# Celestial OPE in self-dual gravity

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In this paper we compute the celestial operator product expansion between two outgoing positive helicity gravitons in the self-dual gravity. It has been shown that the self-dual gravity is a $w_{1+\infty}$-invariant theory whose scattering amplitudes are one-loop exact with all positive helicity gravitons. Celestial $w_{1+\infty}$ symmetry is generated by an infinite tower of (conformally soft) gravitons which are holomorphic conserved currents. We find that at any given order only the descendants of a finite number of $w_{1+\infty}$ currents contribute to the operator product expansion. This is somewhat surprising but, this is consistent with our earlier analysis based on $w_{1+\infty}$ symmetry alone. The phenomenon of truncation also suggests that in some (unknown) formulation the spectrum of conformal dimensions in the dual two dimensional theory can be bounded from below.

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## I. INTRODUCTION

Celestial holography is a conjectured duality between quantum gravity in 4D asymptotically flat spacetime and a quantum field theory on the 2D celestial sphere [1–3]. Symmetries play an important role in this conjectured duality. The Lorentz group in 4D acts on the 2D celestial sphere as the global conformal group. So the dual theory should be a conformal field theory. Motivated by this, a new basis was introduced [2–4] in which the $S$-matrix elements transform like 2D conformal correlators. Besides the two-dimensional global conformal symmetry, celestial conformal field theory has various infinite-dimensional current algebra symmetries [5–34].

Operator product expansion (OPE) is a central tool used to study various aspects of any CFT. In the context of celestial CFT also, OPE played an important role in identifying new symmetries [19,22,23], null states [19,20,35–45] etc. It has also found applications in the bootstrap program [46,47]. Based on the universal singular structure of the tree-level OPE between two positive helicity gravitons, it was shown in [22] that celestial conformal field theory has an infinite tower of soft symmetries which close into $w_{1+\infty}$ algebra [23]. Loop corrections to the tree-level celestial OPEs have been studied in [48,49].

In a previous paper [39], we have studied the implications of the $w_{1+\infty}$ symmetry at the level of OPEs by using representation theory. By studying the subleading terms in the OPE between two positive helicity outgoing gravitons, we have shown that there should exist an infinite number of theories which are invariant under $w_{1+\infty}$ algebra.

In this paper we derive the OPE in one such theory, known as the quantum self-dual gravity [50–54] which was shown to be $w_{1+\infty}$ invariant in [55–57]. Here we do a collinear expansion of the known graviton scattering amplitudes in the self-dual gravity theory and extract the celestial OPE from there. For simplicity, we analyze the 5-point all plus amplitude in self-dual gravity and factorize it in the collinear limit through a 4-point amplitude. The results we obtain are consistent with what we proposed in [39] based on the representation theory of $w_{1+\infty}$. The rest of the paper is organized as follows.

In Sec. II we introduce notations and conventions used in this paper. Section III briefly describes the $w_{1+\infty}$ algebra

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and how the whole tower of the $w_{1+\infty}$ currents can be generated using the two $sl_2(R)$ subalgebras. In Sec. IV we briefly discuss about the scattering amplitudes in quantum self-dual gravity. Section V discusses how to extract the OPE between two positive helicity outgoing gravitons from the 5-point one-loop self-dual amplitude. We start by simplifying the 5-point amplitude in the momentum space and then Mellin transform it to get the celestial amplitudes. We then discuss how to factorize each term order-by-order in the OPE limit of the celestial amplitudes. The null states of the self-dual gravity appearing at various orders of the OPE and the invariance of the OPE under $w_{1+\infty}$ algebra are discussed in Appendixes I and J.

For the sake of completeness of the paper, we give a brief review of the celestial amplitude in Appendix A. In Appendix B, we discuss the parametrization of the 4- and 5-point delta functions which are useful in our context of the OPE expansion. Appendixes C and D discuss how to simplify the 4- and 5-point amplitudes in momentum space using momentum conserving delta functions and various identities of the spinor-helicity brackets. These simplifications are done keeping in mind the fact that we want to factorize the 5-point amplitude in terms of the 4-point amplitude in the OPE expansion. Appendix E deals with the Mellin transformation of the 5-point amplitude. In Appendix F we discuss the conditions on the graviton primary operators under the $w_{1+\infty}$ algebra which follow from the universal structure of the OPE. In Appendix G, we list the transformation properties of all the maximally helicity violating (MHV) null states under the action of $sl_2(R)_V$ and $\overline{sl_2(R)}$ algebras which are required to show the $w_{1+\infty}$-invariance of the self-dual OPE. Appendix H briefly reviews the construction of a general $w_{1+\infty}$-algebra invariant OPE and how one can obtain an infinite family of $w_{1+\infty}$-algebra invariant theories.

