the bulk scalar field, which corresponds to the isotropic limit ($\delta = 1$) of the spin chain, the scalar solution (25) has the logarithmically divergent terms in the asymptotic expansion at $u \sim 0$, whose coefficients involve the digamma function, and the logarithmic corrections to thermodynamic quantities of the isotropic spin chain [19,35] can be reproduced.

IV. DISCUSSION

We have constructed the holographic dual theory of the XXZ quantum spin chain based on the CS gauge theory with a charged scalar field,[10] and calculated the thermodynamic quantities using the holographic techniques, which are completely consistent with the exact results of the spin chain. The surprising exact matching for the holographic dual of the free boson field theory[11] with $c = 1$ can be partly attributed to the CS/CFT duality in the three-dimensional CS gauge theory [22,46–48]. This is, to our best knowledge, the first example which has the precise matching of the physical quantities (including the numerical factors) between the holographic dual theory and the quantum many-body system in condensed matters, and this precise duality can lead to not only new insights from higher dimensional perspectives but also new analytical methods in quantum many-body systems. The nontrivial matching between the bulk scalar action and the free energy of the spin chain can give a physical background to the ODE/IM correspondence [49], which is the correspondence between ordinary differential equations and quantum integrable models, based on the black holes through the holographic duality. Furthermore the holographic dual theory can have the applications to the dynamics of one-dimensional quantum many-body systems in real materials, such as various transports in Tomonaga-Luttinger liquids [5], quantum spin chain [50], and spacetime-emergent materials [51].

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[10] A similar setup has been also discussed for the holographic dual description of the Kondo effect [42–44].

[11] A holographic dual of the free fermion description of the transverse-field Ising chain has been recently discussed in [45].

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