U folds from geodesics in moduli space

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We exploit the presence of moduli fields in the $AdS_3 \times S^3 \times CY_2$, where $CY_2 = T^4$ or $K3$, solution to type IIB superstring theory, to construct a U-fold solution with geometry $AdS_2 \times S^1 \times S^3 \times CY_2$. This is achieved by giving a nontrivial dependence of the moduli fields in $SO(4, n)/SO(4) \times SO(n)$ ($n = 4$ for $CY_2 = T^4$ and $n = 20$ for $CY_2 = K3$), on the coordinate η of a compact direction S^1 along the boundary of AdS_3, so that these scalars, as functions of η, describe a geodesic on the corresponding moduli space. The backreaction of these evolving scalars on spacetime amounts to a splitting of AdS_3 into $AdS_2 \times S^1$ with a nontrivial monodromy along S^1 defined by the geodesic. Choosing the monodromy matrix in $SO(4, n; \mathbb{Z})$, this supergravity solution is conjectured to be a consistent superstring background. We generalize this construction starting from an ungauged theory in $D = 2d$, d odd, describing scalar fields nonminimally coupled to $(d - 1)$ forms and featuring solutions with topology $AdS_d \times S^d$, and moduli scalar fields. We show, in this general setting, that giving the moduli fields a geodesic dependence on the η coordinate of an S^1 at the boundary of AdS_d is sufficient to split this space into $AdS_{d-1} \times S^1$, with a monodromy along S^1 defined by the starting and ending points of the geodesic. This mechanism seems to be at work in the known J-fold solutions in $D = 10$ type IIB theory and hints toward the existence of similar solutions in the type IIB theory compactified on CY_2. We argue that the holographic dual theory on these backgrounds is a $1 + 0$ CFT on an interface in the $1 + 1$ theory at the boundary of the original AdS_3.

I. INTRODUCTION

Solutions of superstring theory of $D = 11$ supergravity with geometry $AdS_d \times M_{int.}$ have been the focus of intense study because of their relevance to the AdS/CFT correspondence. Type IIB superstring theory, in particular, features a number of backgrounds of the form $AdS_d \times S^d \times M_{10-2d}$ which are characterized by moduli fields, i.e. scalar fields which can be assigned an arbitrary constant value throughout spacetime without affecting its geometry. These fields are dual to exactly marginal operators in the dual CFT. The simplest example is $AdS_5 \times S^5$ on which the holographic duality was originally conjectured, and whose moduli fields are the type IIB axion and dilaton fields which correspond to the complexified coupling constant in the dual $\mathcal{N} = 4$ SYM theory. Other examples of such backgrounds, or variants thereof, are the following[1]:

(i) $AdS_3 \times S^3 \times T^4$ describing the near-horizon geometry of a D1-D5 or of an F1-NS5 system, in which the 5-branes are wrapped around the 4-torus T^4;

(ii) $AdS_3 \times S^3 \times K3$ describing the near-horizon geometry of a D1-D5 or of an F1-NS5 system, in which the 5-branes are wrapped around the $K3$ manifold;

(iii) $AdS_5 \times S^5/\mathbb{Z}_k$ describing the near-horizon geometry of a stack of D3-branes at the apex of an orbifold $\mathbb{C}^3/\mathbb{Z}_k$ with $k > 1$ [4,5].

The backgrounds in (i) can be described within type IIB superstring theory compactified on T^4 to $D = 6$, which is described by a maximal six-dimensional supergravity featuring the global symmetry group $SO(5, 5)$ at the classical level. The classical moduli space of the $AdS_3 \times S^3$ solution of the six-dimensional maximal supergravity is $SO(4, 5)/SO(4) \times SO(5)$, $SO(4, 5)$ being the stabilizer in $SO(5, 5)$ of the brane charges.

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[1]For a review of the backgrounds $AdS_3 \times S^3 \times CY_2$, with $CY_2 = T^4$ or $K3$, and of their holographic dual descriptions, see [1–3]. In the present work, we shall not concentrate on the dual CFT side of these backgrounds.

The background in (ii) is a solution to type IIB superstring theory compactified on $K3$, which is described, at the classical level, by a half-maximal six-dimensional supergravity and the classical moduli space is $SO(4, 21)/SO(4) \times SO(21)$.

Finally, in the last case (iii), the classical moduli space is $SU(1, k)/U(k)$ and acts on the k complexified coupling constants of the dual necklace quiver theory [6,7].

All of these moduli occur in the d-dimensional gauged supergravity featuring the AdS_d factor ($d = 3$ and 5 in the above examples) as vacuum solution, as flat directions of the scalar potential.

Assuming a dependence of these scalar fields on the spacetime coordinates will in general affect the geometry of the background in some nontrivial way. In [5,8–10] the authors investigated spacetime-dependent configurations of the moduli fields in the Wick-rotated moduli space within the Euclidean version of the theory. The resulting solutions were described by geodesics in the (pseudo-Riemannian) moduli space, which were classified according to their being lightlike, spacelike, and timelike, and an interpretation of the corresponding backgrounds, in the dual CFT, was worked out.

Here we shall work in a suitable, ungauged Lorentzian theory in $D = 2d$, d odd, exhibiting an $AdS_d \times S^d$ background with moduli fields. We show how to construct an $AdS_{d-1} \times S^1 \times S^d$ geometry from the $AdS_d \times S^d$ one by giving a suitable spatial dependence to the scalar moduli.

The construction proceeds along the following steps. We first compactify one direction in the boundary of AdS_d. Let us denote by η the coordinate of the corresponding S^1. We then assume a suitable subset φ^a of the moduli fields to depend on η only and to describe, in their evolution, a geodesic in the moduli space as η varies in its defining interval of values $\eta \in [0, T]$, T being the length of S^1. As a consequence of the backreaction of the evolving scalar fields on spacetime, the $AdS_d \times S^d$ background is transformed into an $AdS_{d-1} \times S^1 \times S^d$ geometry, with a nontrivial monodromy along S^1. The two solutions cannot be continuously deformed into one another. If we denote by $\mathcal{M}_0 = G_0/H_0$ the moduli space spanned by φ^a, and by O and P the starting and ending points of the geodesic, parametrized by $\varphi^a(0)$ and $\varphi^a(T)$, respectively, the G_0 element g connecting P to O, defines the monodromy: $P = gO$. If we assume that $G_0(\mathbb{Z})$ is a symmetry of the underlying superstring theory reduced to $AdS_d \times S^d$, then by choosing $g \in G_0(\mathbb{Z})$, the backgrounds defined by the initial and final points of the geodesic would be identified from the string perspective and the corresponding supergravity solution could be conjectured to be a consistent superstring background. The component of the metric in the S^1 direction is proportional to the squared "velocity" along the geodesic. Therefore the $AdS_{d-1} \times S^1 \times S^d$ is not continuously connected to $AdS_d \times S^d$ as this velocity is set to zero and the moduli fields to constant values.

A prototype of this background is the nonsupersymmetric J fold [11], with geometry $AdS_4 \times S^1 \times S^5$, characterized by an axiodilaton field evolving along a geodesic in their moduli space $SL(2, \mathbb{R})/SO(2)$. In the corresponding J-fold description, the initial and final points of the geodesic are connected by a monodromy $g = J_n \in SL(2, \mathbb{Z})_{IIB}$, J_n being a hyperbolic element of the type IIB global symmetry group. However, the mechanism analyzed here, of the construction of $AdS_{d-1} \times S^1 \times S^d$ U folds by giving the moduli fields a geodesic dependence on the coordinate of S^1, seems to be at work in the J-fold solutions studied in [11–22] with geometry $AdS_4 \times S^1 \times \tilde{S}^5$, $\tilde{S}^5$ being a deformed 5-sphere. We shall further comment on this in the final Discussion section.

In this paper, we shall focus on solutions of the form $AdS_2 \times S^1 \times S^3$ within type IIB theory compactified on a CY_2 manifold, which can either be T^4 or $K3$. Such backgrounds feature a richer class of monodromies. In the former case, $CY_2 = T^4$, the solutions have an $O(4, 4; \mathbb{Z})$ monodromy, in the latter, $CY_2 = K3$, the monodromy is chosen in $O(4, 20; \mathbb{Z})$. These discrete groups are contained inside the so-called U-duality group which is conjectured to encode all string dualities and to be an exact symmetry of the underlying, though yet unknown, unifying quantum theory of gravity [23]. We shall work extensively on the $CY_2 = T^4$ case and construct explicit U-fold solutions within the effective $\mathcal{N} = (2, 2)$ six-dimensional supergravity.

The paper is organized as follows. In Sec. II we describe the general construction of such $AdS_{d-1} \times S^1 \times S^d$ backgrounds. We start from a $AdS_d \times S^d$ solution of an ungauged model in $D = 2d$, d odd, which describes scalar fields, spanning the symmetric target space of the sigma model, and nonminimally coupled to a set of $(d - 1)$ forms. We discuss, in Sec. II A, the issue of the boundary conditions along S^1 in relation to the geodesic motion of the moduli fields. In Sec. II A 1 we also introduce, on such backgrounds, the so-called χ deformations originally studied, in the context $D = 10$ J-fold solutions, in [15,16,20,22]. To give concrete examples of these solutions, we focus on the type IIB theory compactified on a 4-torus. We start reviewing, in Sec. III, the bosonic sector of the resulting ungauged maximal six-dimensional supergravity. This is the model where we construct explicit U-fold solutions. Then, in Sec. III B we review the known $AdS_3 \times S^3$ solutions and their string interpretation. Starting from these backgrounds, in Sec. III C we build the U fold for a certain choice of the monodromy matrix.

In Appendix A we review the geometric characterization of the type IIB string origin of the scalar and tensor fields in the maximal six-dimensional supergravity. In Appendix B we review the general construction of a black-string solution with $SO(4)$ symmetry and the attractor mechanism at work for the extremal ones. In Appendix C we review the construction of the D1-D5 solution within the maximal

six-dimensional theory, also in the presence of certain, radially evolving, moduli fields. In Appendix D we give the explicit form of the effective potential in the presence of D1-D5 charges, as a function of the string 0-modes, and discuss its extremization.

II. THE GENERAL CONSTRUCTION OF THE $AdS_{d-1} \times S^1 \times S^d$ U FOLD

In this section, we show how to build a new class of solutions with topology $AdS_{d-1} \times S^1 \times S^d$ (with dimension $D = 2d$) by giving the moduli a suitable nontrivial profile. For the sake of concreteness, we take d odd since we have in mind the type IIB backgrounds with $d = 3$ and 5.

We start considering a model, in D-dimensions, describing n nonchiral p forms $C^I_{(p)}$ ($I = 1, \dots n$), with $p = d - 1$, coupled to gravity and to a number of scalar fields ϕ^s, $s = 1, \dots n_s$. The latter are described by a sigma model with homogeneous symmetric, Riemannian target space:

$$M_{\text{scal}} = \frac{G}{H}.$$

We assume the isometry group G of the scalar manifold to admit a pseudo-orthogonal representation $\mathscr{R}_p$ in terms of $O(n, n)$ transformations. Our analysis will be extended to models in which the number n of self-dual and m anti-self-dual d-form field strengths of the p-form fields are different. In this case, we require G to admit a representation in terms of $O(n, m)$ transformations.

In an extended D-dimensional supergravity model, this geometric feature is built in. As is the case in ungauged D-dimensional supergravity, the scalar fields are nonminimally coupled to the tensor ones, that is they couple to their field strengths $H^I_{(d)} \equiv dC^I_{(p)}$ of the latter in their kinetic terms. The general form of the bosonic action we are considering is[2]

$$\mathcal{L}_{2d} = e_{2d} \left[R - \frac{1}{2} \partial_{\hat{\mu}} \phi^r \partial^{\hat{\mu}} \phi^s \mathcal{G}_{rs}(\phi) - \frac{1}{2} \mathcal{I}(\phi)_{IJ} H^I \cdot H^J \right.$$
$$\left. - \frac{1}{2} \mathcal{R}(\phi)_{IJ} H^I \cdot {}^* H^J \right],$$
(2.2)

where $\mathcal{G}_{rs}(\phi) > 0$ is the target space metric of the scalar sigma model, and $e_D = e_{2d} \equiv \sqrt{|\det(g_{\hat{\mu}\hat{\nu}})|}$. The matrices $\mathcal{I}_{IJ}(\phi) > 0$ and $\mathcal{R}_{IJ}(\phi) = -\mathcal{R}_{JI}(\phi)$ are functions of the

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[2]Throughout the paper we adopt the mostly plus notation for the metric. Moreover we define

$$\sigma_{(p)} \cdot \omega_{(d)} \equiv \sigma_{\hat{\mu}_1 \dots \hat{\mu}_d} \omega^{\hat{\mu}_1 \dots \hat{\mu}_d} / d!,$$

$$({}^*\omega)_{\hat{\mu}_1 \dots \hat{\mu}_{D-p}} \equiv \frac{e_D}{p!} \epsilon_{\hat{\mu}_1 \dots \hat{\mu}_{D-p} \hat{\nu}_1 \dots \hat{\nu}_p} \omega^{\hat{\nu}_1 \dots \hat{\nu}_p},$$
(2.1)

where $\epsilon_{0, \dots D-1} = 1$ and $\star^2 \omega_{(p)} = (-1)^{(D-p)p+1} \omega_{(p)}$.

scalar fields and describe the nonminimal coupling between them and the tensor field strengths H^I. We define the dual field strengths as

$$G_{I\hat{\mu}_1 \dots \hat{\mu}_d} = \epsilon_{\hat{\mu}_1 \dots \hat{\mu}_d \hat{\nu}_1 \dots \hat{\nu}_d} \frac{\partial \mathcal{L}_{2d}}{\partial H^I_{\hat{\nu}_1 \dots \hat{\nu}_d}},$$
(2.3)

which, in matrix form and suppressing the spacetime indices, reads

$$G = -\mathcal{I} \star H - \mathcal{R} H.$$
(2.4)

In what follows we treat the field strength and the dual field strength on equal footing and define this column vector

$$\mathbb{H} = (\mathbb{H}^M) = \begin{pmatrix} H^I \\ G_I \end{pmatrix}.$$
(2.5)

From this the following *twisted self-duality condition* follows

$$\star \mathbb{H} = -\Omega \mathcal{M} \mathbb{H},$$
(2.6)

where

$$\Omega = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \mathcal{M}(\phi) \equiv \begin{pmatrix} \mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & -\mathcal{R}\mathcal{I}^{-1} \\ \mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}, \quad (2.7)$$

Ω is the $O(n, n)$-invariant matrix, $\Omega^2 = \mathbb{I}$, and $\mathcal{M}$ is a scalar-dependent pseudo-orthogonal, positive-definite symmetric matrix: $\mathcal{M}\Omega\mathcal{M} = \Omega$. The Maxwell equations and the Bianchi equations read

$$d\mathbb{H} = 0.$$
(2.8)

The Einstein and scalar field equations of motion for this model are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(s)}_{\mu\nu} + T^{(H)}_{\mu\nu},$$

$$D_{\hat{\mu}}(\partial^{\hat{\mu}} \phi^s) = \nabla_{\hat{\mu}}(\partial^{\hat{\mu}} \phi^s) + \tilde{\Gamma}^s_{uv} \partial_{\hat{\mu}} \phi^u \partial^{\hat{\mu}} \phi^v$$
$$= \frac{1}{4} \mathcal{G}^{st} \mathbb{H}^T \partial_t \mathcal{M} \mathbb{H},$$
(2.9)

where the energy-momentum tensors are defined as

$$T^{(s)}_{\hat{\mu}\hat{\nu}} \equiv \frac{1}{2} \mathcal{G}_{rs} \left(\partial_{\hat{\mu}} \phi^r \partial_{\hat{\nu}} \phi^s - \frac{1}{2} g_{\hat{\mu}\hat{\nu}} \partial_{\hat{\rho}} \phi^r \partial^{\hat{\rho}} \phi^s \right),$$

$$T^{(H)}_{\hat{\mu}\hat{\nu}} \equiv \frac{1}{4p!} \mathbb{H}^T_{\hat{\mu}\hat{\mu}_1 \dots \hat{\mu}_p} \mathcal{M}(\phi) \mathbb{H}_{\hat{\nu}}^{\hat{\mu}_1 \dots \hat{\mu}_p},$$
(2.10)

and $\tilde{\Gamma}^s_{uv}(\phi)$ denotes the Levi-Civita connection on the scalar manifold. The scalar kinetic term can also be written in the following form

$$\frac{\Bbbk}{8}\text{Tr}(\mathcal{M}^{-1}\partial\mathcal{M}\mathcal{M}^{-1}\partial\mathcal{M}) = \frac{1}{2}\mathcal{G}_{st}\partial_{\hat{\mu}}\phi^s\partial^{\hat{\mu}}\phi^t, \qquad (2.11)$$

where the constant $\Bbbk$ depends of the matrix representation of $\mathcal{M}$.