## II. NOTATIONS AND CONVENTIONS

In this paper, we will work in the (2, 2) signature spacetime, which is also known as Klein space. The null momentum $p^\mu$ of a massless particle, satisfying the on shell condition $p^2 = 0$, is parametrized as

$$p^\mu = \epsilon q^\mu, \qquad q^\mu = \omega\{1 + z\bar{z}, z + \bar{z}, z - \bar{z}, 1 - z\bar{z}\}, \tag{2.1}$$

where $\epsilon = \pm 1$ for outgoing and incoming particles respectively, $(z, \bar{z})$ are two independent real variables and $\omega$ is any positive number interpreted as the energy of the particle. In Klein space the null infinity takes the form of a Lorentzian torus (known as the celestial torus) times a null line. The Lorentz group in (2, 2) signature is given by $SO(2,2) \simeq \frac{SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R}{\mathbb{Z}_2}$ and acts as the group of conformal transformations on the celestial torus,

$$SL(2,\mathbb{R})_L: \quad z \to \frac{az+b}{cz+d}, \quad \bar{z} \to \bar{z}, \quad ad - bc = 1,$$
$$SL(2,\mathbb{R})_R: \quad \bar{z} \to \frac{\bar{a}\,\bar{z} + \bar{b}}{\bar{c}\,\bar{z} + \bar{d}}, \quad z \to z, \quad \bar{a}\,\bar{d} - \bar{b}\,\bar{c} = 1. \tag{2.2}$$

In our conventions the spinor-helicity variables are given by

$$\langle ij \rangle = 2\epsilon_i \epsilon_j \sqrt{\omega_i \omega_j} z_{ij}, [ij] = 2\sqrt{\omega_i \omega_j} \bar{z}_{ij}, \tag{2.3}$$

where $z_{ij} = z_i - z_j$ and we also have $2p_i \cdot p_j = -\langle ij \rangle [ij]$.

## III. REVIEW OF $w_{1+\infty}$ ALGEBRA

We start by reviewing the $w_{1+\infty}$ algebra which follows from the universal singular terms in the OPE between two positive helicity outgoing gravitons. Let $G^+_\Delta(z, \bar{z})$ denote the positive helicity graviton conformal primary operator of dimension $\Delta$ at the point $(z, \bar{z})$ on the celestial torus. The universal singular terms in the OPE are given by [22]

$$G^+_{\Delta_1}(z_1, \bar{z}_1) G^+_{\Delta_2}(z_2, \bar{z}_2) = -\frac{\bar{z}_{12}}{z_{12}} \sum_{n=0}^{\infty} B(\Delta_1 - 1 + n, \Delta_2 - 1) \times \frac{\bar{z}_{12}{}^n}{n!} \bar{\partial}^n G^+_{\Delta_1+\Delta_2}(z_2, \bar{z}_2) \tag{3.1}$$

Let us define an infinite family of positive helicity conformally soft [58–64] gravitons [22] as,[1]

$$H^k(z, \bar{z}) = \lim_{\Delta \to k} (\Delta - k) G^+_\Delta(z, \bar{z}), \quad k = 1, 0, -1, -2, \ldots, \tag{3.2}$$

with weights $(\frac{k+2}{2}, \frac{k-2}{2})$. It follows from the OPE (3.1) that we can introduce the following truncated mode expansion:

$$H^k(z, \bar{z}) = \sum_{m=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_m(z)}{\bar{z}^{m+\frac{k-2}{2}}}, \tag{3.3}$$

and the modes $H^k_m(z)$ are the conserved holomorphic currents. The currents $H^k_m(z)$ can be further mode expanded in the $z$-variable to get,

$$H^k_m(z) = \sum_{\alpha \in \mathbb{Z} - \frac{k+2}{2}} \frac{H^k_{\alpha,m}}{z^{\alpha+\frac{k+2}{2}}} \tag{3.4}$$

and one can show [22] that the modes $H^k_{\alpha,m}$ satisfy the algebra,[2]

[1]In (3.2) and the equations following this, the index $k$ starts from 1 instead of 2 [23]. $H^2(z, \bar{z})$ is a central term and we take it to be zero because $G^+_\Delta$ has no pole at $\Delta = 2$. This has the consequence that supertranslations commute.

[2]Here we are assuming that $\kappa = \sqrt{32\pi G_N} = 2$.

$$[H^k_{\alpha,m}, H^l_{\beta,n}] = -[n(2-k) - m(2-l)] \times \frac{(\frac{2-k}{2} - m + \frac{2-l}{2} - n - 1)!}{(\frac{2-k}{2} - m)!(\frac{2-l}{2} - n)!} \times \frac{(\frac{2-k}{2} + m + \frac{2-l}{2} + n - 1)!}{(\frac{2-k}{2} + m)!(\frac{2-l}{2} + n)!} H^{k+l}_{\alpha+\beta,m+n}. \quad (3.5)$$

This is called the holographic symmetry algebra (HSA). Now if we make the following redefinition (or discrete light transformation) [23]:

$$w^p_{\alpha,m} = \frac{1}{2}(p-m-1)!(p+m-1)! H^{-2p+4}_{\alpha,m} \quad (3.6)$$

then (3.5) turns into the $w_{1+\infty}$ algebra[3]

$$[w^p_{\alpha,m}, w^q_{\beta,n}] = [m(q-1) - n(p-1)] w^{p+q-2}_{\alpha+\beta,m+n}, \quad (3.7)$$

where $p = \frac{3}{2}, 2, \frac{5}{2}, \cdots$ and $1 - p \leq m \leq p - 1$.[4]

For our purpose it is more convenient to work with the HSA (3.5) rather than the $w_{1+\infty}$ algebra. However, we continue to refer to the HSA as the $w$ algebra.

Now, in [39], it was shown that the whole tower of the $w$ currents can be generated using the two $sl_2(R)$ subalgebras. One of them is $sl_2(R)_V$[5] generated by the operators $\{H^1_{-\frac{1}{2},-\frac{1}{2}}, H^0_{0,0}, H^{-1}_{\frac{1}{2},\frac{1}{2}}\}$,

$$\left[H^0_{0,0}, H^1_{-\frac{1}{2},-\frac{1}{2}}\right] = H^1_{-\frac{1}{2},-\frac{1}{2}},$$
$$\left[H^0_{0,0}, H^{-1}_{\frac{1}{2},\frac{1}{2}}\right] = -H^{-1}_{\frac{1}{2},\frac{1}{2}},$$
$$\left[H^1_{-\frac{1}{2},-\frac{1}{2}}, H^{-1}_{\frac{1}{2},\frac{1}{2}}\right] = -H^0_{0,0}. \quad (3.8)$$

The other $sl_2(R)$ subalgebra is generated by the global (Lorentz) conformal transformations $\{H^0_{0,1}, H^0_{0,0}, H^0_{0,-1}\}$. We call this $\overline{sl_2(R)}$ because this acts only on the $\bar{z}$ coordinate. Now the $w$ symmetry is generated by the infinite number of soft currents $\{H^k_p(z)\}$ where $k = 1, 0, -1, -2, \ldots$ is the dimension ($\Delta$) of the soft operator and $\frac{k-2}{2} \leq p \leq -\frac{k-2}{2}$. For a fixed $k$, the soft currents $\{H^k_{-\frac{k-2}{2}}(z), \ldots, H^k_{\frac{k-2}{2}}(z)\}$ transform in a spin-$(\frac{2-k}{2})$ representation of the $\overline{sl_2(R)}$.

Now let us consider the currents $\{H^1_{\frac{1}{2}}, H^0_1, \ldots, H^k_{\frac{2-k}{2}}, \ldots\}$ with the *lowest* $\overline{sl_2(R)}$ weights. These currents transform in an irreducible highest weight representation of the $sl_2(R)_V$.

[3]This is the wedge subalgebra of $w_{1+\infty}$.

[4]Again we let $p$ run from $\frac{3}{2}$ instead of 1 because $w^1 = 0$.

[5]Here V stands for vertical. Please see Fig. 1 for an explanation.