It is straightforward to prove, using the above expression for $T^{(H)}$, that the trace of this tensor vanishes:

$$T^{(H)}{}_{\hat{\mu}}{}^{\hat{\mu}} = \frac{1}{8}\mathbb{H}_{\hat{\mu}}^T\mathcal{M}\mathbb{H}^{\hat{\mu}} = \frac{1}{8}\mathbb{H}^T\mathcal{M}\wedge\star\mathbb{H}$$
$$= -\frac{1}{8}\mathbb{H}^T\wedge\mathcal{M}\Omega\mathcal{M}\mathbb{H} = -\frac{1}{8}\mathbb{H}^T\wedge\Omega\mathbb{H} = 0. \qquad (2.12)$$

The global symmetry of the above equations in supergravity is the invariance under the two-fold action of the group G on the scalar fields as the isometry group of the scalar manifold, and on the field strengths $\mathbb{H}$ through a pseudo-orthogonal representation $\mathscr{R}_p$ (we suppress all pseudo-orthogonal indices)[3]:

$$g\in G \xrightarrow{\mathscr{R}_p} \mathscr{R}_p[g]\in O(n,n),$$
$$\mathbb{H} \to \mathbb{H}' = \mathscr{R}_p[g]\mathbb{H},$$
$$\mathcal{M}(\phi) \to \mathcal{M}(\phi') = \mathscr{R}_p[g]^{-T}\mathcal{M}(\phi)\mathscr{R}_p[g]^{-1}. \qquad (2.13)$$

In extended supergravities the matrix $\mathcal{M}(\phi)$ obeys the above transformation property since it is expressed in terms of the coset representative $\mathbb{L}(\phi^s)$ of M_{scal} in the representation $\mathscr{R}_p$ as follows[4]:

$$\mathcal{M}(\phi) = \mathscr{R}_p[\mathbb{L}(\phi)]^{-T}\mathscr{R}_p[\mathbb{L}(\phi)]^{-1}. \qquad (2.14)$$

The transformation property (2.13) of $\mathcal{M}(\phi)$ then follows from the action of an isometry element $g\in G$ on ϕ^s:

$$g\mathbb{L}(\phi) = \mathbb{L}(\phi')h(g,\phi), \qquad h(g,\phi)\in H \qquad (2.15)$$

and the fact that $\mathscr{R}_p[h(g,\phi)]\in O(n)\times O(n)$. We now consider solitonic solutions described only by the scalar and the tensor fields, with spacetime geometry $M_d\times_w S^d$, where we will choose $M_d = \text{AdS}_{d-1}\times S^1$. The metric will read

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{ij}d\xi^i d\xi^j$$
$$= v_1^2 ds^2_{\text{AdS}_{d-1}} + v_3^2 d\eta^2 + v_2^2 ds^2_{S^d}, \qquad (2.16)$$

where $\mu = 0,\ldots,d-1$ and $i = 1,\ldots,d$, $ds^2_{\text{AdS}_{d-1}}$ is the metric of an AdS_{d-1} space of radius 1 and $ds^2_{S^d}$ is the metric of a S^d-sphere of radius 1. The coefficients v_1, v_2, v_3 are constant and we have split the M_d coordinates

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[3]For the sake of simplicity, we use the shorthand notation $A^{-T}\equiv(A^{-1})^T$.
[4]Here we describe the scalar manifold as a left-coset space.

$(x^\mu) = (x^\alpha, \eta)$, where $\alpha = 0\ldots,d-2$ and $x^{d-1} = \eta$. We shall also define

$$\tilde{e}_d = \sqrt{|\det(g_{\mu\nu})|}, \qquad e_d = \sqrt{\det(g_{ij})}.$$

The radius v_2 of the d-sphere will also be denoted by L.

The general ansatz for the tensor field strengths reads

$$\mathbb{H} = -\Omega\mathcal{M}\Gamma\boldsymbol{\epsilon}_{M_d} + \Gamma\boldsymbol{\epsilon}_{S^d}, \qquad (2.17)$$

where $\boldsymbol{\epsilon}_M$, $\boldsymbol{\epsilon}_{S^d}$ denote the volume densities of the manifold M_d and S^d:

$$\boldsymbol{\epsilon}_M \equiv \frac{\tilde{e}_d}{d!\,L^d}\epsilon_{\mu_1\ldots\mu_d}dx^{\mu_1}\wedge\ldots dx^{\mu_d},$$
$$\boldsymbol{\epsilon}_{S^d} \equiv \frac{e_d}{d!L^d}\epsilon_{i_1\ldots i_d}d\xi^{i_1}\wedge\ldots d\xi^{i_d},$$

where we recall that L is the radius v_2 of the d-sphere. We have also denoted by $\Gamma = (\Gamma^M)$ the pseudo-orthogonal vector of charges:

$$\Gamma^M \equiv \frac{1}{S_{S^d}}\int_{S^d}\mathbb{H}^M, \qquad (2.18)$$

where S_{S^d} is the surface area of a d-sphere of radius 1. This ansatz is covariant under (2.13) provided the charge vector is transformed accordingly:

$$g\in G, \qquad \Gamma \to \Gamma' = \mathscr{R}_p[g]\Gamma. \qquad (2.19)$$

Quantum effects would restrict Γ^M to belong to an even, unimodular charge lattice $\boldsymbol{\Gamma}^{n,n}$ and this would, in turn, break the global symmetry group G to a discrete subgroup $G(\mathbb{Z}) \sim \mathscr{R}_p[G]\cap O(n,n;\mathbb{Z})$ preserving the lattice $\boldsymbol{\Gamma}^{n,n}$. In the effective supergravity description of superstring/M-theories, this group $G(\mathbb{Z})$ was conjectured to encode the known string dualities [23]. Field configurations connected by $G(\mathbb{Z})$ should be identified from the string theory point of view. The values of the D-dimensional scalar fields defining inequivalent string backgrounds should then span the manifold:

$$G(\mathbb{Z})\backslash G/H.$$

In chiral models, in which the number n of self-dual and the number m of anti-self-dual d-form field strengths in D dimensions are different, as is the case of type IIB superstring theory compactified in $K3$ to $D = 6$, the duality action of the global symmetry group G on the d-form field strengths is implemented by transformations in $O(n,m)$: $\mathscr{R}_p[G]\subset O(n,m)$. Quantum corrections break $O(n,m)$ to $O(n,m;\mathbb{Z})$ leaving the even, unimodular charge lattice $\boldsymbol{\Gamma}^{n,m}$ invariant. The global symmetry group is consequently

broken to $G(\mathbb{Z})$. Our construction also applies to this more general setting.

In the remainder of this paragraph, we shall focus on the classical description and thus consider the global symmetry group G to be continuous.

As for the space-time dependence of the scalar fields, we need first to define the solution's moduli. We define the *little group*, or *stabilizer*, of the charge vector Γ the maximal subgroup G_ℓ of G such that

$$\forall g \in G_\ell: \quad \mathscr{R}_p[g]\Gamma = \Gamma. \qquad (2.20)$$

If H_ℓ denotes the maximal compact subgroup of G_ℓ, the moduli fields φ^u, $u = 1, \ldots, n_\ell$, are defined as the subset of scalar fields ϕ^s spanning the submanifold,

$$M_\ell \equiv \frac{G_\ell}{H_\ell}, \qquad (2.21)$$

of M_{scal} at the origin. We split the scalar fields accordingly:

$$\{\phi^s\} = (\varphi^u, \phi^k), \quad u = 1, \ldots, n_\ell, \quad k = 1, \ldots, n_s - n_\ell,$$

by defining the coset representative of M_{scal} as [24]

$$\mathbb{L}(\phi^s) = \mathbb{L}_\ell(\varphi^u)\hat{\mathbb{L}}(\phi^k), \qquad \mathbb{L}_\ell(\varphi^u) \in \frac{G_\ell}{H_\ell}, \qquad (2.22)$$

where the scalars $\hat{\phi} \equiv (\phi^k)$ are defined as follows. Consider the Cartan decomposition of the Lie algebra $\mathbb{G}$ of G into the Lie algebra $\mathbb{H}$ generating H and the coset space $\mathbb{K}$ isomorphic to the tangent space to M_{scal} at the origin:

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K}. \qquad (2.23)$$

The latter space $\mathbb{K}$ supports a representation of H under its adjoint action. This representation is completely reducible under the adjoint action of $H_\ell \subset H$ and $\mathbb{K}$ splits accordingly in H_ℓ-invariant subspaces

$$\mathbb{K} = \mathbb{K}_\ell \oplus \hat{\mathbb{K}}, \qquad (2.24)$$

where $\mathbb{K}_\ell$ is the coset space of G_ℓ/H_ℓ, while $\hat{\mathbb{K}}$ supports a representation of H_ℓ. We define

$$\hat{\mathbb{L}}(\phi^k) \in e^{\hat{\mathbb{K}}}. \qquad (2.25)$$

We note that the scalar fields φ^u span a symmetric submanifold G_ℓ/H_ℓ of M_{scal} only at $\phi^k = 0$. For fixed $\hat{\phi} = (\phi^k) \neq (0)$, φ^u span a space which is not a symmetric submanifold G_ℓ/H_ℓ of M_{scal} since this space does not support a transitive action of G_ℓ. Indeed, using the shorthand notation $\phi \equiv (\phi^s)$ and $\varphi_\ell \equiv (\varphi^u)$, the action of an element $g_\ell \in G_\ell$ on these scalars also affects $\hat{\phi}$:

$$g_\ell\mathbb{L}(\phi) = g_\ell\mathbb{L}_\ell(\varphi_\ell)\hat{\mathbb{L}}(\hat{\phi}) = \mathbb{L}_\ell(\varphi'_\ell)\hat{\mathbb{L}}(\hat{\phi}')h_\ell, \qquad (2.26)$$

where $h_\ell \in H_\ell$ is the compensating transformation in

$$g_\ell\mathbb{L}_\ell(\varphi_\ell) = \mathbb{L}_\ell(\varphi'_\ell)h_\ell$$

and

$$\hat{\mathbb{L}}(\hat{\phi}') \equiv h_\ell\hat{\mathbb{L}}(\hat{\phi})h_\ell^{-1}.$$

Since, for our analysis, we are interested in a symmetric submanifold M_0 of M_{scal} at fixed, nonvanishing values $\hat{\phi}_* \equiv (\phi_*^k)$ of $\hat{\phi} = (\phi^k)$ (to be identified with the extremum of the scalar potential) we will have to restrict the moduli fields to a symmetric submanifold G_0/H_0 of G_ℓ/H_ℓ ($G_0 \subset G_\ell$, $H_0 \subset H_\ell$) characterized by the property that $[G_0, \mathbb{L}(\hat{\phi}_*)] = 0$. To this end, we first reduce $\hat{\phi}_*$ to its simplest form (*normal form*), by acting on it using H_ℓ and then define G_0 to be a noncompact, semisimple subgroup of G_ℓ commuting with $\mathbb{L}(\hat{\phi}_*)$. The manifold $M_0 = G_0/H_0$, H_0 being the maximal compact subgroup of G_0 is now a symmetric submanifold of M_{scal} at $\hat{\phi} = \hat{\phi}_*$ and, as such, it is *totally geodesic*.[5] Let us denote a point in it by $\varphi_0 \equiv (\varphi^a)$, $a = 1, \ldots, n_m$, where $n_m < n_\ell$ is the dimension of M_0.

Using the ansatz (2.17) and the property of $\mathcal{M}$ of being pseudo-orthogonal symmetric, we can write

$$\mathbb{H}^T \frac{\partial}{\partial\phi^s}\mathcal{M}\mathbb{H} = 4\frac{\partial}{\partial\phi^s}V(\phi, \Gamma)L^{-2d}, \qquad (2.27)$$

where

$$V(\phi, \Gamma) \equiv \frac{1}{2}\Gamma^T\mathcal{M}(\phi)\Gamma. \qquad (2.28)$$

The scalar field equations, on the ansatz, read

$$\begin{aligned} D_{\hat{\mu}}(\partial^{\hat{\mu}}\phi^s) &= \nabla_{\hat{\mu}}(\partial^{\hat{\mu}}\phi^s) + \tilde{\Gamma}_{uv}^s\partial_{\hat{\mu}}\phi^u\partial^{\hat{\mu}}\phi^v \\ &= \mathcal{G}^{st}\partial_t V \, L^{-2d}. \end{aligned} \qquad (2.29)$$

Notice that, in the chosen parametrization defined by the coset representative of the form (2.22), the scalar potential is independent of the moduli fields φ^u. Indeed, by using Eqs. (2.14) and (2.20):

$$V(\varphi^s, \Gamma^M) = V(\varphi^u, \phi^k, \Gamma^M) = V(\phi^k, \Gamma^M).$$

Let us define $\hat{\phi}_* \equiv (\phi_*^k)$ to be the value of the nonmoduli fields ϕ^k which extremize the scalar potential V:

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[5]A totally geodesic submanifold M_0 of a Riemannian manifold M_{scal} is characterized by the property that a geodesic in M_{scal} originating in a point of M_0 and initially tangent to M_0 lies entirely in the submanifold itself.

$$\frac{\partial V}{\partial \phi^k}\bigg|_{\phi^k=\phi^k_*} = 0 \Rightarrow \phi^k_* = \phi^k_*(\Gamma). \qquad (2.30)$$

The fixed values $\hat{\phi}_* = (\phi^k_*)$ only depend on the quantized charges and are defined modulo the action of a H_ℓ transformation. In light of our previous discussion, we fix the action of H_ℓ to bring $\hat{\phi}_*$ to its normal form and then restrict the moduli fields to the coordinates φ^a, $a = 1, \ldots, n_m$, of the symmetric submanifold $M_0 = G_0/H_0$. The value of the scalar potential at the minimum is denoted by V_* and only depends on the quantized charges Γ^M:

$$V_*(\Gamma) = V(\phi^k_*, \Gamma) > 0.$$

We complete our ansatz by choosing

$$\phi^k(x) \equiv \phi^k_*, \qquad \varphi^a(x) = \varphi^a(\eta), \qquad (2.31)$$

where η is the coordinate parametrizing S^1 in $\mathrm{M}_d = \mathrm{AdS}_{d-1} \times S^1$. All other scalar fields are set to zero. The scalar field equation reduces to the geodesic equation for $\varphi^a(\eta)$ on M_0:

$$\ddot{\varphi}^a + \tilde{\Gamma}^a_{\ bc}\dot{\varphi}^b\dot{\varphi}^c = g_{\eta\eta} \mathcal{G}^{ak} \frac{\partial V}{\partial \phi^k}\bigg|_{\phi^k=\phi^k_*} L^{-2d} = 0, \qquad (2.32)$$

where we have used the fact that V is independent of φ^a and have defined $\dot{\varphi} \equiv d\varphi/d\eta$. Having assumed the nonmoduli fields ϕ^k to be constant and equal to the values that minimize V, all the derivatives of ϕ^k on the solution vanish and the corresponding field equations read

$$g^{\eta\eta}\tilde{\Gamma}^k_{\ bc}\dot{\varphi}^b\dot{\varphi}^c = L^{-2d}\mathcal{G}^{kk'}\frac{\partial V}{\partial \phi^{k'}}\bigg|_{\phi^k=\phi^k_*} L^{-2d} = 0,$$

which is satisfied provided $\tilde{\Gamma}^k_{\ bc} = 0$ for $\phi^k = \phi^k_*$ and φ^a generic. This condition follows from the property of the moduli space G_0/H_0 of being a totally geodesic submanifold of G/H.[6][7]

The moduli fields $\varphi^a(\eta)$, therefore, describe a geodesic in M_0. Let us denote by κ the corresponding line element (or "velocity") along it:

$$\kappa^2 = \frac{1}{2}\mathcal{G}_{ab}\dot{\varphi}^a\dot{\varphi}^b. \qquad (2.33)$$

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[6]Notice indeed that the coset space $\mathbb{K}_0 \subset \mathbb{K}$ of G_0/H_0 is also contained in the tangent space $\mathbb{K}_* \equiv \hat{\mathbb{L}}(\phi^k_*)\mathbb{K}\hat{\mathbb{L}}(\phi^k_*)^{-1}$ to M_{scal} at the point $\varphi^u = 0, \phi^k = \phi^k_*$ since $\mathbb{K}_0 = \hat{\mathbb{L}}(\phi^k_*)\mathbb{K}_0\hat{\mathbb{L}}(\phi^k_*)^{-1} \subset \mathbb{K}_*$. Being M_0 symmetric, it follows that $[[\mathbb{K}_0, \mathbb{K}_0], \mathbb{K}_0] \subset \mathbb{K}_0$, namely $\mathbb{K}_0$ is a *Lie triple system* and thus M_0 is totally geodesic [25].
[7]By the same token, also the field equations for the moduli fields φ^u, $u \neq a$, which are set to zero, are satisfied since $\tilde{\Gamma}^u_{\ bc} = 0$, $u \neq a$.

The reader can verify that the tensor-field equations $d\mathbb{H}^M = 0$ are satisfied by the ansatz.

As far as the Einstein equations are concerned, the only nonvanishing components of the Riemann and Ricci tensors are

$$R_{\alpha\beta\gamma\delta} = -v_1^{-2}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \Rightarrow,$$

$$R_{\alpha\beta} = R_{\alpha\gamma\beta}{}^\gamma = -\frac{(d-2)}{v_1^2}g_{\alpha\beta},$$

$$R_{ijkl} = v_2^{-2}(g_{ik}g_{jl} - g_{il}g_{jk}) \Rightarrow,$$

$$R_{ij} = R_{ikj}{}^k = \frac{(d-1)}{v_2^2}g_{ij}. \qquad (2.34)$$

The Einstein equation can be conveniently recast in the form

$$R_{\hat{\mu}\hat{\nu}} = \frac{1}{2}\mathcal{G}_{rs}\partial_{\hat{\mu}}\phi^r\partial_{\hat{\nu}}\phi^s + T^{(H)}_{\hat{\mu}\hat{\nu}}. \qquad (2.35)$$

The components of $T^{(H)}$ are

$$T^{(H)}_{\alpha\beta} = -\frac{1}{2}V_*L^{-2d}g_{\alpha\beta},$$

$$T^{(H)}_{\eta\eta} = -\frac{1}{2}V_*v_3^2L^{-2d},$$

$$T^{(H)}_{ij} = \frac{1}{2}V_*L^{-2d}g_{ij}. \qquad (2.36)$$

Taking into account the contribution to the energy-momentum tensor coming from the geodesic motion of the moduli fields φ^a, the Einstein equations imply the following relations:

$$v_1^2 = \frac{2(d-2)L^{2d}}{V_*}, \quad v_2^2 = L^2 = \frac{2(d-1)L^{2d}}{V_*}, \quad v_3^2 = \frac{2\kappa^2 L^{2d}}{V_*}. \qquad (2.37)$$

From the second equation, it follows that

$$L^{2(d-1)} = \frac{V_*}{2(d-1)},$$

and, in terms of L, the radii v_1 of AdS_{d-1} and v_3 of S^1 read

$$v_1 = \sqrt{\frac{d-2}{d-1}}L, \qquad v_3 = \frac{\kappa}{\sqrt{d-1}}L.$$

We notice that the radius v_3 of S^1 is proportional to the velocity along the geodesic, that is its constant unit measure κ.