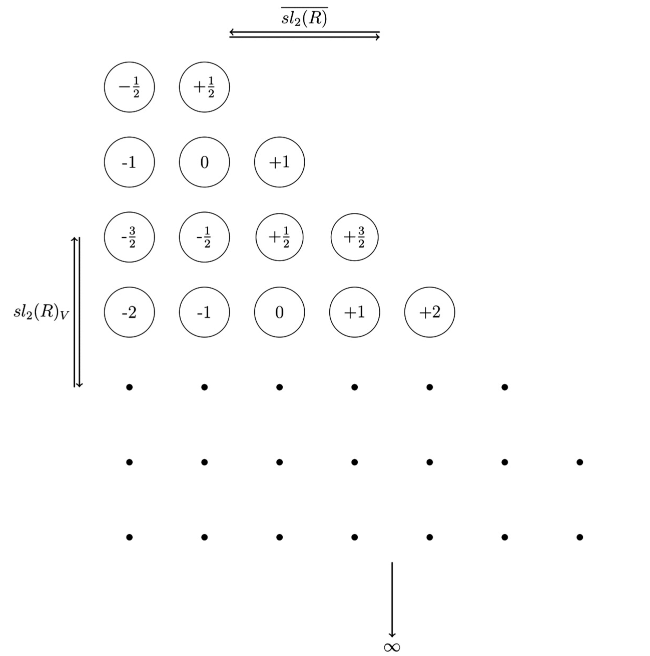


FIG. 1. The figure shows the soft currents. The rows and the columns are indexed by the $\overline{sl_2(R)}$ weights and the dimension ($\Delta = k = 1, 0, -1, -2, \ldots$) of the conformally soft graviton $H^k(z, \bar{z})$, which generates the currents sitting in a row, respectively. $\overline{sl_2(R)}$ acts horizontally along a row and $sl_2(R)_V$ acts vertically along a column. In this way they generate the whole symmetry algebra starting from the current $H^1_{\frac{1}{2}}(z)$ on the top-left corner.

This can be seen from the following commutation relations following from (3.5),

$$\left[H^{-1}_{\frac{1}{2},\frac{1}{2}}, H^k_{\alpha,\frac{2-k}{2}}\right] = -\frac{1}{2}(k-2)(k-3) H^{k-1}_{\alpha+\frac{1}{2},\frac{2-(k-1)}{2}},$$
$$\left[H^0_{0,0}, H^k_{\alpha,\frac{2-k}{2}}\right] = (k-2) H^k_{\alpha,\frac{2-k}{2}},$$
$$\left[H^1_{-\frac{1}{2},-\frac{1}{2}}, H^k_{\alpha,\frac{2-k}{2}}\right] = -H^{k+1}_{\alpha-\frac{1}{2},\frac{2-(k+1)}{2}}. \quad (3.9)$$

Therefore, starting from the current $H^1_{\frac{1}{2}}(z)$ we can generate any other $w$ current by the combined action of the $\overline{sl_2(R)}$ and $sl_2(R)_V$ (Fig. 1).

## IV. SCATTERING AMPLITUDES IN QUANTUM SELF-DUAL GRAVITY

In this section, following [51] we briefly review the all-plus helicity scattering amplitudes in quantum self-dual gravity. In (2, 2) signature, self-duality translates into the following condition on the Riemann tensor:

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta} R_{\alpha\beta\rho\sigma}, \quad (4.1)$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric tensor with $\epsilon^{0123} = +1$. In order to maintain the reality condition on the fields, the self-dual gravity is described in either (2, 2) or (0, 4) signature. In Lorentzian (1, 3) signature, the condition (4.1) acquires an extra factor of $i$ and contain no real solutions. At the classical level, the linearized self-dual solutions consist of positive helicity plane waves.

In this paper, we are interested in the collinear behaviour of gravitons in the self-dual gravity. At the tree level, we have only one nontrivial amplitude; the three point $\overline{\text{MHV}}$ amplitude where, only one external graviton has negative helicity and other two have positive helicity. The appearance of the negative helicity graviton in the three point $\overline{\text{MHV}}$ amplitude can be explained from the fact that the action contains a Lagrange multiplier which is physically interpreted as the negative helicity graviton.

At the one loop level, the only nonzero amplitudes are the ones with all plus helicity gravitons with the minimum number of gravitons being four. These amplitudes are both UV and IR finite. The only divergences of these amplitudes are collinear and soft divergences. Our interest in this theory stems from the fact that this is a nontrivial quantum theory which is known to be $w$ invariant.

The one loop all-plus $n$-graviton stripped amplitude in self-dual gravity is given by [51]

$$A_n(1^+, 2^+, \ldots, n^+) = -\frac{i}{(4\pi)^2 960}\left(-\frac{\kappa}{2}\right)^n \sum_{\substack{1\le a<b\le n \\ M,N}} h(a, M, b) h(b, N, a) \mathrm{tr}^3[aMbN], \tag{4.2}$$

where $a$ and $b$ are the external legs and $M$ and $N$ are two sets such that $M \cup N = 1, \ldots a-1, a+1, \ldots b-1, b+1, \ldots n$ and $M \cap N = \phi$. The sum is over all possible $(a, b)$ and $(M, N)$, where $(M, N)$ and $(N, M)$ are not distinguished. The trace is defined as

$$\mathrm{tr}[aMbN] = \langle a|K_M|b]\langle b|K_N|a] + [a|K_M|b\rangle[b|K_N|a\rangle, \tag{4.3}$$

where $K_M = \sum_{i\in M} k_i$. The "half-soft" function $h$ is given by

$$h(a, \{1, 2, \ldots, n\}, b) = \frac{[12]}{\langle 12\rangle}\frac{\langle a|K_{1,2}|3]\langle a|K_{1,3}|4]\cdots\langle a|K_{1,n-1}|n]}{\langle 23\rangle\langle 34\rangle\cdots\langle n-1, n\rangle\langle a1\rangle\langle a2\rangle\cdots\langle an\rangle\langle 1b\rangle\langle nb\rangle} + \mathcal{P}(2, 3, \ldots, n), \tag{4.4}$$

where $K_{1,m} = \sum_{i=1}^{m} k_i$ and $\mathcal{P}(2, 3, \ldots, n)$ represents all permutations keeping the first leg fixed. Throughout this paper we will set $\kappa = 2$.