This solution is to be compared to the known one with geometry $\mathrm{AdS}_d \times S^d$ in which

$$\phi^k = \phi_*^k(\Gamma) = \text{const}, \qquad \varphi^u = \text{const}.$$

In that case, the ansatz for the metric reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{ij}d\xi^i d\xi^j = v_1^2 ds_{\text{AdS}_d}^2 + v_2^2 ds_{S^d}^2, \quad (2.38)$$

and that for the tensor field strengths has the same general form (2.17). For this solution, we find

$$v_1^2 = v_2^2 = L^2 = \frac{2(d-1)L^{2d}}{V_*} \Rightarrow v_1 = L = \left(\frac{V_*}{2(d-1)}\right)^{\frac{1}{2(d-1)}}. \tag{2.39}$$

Notice that the $\text{AdS}_{d-1} \times S^1 \times S^d$ discussed above and the known $\text{AdS}_d \times S^d$ solution are not continuously connected. Indeed setting $\kappa = 0$ amounts in the former to shrinking S^1 to zero radius. We may say that the $\text{AdS}_{d-1} \times S^1 \times S^d$ solution may be obtained from the $\text{AdS}_d \times S^d$ by compactifying one direction on the boundary of AdS_d and giving the moduli fields φ^a a geodesic dependence on the coordinate of the corresponding S^1. This procedure amounts to performing a Scherk-Schwarz reduction on S^1 with a twist matrix described by a *hyperbolic element* of G_0, as we will show below. It is the backreaction of the evolving scalar fields on space-time that deforms AdS_d into $\text{AdS}_{d-1} \times S^1$.

It is interesting to note that this construction can be generalized. We can take k directions along the boundary of AdS_d and compactify them, allowing the moduli to depend on the coordinates $\eta_1, \ldots, \eta_k$ along these k directions:

$$\varphi^a = \varphi^a(\boldsymbol{\eta}) = \varphi^a(\eta_1, \ldots, \eta_k),$$

where $\boldsymbol{\eta} \equiv (\eta_1, \ldots, \eta_k)$, while the nonmoduli fields ϕ^k are still fixed at their attractor values ϕ_*^k. We assume $d - k > 1$. Let us further assume that there is a symmetric submanifold of G_ℓ/H_ℓ consisting of the product of k factors:

$$M_0 = \frac{G_0}{H_0} = \frac{G_1}{H_1} \times \frac{G_2}{H_2} \times \ldots \times \frac{G_k}{H_k}, \tag{2.40}$$

where the elements of $G_0 = G_1 \times \cdots \times G_k$ are required to commute with $\hat{\mathbb{L}}(\hat{\phi}_*)$. Then we take $\varphi^a(\eta_1, \ldots, \eta_k)$ to describe k geodesics, one within each factor $\frac{G_j}{H_j}$, $j = 1, \ldots, k$, so that

$$\frac{1}{2}\mathcal{G}_{ab}(\varphi(\boldsymbol{\eta}))\partial_{\eta_i}\varphi^a \partial_{\eta_j}\varphi^b = \delta_{ij}\kappa_i^2.$$

The backreaction of the moduli on the background yields a spacetime with geometry:

$$\text{AdS}_{d-k} \times \underbrace{S^1 \times \ldots \times S^1}_{k} \times S^d. \tag{2.41}$$

Writing the metric in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{ij}d\xi^i d\xi^j$$

$$= v_1^2 ds_{\text{AdS}_{d-k}}^2 + \sum_{j=1}^k v_{2+j}^2 d\eta_j^2 + v_2^2 ds_{S^d}^2, \tag{2.42}$$

where $\eta_1, \ldots, \eta_k$ parametrize the k circles, we find, from the Einstein equation

$$v_1^2 = \frac{2(d-k-1)L^{2d}}{V_*}, \qquad v_2^2 = L^2 = \frac{2(d-1)L^{2d}}{V_*},$$

$$v_{j+2}^2 = \frac{2\kappa_j^2 L^{2d}}{V_*}, \qquad j = 1, \ldots k. \tag{2.43}$$

Having assumed $d - k > 1$, v_1 is nonvanishing. The above equations imply

$$L^{2(d-1)} = \frac{V_*}{2(d-1)}, \qquad v_1 = \sqrt{\frac{d-1-k}{d-1}}L,$$

$$v_{j+2} = \frac{\kappa_j}{\sqrt{d-1}}L.$$

In this section, we proved that the general ansatz describing the new solutions satisfies the field equations. In this, no role is played by the global properties of the solutions themselves. In particular, we could have also considered our backgrounds in the form

$$\text{AdS}_{d-k} \times \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{k} \times S^d. \tag{2.44}$$

For $d = 5$ and $k = 1$ the background is a singular instance of Janus solutions [26–29] in type IIB superstring theory.

However, since the geodesics on the moduli space G_0/H_0 are noncompact, if the affine parameter η_j varies in $\mathbb{R}$, the corresponding geodesic would stretch to the boundary of the manifold and the solution would not be regular. Considering the product of k circles S^1, we need to specify, in a consistent way, the corresponding boundary conditions of the bosonic fields.[8] We shall expand on this issue, within superstring theory, in the next subsection.

Let us conclude this section by noting that if we started from the background $\text{EAdS}_d \times S^d$ in the Euclidean version of the model, the moduli space would be a pseudo-Riemannian, Wick-rotated version of the one in the Lorentzian theory. As such it can also describe timelike geodesics for which

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[8]Alternatively, we could consider a product of k finite intervals (I^k) or of k circles.

$$\frac{1}{2}\mathcal{G}_{ab}\dot{\varphi}^a\dot{\varphi}^b = -\kappa^2 < 0. \tag{2.45}$$

These geodesics would allow for a decomposition of the d-sphere yielding geometries of the form

$$\text{EAdS}_d \times \underbrace{S^1 \times \ldots \times S^1}_{k} \times S^{d-k}. \tag{2.46}$$

A. Boundary conditions on S^1

Consider the solution with spacetime of the form $\text{AdS}_{d-1} \times S^1 \times S^d$ within superstring theory. Let us denote by T the length of S^1 so that $\eta \in [0, T]$. Since the dependence of the moduli fields $\varphi^a(\eta)$ on η describes a geodesic on M_0 which connects two distinct points as we move around the circle S^1, consistency of the background as a string theory solution requires the *monodromy* matrix connecting the initial and final points to be a string duality, namely to belong to $G_0(\mathbb{Z})$. More precisely, suppose $\varphi^a(\eta)$ describe a geodesic on M_0 connecting an initial point, which we can choose to be the origin of the moduli space $\varphi^a(0) = 0$, to a final point in M_0 with coordinate $\varphi^a(T)$. The coset representative along the geodesic defines an η-dependent "twist" matrix in G_0/H_0:

$$\mathcal{A}(\eta) \equiv \mathbb{L}_0(\varphi^a(\eta)). \tag{2.47}$$

The monodromy matrix reads

$$\mathfrak{M} \equiv \mathcal{A}(T)\mathcal{A}(0)^{-1} \tag{2.48}$$

and maps the coset representative in M_{scal} at $\eta = 0$ to that at $\eta = T$:

$$\mathfrak{M}\mathbb{L}(\varphi^a(0), \hat{\phi}_*) = \mathbb{L}(\varphi^a(T), \hat{\phi}_*). \tag{2.49}$$

This matrix is defined modulo a compensating transformation in the isotropy group of $\varphi^a(0)$ in M_0:

$$\mathfrak{M} \sim \mathfrak{M}h_0, \qquad h_0 \in \mathbb{L}_0(\varphi^a(0))H_0\mathbb{L}_0(\varphi^a(0))^{-1}. \tag{2.50}$$

We require one representative of the equivalence class of $\mathfrak{M}$ to be an integer matrix:

$$\mathfrak{M} \in G_0(\mathbb{Z}). \tag{2.51}$$

A conjugation of $\mathfrak{M}$ by an element of $G_0(\mathbb{Z})$ can be reabsorbed by a redefinition of the scalar fields φ^a in the background so that it is unphysical. The monodromy matrix is therefore defined by conjugacy classes of hyperbolic elements in $G_0(\mathbb{Z})$.

The geodesic on M_0 is uniquely defined by the initial point $\varphi_0(0) \equiv (\varphi^a(0))$ and by a velocity vector $\mathbb{Q} \in T_{\varphi_0(0)}(M_0)$. The latter can be described by a matrix of the form

$$\mathbb{Q} = \mathbb{L}_0(\varphi_0(0))\mathbb{Q}_0\mathbb{L}_0(\varphi_0(0))^{-1},$$

where $\mathbb{Q}_0$, being an element of the coset space of M_0 in a suitable basis of the chosen real matrix representation, is a symmetric matrix: $\mathbb{Q}_0 = \mathbb{Q}_0^T$. The moduli along the corresponding geodesic $\varphi_0(\eta) \equiv (\varphi^a(\eta))$ are solutions to the matrix equation:

$$\begin{aligned}
\mathcal{M}_0(\varphi_0(\eta)) &\equiv \mathbb{L}_0(\varphi_0(\eta))\mathbb{L}_0(\varphi_0(\eta))^T \\
&= \mathcal{M}_0(\varphi_0(0))e^{\mathbb{Q}^T\eta} \\
&= \mathbb{L}_0(\varphi_0(0))e^{\mathbb{Q}_0\eta}\mathbb{L}_0(\varphi_0(0))^T. \tag{2.52}
\end{aligned}$$

In terms of $\mathbb{Q}_0$ the velocity κ along the geodesic reads

$$\kappa^2 = \frac{\text{k}}{8}\text{Tr}(\mathbb{Q}_0^2). \tag{2.53}$$

We can restrict ourselves to geodesics originating in the origin of M_0, $\varphi^a(0) = 0$, where $\mathbb{L}_0(\varphi_0(0)) = \mathbb{L}_0(0) = \mathbf{1}$. Having chosen $\mathfrak{M} \in G_0(\mathbb{Z})$, then T and $\mathbb{Q}_0$ have to be fixed so that

$$\mathcal{M}_0(\varphi_0(T)) = \mathfrak{M}\mathcal{M}_0(\varphi_0(0))\mathfrak{M}^T = \mathfrak{M}\mathfrak{M}^T = e^{\mathbb{Q}_0T}. \tag{2.54}$$

In the more general solution (2.41), we have a monodromy matrix $\mathfrak{M}_j$ associated with each of the k 1-cycles. Consistency as a string background requires, for each of these matrices,

$$\mathfrak{M}_j \in G_j(\mathbb{Z}) \subset G(\mathbb{Z}), \qquad j = 1, \ldots, k. \tag{2.55}$$

The solutions described here are instances of U folds [30]. They feature noncontractible 1-cycles along which there is a monodromy matrix in a string duality in $G(\mathbb{Z})$. In general, this duality is a combination of S and T dualities and it is referred to as U duality.

1. Deformations of the global geometry of the solution

Let us discuss a deformation of the global geometry of the solution effected by introducing suitable metric moduli. Since d is an odd number, it is convenient to choose the following parametrization for S^d:

$$\{\xi^i\} = \{\mu_I, \phi_I\}, \quad i = 1, \ldots, d, \quad I = 1, \ldots, \text{p} = \frac{d+1}{2}, \tag{2.56}$$

where $\mu_I \geq 0, \phi_I \in [0, 2\pi)$, in which its metric reads

$$ds_{S^d}^2 = L^2 \sum_{I=1}^{\text{p}}(d\mu_I^2 + \mu_I^2 d\phi_I^2). \tag{2.57}$$

The coordinates μ^I are subject to the constraint $\sum_{I=1}^{\text{p}}\mu_I^2 = 1$. Let us now perform, on the solution discussed in the previous sections, the following local change of variables:

$$\phi_I \to \phi_I' = \phi_I + \chi_I \eta.$$

The resulting solution is locally, though not globally, equivalent to the original one, as discussed in [15,16,20,22]. The parameters χ_I are metric deformation of the manifold $S^1 \times S^d$ which can be understood as follows. Consider the *local* torus

$$T^{\mathrm{p}+1} = S_{\phi_1}^1 \times S_{\phi_2}^1 \times \cdots \times S_\eta^1,$$

where $S_{\phi_I}^1$ are the local circles parametrized by ϕ_I and S_η^1 is the circle parametrized by η. For $\chi_I = 0$, $T^{\mathrm{p}+1}$ can be described as the quotient

$$T^{\mathrm{p}+1} = \mathbb{R}^{\mathrm{p}+1}/\Lambda_{\mathrm{cubic}},$$

where Λ_{cubic} is a cubic lattice generated by an orthonormal basis of vectors $\mathbf{u}_I$, $\mathbf{u}$. The position vector $\mathbf{x}$ of a point in $T^{\mathrm{p}+1}$ is defined modulo a vector in the lattice (summation over the repeated index I is understood):

$$\mathbf{x} = \frac{\phi_I}{2\pi}\mathbf{u}_I + \frac{\eta}{T}\mathbf{u} \sim \mathbf{x} + n_I\mathbf{u}_I + n\mathbf{u}; \quad n_I, n \in \mathbb{Z}, \quad (2.58)$$

which implies $\phi_I \sim \phi_I + 2\pi$ and $\eta \sim \eta + T$.

After the deformation, that is for $\chi_I \neq 0$, the local torus is described as

$$T^{\mathrm{p}+1} = \mathbb{R}^{\mathrm{p}+1}/\Lambda[\chi_I],$$

where $\Lambda[\chi_I]$ is a lattice generated by the nonorthogonal basis $\{\tilde{\mathbf{u}}_I, \tilde{\mathbf{u}}\}$, where

$$\tilde{\mathbf{u}}_I \equiv \mathbf{u}_I, \qquad \tilde{\mathbf{u}} \equiv \frac{\chi_I T}{2\pi}\mathbf{u}_I + \mathbf{u}.$$

Now the position vector of a point on the local torus is subject to the identification:

$$\mathbf{x} = \frac{\phi_I}{2\pi}\mathbf{u}_I + \frac{\eta}{T}\tilde{\mathbf{u}} \sim \mathbf{x} + n_I\mathbf{u}_I + n\tilde{\mathbf{u}}, \qquad n_i, n \in \mathbb{Z}, \quad (2.59)$$

which implies the same identifications on ϕ_i, η given earlier. Defining instead new coordinates ϕ_I', η' with respect to the old orthonormal basis

$$\mathbf{x} = \frac{\phi_I'}{2\pi}\mathbf{u}_I + \frac{\eta'}{T}\mathbf{u},$$

we have the following relation:

$$\phi_I' = \phi_I + \chi_I \eta, \qquad \eta' = \eta.$$

While in the coordinates $\{\phi_I, \eta\}$ the parameters χ_I appear in the metric, in the coordinates $\{\phi_I', \eta\}$, they only appear in *twisted* identifications:

$$\phi_I' \sim \phi_I' + 2\pi, \qquad \eta \sim \eta + T \wedge \phi_I' \sim \phi_I' + \chi_I T. \quad (2.60)$$

The deformations χ_I break the isometry group $SO(d+1)$ of S^d to its maximal torus $U(1)^{\mathrm{p}}$ acting on ϕ_I as shift isometries.

This global deformation was originally studied in [15,16,20,22] in more sophisticated variants of the U folds described here, for $d = 5$, and the parameters χ_I corresponded to exactly marginal deformations of the dual Supersymmetric conformal field theory (SCFT) at the boundary.

From the above characterization, it follows that χ_I are periodic since

$$\chi_I \to \chi_I \pm \frac{2\pi}{T} \Rightarrow \tilde{\mathbf{u}}_I \to \tilde{\mathbf{u}}_I, \quad \tilde{\mathbf{u}} \to \tilde{\mathbf{u}} \pm \tilde{\mathbf{u}}_I \in \Lambda[\chi_I], \quad (2.61)$$

so that $\chi_I \sim \chi_I \pm \frac{2\pi}{T}$.

B. Examples in type IIB superstring theory

1. Type IIB in $D = 10$

The first setting where our construction can be applied is type IIB theory in ten dimensions [31]. The global symmetry group of the classical low-energy supergravity description is $G = SL(2, \mathbb{R})_{\mathrm{IIB}}$ within which only $G(\mathbb{Z}) = SL(2, \mathbb{Z})_{\mathrm{IIB}}$ is a symmetry of the superstring theory. It is known that this theory admits maximally supersymmetric background of the form $AdS_5 \times S^5$ in which the three-form field strengths vanish and the self-dual five-form field strength $\hat{F}_{(5)} = {}^* \hat{F}_{(5)}$ has a nonvanishing flux, related to the radii of the AdS_5 and S^5. The construction discussed earlier applies with $d = 5$, $p = 4$. In this case the representation is $\mathscr{R}_p = \mathscr{R}_4 = \mathbf{1}$, namely it is the singlet representation since the electric and magnetic five-form field strengths coincide $G = F = \mathbb{H} = \hat{F}_{(5)}$, $\mathcal{M}(\phi) = 1$. Moreover, $G_0 = G$ and the scalar manifold

$$M_{\mathrm{scal}} = \frac{G}{H} = \frac{SL(2, \mathbb{R})_{\mathrm{IIB}}}{SO(2)}$$

coincides with the moduli space $M_0 = G_0/H_0$ and describes the axion $C_{(0)}$ and the dilaton ϕ. In our previous notation we identify $\varphi^a = (\phi, C_{(0)})$ and the (left) coset representative in the $\mathbf{2}$ of $SL(2, \mathbb{R})$ is $\mathbb{L} = \mathbb{L}_0(\varphi^a) = L_{(2)}(\phi, C_{(0)})$, where $L_{(2)}$ is defined in Eq. (3.6) below.[9]

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[9]In this representation, the action from the left of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with $ad - bc = 1$, amounts to the following transformation on the complex axion-dilaton field $\rho = C_{(0)} + ie^{-\phi}$: $\rho \to \rho' = (d\rho + c)/(b\rho + a)$.

The background with geometry $AdS_4 \times S^1 \times S^5$ was constructed in [11]. The axion and the dilaton describe a geodesic on M_{scal}. This background is nonsupersymmetric and unstable. The solution in [11] can be generalized by considering a geodesic in the moduli space originating in a generic point $\phi(0), C_{(0)}(0)$ with velocity κ:

$$e^{\phi(\eta)} = \cosh(\sqrt{2}\kappa\eta)e^{\phi(0)},$$
$$C_{(0)}(\eta) = e^{-\phi(0)}\tanh(\sqrt{2}\kappa\eta) + C_{(0)}(0). \qquad (2.62)$$

This solution, however, does not satisfy the appropriate boundary conditions for it to be a possible solution to type IIB superstring theory: the monodromy matrix as $\eta \to \eta + T$ must be in $SL(2, \mathbb{Z})_{IIB}$. Choosing, for instance,[10]

$$\mathfrak{M} = J_n \equiv -\mathcal{S}\mathcal{T}^n = \begin{pmatrix} n & 1 \\ -1 & 0 \end{pmatrix}, \quad n \in \mathbb{Z}, n > 0, \quad (2.63)$$

and $C_{(0)} = \phi(0) = 0$, solving Eq. (2.54), we find the following solution:

$$e^{\phi(\eta)} = \frac{n\sinh(\sqrt{2}\kappa\eta)}{\sqrt{n^2 + 4}} + \cosh(\sqrt{2}\kappa\eta),$$
$$C_{(0)}(\eta) = -\frac{2\sinh(\sqrt{2}\kappa\eta)}{\sqrt{n^2 + 4}\cosh(\sqrt{2}\kappa\eta) + n\sinh(\sqrt{2}\kappa\eta)},$$
$$T = \frac{1}{\sqrt{2}\kappa}\cosh^{-1}\left(\frac{n^2}{2} + 1\right). \qquad (2.64)$$

Defining the complex field $\rho = C_{(0)} + ie^{-\phi}$ one can verify that

$$\rho(T) = -\frac{1}{\rho(0) + n}, \qquad (2.65)$$

which is the effect of the transformation $J_n = -\mathcal{S}\mathcal{T}^n$ on $\rho(0)$. Notice that, on this solution

$$\mathfrak{M} = \mathcal{A}(T)\mathcal{A}(0)^{-1}h_0 = L_{(2)}(\phi(T),$$
$$C_{(0)}(T))L_{(2)}(\phi(0), \qquad C_{(0)}(0))^{-1}h_0,$$

where the compensating transformation h_0 reads

$$h_0 = \frac{1}{\sqrt{1 + n^2}}\begin{pmatrix} n & 1 \\ -1 & n \end{pmatrix} \in SO(2). \qquad (2.66)$$

We could start from the background of the form [4,6,7]

$$AdS_5 \times S^5/\mathbb{Z}_k, \qquad (2.67)$$

which preserves 16 supercharges. The classical supergravity description features the following moduli space:

$$M_{scal} = \frac{SU(1, k)}{U(1) \times SU(k)}. \qquad (2.68)$$

It describes k complex scalars dual to the complexified coupling constants τ_i of the dual $\mathcal{N} = 2$ supersymmetric A_{k-1} quiver gauge theory. Also in this case $\mathscr{R}_p = \mathscr{R}_4 = \mathbf{1}$, $G = F = \mathbb{H} = \hat{F}_{(5)}$, $\mathcal{M}(\phi) = 1$, and $G_0 = G = SU(1, k)$. Applying our construction to this situation would require the monodromy matrix to belong to the quantum duality which generalizes the $SL(2, \mathbb{Z})$ symmetry for the $k = 1$ case to $k > 1$. For a discussion on the definition of this group and its relation to the classical symmetry $SU(1, k)$ of the supergravity moduli space see, for instance, [6,32].

2. Type IIB on T^4 or K3

The six-dimensional theory resulting from the compactification of type IIB superstring theory on a 4-torus T^4 is described, in its low-energy limit, by the maximal $\mathcal{N} = (2, 2)$ six-dimensional supergravity [33]. This theory features a solution with spacetime $AdS_3 \times S^3$, describing the near-horizon geometry of a system of D1-D5-branes with two common Neumann directions (at the boundary of AdS_3) and the 5-branes wrapping the 4-torus. Alternatively, in an S-dual picture, the same background describes the near-horizon geometry of an F1-NS5 system.

The global symmetry group G of the classical six-dimensional supergravity is $spin(5, 5)$, double cover of $SO(5, 5)$. Quantum effects break this group to $G(\mathbb{Z}) = spin(5, 5; \mathbb{Z})$, which is conjectured to encode superstring U dualities. This group acts on the charge lattice $\Gamma^{5,5}$. The bosonic sector of the theory describes $n = 5$ two-forms, 25 scalar fields in the coset (in the classical theory)

$$M_{scal} = \frac{G}{H} = \frac{spin(5, 5)}{[(Spin(5) \times spin(5))/\mathbb{Z}_2]},$$

and 16 vector fields in the **16** of $spin(5, 5)$. The ten self-dual and anti-self-dual components of the three-form field strengths $\mathbb{H}^M_{\hat{\mu}\hat{\nu}\hat{\rho}}$, $M = 1\ldots, 10$, transform under the action of the classical global symmetry group $SO(5, 5)$ in the representation $\mathscr{R}_2 = \mathbf{10}$. On the bosonic backgrounds under consideration, the vector fields vanish. Since we shall be focusing on the bosonic sector only, the global symmetry group G will be described as $SO(5, 5)$. The

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[10]In the definition of J_n, $\mathcal{S} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathcal{T} \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. J_n is an elliptic element of $SL(2, \mathbb{Z})$ for $n = 0, 1$, parabolic for $n = 2$ and hyperbolic for $n > 2$. It was shown that hyperbolic elements of $SL(2, \mathbb{Z})$, modulo conjugations, can be either brought to the standard form J_n, $n > 2$, or to coincide with sporadic monodromies, see [30].

charge vector Γ^M of the D1-D5 system has little group $G_\ell = \mathrm{SO}(4,5) \subset G$ in the classical supergravity. The coset space $\mathbb{K}$ of M_{scal}, under the adjoint action of $H_\ell = \mathrm{SO}(4) \times \mathrm{SO}(5)$ splits into $\mathbb{K}_\ell \oplus \hat{\mathbb{K}}$ in the representation $(\mathbf{4}, \mathbf{5}) \oplus (\mathbf{1}, \mathbf{5})$, see Eq. (2.24). The nonmoduli scalar fields $\hat{\phi}^k$ are chosen to be parameters of $\hat{\mathbb{K}}$ in the $(\mathbf{1}, \mathbf{5})$ of H_ℓ. As discussed earlier, we fix $\mathrm{SO}(5)$ to rotate $\hat{\phi}^k$ to their simplest form. This corresponds to the parameter of a Cartan generator in the coset space. Next, we consider a group G_0 commuting with the corresponding element $\hat{\mathbb{L}}(\hat{\phi}^k)$. Our choice for G_0 needs not to be maximal. The maximal choice of G_0 is $\mathrm{SO}(4,4)$ and thus of the moduli space $M_0 = G_0/H_0 = \mathrm{SO}(4,4)/\mathrm{SO}(4) \times \mathrm{SO}(4)$.

The D5-branes of the D1-D5 system wrap the whole T^4 and the D1, D5 charges d_1, d_5 belong to a $\Gamma^{1,1}$ sublattice of $\Gamma^{5,5}$, which thus decomposes as [34]

$$\Gamma^{5,5} = \Gamma^{1,1} \oplus \Gamma^{4,4}.$$

The subgroup G_0 acts on $\Gamma^{4,4}$ leaving $\Gamma^{1,1}$ invariant.

Choosing one direction along AdS_3 to be compact and fixing the dependence $\varphi^a(\eta)$ of the moduli fields φ^a in the space G_0/H_0 on the compact boundary coordinate η, to describe a geodesic in G_0/H_0, the backreaction of these moduli deforms $\mathrm{AdS}_3 \times S^3$ into $\mathrm{AdS}_2 \times S^1 \times S^3$ as discussed earlier. Consistency as a string background requires the monodromy matrix $\mathfrak{M}$, connecting the starting and ending points of the geodesic, to be a *hyperbolic* element of $\mathrm{SO}(4,4;\mathbb{Z})$:

$$\mathfrak{M} \equiv \mathbb{L}_0(\varphi^a(T))\mathbb{L}_0(\varphi^a(0))^{-1} \in \mathrm{SO}(4,4;\mathbb{Z}). \quad (2.69)$$

We could also choose the geodesic describing the moduli, to lie in a nonmaximal symmetric subspace of M_{scal} such as

$$M_0 = \frac{G_0}{H_0} = \left(\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}\right)^4 \subset \frac{\mathrm{SO}(4,4)}{\mathrm{SO}(4) \times \mathrm{SO}(4)} \subset M_{\mathrm{scal}}.$$

In this particular case, $\mathfrak{M}$ has the general form

$$\mathfrak{M} = \mathfrak{M}_1 \mathfrak{M}_2 \mathfrak{M}_3 \mathfrak{M}_4, \quad (2.70)$$

where $\mathfrak{M}_\mathrm{j}$, $\mathrm{j} = 1,\ldots,4$, is an element of the $\mathrm{SL}(2,\mathbb{Z})$ subgroup of the corresponding $\mathrm{SL}(2,\mathbb{R})$ factor in G_0. We shall expand on this class of U-fold solutions in the maximal six-dimensional theory in the next section.

Considering type IIB superstring theory compactified on $K3$, this is a six-dimensional (2,0) ungauged supergravity [1]. The group G is $O(5,21)$ and the charge lattice is $\Gamma^{5,21}$. Considering a D1-D5-brane system in which the D5-branes wrap the whole $K3$, the little group G_ℓ of corresponding charge vector Γ^M is $O(4,21)$, yielding 84 moduli fields in $G_\ell/H_\ell = \mathrm{SO}(4,21)/\mathrm{SO}(4) \times \mathrm{SO}(21)$. According to our construction, we do not consider geodesics within this

manifold, but rather within a smaller one $M_0 = G_0/H_0$. The maximal choice of G_0 is $O(4,20)$. The D1, D5 charges d_1, d_5 belong to a $\Gamma^{1,1}$ sublattice of $\Gamma^{5,21}$, which thus decomposes as [34]

$$\Gamma^{5,21} = \Gamma^{1,1} \oplus \Gamma^{4,20}.$$

The subgroup G_0 acts on $\Gamma^{4,20}$ leaving $\Gamma^{1,1}$ invariant. The only nonvanishing moduli fields φ^a in the solution are required to describe a geodesic in G_0/H_0. The monodromy matrix is then chosen as follows:

$$\mathfrak{M} \in G_0(\mathbb{Z}) \subset O(4,20;\mathbb{Z}). \quad (2.71)$$

III. THE BOSONIC SECTOR OF THE $\mathcal{N} = (\mathbf{2},\mathbf{2})$, $D = 6$ THEORY

In this section, we shall describe an instance of U fold of the form $\mathrm{AdS}_2 \times S^1 \times S^3$, obtained applying the construction discussed in Sec. II to the maximal six-dimensional model obtained from type IIB supergravity upon compactification on a 4-torus and dualization of all forms are dualized to lower-order ones. As pointed out above, this model, in its classical limit, features a scalar manifold of the form

$$\mathcal{M}_{\mathrm{scal}} = \frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5) \times \mathrm{SO}(5)}. \quad (3.1)$$

Let us start reviewing the general mathematical description of the model. We recall that the scalar fields originating from toroidal dimensional reduction to $D = 6$ consist in the moduli of the internal metric G_{ij}, the axion-dilaton field $\rho = C_{(0)} + ie^{-\phi}$, the scalars $B_{ij}^\alpha = (C_{ij}, -B_{ij})$, $\alpha = 1,2$, originating from the ten-dimensional two-forms $C_{(2)}, B_{(2)}$ and the scalar $C_{ijkl} = c\epsilon_{ijkl}$ from the four-form. As mentioned earlier, the five two-form fields and their duals transform, under the global symmetry group $\mathrm{SO}(5,5)$, in its fundamental representation $\mathscr{R}_{(2)} = \mathbf{10}$. They are $B_{\mu\nu}^\alpha = (C_{\mu\nu}, -B_{\mu\nu})$, their duals $\tilde{B}_{\alpha\mu\nu} = (\tilde{C}_{\mu\nu}, -\tilde{B}_{\mu\nu})$ and the six $C_{ij\mu\nu}$. The identification of the $D = 6$ scalar fields and two-forms with the above type IIB fields is effected by decomposing the adjoint and the fundamental representations of $\mathrm{SO}(5,5)$ with respect to the subgroup $\mathrm{SL}(4,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$, where $\mathrm{SL}(4,\mathbb{R}) \times \mathrm{SO}(1,1)$ is the group of acting transitively on the metric moduli of the torus:

$$\mathrm{Adj}(\mathrm{SO}(5,5)) \to \mathrm{Adj}(\mathrm{GL}(4,\mathbb{R})) + \mathrm{Adj}(\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}})$$
$$+ (\mathbf{6}, \mathbf{2})_{+1} + (\mathbf{1}, \mathbf{1})_{+2}$$
$$+ (\mathbf{6}', \mathbf{2})_{-1} + (\mathbf{1}, \mathbf{1})_{-2},$$
$$\mathbf{10} \to (\mathbf{1}, \mathbf{2})_{-1} + (\mathbf{6}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{2})_{+1}, \quad (3.2)$$

where the grading refers to the SO(1, 1) factor. The scalars B_{ij}^α parametrize the nilpotent generators in the $(\mathbf{6},\mathbf{2})_{+1}$, c the highest-grading generator $(\mathbf{1},\mathbf{1})_{+2}$. As for the two-forms, $B_{\mu\nu}^\alpha$ and $\tilde{B}_{\alpha\mu\nu}$ are defined by the $(\mathbf{1},\mathbf{2})_{-1}$ and $(\mathbf{1},\mathbf{2})_{+1}$ representations in the branching of the $\mathbf{10}$, while $C_{ij\mu\nu}$ transform in the $(\mathbf{6},\mathbf{1})_0$.

Besides these fields, the bosonic sector also consists of 16 vector fields which, however, will not play a role in our analysis.

Below we describe the subsector of the maximal $D = 6$ theory describing the graviton, the scalar fields, and the two-forms. We shall collectively denote by $C_{(2)}^I$, $I = 1, \ldots, 5$ the tensor fields and by ϕ^s the 25 scalar fields. The corresponding Lagrangian (2.2) reads

$$\mathcal{L}_6 = e_6 R_6 - \frac{e_6}{12}(\mathcal{I}_{IJ}H_{\hat\mu\hat\nu\hat\rho}^I H^{J\hat\mu\hat\nu\hat\rho} + \mathcal{R}_{IJ}H_{\hat\mu\hat\nu\hat\rho}^I {}^\star H^{J\hat\mu\hat\nu\hat\rho})$$
$$- e_6 \frac{\mathrm{Tr}}{8}(\mathcal{M}^{-1}\partial\mathcal{M}\mathcal{M}^{-1}\partial\mathcal{M}), \qquad (3.3)$$

where $\hat\mu = 0, \ldots, 5$ and $\mathrm{I}, \mathrm{J} = 1, \ldots, n$. The matrices $\mathcal{I}(\phi) = (\mathcal{I}(\phi)_{IJ}) = \mathcal{I}(\phi)^T > 0$ and $\mathcal{R}(\phi) = (\mathcal{R}(\phi)_{IJ}) = -\mathcal{R}(\phi)^T$ were introduced earlier in Sec. II, together with the symmetric, positive-definite O(5, 5)-matrix $(\mathcal{M}) = \mathcal{M}_{MN}$, where $M, N = 1, \ldots, 10$, see Eq. (2.7).

A. An explicit parametrization of the scalar manifold

In this section, we review the explicit description of the scalar manifold M_{scal} in terms of the so-called *solvable* or *Borel* one [35,36] parametrization, in which the type IIB origin of the 25 scalar fields is manifest. A detailed account of this parametrization, for this model, is also given in [37]. Here we use, for the description of the type IIB theory in $D = 10$, the conventions defined in Appendix B of [11].

The basis of the $\mathscr{R}_{(2)} = \mathbf{10}$ is chosen as follows[11]:

$$V^M = (V^\alpha, V_{ij}, V_\alpha), \qquad (3.4)$$

where the components $V^\alpha, V_{ij}, V_\alpha$ transform in the $(\mathbf{1},\mathbf{2})_{-1}$, $(\mathbf{6},\mathbf{1})_0$, and $(\mathbf{1},\mathbf{2})_{+1}$, respectively. In this basis, the SO(5, 5) invariant matrix in the fundamental reads

$$\Omega_{MN} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{0} & \delta_\beta^\alpha \\ \mathbf{0} & \epsilon_{ijkl} & \mathbf{0} \\ \delta_\beta^\alpha & \mathbf{0} & \mathbf{0} \end{pmatrix}. \qquad (3.5)$$

In the solvable parametrization of the scalar manifold the scalar fields are defined as parameters of the Borel subalgebra of $\mathfrak{so}(5, 5)$.

We start by discussing the axion-dilaton system and then describe the rest of the scalar fields.

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[11]In this representation the coefficient $\Bbbk$ in (2.11) is equal to 1.