## V. GRAVITON-GRAVITON OPE FROM SELF-DUAL AMPLITUDES

In this section, we take the 4- and 5-point all plus amplitudes and express them in the conformal primary basis by (modified) Mellin transformation. Then we take the (collinear) OPE limit $(z_{45} \to 0, \bar{z}_{45} \to 0)$ in the 5-point amplitude with the aim of factorizing it into some differential operators acting on the 4-point amplitude at every order in the $(z_{45}, \bar{z}_{45})$ expansion. Let us now closely look at the 4-point amplitude, first in momentum space and then in Mellin space.[6]

[6]For the sake of convenience of the reader we have moved some of the intermediate steps in the calculations to the Appendix. We have refereed to the Appendix in the main text whenever necessary.

### A. 4-Point momentum space amplitude

From (4.2), the 4-point amplitude is given by

$$A_4(1^+, 2^+, 3^+, 4^+) = -\frac{i}{(4\pi)^2 960} B_4, \tag{5.1}$$

where

$$B_4 = \sum_{\substack{1\le a<b\le 4 \\ M,N}} h(a, M, b) h(b, N, a) \mathrm{tr}^3[aMbN]. \tag{5.2}$$

Using the explicit expressions for the trace and the "half-soft" functions, $B_4$ can be easily evaluated and then simplified to get (see Appendix C for details),

$$B_4 = -2^4\left[\frac{\langle 13\rangle\langle 23\rangle([13][23])^3}{\langle 15\rangle^2\langle 25\rangle^2} + (2 \leftrightarrow 3) + (1 \leftrightarrow 3)\right], \tag{5.3}$$

where we have relabeled 4 as 5. In terms of $(\omega, z, \bar{z})$ variables, the above equation becomes

$$B_4 = -2^8\left[\epsilon_1\epsilon_2\frac{\omega_1\omega_2\omega_3^4}{\omega_5^2}\frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2z_{25}^2} + (2\leftrightarrow 3) + (1\leftrightarrow 3)\right].$$

This is the form of the 4-point momentum space amplitude that we use in evaluating the Mellin transform and other manipulations.

### B. 4-Point Mellin amplitude

The modified Mellin transform[7] of the $n$-point amplitude is given by

$$\mathcal{M}_n(\{u_i, z_i, \bar{z}_i, h_i, \bar{h}_i\}) = \prod_{i=1}^{n}\int_0^\infty d\omega_i \omega_i^{\Delta_i - 1} e^{-i\sum_{i=1}^{n}\epsilon_i\omega_i u_i} A_n(\{\omega_i, z_i, \bar{z}_i, \sigma_i\}) \tag{5.4}$$

where $u$ can be thought of as a time coordinate and $\epsilon_i = \pm 1$ for an outgoing (incoming) particle. Note that $A_n(\{\omega_i, z_i, \bar{z}_i, \sigma_i\})$ in (5.4) is the full momentum space amplitude including the momentum conserving delta function. Using (5.4) we now Mellin transform the 4-point momentum space amplitude (5.1). Using the parametrization of 4-point delta function given by (B2), we get the full 4-point Mellin amplitude as

$$\begin{aligned}\mathcal{M}_4(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 5^+_{\Delta_5}) &= \frac{i}{(4\pi)^2 960}2^6\frac{\Gamma(\Delta')}{(i\mathcal{D})^{\Delta'}}\delta(x-\bar{x})\prod_{k=1}^{3}(\epsilon_k\sigma_{k,1})^{\Delta_k - 1}\\ &\quad\times[\mathcal{N}_4 + \mathcal{N}_4(1\leftrightarrow 3) + \mathcal{N}_4(2\leftrightarrow 3)],\end{aligned} \tag{5.5}$$

where $\Delta' = \Delta_1 + \Delta_2 + \Delta_3 + \Delta_5$ and

$$\begin{aligned}\mathcal{N}_4 &= \sigma_{1,1}\sigma_{2,1}\sigma_{3,1}^4\frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2z_{25}^2}\\ \mathcal{D} &= \sum_{k=1}^{3}\sigma_{k,1}u_{k5}.\end{aligned} \tag{5.6}$$

$\mathcal{N}_4(1\leftrightarrow 3)$ and $\mathcal{N}_4(2\leftrightarrow 3)$ corresponds to $\mathcal{N}_4$ with the points (1, 3) and (2, 3) interchanged, respectively. The expressions for $\sigma_{i,j}$ are given in Appendix B. Note that when we interchange the points (1, 2, 3) in $\mathcal{N}_4$, only the first subscript in $\sigma_{i,j}$ changes, second one remains unchanged.

### C. 5-point amplitude in self-dual gravity

The 5-point one-loop all plus helicity stripped amplitude (without the momentum conservation delta function) is given by

$$A_5(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i}{(4\pi)^2 960}B_5, \tag{5.7}$$

where

$$B_5 = \sum_{\substack{1\leq a<b\leq 5\\ M,N}} h(a, M, b)h(b, N, a)\mathrm{tr}^3[aMbN]. \tag{5.8}$$

The above expression consists of 30 distinct terms in total. The expression of $B_5$ has been explicitly computed and simplified in the Appendix D. Its simplified form gives,

[7]We use the modified Mellin transformation [4,65] because the original Mellin transformation [2,3] diverges for graviton scattering amplitudes. Introduction of $u$-dependent phase factors regulate these UV divergence while preserving all the symmetries of the theory. Graviton-graviton OPE can be extracted from the modified Mellin amplitude and since the OPE has no divergence in the $u$ direction, one can get the standard celestial OPE by setting all the $u_i$s to zero at the end. This method has been applied in the past. See for example [18,19].

$$B_5 = -8\left[\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle} + \frac{[24]\langle 13\rangle\langle 35\rangle([13][35])^3}{\langle 24\rangle\langle 12\rangle\langle 14\rangle\langle 25\rangle\langle 45\rangle} + \frac{[15]\langle 23\rangle\langle 34\rangle([23][34])^3}{\langle 15\rangle\langle 21\rangle\langle 25\rangle\langle 14\rangle\langle 54\rangle}\right.$$
$$\frac{[14]\langle 23\rangle\langle 35\rangle([23][35])^3}{\langle 14\rangle\langle 21\rangle\langle 24\rangle\langle 15\rangle\langle 45\rangle} + \frac{[45]\langle 13\rangle\langle 23\rangle([13][23])^3}{\langle 45\rangle\langle 14\rangle\langle 15\rangle\langle 42\rangle\langle 52\rangle} + \frac{[34]\langle 25\rangle\langle 15\rangle([15][25])^3}{\langle 34\rangle\langle 13\rangle\langle 14\rangle\langle 32\rangle\langle 42\rangle}$$
$$\frac{[35]\langle 14\rangle\langle 24\rangle([14][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 32\rangle\langle 52\rangle} + \frac{[12]\langle 34\rangle\langle 35\rangle([34][35])^3}{\langle 12\rangle\langle 41\rangle\langle 42\rangle\langle 15\rangle\langle 25\rangle} + \frac{[12]\langle 35\rangle\langle 45\rangle([35][45])^3}{\langle 12\rangle\langle 31\rangle\langle 32\rangle\langle 14\rangle\langle 24\rangle}$$
$$\left.\frac{[12]\langle 34\rangle\langle 45\rangle([34][45])^3}{\langle 12\rangle\langle 31\rangle\langle 32\rangle\langle 15\rangle\langle 25\rangle}\right] + (1\leftrightarrow 3) + (2\leftrightarrow 3). \tag{5.9}$$

To avoid complication, we will not write down the Mellin transformation for the full 5-point amplitude. Rather, we will first expand the 5-point amplitude around $z_{45}=0$, $\bar{z}_{45}=0$ in momentum space and then Mellin transform the individual terms in that expansion.

### D. Expansion of the 5-point amplitude around $z_{45} = \bar{z}_{45} = 0$ in momentum space

By parametrizing (5.9) in terms of $\{\omega, z, \bar{z}\}$ one may think that there are holomorphic singularities in the limit $z_4 \to z_5$ which goes like $\frac{1}{z_{45}}$. But this is not true. Clubbing together all the twelve singular-looking terms, and rewriting them gives contributions only at leading $\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})$ and higher orders (see Appendix D for details). By parametrizing $\omega_4 = t\omega_P, \omega_5 = (1-t)\omega_P$ we arrange all the terms in (5.9) in the following way:

$$B_5 = -2^7\frac{\omega_P}{t(1-t)}\left(\frac{\bar{z}_{45}}{z_{45}}T_L + T_{\mathcal{O}(1)} + \bar{z}_{45}T_{\bar{z}}\right) + \text{Higher-Order Terms}, \tag{5.10}$$