1. Axion-dilaton system

The dilaton ϕ and the axion field $C_{(0)}$, in the ten-dimensional theory, span the coset $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}/\mathrm{SO}(2)$. The corresponding coset representative is chosen of the following form:

$$L_{(2)} = (L_\alpha{}^\beta) = \begin{pmatrix} e^{\phi/2} & 0 \\ C_{(0)}\, e^{\phi/2} & e^{-\phi/2} \end{pmatrix}. \qquad (3.6)$$

This allows us to define the matrix

$$\mathbf{m}_{\alpha\beta} \equiv (L_{(2)}L_{(2)}^T)_{\alpha\beta} = \frac{1}{\mathrm{Im}(\rho)} \begin{pmatrix} 1 & \mathrm{Re}(\rho) \\ \mathrm{Re}(\rho) & |\rho|^2 \end{pmatrix}. \qquad (3.7)$$

2. The SO(5,5)/SO(5) × SO(5) manifold

As mentioned above, the scalars B_{ij}^α and c parametrize the generators t_α^{ij}, t of the Borel subalgebra of $\mathfrak{so}(5, 5)$ in the representations $(\mathbf{6},\mathbf{2})_{+1}$ and $(\mathbf{1},\mathbf{1})_{+2}$ of $\mathrm{GL}(4,\mathbb{R})$, respectively. In the basis (3.4), these generators read

$$\mathbb{B} = \frac{1}{2}B_{ij}^\alpha t_\alpha^{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_{\alpha ij} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -B_\alpha{}^{ij} & \mathbf{0} \end{pmatrix},$$

$$\mathbb{C} = ct = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ c\epsilon_{\alpha\beta} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \qquad (3.8)$$

where

$$B_{\alpha ij} \equiv \epsilon_{\alpha\beta}B_{ij}^\beta = (B_{ij}, C_{ij}), \qquad B_\alpha{}^{ij} \equiv \frac{1}{2}\epsilon^{ijkl}B_{\alpha kl}. \qquad (3.9)$$

Next we write the coset representatives $\mathbb{L}_{(2)}, \mathbb{L}_{(4)}$ of $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}/\mathrm{SO}(2)$ and $\mathrm{GL}(4,\mathbb{R})/\mathrm{SO}(4)$ in the same representation:

$$\mathbb{L}_{(2)} = \begin{pmatrix} L_{(2)}^{-T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & L_{(2)} \end{pmatrix},$$

$$\mathbb{L}_{(4)} = \begin{pmatrix} G^{-\frac{1}{4}}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2G^{-\frac{1}{4}}E_{[i}{}^k E_{j]}{}^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G^{\frac{1}{4}}\mathbf{1} \end{pmatrix}, \qquad (3.10)$$

where $G \equiv \det(G_{ij})$, the metric G_{ij} being in the $D = 10$ Einstein frame and $E_i{}^k$ being the vielbein matrix: $G_{ij} = E_i{}^k E_j{}^k$ (summation over k is understood). The $\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5)\times\mathrm{SO}(5)}$ coset representative is then written as

$$\mathbb{L}^M{}_N = e^{\mathbb{C}} \cdot e^{\mathbb{B}} \cdot \mathbb{L}_{(4)} \cdot \mathbb{L}_{(2)}. \qquad (3.11)$$

We can now compute the components of $\mathcal{M} = \mathbb{L}\mathbb{L}^T$:

$$\mathcal{M}^{\alpha\beta} = G^{-\frac{1}{2}}\mathbf{m}^{\alpha\beta},$$

$$\mathcal{M}_{ij}{}^{\beta} = G^{-\frac{1}{2}}\mathbf{m}^{\alpha\beta} B_{\alpha ij},$$

$$\mathcal{M}_{\alpha}{}^{\beta} = G^{-\frac{1}{2}}c_{\alpha\sigma}\mathbf{m}^{\sigma\beta},$$

$$\mathcal{M}_{ij,kl} = G^{-\frac{1}{2}}(2G_{i[k}G_{l]j} + \mathbf{m}^{\alpha\beta}B_{\alpha ij}B_{\beta,kl}),$$

$$\mathcal{M}_{\alpha ij} = G^{-\frac{1}{2}}(c_{\alpha\beta}\mathbf{m}^{\beta\sigma}B_{\sigma ij} - B_{\alpha}{}^{kl}G_{ki}G_{lj}),$$

$$\mathcal{M}_{\alpha\beta} = G^{-\frac{1}{2}}\left(c_{\alpha\sigma}c_{\beta\gamma}\mathbf{m}^{\sigma\gamma} + \frac{1}{2}B_{\alpha}{}^{ij}B_{\beta}{}^{kl}G_{ik}G_{jl} + G\mathbf{m}_{\alpha\beta}\right),$$

$$(3.12)$$

where

$$c_{\alpha\beta} \equiv c\epsilon_{\alpha\beta} - \frac{1}{4}B_{\alpha ij}B_{\beta}{}^{ij}. \qquad (3.13)$$

We refer to Appendix A for more details about the solvable parametrization of the scalar manifold as well as the description of the tensor fields in terms of the $\mathfrak{so}(5,5)$ weights of the **10**.

B. AdS$_3 \times S^3$ solution of type IIB theory on T^4

In Appendix C we present a general discussion of the static black-string solutions in the presence of nontrivial moduli. A known instance is the D1-D5 system in which the D1- and D5-branes have only two common Neumann directions, defining the worldsheet of the black string in $D = 6$, and the D5-branes wrap the whole T^4. This solution is of particular importance in string theory since it provided the setting for the first successful black hole entropy calculation through string microstate counting [38].

The charge vector reads

$$\Gamma^M = (d_5, 0, 0, 0, 0, 0, 0, 0, d_1, 0), \qquad (3.14)$$

where d_5, d_1 are taken to be both positive and denote the D5- and the D1-brane charges, respectively. Through a (charge-dependent) change of basis, see Appendix D, the above vector can be brought to the form

$$\Gamma'^M = (\sqrt{2|d_1 d_5|}, 0, 0, 0, 0, 0, 0, 0, 0, 0). \qquad (3.15)$$

In this new basis, the invariant pseudo-orthogonal metric reads

$$\Omega' = \text{diag}(+1, +1, +1, +1, +1, -1, -1, -1, -1, -1). \qquad (3.16)$$

The above form of Γ' makes its little group $G_\ell = \text{SO}(4,5)$ manifest: it is the subgroup of G acting nontrivially only on the entries 2, 3, …, 10. The matrix representation of G_ℓ in

the original basis (3.14) is d_1, d_5 dependent. As explained in Secs. II and II B, the coset space $\mathbb{K}$ of M_{scal} splits into $\mathbb{K}_\ell$ and $\hat{\mathbb{K}}$ supporting the representations $(\mathbf{4}, \mathbf{5})$ and $(\mathbf{1}, \mathbf{5})$ of $H_\ell = \text{SO}(4) \times \text{SO}(5)$. The nonmoduli fields ϕ^k parametrize the five-dimensional $\hat{\mathbb{K}}$ and can be brought to a normal form $\phi^k = (g, 0, 0, 0, 0)$, through the action of $\text{SO}(5)$, in which the only nonvanishing scalar g parametrizes the Cartan generator:

$$\mathfrak{h} = \text{diag}(1, 0, 0, 0, 0, 0, 0, 0, -1, 0). \qquad (3.17)$$

The scalar field g is identified as follows: $e^g = e^\phi G^{\frac{1}{2}}$, where $G \equiv \det(G_{ij})$, see Appendix A.

The effective potential V for the D1-D5 system, and its extremization, is discussed in Appendix D. The maximal subgroup of $\text{SO}(5,5)$ commuting with $\mathfrak{h}$ is $\text{SO}(4,4)$, which represents the maximal choice of $G_0 \subset G_\ell$. The maximal moduli space M_0 is

$$M_0 = \frac{\text{SO}(4,4)}{\text{SO}(4) \times \text{SO}(4)},$$

parametrized by C_{ij} and $\tilde{G}_{ij} = e^{-\frac{\phi}{2}}G_{ij}$, see Appendix A. Had we chosen the system of F1 and NS5-branes dual to the D1-D5 one, the maximal moduli space M_0 would be spanned by $B_{ij}, G_{ij}^{(s)} \equiv e^{\frac{\phi}{2}}G_{ij}$, where $G_{ij}^{(s)}$ is the torus metric in the string frame. The $\text{O}(4,4;\mathbb{Z})$ global symmetry group would, in this case, encode the T dualities along directions of T^4, and transformations in $\text{O}(4,4;\mathbb{Z})/\text{SO}(4,4;\mathbb{Z})$ would map the type IIB and type IIA descriptions of the same F1-NS5 solution.

Coming back to the D1-D5 system, when we only switch on the moduli fields C_{ij}, $\tilde{G}_{ij}$ and the field g, the matrix $\mathcal{M}$ is the product of two commuting symmetric matrices:

$$\mathcal{M}(\varphi^a, g) = \mathcal{M}_0(\varphi^a)\hat{\mathcal{M}}(g), \qquad (3.18)$$

where

$$\hat{\mathcal{M}}(g) = \text{diag}(e^g, 1, 1, 1, 1, 1, 1, 1, 1, e^{-g}, 1), \qquad (3.19)$$

and $\Gamma^T \mathcal{M}(\varphi^a, g)\Gamma = \Gamma^T \hat{\mathcal{M}}(g)\Gamma$, so that the effective potential reads

$$V = \frac{1}{2}(d_1^2 e^{-g} + d_5^2 e^g). \qquad (3.20)$$

By further fixing the moduli fields $C_{ij} = 0, \tilde{G}_{ij} = \delta_{ij}$, we have $g = 2\phi$ and the solution takes the known form (see Appendix C 1)

$$ds^2 = (Z_1 Z_5)^{-\frac{1}{2}}(-dt^2 + dx^2) + (Z_1 Z_5)^{\frac{1}{2}}(dx^i dx^i), \qquad (3.21)$$

$$e^g = e^{2\phi} = \log\left(\frac{Z_1}{Z_5}\right),$$

$$dx^i dx^i = dr^2 + r^2 d\Omega_3^2, \qquad Z_1 = 1 + \frac{d_1}{2r^2}, \qquad Z_5 = 1 + \frac{d_5}{2r^2}.$$

$$(3.22)$$

To avoid the factor $1/2$ in the harmonic functions, it can be convenient to define rescaled charges: $d_1 = 2Q_1$, $d_5 = 2Q_5$. The charges d_1 and d_5 are defined in Eq. (D1). At the horizon the Z functions become

$$Z_1 = \frac{d_1}{2r^2} = \frac{Q_1}{r^2}, \qquad Z_5 = \frac{d_5}{2r^2} = \frac{Q_5}{r^2}. \quad (3.23)$$

This background is $AdS_3 \times S^3$, which is the throat of the D1-D5 system. The nonmoduli scalars are attracted toward configurations that extremize the potential. We give a general discussion in Appendix D. In this particular case, in which we set $C_{ij} = 0$, $\tilde{G}_{ij} = \delta_{ij}$, the scalar field $g = 2\phi$ is fixed at the horizon, through the attractor mechanism, to the value

$$g = 2\phi = \log\left(\frac{d_1}{d_5}\right) = \log\left(\frac{Q_1}{Q_5}\right). \quad (3.24)$$

At the minimum the effective potential has the value

$$V_* = d_1 d_5. \quad (3.25)$$

The near-horizon metric is conveniently written in the following form:

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2) + \frac{L^2}{r^2}dr^2$$
$$+ L^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\omega^2)], \quad (3.26)$$

where $L^2 = \frac{\sqrt{d_1 d_5}}{2}$, as obtained in Eqs. (2.39), (3.25).

C. $AdS_2 \times S^1 \times S^3$ U folds

In this section, we apply the general construction of Sec. II to the construction of U-fold solutions with geometry $AdS_2 \times S^1 \times S^3$ in type IIB theory on T^4. As discussed in Sec. II, we only switch on the field $e^g = e^\phi G^{\frac{1}{2}}$ and the moduli fields φ^a spanning either the manifold $SO(4,4)/[SO(4) \times SO(4)]$, parametrized by $\tilde{G}_{ij} = e^{-\frac{\phi}{2}}G_{ij}$, C_{ij}, or a symmetric submanifold $M_0 = G_0/H_0$ of it. We start with a metric of the form

$$ds^2 = -k_1 \frac{r^2}{L^2}dt^2 + k_1 \frac{L^2}{r^2}dr^2 + k_2 d\eta^2$$
$$+ L^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\omega^2)], \quad (3.27)$$

where k_1, k_2 are positive constants and we compactify the η direction: $\eta \in [0, T]$. Just as in the D1-D5 solution, we turn on the charges d_1, d_5. The ansatz for the three-form field strength is again

$$\mathbb{H} = -\gamma \Omega \mathcal{M} \Gamma dt \wedge d\eta \wedge dr + \zeta(\theta, \psi)\Gamma d\psi \wedge d\theta \wedge d\omega, \quad (3.28)$$

which is duality covariant. On the three-form field we need to impose the twisted self-duality condition, the Maxwell equations, and the Bianchi equations, which read respectively

$$\star \mathbb{H} = -\Omega \mathcal{M} \mathbb{H}, \qquad d\mathbb{H} = 0. \quad (3.29)$$

These are all solved if we pick

$$\gamma = \frac{k_1\sqrt{k_2}}{L^3}, \qquad \zeta(\theta, \psi) = \sin\theta\sin^2\psi. \quad (3.30)$$

The moduli φ^a in M_0 are switched on along a geodesic arc,

$$\varphi^a = \varphi^a(\eta), \quad (3.31)$$

with velocity κ and affine parameter η, so that

$$\frac{1}{2}\mathcal{G}_{ab}\partial_\eta\varphi^a\partial_\eta\varphi^b = \kappa^2. \quad (3.32)$$

Thanks to the ansatz for the field strength $\mathbb{H}$ (3.28), the equation of motion for the scalars has the same form as Eq. (2.9), where $V(\phi, \Gamma) = V(g, \Gamma)$ is given in Eq. (3.20). The scalar equation of motion is solved if we extremize the potential as

$$g = \log(e^\phi G^{\frac{1}{2}}) = \log\left(\frac{d_1}{d_5}\right) = \text{const}, \quad (3.33)$$

since we forced the moduli fields φ^a to evolve in η along a geodesic.

The last equation to check is the trace-reversed Einstein equation. The computation of the energy-momentum part which involves the three-form $\mathbb{H}$ gives

$$T_{\hat{\mu}\hat{\nu}}^{(H)} = \frac{d_1 d_5}{2L^4}\text{diag}\left(\frac{k_1 r^2}{L^4}, -\frac{k_2}{L^2}, -\frac{k_1}{r^2}, 1, \sin^2(\psi), \sin^2(\theta)\sin^2(\psi)\right), \quad (3.34)$$

where the second element on the diagonal is the (η, η) component of the tensor and has to be canceled, in the Einstein equation, by the only nonvanishing component of the scalar energy-momentum tensor, which equals the left-hand side of Eq. (3.32), namely κ^2. The Einstein equation then completely fixes the coefficients as follows:

$$k_1 = \frac{1}{2}, \qquad k_2 = \frac{\kappa^2 \sqrt{d_1 d_5}}{4}, \qquad L^2 = \frac{\sqrt{d_1 d_5}}{2}. \qquad (3.35)$$

To give an instance of a geodesic curve described by string 0-modes, we can restrict the moduli fields φ^a to the submanifold

$$M_0 = \frac{G_0}{H_0} = \left(\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)} \right)^2, \qquad (3.36)$$

which is obtained by restricting the moduli $\tilde{G}_{ij} = e^{-\frac{\phi}{2}} G_{ij}, C_{ij}$ to the following nonvanishing fields:

$$e^{-\phi_{12}} \equiv \tilde{G}_{11} = \tilde{G}_{22}, \quad e^{-\phi_{34}} \equiv \tilde{G}_{33} = \tilde{G}_{44}, C_{12}, C_{34}. \quad (3.37)$$

The metric of the scalar manifold restricted to g, $\phi_{12}, \phi_{34}, C_{12}, C_{34}$ reads

$$d\tilde{s}^2 = \frac{dg^2}{2} + d\phi_{12}^2 + d\phi_{34}^2 + e^{2\phi_{12}} \, dC_{12}^2$$
$$+ e^{2\phi_{34}} \, dC_{34}^2. \qquad (3.38)$$

The geodesic can be chosen as the product of two geodesics spanned by ϕ_{12}, C_{12} and by ϕ_{34}, C_{34}, respectively, unfolding in the two factors of M_0. The monodromy matrix $\mathfrak{M}$ will be the product $\mathfrak{M} = \mathfrak{M}_1 \mathfrak{M}_2$, of two elements $\mathfrak{M}_1$, $\mathfrak{M}_2$ in the two factors $\mathrm{SL}(2, \mathbb{Z})$ in $G_0(\mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})^2$. Choosing, for instance,

$$\mathfrak{M}_1 = J_{\mathrm{n}_1}, \qquad \mathfrak{M}_2 = J_{\mathrm{n}_2},$$

the geodesic reads

$$e^{\phi_{12}(\eta)} = \frac{\sqrt{\mathrm{n}_1^2 + 4} \cosh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_1^2 + 2))}{T}\right) + \mathrm{n}_1 \sinh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_1^2 + 2))}{T}\right)}{\sqrt{\mathrm{n}_1^2 + 4}},$$

$$C_{12}(\eta) = -\frac{2 \sinh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_1^2 + 2))}{T}\right)}{\sqrt{\mathrm{n}_1^2 + 4} \cosh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_1^2 + 2))}{T}\right) + \mathrm{n}_1 \sinh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_1^2 + 2))}{T}\right)},$$

$$e^{\phi_{34}(\eta)} = \frac{\sqrt{\mathrm{n}_2^2 + 4} \cosh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_2^2 + 2))}{T}\right) + \mathrm{n}_2 \sinh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_2^2 + 2))}{T}\right)}{\sqrt{\mathrm{n}_2^2 + 4}},$$

$$C_{34}(\eta) = -\frac{2 \sinh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_2^2 + 2))}{T}\right)}{\sqrt{\mathrm{n}_2^2 + 4} \cosh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_2^2 + 2))}{T}\right) + \mathrm{n}_2 \sinh\left(\frac{\eta \cosh^{-1}(\frac{1}{2}(\mathrm{n}_2^2 + 2))}{T}\right)},$$

which satisfy the equations explicitly presented in Appendix C, after the appropriate coordinate transformations to bring the target metric (3.38) in the form (C1). It is straightforward to verify that, as $\eta \to \eta + T$ the complexified fields $\rho_{12} \equiv C_{12} + ie^{-\phi_{12}}, \rho_{34} \equiv C_{34} + ie^{-\phi_{34}}$ transform under the action of the monodromies J_{n_1}, J_{n_2}, respectively:

$$\rho_{12} \to -\frac{1}{\rho_{12} + \mathrm{n}_1}, \qquad \rho_{34} \to -\frac{1}{\rho_{34} + \mathrm{n}_2}.$$