where

$$T_L = \left[\epsilon_1\frac{z_{12}z_{25}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2 z_{35}^2}\frac{\omega_1\omega_2^4}{\omega_3^2} + \epsilon_2\frac{z_{12}z_{15}\bar{z}_{12}^3\bar{z}_{15}^3}{z_{23}^2 z_{35}^2}\frac{\omega_1^4\omega_2}{\omega_3^2} + \epsilon_3\frac{z_{13}z_{15}\bar{z}_{13}^3\bar{z}_{15}^3}{z_{23}^2 z_{25}^2}\frac{\omega_1^4\omega_3}{\omega_2^2}\right] + \left[\epsilon_1\epsilon_2\frac{z_{13}z_{23}\bar{z}_{13}^3\bar{z}_{23}^3}{z_{15}^2 z_{25}^2}\frac{\omega_1\omega_2\omega_3^4}{\omega_P^3} + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]. \tag{5.11}$$

The expressions for $T_{\mathcal{O}(1)}, T_{\bar{z}}$ and the detailed calculation about how we arrived at these expressions are given in the Appendix D. The point we want to emphasize here is that, (5.10) is the expansion of the 5-point amplitude around $z_{45} = \bar{z}_{45} = 0$ in the momentum space. One should not confuse it with the OPE expansion on the celestial torus, which will be done in the following subsections. The terms $T_L, T_{\mathcal{O}(1)}$, and $T_{\bar{z}}$ contain energy factors $\{\omega_1,\omega_2,\omega_3\}$ which will contribute to the OPE expansion after Mellin transformation. On top of that we have 5-point momentum conserving delta functions as well as other factors in the Mellin integral, all of which will contribute in the OPE limit of the 5-point Mellin amplitude. (5.10) is just a neat way of organizing the 5-point momentum space amplitude, which allows us to easily extract the OPE from the 5-point celestial amplitude.

### E. Mellin transformation of the 5-point amplitude and extracting the graviton-graviton OPE

Let us start with the modified Mellin transformation of $B_5$ given by

$$\tilde{B}_5 = \int_0^\infty \prod_{i=1}^5 d\omega_i\omega_i^{\Delta_i-1}e^{-i\sum_{i=1}^5\epsilon_i\omega_i u_i}B_5\delta^{(4)}\left(\sum_{i=1}^5\epsilon_i\omega_i q_i\right). \tag{5.12}$$

In the above equation for $B_5$, we use the expansion (5.10). Then using the 5-point delta function parametrization given in the Appendix B 2, we can extract each term in the OPE factorization in the Mellin space. We now discuss the terms order-by-order in the OPE expansion in Mellin space.

#### 1. Leading order

For convenience let us take $\epsilon_4 = \epsilon_5 = +1$. Then the leading-order term in (5.12) is given by

$$\tilde{B}_5|_{\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})} = -2^6\frac{\bar{z}_{45}}{z_{45}}B(\Delta_4-1,\Delta_5-1)\frac{\Gamma(\Delta)}{(i\mathcal{D})^\Delta}\delta(x-\bar{x}) \times\prod_{k=1}^3(\epsilon_k\sigma_{k,1})^{\Delta_k-1}[\mathcal{N}_4 + \mathcal{N}_4(2\leftrightarrow 3) + \mathcal{N}_4(1\leftrightarrow 3)], \tag{5.13}$$

where $\Delta = \sum_{i=1}^5\Delta_i$. This has been derived in detail in Appendix E. Finally, taking care of the prefactors, we can write down the Mellin transformation of the complete 5-point amplitude $A_5$ (5.7) at $\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})$,

$$\mathcal{M}_5|_{\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})} = -\frac{i}{(4\pi)^2 960} 2^6 B(\Delta_4 - 1, \Delta_5 - 1) \frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}} \delta(x - \bar{x}) \frac{\bar{z}_{45}}{z_{45}} \prod_{k=1}^{3} (\epsilon_k \sigma_{k,1})^{\Delta_k - 1} \times [\mathcal{N}_4 + \mathcal{N}_4(2 \leftrightarrow 3) + \mathcal{N}_4(1 \leftrightarrow 3)]. \tag{5.14}$$

This gives us the 5-point Mellin amplitude at leading order. In terms of the 4-point Mellin amplitude $\mathcal{M}_4(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 5^+_{\Delta_4+\Delta_5})$ given by (5.5), we can write (5.14) as follows:

$$\mathcal{M}_5(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 4^+_{\Delta_4}, 5^+_{\Delta_5}) = -\frac{\bar{z}_{45}}{z_{45}} B(\Delta_4 - 1, \Delta_5 - 1) \mathcal{M}_4(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 5^+_{\Delta_4+\Delta_5}) + \cdots. \tag{5.15}$$

Thus, at the level of OPE we have

$$G^+_{\Delta_4}(z_4, \bar{z}_4) G^+_{\Delta_5}(z_5, \bar{z}_5) = -\frac{\bar{z}_{45}}{z_{45}} B(\Delta_4 - 1, \Delta_5 - 1) G^+_{\Delta_4+\Delta_5}(z_5, \bar{z}_5) + \cdots. \tag{5.16}$$

This matches with the well known answer [66] and provides a basic sanity check for our calculation.