In this case, we have

$$\kappa^2 = \frac{1}{2T^2} \left(\cosh^{-1} \left(\frac{\mathrm{n}_1^2}{2} + 1 \right) + \cosh^{-1} \left(\frac{\mathrm{n}_2^2}{2} + 1 \right) \right). \quad (3.39)$$

Along the same lines, we can construct the $\mathrm{AdS}_2 \times S^1 \times S^3$ T-fold starting from the near horizon geometry of an F1-NS5 system with charges

$$\Gamma^M = (0, n_5, 0, 0, 0, 0, 0, 0, 0, f_1).$$

This solution is obtained by applying an S-duality $\mathcal{S} \in \mathrm{SL}(2, \mathbb{Z})_{\mathrm{IIB}}$ to the D1-D5 *U* fold. As pointed out earlier, the maximal choice $\mathrm{SO}(4, 4)/\mathrm{SO}(4) \times \mathrm{SO}(4)$ for the moduli space M_0 is spanned by $B_{ij}, G_{ij}^{(s)} \equiv e^{\frac{\phi}{2}} G_{ij}$, where $G_{ij}^{(s)}$ is the torus metric in the string frame, while

$$\hat{\mathcal{M}} = \mathrm{diag}(1, e^g, 1, 1, 1, 1, 1, 1, 1, e^{-g}), \quad (3.40)$$

with $e^g = e^{-2\phi_6} \equiv e^{-\phi} G^{\frac{1}{2}}$, see Appendix A. Since now the monodromy matrix is an element of $\mathrm{O}(4, 4; \mathbb{Z})$, which is

now the T-duality group associated with the internal 4-torus, the resulting U fold is a T fold.

We can choose a monodromy matrix $\mathfrak{M} \in O(4, 4; \mathbb{Z})/SO(4, 4; \mathbb{Z})$, which is not connected continuously to the identity. This can be achieved by describing S^1 through two open patches: one consisting of S^1 minus the point $\eta = 0$, the other obtained by subtracting to S^1 the point $\eta = T/2$. Within each patch the evolution of the scalar fields is geodesic. However, in the interval $(0, T/2)$ the moduli fields are described in both patches by[12] $\mathbb{L}_0(\varphi^a(\eta))$ while in $(T/2, T)$ they are described in the first patch by $\mathbb{L}_0(\varphi^a(\eta))$ and in the second one by $\mathcal{O} \cdot \mathbb{L}_0(\varphi^a(\eta))$, where $\mathcal{O} \in O(4, 4; \mathbb{Z})/SO(4, 4; \mathbb{Z})$, $\mathcal{O}^2 = \mathbf{1}$. The twist matrix has the following property: $\mathcal{A}(0) = \mathbb{L}_0(\varphi^a(0))$, $\mathcal{A}(T) = \mathcal{O} \cdot \mathbb{L}_0(\varphi^a(T))$, so that $\mathfrak{M}$, given by Eq. (2.48), is in $O(4, 4; \mathbb{Z})/SO(4, 4; \mathbb{Z})$. This T-fold solution with NS-NS charges is therefore described in one patch within the type IIB theory and in the other one within the type IIA one, the transition function being a pseudo-orthogonal matrix with a negative determinant.

In the next subsection, we review the action of a duality transformation on the general, duality covariant, ansatz discussed in Sec. II.

1. Duality

Let us discuss here the action of the duality symmetry on a generic solution. Consider a duality transformation

$$\mathcal{O} = (\mathcal{O}^M{}_N) \in SO(5, 5) \Leftrightarrow \mathcal{O}\Omega\mathcal{O}^T = \Omega. \qquad (3.41)$$

Its action on the scalar fields is described by the transformation property of the matrix $\mathcal{M}(\phi)$:

$$\phi \equiv (\phi^s) \xrightarrow{\mathcal{O}} \phi' \equiv (\phi'^s) \Leftrightarrow \mathcal{O}^{-T}\mathcal{M}(\phi)\mathcal{O}^{-1}$$
$$= \mathcal{M}(\phi'), \qquad (3.42)$$

while the corresponding action on $\mathbb{H} \equiv (\mathbb{H}^M)$ reads

$$\mathbb{H}' = \mathcal{O}\mathbb{H} = -\Omega\mathcal{O}^{-T}\mathcal{M}(\phi)\mathcal{O}^{-1}\mathcal{O}\Gamma\boldsymbol{\epsilon}_{(3)} + \mathcal{O}\Gamma\tilde{\boldsymbol{\epsilon}}_{(3)}$$
$$= -\Omega\mathcal{M}(\phi')\Gamma'\boldsymbol{\epsilon}_{(3)} + \Theta\Gamma'\tilde{\boldsymbol{\epsilon}}_{(3)}. \qquad (3.43)$$

A particular instance of duality symmetry, which we shall use in the sequel, is S duality, implemented, in the basis we are using, by the following matrix:

$$\Theta = \begin{pmatrix} \theta^\alpha{}_\beta & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\delta^{kl}_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{\theta}_\alpha{}^\beta \end{pmatrix}, \qquad (3.44)$$

where

---
[12]Recall that $\mathbb{L}_0$ is in the identity sector of the isometry group.

$$\theta^\alpha{}_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \tilde{\theta}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (3.45)$$

such that $\tilde{\theta}_\alpha{}^\beta = (\theta^{-1})^\beta{}_\alpha$. Besides being pseudo-orthogonal, Θ is orthogonal as well: $\Theta\Theta^T = \mathbf{1}$. From Eq. (3.42) we deduce the following transformation rule for the scalar fields:

$$B'_\alpha = \tilde{\theta}_\alpha{}^\beta B_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} B_2 \\ -B_1 \end{pmatrix}, \qquad (3.46)$$

$$c'\epsilon_{\alpha\sigma} = c(\theta^{-1})^\beta{}_\alpha\epsilon_{\beta\gamma}\tilde{\theta}_\sigma{}^\gamma = c\epsilon_{\alpha\sigma}, \qquad (3.47)$$

$$\rho' = -\frac{1}{\rho}, \qquad (3.48)$$

all other fields being invariant.

IV. FINAL REMARKS AND DISCUSSION OF THE SOLUTION

In this work, we discuss the general construction of a U-fold solution with $AdS_{d-1} \times S^1 \times S^d$ geometry and a monodromy along S^1 in the duality group, within theories admitting a $AdS_d \times S^d$ background with moduli fields. More in general we build U-fold backgrounds of the form

$$AdS_{d-k} \times \underbrace{S^1 \times \cdots \times S^1}_{k} \times S^d, \qquad (4.1)$$

with $d - k > 1$.

This was achieved by introducing an appropriate spatial dependence of the scalar moduli along a compact direction which, in the original $AdS_d \times S^d$ solution, is in the boundary of AdS_d. The effect of the backreaction of the evolving moduli on the geometry is to split AdS_d into $AdS_{d-1} \times S^1$. The geodesic arc connects two distinct points in the classical moduli space. Consequently, the consistency of this background in superstring theory requires the initial and ending points to be identified in the string moduli space.

Since the dependence of the fields, in the U-fold solution, on the S^1-coordinate η is factorized in the action of a twist matrix $\mathcal{A}(\eta) \in G_0$ on the fields, it is natural to expect this background to be a solution of the $D = 2d$-dimensional model compactified along S^1 á la Cremmer-Scherk-Schwarz [39] to $(2d - 1)$ dimensions, with a twist matrix $\mathcal{A}(\eta)$ in the global symmetry group G_0. Indeed the only fields in the solution transforming nontrivially under G_0 are the moduli fields φ^a. We extensively discussed, as a concrete application of our general construction, type IIB superstring theory compactified on T^4 to six dimensions, and utilized the moduli fields present in the $AdS_3 \times S^3 \times T^4$ solution to create a U fold with geometry $AdS_2 \times S^1 \times S^3 \times T^4$. This construction involves

introducing a nontrivial geodesic dependence of the moduli fields in $SO(4,4)/SO(4) \times SO(4)$, or in a symmetric submanifold M_0 thereof, on a compact direction S^1 along the boundary of AdS_3.

Cremmer-Scherk-Schwarz reduction of maximal $\mathcal{N} = (2,2)$ six-dimensional supergravity to five dimensions admit Minkowski five-dimensional vacua only if the twist matrix is compact, that is in $\mathrm{spin}(5) \times \mathrm{spin}(5)$. In this case the reduction was originally studied in [40,41] and, more recently, in [37,42] in relation to the study of black hole solutions. Our setting, however, is substantially different since it relies on the moduli fields describing a geodesic in $M_0 = G_0/H_0$, which requires the twist matrix $\mathcal{A}(\eta) \in G_0/H_0$ to be intrinsically noncompact. This is necessary for the evolving moduli to backreact on the geometry through a nonvanishing component of the energy-momentum tensor: $T_{\eta\eta}^{(S)} \neq 0$.

The general mechanism analyzed here in detail, underlying the construction of $AdS_{d-1} \times S^1 \times S^d$ U folds, seems to be at work in the more sophisticated class of J-fold solutions with geometry $AdS_4 \times S^1 \times \tilde{S}^5$, which have recently attracted considerable interest [11–22,43]. All these solutions correspond to AdS_4 vacua of maximal four-dimensional supergravity with gauge group $\mathcal{G} = [SO(6) \times SO(1,1)] \ltimes T^{12}$. Aside from the nonsupersymmetric J fold with $SO(6)$ symmetry mentioned earlier and found in [11], which fits the general class discussed in Sec. II, in all other solutions the axion-dilaton system features a dependence on the coordinates ξ^i of the deformed 5-sphere $\tilde{S}^5$ and on η. The former is sourced by three-form field strengths while the latter describes a geodesic arc in the moduli space, as noticed in [17]. In the present work, we have highlighted this common feature of all these solutions and generalized it to the case $d \neq 5$, showing that the geodesic dependence of the moduli fields on the η coordinate of an S^1 at the boundary of AdS_d is sufficient to split this space into $AdS_{d-1} \times S^1$, where the ratio of the warp factors multiplying the metric AdS_{d-1} and $d\eta^2$ is a suitable constant.

An important issue to address is the supersymmetry and stability of the $AdS_2 \times S^1 \times S^3$ solution discussed in this work. Secondly, it would be of considerable interest to construct variants of these solutions, with geometry $AdS_2 \times S^1 \times \tilde{S}^3$, where $\tilde{S}^3$ is a deformed 3-sphere, in which the moduli fields φ^a, aside from the geodesic dependence on η, also feature a dependence on the coordinates of $\tilde{S}^3$ sourced by two-form field strengths. These new solutions would be the counterpart in $D = 6$ of the $\mathcal{N} = 1, 2$, and 4 J folds in $D = 10$. In analogy with the latter solutions, we may argue that these $AdS_2 \times S^1 \times \tilde{S}^3$ U folds should be dual to the IR limit of a $1 + 1$ SCFT compactified along one spatial direction, with an interface localized along this direction and an $O(4,4;\mathbb{Z})$ monodromy acting on the exactly marginal operators dual to φ^a, when crossing the

interface. Although we do not expect the solutions $AdS_2 \times S^1 \times S^3$ to be supersymmetric, we expect their variants $AdS_2 \times S^1 \times \tilde{S}^3$ to preserve some supersymmetry, in analogy with the $D = 10$ J-fold counterparts. It would also be interesting to relate generalizations of our backgrounds to a compactified version of the Janus solutions found in [44] which were also put in correspondence with an interface CFT.

Similar to the Janus solutions, the approach outlined in this paper is relevant for the supersymmetric wormhole solutions recently found in [45]. The existence of moduli, or scalar fields becoming moduli near the wormhole's neck might be a crucial element in constructing such wormhole solutions.

We leave these projects to a future investigation.

APPENDIX A: TYPE IIB SUPRGRAVITY ON T^4: GEOMETRIC CHARACTERIZATION OF THE SCALAR AND TENSOR FIELDS

In the solvable parametrization of the scalar manifold, the ten-dimensional dilaton ϕ and the radial moduli σ_i, $i = 1, \ldots, 4$, of T^4, parametrize the Cartan subalgebra of $\mathfrak{so}(5,5)$ while the remaining scalar fields are parameters of the shift generators E_α corresponding to the positive roots $\boldsymbol{\alpha}$. The latter are conveniently expressed in terms of the simple roots which, in a suitable orthonormal basis $\boldsymbol{\epsilon}_i, \boldsymbol{\epsilon}_5$, are chosen to have the following form:

$$\boldsymbol{\alpha}_1 = \boldsymbol{\epsilon}_1 - \boldsymbol{\epsilon}_2, \boldsymbol{\alpha}_2 = \boldsymbol{\epsilon}_2 - \boldsymbol{\epsilon}_3, \boldsymbol{\alpha}_3 = \boldsymbol{\epsilon}_3 - \boldsymbol{\epsilon}_4, \boldsymbol{\alpha}_4 = \boldsymbol{\epsilon}_3 + \boldsymbol{\epsilon}_4, \boldsymbol{\alpha}_5$$

$$= \mathbf{a} = -\frac{1}{2}\left(\sum_{i=1}^4 \boldsymbol{\epsilon}_i\right) + \boldsymbol{\epsilon}_5. \tag{A1}$$

The generators $t_\alpha{}^{ij}, t$ of the solvable Lie algebra generating the coset, which enter the definition (3.11) of the coset representative, correspond to the following shift-generators E_α:

$$t_{\alpha=1}{}^{ij} = E_{\mathbf{a}+\boldsymbol{\epsilon}_i+\boldsymbol{\epsilon}_j}, \qquad t_{\alpha=2}{}^{ij} = E_{\boldsymbol{\epsilon}_i+\boldsymbol{\epsilon}_j}, \qquad t = E_{\mathbf{a}+\boldsymbol{\epsilon}_1+\boldsymbol{\epsilon}_2+\boldsymbol{\epsilon}_3+\boldsymbol{\epsilon}_4}. \tag{A2}$$

The six positive roots $\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_j$, $i < j$, are the positive roots of the $GL(4,\mathbb{R})$ group of the 4-torus and enter the definition of the vielbein matrix $\mathbf{E} \equiv (E_i{}^k)$. The latter can be written as

$$\mathbf{E} = \hat{\mathbf{E}} \cdot \mathbf{R},$$

where

$$\hat{\mathbf{E}} = \prod_{i<j} e^{\gamma_i{}^j E_{\epsilon_i - \epsilon_j}}, \qquad \mathbf{R} \equiv (\delta_j^i e^{\sigma_j}), \qquad \text{(A3)}$$

and the generators $E_{\epsilon_i - \epsilon_j}$ are meant, in the above formula, as 4×4 nilpotent matrices in the **4** of $GL(4, \mathbb{R})$. The dependence of the coset representative on the dilatonic scalars ϕ, σ_i is through the matrix

$$\mathbb{L}_{\text{dil}} \equiv \exp\left(\sum_{i=1}^{4} \sigma_i \left(H_{\epsilon_i} + \frac{1}{2} H_{\epsilon_5} \right) - \frac{\phi}{2} H_{\mathbf{a}} \right). \qquad \text{(A4)}$$

The dependence of the kinetic term of the nondilatonic scalar $(\gamma_i{}^j, B^\alpha{}_{ij}, c_4)$ on the dilatonic ones, in the Lagrangian, is through the characteristic exponential $e^{-2\mathbf{h} \cdot \alpha}$, where

$$\mathbf{h} \equiv \left(\sum_{i=1}^{4} \sigma_i \left(\epsilon_i + \frac{1}{2} \epsilon_5 \right) - \frac{\phi}{2} \mathbf{a} \right),$$

and α is the positive root associated with the scalar.

Similarly to the nondilatonic scalars we can associate each of the ten three-form field-strength $\mathbb{H}^M$ with weights $\mathbf{w}^M$ of the **10** of $\mathfrak{so}(5,5)$ which, the chosen conventions, read

$$\mathbf{w}^M = (\mathbf{w}^\alpha, \mathbf{w}_{ij}, \mathbf{w}_\alpha), \qquad \text{(A5)}$$

where

$$\mathbf{w}^\alpha = (\mathbf{w}_0, -\epsilon_5), \qquad \mathbf{w}_{ij} = \mathbf{w}_0 + \epsilon_i + \epsilon_j,$$

$$\mathbf{w}_\alpha = -\mathbf{w}^\alpha, \qquad \mathbf{w}_0 \equiv -\frac{1}{2} \sum_{i=1}^{4} \epsilon_i. \qquad \text{(A6)}$$

The dependence of the kinetic term of $\mathbb{H}^M$ on the dilatonic scalars in the Lagrangian is through the characteristic exponential $e^{-2\mathbf{h} \cdot \mathbf{w}}$.

The d_5, d_1 charges are associated with the weights $\pm \mathbf{w}_0$. The Cartan generator $\mathfrak{h}$ in (3.17) is defined by the component of $\mathbf{h}$ along $\mathbf{W}_0$ and is spanned by the scalar g:

$$e^g = e^{-\mathbf{h} \cdot \mathbf{w}_0} = g_s G^{\frac{1}{2}}.$$

The roots of $SO(5,5)$ which are orthogonal to $\mathbf{w}_0$ define an $SO(4,4)$ algebra representing the maximal choice of G_0, subgroup of G which commutes with $\mathfrak{h}$. The positive roots of $SO(4,4)$ and the corresponding nondilatonic scalars are

$$\mathbf{a} + \epsilon_i + \epsilon_j \leftrightarrow C_{ij},$$

$$\epsilon_i - \epsilon_j, (i < j) \leftrightarrow \gamma_i{}^j.$$

As for the dilatonic scalar fields, these are the components of $\mathbf{h}$ along the orthonormal basis $\mathbf{v}_i \equiv \frac{\mathbf{a}}{2} + \epsilon_i$, orthogonal to $\mathbf{w}_0$:

$$\tilde{\sigma}_i = \mathbf{h} \cdot \mathbf{v}_i = \sigma_i - \frac{\phi}{4}. \qquad \text{(A7)}$$

The scalars $\gamma_i{}^j, \tilde{\sigma}_i$ define the components of the matrix $\tilde{G}_{ij} \equiv e^{-\frac{\phi}{2}} G_{ij}$.

The NS-NS charges n_5, f_1, associated with the NS 5-brane and the fundamental string, respectively, correspond to the weights $\mp \epsilon_5$. The scalar g parametrizing $\mathfrak{h}$ and entering Eq. (3.40) is now given by

$$e^g = e^{\mathbf{h} \cdot \epsilon_5} = g_s^{-1} G^{\frac{1}{2}} = e^{-2\phi_6},$$

where the six-dimensional dilaton is defined as $e^{\phi_6} \equiv e^\phi \det (G_{ij}^{(s)})^{-\frac{1}{4}}$ and $G_{ij}^{(s)} \equiv e^{\frac{\phi}{2}} G_{ij}$ is the torus metric in the string frame. The dilatonic scalar fields in the $SO(4,4)/SO(4) \times SO(4)$ coset are given by the components of $\mathbf{h}$ along the orthonormal basis

$$\mathbf{v}_i \equiv \epsilon_i, \quad i = 1, \dots, 4.$$

They are

$$\sigma_i^{(s)} \equiv \mathbf{h} \cdot \mathbf{v}_i = \sigma_i + \frac{\phi}{4},$$

and are the radial moduli in the string frame. On the other hand, the positive roots which are orthogonal to ϵ_5 are

$$\epsilon_i + \epsilon_j \leftrightarrow B_{ij},$$

$$\epsilon_i - \epsilon_j, (i < j) \leftrightarrow \gamma_i{}^j.$$

This proves that the manifold $SO(4,4)/SO(4) \times SO(4)$ is spanned by the moduli $G_{ij}^{(s)}, B_{ij}$.

APPENDIX B: THE TYPE IIB STATIC BLACK STRING SOLUTIONS

Let us study the most general ansatz for a static black-string solution with $SO(4)$ symmetry. The corresponding metric has the following general form:

$$ds^2 = A(r)^2(-dt^2 + dx^2) + A(r)^{-2} dr^2$$
$$+ B(r)^2 [d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\omega^2)], \quad \text{(B1)}$$

where r, ψ, θ, ω are the polar coordinates of the four-dimensional space transverse to the string world volume and the last three angles parametrize a 3-sphere S^3. The coordinates $x^{\hat{\mu}}$, $\hat{\mu} = 0, \dots, 6$ naturally split into

$$x^\mu = \{t, x, r\}, \qquad x^{\tilde\mu} = \{\psi, \theta, \omega\},$$

where $\mu = 0, 1, 2$ and $\tilde\mu = 3, 4, 5$.

The Ricci tensor is

$$R_{tt} = -R_{xx} = A^2\left(AA'' + 3\frac{AA'B'}{B} + 2A'^2\right),$$

$$R_{rr} = \frac{1}{A^2B}[-3A(A'B' + AB'') - 2B(AA'' + A'^2)],$$

$$R_{\psi\psi} = \frac{R_{\theta\theta}}{\sin^2\psi} = \frac{R_{\omega\omega}}{\sin^2\psi\sin^2\theta}$$
$$= -3ABA'B' - A^2[(BB'' + 2B'^2)].$$

We emphasize that this ansatz described the most general static black-string solution coupled to any number of scalar fields. The constant charge vector $\Gamma \equiv (\Gamma^M)$ is defined as

$$\Gamma^M = \frac{1}{2\pi^2}\int_{S^3} \mathbb{H}^M,$$

and characterizes the solution. Let us consider the already introduced (2.17) ansatz for the tensor field

$$\mathbb{H} = -\xi(r)\Omega\mathcal{M}\Gamma dt \wedge dx \wedge dr + \chi(\theta, \psi)\Gamma d\psi \wedge d\theta \wedge d\omega,$$
$$\text{(B2)}$$

and this solves the twisted self-duality condition, the Maxwell equations, and the Bianchi equations if

$$\xi(r) = \frac{A}{B^3}, \qquad \chi(\theta, \psi) = \sin^2\psi\sin\theta. \quad \text{(B3)}$$

With this choice, the energy-momentum tensor reads

$$T_{tt} = -T_{xx} = -A^4T_{rr} = \frac{A^2}{2B^6}V, \quad \text{(B4)}$$

$$T_{\psi\psi} = \frac{V}{2B^4}. \quad \text{(B5)}$$

On the inverse trace of Einstein's equation

$$R_{\hat\mu\hat\nu} = T^{(H)}_{\hat\mu\hat\nu} + \frac{1}{2}\mathcal{G}_{ts}\partial_{\hat\mu}\phi^t\partial_{\hat\nu}\phi^s, \quad \text{(B6)}$$

we impose the consistency requirement $\phi = \phi(r)$ and the equations explicitly read

$$R^t_t = R^x_x = -\frac{V}{2B^6}, \quad \text{(B7)}$$

$$R^r_r = -\frac{V}{2B^6} + \frac{1}{2}\mathcal{G}_{ts}\partial_r\phi^t\partial^r\phi^s, \quad \text{(B8)}$$

$$R^\psi_\psi = R^\theta_\theta = R^\omega_\omega = \frac{V}{2B^6}. \quad \text{(B9)}$$

These equations force some consistency relations such that

$$R^t_t = R^x_x, \qquad R^\psi_\psi = R^\theta_\theta = R^\omega_\omega, \quad \text{(B10)}$$

which are already verified, but there is another condition that we still have to impose, i.e.

$$R^t_t + R^\psi_\psi = 0 \rightarrow \frac{1}{2}(A^2B^2)'' + (AB)'^2 - 2 = 0. \quad \text{(B11)}$$

Posing $AB = \pm\sqrt{u}$ this condition becomes

$$\frac{d^2u}{dr^2} + \frac{1}{2u}\left(\frac{du}{dr}\right)^2 - 4 = 0. \quad \text{(B12)}$$

To solve this we define

$$\Omega(u) = \frac{du}{dr}, \qquad \dot\Omega = \frac{d}{du}\Omega, \quad \text{(B13)}$$

and we get the Bernoulli equation

$$\Omega\dot\Omega + \frac{\Omega^2}{2u} - 4 = 0, \quad \text{(B14)}$$

with solutions

$$\Omega = \pm\sqrt{\frac{a + 4u^2}{u}}. \quad \text{(B15)}$$

We choose the "+" sign and, from Eq. (B13), we have

$$r - c = \int \frac{du}{\sqrt{\frac{a+4u^2}{u}}} = \frac{2u\sqrt{\frac{4u^2}{a}+1}}{3\sqrt{\frac{a}{u}+4u}}\,{}_2F_1\left(\frac{1}{2}, \frac{3}{4}, \frac{7}{4}, \frac{-4u^2}{a}\right),$$

$$u = A(r)^2B(r)^2. \quad \text{(B16)}$$

The extremal solutions are recovered with $a \rightarrow 0$. In fact, this gives

$$(r - c)^2 = A(r)^2B(r)^2, \quad \text{(B17)}$$

from which we can recognize the structure of the double coincident horizon at $r = c$.

At radial infinity, r and u go to $+\infty$, while, near the horizon $r \sim c$ and $u \sim 0$. In these parametrizations, however, the inner and outer horizons do not appear. Now with a little abuse of notation we regard A and B as functions of u and we trade B with u in the metric:

$$ds^2 = A(u)^2(-dt^2 + dx^2)$$
$$+ uA(u)^{-2}\left(\frac{du^2}{a + 4u^2} + d\Omega^2_{S^3}\right). \quad \text{(B18)}$$

The affine parameter τ can be defined by the differential equation

$$\frac{d\tau}{du} = \frac{1}{u\sqrt{a + 4u^2}}, \tag{B19}$$

which can be solved as follows:

$$u(\tau) = -\frac{\sqrt{a}}{2\sinh(\sqrt{a}\tau)}. \tag{B20}$$

Notice that we have chosen the sign of $u(\tau)$ so that, since $u \geq 0$, τ will be nonpositive. At radial infinity $(u \to \infty)$ $\tau \to 0^-$ and near the horizon $(u \to 0)$ $\tau \to -\infty$.

In terms of the affine parameter and defining $A(u) = e^{U(\tau)}$ the nonextremal metric is

$$\begin{aligned}
ds^2 &= e^{2U}(-dt^2 + dx^2) \\
&+ \frac{1}{2}\frac{\sqrt{a}}{\sinh(\sqrt{a}\tau)}e^{-2U(\tau)}\left(\frac{1}{4}\frac{a}{\sinh^2(\sqrt{a}\tau)}d\tau^2 + d\Omega_{S^3}^2\right).
\end{aligned} \tag{B21}$$

Regularity requires $a > 0$. Einstein's equations and the scalar fields equations are

$$\ddot{U} = e^{4U}\frac{V}{2}, \tag{B22}$$

$$\ddot{\phi}^s + \tilde{\Gamma}^s_{tu}\dot{\phi}^t\dot{\phi}^u = e^{4U}\mathcal{G}^{st}\partial_s V, \tag{B23}$$

$$\frac{3a}{8} = \dot{U}^2 - e^{4U}\frac{V}{4} + \frac{1}{8}\mathcal{G}_{ts}\dot{\phi}^t\dot{\phi}^s, \tag{B24}$$

where the dots are the derivatives with respect to the affine parameter τ. The first two equations can be deduced from the effective action

$$S_{\text{eff}} = \int d\tau \left(\dot{U}^2 + \frac{1}{8}G_{su}\dot{\phi}^s\dot{\phi}^u + e^{4U}\frac{V}{4}\right), \tag{B25}$$

while the third one can be interpreted as a Hamiltonian constraint, with energy $\frac{3a}{8}$.

The near horizon limit can be obtained by taking $\tau \to -\infty$ and the metric reads

$$\begin{aligned}
ds^2 &= e^{2U}(-dt^2 + dx^2) \\
&+ \sqrt{a}e^{\sqrt{a}\tau}e^{-2U(\tau)}\left(ae^{-2\sqrt{a}\tau}d\tau^2 + d\Omega_{S^3}^2\right).
\end{aligned} \tag{B26}$$

To have a finite horizon area, we must require the following behavior:

$$e^{-2U} \sim \left(\frac{A_H}{2\pi^2}\right)^{2/3}\frac{1}{\sqrt{a}e^{\sqrt{a}\tau}}, \quad \text{as } \tau \to \infty. \tag{B27}$$

The two horizons can be described by changing the radial variable into a new one ρ defined through the relation

$$\sinh^2(\sqrt{a}\tau) = \frac{a}{(\rho - \rho_0)^2 - a} = \frac{a}{4u^2}, \tag{B28}$$

which can be solved in the two variables as follows:

$$\begin{aligned}
\rho &= \rho_0 - \sqrt{a}\coth(\sqrt{a}\tau), \\
\tau &= \frac{1}{2\sqrt{a}}\log\left(\frac{\rho - \rho_+}{\rho - \rho_-}\right),
\end{aligned} \tag{B29}$$

where we have denoted by $\rho_\pm$ the radial location of the two horizons defined as

$$\rho_\pm \equiv \rho_0 \pm \sqrt{a}. \tag{B30}$$

Therefore, the metric near the horizon reads

$$\begin{aligned}
ds^2 &= \left(\frac{A_H}{2\pi^2}\right)^{-2/3}\sqrt{a}e^{\sqrt{a}\tau}(-dt^2 + dx^2) \\
&+ \left(\frac{A_H}{2\pi^2}\right)^{2/3}(ae^{2\sqrt{a}\tau}d\tau^2 + d\Omega_{S^3}^2).
\end{aligned} \tag{B31}$$

The existence of a timelike killing vector $\xi = \partial_t$ guarantees the existence of the Komar mass for this class of solutions, which is given by

$$M = \frac{c^2}{8\pi G}\int_{S_\infty^3}\sqrt{g}|_{\theta,\phi,\omega}\epsilon_{\theta\phi\omega\mu\nu}\nabla^\mu\xi^\nu = \frac{\pi c^2}{4G}\lim_{\tau\to 0^-}\dot{U}. \tag{B32}$$

This allows us to fix the following conditions at infinity:

$$U(0) = 0, \qquad \dot{U}(0) = M\frac{4G}{\pi c^2}. \tag{B33}$$

1. Attractor mechanism for extremal solutions

If $a = 0$ we have

$$\tau = -\frac{1}{2u}, \qquad \rho - \rho_0 = -\frac{1}{\tau} = 2r^2, \tag{B34}$$

and the two horizons coincide, $\rho_\pm = \rho_0$. The metric reads

$$ds^2 = e^{2U}(-dt^2 + dx^2) + \frac{1}{(-2\tau)}e^{-2U}\left(\frac{d\tau^2}{4\tau^2} + d\Omega_{S^3}^2\right). \tag{B35}$$

The horizon is again at $u \to 0$ and $\tau \to -\infty$ and the near horizon behavior of the function e^{-U} is now given by

$$e^{-2U} = \left(\frac{A_H}{2\pi^2}\right)^{2/3}(-2\tau), \quad \text{as } \tau \to \infty. \tag{B36}$$

Since the area of S_3 is $A_H = 2\pi^2 L^3$, we can also replace A_H with the radius of the 3-sphere L. The coordinate which defines the proper spatial distance from the horizon is

$$d\zeta = \frac{e^{-U}}{\sqrt{(-8\tau^3)}} d\tau = -\frac{L}{2} d\log(-\tau). \qquad (B37)$$

Thanks to the regularity condition in Eq. (B36) and fixing a not-useful constant we find

$$\zeta = -\frac{L}{2}\log(-\tau) + \text{const}, \qquad (B38)$$

while the position of the horizon is now at $\zeta \to -\infty$. Again, we require the scalar fields to be regular at the horizon

$$\lim_{\zeta \to -\infty} \phi^s(\zeta) = \phi_*^s < \infty. \qquad (B39)$$

If the functions ϕ^s are uniformly continuous this request necessarily implies that the derivatives

$$\lim_{\zeta \to -\infty} \frac{d^k}{d\zeta^k}\phi^s = 0. \qquad (B40)$$

This implies

$$\lim_{\tau \to -\infty} \tau \frac{d}{d\tau}\phi^s = \lim_{\tau \to -\infty} \tau^2 \frac{d^2}{d\tau^2}\phi^s = \ldots = 0. \qquad (B41)$$

The scalar field equation (B23) can be rewritten as

$$\tau^2 \partial_\tau^2 \phi^s + \tilde\Gamma^s_{uv}(\tau\partial_\tau\phi^u)(\tau\partial_\tau\phi^v) = \mathcal{G}^{st}\partial_t V L^{-6}. \qquad (B42)$$

Taking the limit at the near horizon and applying the rule for regularity (B41) we get the *attractor mechanism*[13] equation,

$$\lim_{\phi^s \to \phi_*^s} \partial_t V = 0. \qquad (B43)$$

The scalar fields which V depends on are attracted toward fixed value extremizing the potential, while the others are called flat directions. Therefore, the attractor mechanism forces the effective potential V to have an extremum at the horizon, namely for $\phi^s = \phi_*^s$:

$$\Gamma^T \partial_s \mathcal{M}\Gamma|_{\text{horizon}} = 0. \qquad (B44)$$

We can also find the explicit critical value of the black hole potential at the horizon. To do this, we note that the function U near the horizon is explicitly given by

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[13]The attractor mechanism was first found for $D = 4$ asymptotically flat extremal black holes in [46,47].

$$U = -\frac{1}{2}\log(L^2(-2\tau)). \qquad (B45)$$

If we insert this result inside Eqs. (B22)–(B24) we get that the value of the black hole potential V at the horizon is fixed to be

$$V_{\text{horizon}} = V_* = 4L^4. \qquad (B46)$$

Restoring the coordinate r, and setting $c = 0$, the metric at the horizon takes the form

$$ds^2 = \frac{r^2}{r_H^2}(-dt^2 + dx^2) + \frac{L^2}{r^2}dr^2 + L^2 d\Omega_{S^3}, \qquad (B47)$$

which describes an $\text{AdS}_3 \times S^3$ geometry.

APPENDIX C: STRING SOLUTIONS IN THE DOUBLE COMMUTING $\frac{\text{SL}(2,\mathbb{R})}{\text{SO}(2)}$ TRUNCATION

We can now write the solutions of Eqs. (B22)–(B24) by minimizing the potential V, as discussed in Appendix D. After this minimization, to solve the equations we just need the moduli to move along geodesics on the target space. Here we discuss the case of two commuting $\text{SL}(2,\mathbb{R})$ of the scalar manifold, for which the potential and the metric reads

$$d\tilde{s} = d\phi^2 + d\psi^2 + d\varphi^2 + e^{\varphi-\sqrt{2}\psi+\phi}(dC_{12})^2 + e^{\varphi+\sqrt{2}\psi+\phi}(dC_{34})^2, \qquad (C1)$$

$$V = \frac{e^\varphi}{2}(d_1^2 e^{-\phi} + d_5^2 e^{\phi-2\varphi}), \qquad (C2)$$

where $B_{12}^1 = C_{12}$ and $B_{34}^1 = C_{34}$. From the perspective of the 10D, the metric in Einstein's frame is

$$ds_{10}^2 = e^{2\alpha\varphi}ds_6^2 + e^{2\beta\varphi}(e^{2\gamma\psi}d\theta_1^2 + e^{2\gamma\psi}d\theta_2^2 + e^{-2\gamma\psi}d\theta_3^2 + e^{-2\gamma\psi}d\theta_4^2), \qquad (C3)$$

$$C_2 = \tilde{C}_2 + C_{12}d\theta_1 \wedge d\theta_2 + C_{34}d\theta_3 \wedge d\theta_4, \qquad (C4)$$

where $\alpha = 1/4 = -\beta$ and $\gamma^2 = 1/8$. In this case the einstein equations are directly from Eqs. (B22)–(B24):

$$\ddot{U}(\tau) - \frac{1}{4}e^{\varphi(\tau)+4U(\tau)}(d_1^2 e^{-\phi(\tau)} + d_5^2 e^{\phi(\tau)-2\varphi(\tau)}) = 0,$$

$$\ddot{\phi}(\tau) + \frac{1}{2}e^{\varphi(\tau)+4U(\tau)}(-d_5^2 e^{\phi(\tau)-2\varphi(\tau)} + d_1^2 e^{\phi(\tau)})$$

$$- \frac{1}{2}e^{\varphi(\tau)-\sqrt{2}\psi(\tau)+\phi(\tau)}(\dot{C}_{12}(\tau)^2 + e^{2\sqrt{2}\psi(\tau)}\dot{C}_{34}(\tau)^2) = 0,$$

$$\ddot{\varphi}(\tau) + \frac{1}{2}e^{4U(\tau)}(d_5^2 e^{\phi(\tau)-\varphi(\tau)} - d_1^2 e^{+\varphi(\tau)-\phi(\tau)})$$

$$- \frac{1}{2}e^{\varphi(\tau)-\sqrt{2}\psi(\tau)+\phi(\tau)}(\dot{C}_{12}(\tau)^2 + e^{2\sqrt{2}\psi(\tau)}\dot{C}_{34}(\tau)^2) = 0,$$

$$\ddot{\psi}(\tau) + \frac{1}{\sqrt{2}}e^{\varphi(\tau)-\sqrt{2}\psi(\tau)+\phi(\tau)}(\dot{C}_{12}(\tau)^2 - e^{2\sqrt{2}\psi(\tau)}\dot{C}_{34}(\tau)^2) = 0,$$

$$\ddot{C}_{12}(\tau) + \dot{C}_{12}(\tau)(\dot{\varphi}(\tau) - \sqrt{2}\dot{\psi}(\tau) + \dot{\phi}(\tau)) = 0,$$

$$\ddot{C}_{34}(\tau) + \dot{C}_{34}(\tau)(\dot{\varphi}(\tau) + \sqrt{2}\dot{\psi}(\tau) + \dot{\phi}(\tau)) = 0. \tag{C5}$$

In order to separate the moduli from the other scalars we introduce the functions

$$\varphi = \frac{h(\tau) - g(\tau)}{2}, \qquad \phi = \frac{h(\tau) + g(\tau)}{2}; \tag{C6}$$

therefore, the system totally splits into two different sets of equations. This is due to the diagonal form of the metric, see Eq. (C1), in this particular truncation. The first equations are for the scalars which are not modules,

$$\ddot{U}(\tau) - \frac{1}{4}e^{4U(\tau)-g(\tau)}(d_1^2 + d_5^2 e^{2g(\tau)}) = 0, \tag{C7}$$

$$\ddot{g}(\tau) + e^{4U(\tau)-g(\tau)}(d_1^2 - d_5^2 e^{2g(\tau)}) = 0, \tag{C8}$$

and the second system is for the moduli

$$\ddot{h}(\tau) - e^{h(\tau)-\sqrt{2}\psi(\tau)}(\dot{C}_{12}(\tau)^2 + e^{2\sqrt{2}\psi(\tau)}\dot{C}_{34}(\tau)^2) = 0, \tag{C9}$$

$$\ddot{\psi}(\tau) + \frac{1}{\sqrt{2}}e^{h(\tau)-\sqrt{2}\psi(\tau)}(\dot{C}_{12}(\tau)^2 - e^{2\sqrt{2}\psi(\tau)}\dot{C}_{34}(\tau)^2) = 0, \tag{C10}$$

$$\ddot{C}_{12}(\tau) + \dot{C}_{12}(\tau)(\dot{h}(\tau) - \sqrt{2}\dot{\psi}(\tau)) = 0, \tag{C11}$$

$$\ddot{C}_{34}(\tau) + \dot{C}_{34}(\tau)(\dot{h}(\tau) + \sqrt{2}\dot{\psi}(\tau)) = 0. \tag{C12}$$

1. D1-D5 solution

When the moduli are equal to zero, to solve the first system we can define the superpotential to be

$$W = e^{-\frac{g(\tau)}{2}}(d_1 + d_5 e^{g(\tau)}). \tag{C13}$$

Thanks to this definition we can prove that the system can be cast into a first-order system

$$\dot{g}(\tau) - 2e^{2U(\tau)}\partial_{g(\tau)}W = 0, \tag{C14}$$

$$\dot{U}(\tau) - \frac{We^{2U(\tau)}}{4} = 0. \tag{C15}$$

Now it is easy to see that the solution is given by

$$U(\tau) = -\frac{1}{4}\log[(1 - d_1\tau)(1 - d_5\tau)], \tag{C16}$$

$$-\varphi(\tau) = \phi(\tau) = \frac{1}{2}\log\left[\frac{1 - d_1\tau}{1 - d_5\tau}\right], \tag{C17}$$

$$\psi(\tau) = c_{12}(\tau) = c_{34}(\tau) = 0, \tag{C18}$$

for which the metric reads

$$ds^2 = \frac{(-dt^2 + dx^2)}{\sqrt{(-d_1\tau + 1)(-d_5\tau + 1)}} + \frac{\sqrt{(-d_1\tau + 1)(-d_5\tau + 1)}}{(-2\tau)}\left(\frac{d\tau^2}{4\tau^2} + d\Omega_{S^3}^2\right). \tag{C19}$$

We can now define the usual radial coordinate $\tau = (-2r^2)^{-1}$ and we end up with the usual form of the D1-D5 system,

$$ds^2 = (Z_1 Z_5)^{-\frac{1}{2}}(-dt^2 + dx^2) + (Z_1 Z_5)^{\frac{1}{2}}(dx^i dx^i), \tag{C20}$$

$$dx^i dx^i = dr^2 + r^2 d\Omega_3^2, \quad Z_1 = 1 + \frac{Q_1}{r^2}, \quad Z_5 = 1 + \frac{Q_5}{r^2}, \tag{C21}$$

where $Q_1 = d_1/2$ and $Q_5 = d_5/2$. At the horizon the Z functions become

$$Z_1 = \frac{Q_1}{r^2}, \qquad Z_5 = \frac{Q_5}{r^2}, \tag{C22}$$

and the physical distance from the horizon is

$$\rho = \log\left(\frac{r}{\ell}\right). \tag{C23}$$

This background is $\text{AdS}_3 \times S^3$, which is the throat of the D1-D5 system. Now, the scalars are attracted toward configurations that extremize this potential, called critical points,

$$\partial_\phi V = 0 \to V = V_* = d_1 d_5. \tag{C24}$$

Since we are in the near-horizon region where this critical point is reached by the scalar fields, we have $V = V_*$ and the energy-momentum tensor for the scalars is zero

$T^{(S)}_{\hat{\mu}\hat{\nu}} = 0$. The near-horizon metric is conveniently written in the following form:

$$ds^2 = \frac{r^2}{L^2}(-dt^2 + dx^2) + \frac{L^2}{r^2}dr^2$$
$$+ L^2[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\omega^2)], \quad \text{(C25)}$$

where $L^2 = \frac{\sqrt{d_1 d_5}}{2}$. Einstein's equations boil down to

$$G^\nu_\mu = -\frac{1}{2}V_0\delta^\nu_\mu = T^{(H)\nu}_\mu, \quad G^{\bar\nu}_{\bar\mu} = \frac{1}{2}V_0\delta^{\bar\nu}_{\bar\mu} = T^{(H)\bar\nu}_{\bar\mu}, \quad \text{(C26)}$$

solved by the condition $L^2 = \sqrt{Q_1 Q_5}$.

2. D1-D5 solution with nontrivial moduli

The equations for the moduli (C9)–(C12) are consistent with all of them to be zero; this was the matter of the last Sec. C 1 which describes the D1-D5 system. In this subsection, we consider a solution in which some of the moduli fields are allowed to evolve in the radial variable τ. The resulting solution, however, is not related to the general construction considered in this work, in which the evolution of the moduli is on one of the boundary coordinates instead.

We can still keep part of the solution we wrote before,

$$U(\tau) = -\frac{1}{4}\log[(1 - d_1\tau)(1 - d_5\tau)], \quad \text{(C27)}$$

$$g(\tau) = \log\left[\frac{1 - d_1\tau}{1 - d_5\tau}\right], \quad \text{(C28)}$$

while the moduli remain to be fixed now. If we now introduce

$$s(\tau) = h(\tau) + \sqrt{2}\psi(\tau), \quad t(\tau) = h(\tau) - \sqrt{2}\psi(\tau), \quad \text{(C29)}$$

the equations decouple into two independent systems:

$$\ddot{t}(\tau) - 2e^{t(\tau)}\dot{c}_{12}(\tau)^2 = 0, \quad \ddot{c}_{12}(\tau) + \dot{c}_{12}(\tau)\dot{t}(\tau) = 0, \quad \text{(C30)}$$

$$\ddot{s}(\tau) - 2e^{s(\tau)}\dot{c}_{34}(\tau)^2 = 0, \quad \ddot{c}_{34}(\tau) + \dot{c}_{34}(\tau)\dot{s}(\tau) = 0. \quad \text{(C31)}$$

We can directly integrate c_{12} and c_{34} as

$$c_{12}(\tau) = c_1 \int e^{-t(\tau)} d\tau,$$

$$c_{34}(\tau) = c_3 \int e^{-s(\tau)} d\tau, \quad \text{(C32)}$$

while the remaining equations are

$$\ddot{t}(\tau) = 2c_1^2 e^{-t(\tau)}, \quad \ddot{s}(\tau) = 2c_3^2 e^{-s(\tau)}, \quad \text{(C33)}$$

where c_1 and c_3 are constants. The whole system can now be integrated and the solution reads

$$t = \log\left[-2c_1^2(\cosh(\tau) - 1)\right], \quad c_{12} = \frac{\coth\left(\frac{\sqrt{6a}}{2}\tau\right)}{2c_1},$$

$$s = \log\left[-2c_3^2(\cosh(\tau) - 1)\right], \quad c_{34} = \frac{\coth\left(\frac{\sqrt{6a}}{2}\tau\right)}{2c_3}.$$

In terms of the field appearing in the truncation given by (C1), the solution is

$$\varphi = \frac{1}{4}\left[-2\log\left(\frac{1 - d_1\tau}{1 - d_5\tau}\right)\right.$$
$$\left. + \log\left(4c_1^2 c_3^2(\cosh(\sqrt{6a}\tau) - 1)^2\right)\right],$$

$$\phi = \frac{1}{4}\left[2\log\left(\frac{1 - d_1\tau}{1 - d_5\tau}\right)\right.$$
$$\left. + \log\left(4c_1^2 c_3^2(\cosh(\sqrt{6a}\tau) - 1)^2\right)\right],$$

$$\psi = \frac{\log\left(-\frac{c_3}{c_1}\right)}{\sqrt{2}}, \quad c_{12} = \frac{\coth\left(\frac{\sqrt{6a}}{2}\tau\right)}{2c_1}, \quad c_{34} = \frac{\coth\left(\frac{\sqrt{6a}}{2}\tau\right)}{2c_3}.$$

This solution is nonextremal. We can directly build other solutions using the S-duality transformation given by Eqs. (3.46)–(3.48).

APPENDIX D: EXTREMIZATION OF THE EFFECTIVE SCALAR POTENTIAL

In this section, we wish to discuss the extremization of $V(\phi, \Gamma)$. To this end, we first need to explicitly construct $\mathcal{M}$ in terms of the scalar fields by using the solvable parametrization of the coset manifold. Using the explicit dependence of $\mathcal{M}_{MN}$ on the dimensionally reduced type IIB fields we can derive the moduli space of the $\text{AdS}_3 \times \text{S}^3$ background. We can derive the expression for the potential V by first writing the charge vector Γ^M in components:

$$\Gamma^M = (\, n^\alpha \quad D_{ij} \quad n_\alpha \,), \quad \text{(D1)}$$

where $n^\alpha = (d_5, n_5)$, $n_\alpha = (d_1, f_1)$, being d_1, d_5 the D1, D5 charges and f_1, n_5 the charges of the fundamental string and of the NS 5-brane, respectively, while D_{ij} are the D3-brane charges. We then restrict ourselves to the D1-D5 charges and obtain

$$2V = \Gamma^M \mathcal{M}_{MN}\Gamma^N = \frac{G^{-\frac{1}{2}}}{g_s}\left(d_1 - \frac{1}{8}\epsilon^{ijkl}B_{ij}B_{kl}d_5\right)^2$$

$$+ g_s G^{-\frac{1}{2}}\left[\left(d_1 - \frac{1}{8}\epsilon^{ijkl}B_{ij}B_{kl}d_5\right)C_{(0)}\right.$$

$$\left. - \left(c - \frac{1}{8}\epsilon^{ijkl}B_{ij}C_{kl}\right)d_5\right]^2$$

$$+ g_s G^{\frac{1}{2}}\left(\frac{1}{2g_s}B_{ij}B_{kl}G^{ik}G^{jl} + 1\right)d_5^2. \quad \text{(D2)}$$

The mass formula for the D1-D5 system given in [48] is obtained by adding an invariant, constant-shift term to the potential V, that is $\Gamma^M \Omega_{MN} \Gamma^N / 2$. The minimizing conditions over the moduli space are the following:

$$c - \frac{1}{8} \epsilon^{ijkl} B_{ij} C_{kl} = g_s G^{\frac{1}{2}} C_{(0)}, \tag{D3}$$

$$g_s G^{\frac{1}{2}} + \frac{1}{8} \epsilon^{ijkl} B_{ij} B_{kl} = \frac{d_1}{d_5}, \tag{D4}$$

$$G^{\frac{1}{2}} B_{ij} G^{ik} G^{jl} = \frac{1}{2} B_{ij} \epsilon^{ijkl}. \tag{D5}$$

With these conditions the potential at the minimum is

$$V_* = d_1 d_5, \tag{D6}$$

which, after the mentioned shift, becomes exactly double. The initial space for the scalar manifold is $\mathrm{SO}(5,5)/(\mathrm{SO}(5) \times \mathrm{SO}(5))$; due to the attractor mechanism, five of these scalars are fixed at the minimum of the potential. These conditions are explicitly given in (D3), (D4), and (D5). Then, the scalar manifold reduces to $\mathrm{SO}(4,5)/(\mathrm{SO}(4) \times \mathrm{SO}(5))$ as required by CFT duality (see Ref. [49]). To see this, let us show that $\mathrm{SO}(4,5)$ is the little group of the charge vector Γ^M when only the D1-D5 charges are switched on:

$$\Gamma^M = (d_5, 0, 0, 0, 0, 0, 0, 0, d_1, 0). \tag{D7}$$

Supersymmetry requires $d_1 d_5 > 0$. Let us perform the following duality transformation:

$$\Gamma^M \rightarrow \Gamma'^M = \mathcal{O}^M{}_N \Gamma^N, \tag{D8}$$

where

$$\mathcal{O}^M{}_N = \mathrm{diag}\left(\sqrt{\frac{d_1}{d_5}}, 1, 1, 1, 1, 1, 1, 1, \sqrt{\frac{d_5}{d_1}}, 1 \right) \in \mathrm{SO}(5,5). \tag{D9}$$

The new charge vector reads

$$\Gamma'^M = \sqrt{d_1 d_5}(1, 0, 0, 0, 0, 0, 0, 0, 1, 0). \tag{D10}$$

Changing the basis of the representation space by a Cayley transformation

$$\mathcal{C}^M{}_N = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}, $$

the invariant matrix becomes

$$\Omega \rightarrow \mathcal{C}^t \Omega \mathcal{C} = \mathrm{diag}(1,1,1,1,1,-1,-1,-1,-1,-1) \tag{D11}$$

and the charge vector acquires the following form:

$$\Gamma \rightarrow \mathcal{C}\Gamma = \sqrt{2d_1 d_5} \mathrm{diag}(1, 0, 0, 0, 0, 0, 0, 0, 0, 0). \tag{D12}$$

From the above form of the charge vector, it is straightforward to identify the $\mathrm{SO}(4,5)$ subgroup of $\mathrm{SO}(5,5)$ which leaves it invariant. It consists of those matrices which have a trivial action of the first entry of the vector.

In the non-BPS case in which $d_1 d_5 < 0$ (e.g. D1-anti-D5 system), the same transformation yields the vector:

$$\mathcal{C}\Gamma = \sqrt{2|d_1 d_5|} \mathrm{diag}(0, 0, 0, 0, 0, 0, 0, 0, 1, 0). \tag{D13}$$

The stabilizer being still $\mathrm{SO}(4,5)$. This orbit, as opposed to the BPS one, is characterized by the invariant property $\Gamma^t \Omega \Gamma < 0$.

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