### 2. $\mathcal{O}(1)$ terms

Now we turn our attention to the $\mathcal{O}(1)$ terms in the 5-point Mellin amplitude. This is one of the main results of our paper. The complete expression for the 5-point Mellin amplitude at $\mathcal{O}(1)$ is given by (E13)

$$\mathcal{M}_5|_{\mathcal{O}(1)} = -\frac{i}{(4\pi)^2 960} 2^5 \frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}} \prod_{i=1}^{3} (\epsilon_i \sigma_{i,1})^{\Delta_i - 1} \sum_{k=0}^{4} B(\Delta_4 + k - 1, \Delta_5 - 1) \mathcal{F}^{(1)}_k(\{\epsilon_i, z_i, \bar{z}_i\}) \delta(x - \bar{x}), \tag{5.17}$$

where $\mathcal{F}^{(1)}_k(\{\epsilon_i, z_i, \bar{z}_i\})$ are some functions of its arguments, but their explicit expressions are not important for OPE factorizations. Now we take the leading conformal soft limit $\Delta_4 \to 1$ in the above equation to get

$$\lim_{\Delta_4 \to 1} (\Delta_4 - 1) \mathcal{M}_5|_{\mathcal{O}(1)} = -\frac{i}{(4\pi)^2 960} 2^5 \frac{\Gamma\left(\sum_{i=1, i\neq 4}^{5} \Delta_i + 1\right)}{(i\mathcal{D})^{\sum_{i=1, i\neq 4}^{5} \Delta_i + 1}} \prod_{i=1}^{3} (\epsilon_i \sigma_{i,1})^{i\lambda_i} \mathcal{F}^{(1)}_0(\{\epsilon_i, z_i, \bar{z}_i\}) \delta(x - \bar{x}). \tag{5.18}$$

Only the $k = 0$ term in the sum in (5.17) survives because, in the $\Delta_4 \to 1$ limit, $B(\Delta_4 + k - 1, \Delta_5 - 1)$ is nonsingular for all $k > 0$.

On the other hand, from the leading soft-graviton theorem we know that

$$\lim_{\Delta_4 \to 1} (\Delta_4 - 1) \mathcal{M}_5|_{\mathcal{O}(1)} = \mathcal{H}^1_{-\frac{3}{2}, \frac{1}{2}}(5) \mathcal{M}_4(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 5^+_{\Delta_5}). \tag{5.19}$$

To make things transparent, we have used $\mathcal{H}$-notations when the soft modes are acting on the Mellin amplitudes as differential operators and the number 5 in the argument of $\mathcal{H}$ denotes that it is a descendant of the 5th conformal graviton primary. The consistency of the two equations (5.18) and (5.19) implies that

$$-\frac{i}{(4\pi)^2 960} 2^5 \frac{\Gamma\left(\sum_{k=1, k\neq 4}^{5} \Delta_k + 1\right)}{(i\mathcal{D})^{\sum_{k=1, k\neq 4}^{5} \Delta_k + 1}} \prod_{k=1}^{3} (\epsilon_k \sigma_{k,1})^{i\lambda_k} \mathcal{F}^{(1)}_0(\{\epsilon_i, z_i, \bar{z}_i\}) \delta(x - \bar{x}) = \mathcal{H}^1_{-\frac{3}{2}, \frac{1}{2}}(5) \mathcal{M}_4(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 5^+_{\Delta_5}). \tag{5.20}$$

Now, we can replace $\Delta_5$ by $\Delta_4 + \Delta_5 - 1$ in (5.20) and then use it in (5.17) to get

$$\mathcal{M}_5|_{\mathcal{O}(1)} = B(\Delta_4 - 1, \Delta_5 - 1) \mathcal{H}^1_{-\frac{3}{2}, \frac{1}{2}}(5) \mathcal{M}_4(1^+_{\Delta_1}, 2^+_{\Delta_2}, 3^+_{\Delta_3}, 5^+_{\Delta_4+\Delta_5-1}) - \frac{i}{(4\pi)^2 960} 2^5 \frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}} \prod_{i=1}^{3} (\epsilon_i \sigma_{i,1})^{\Delta_i - 1} \sum_{k=1}^{4} B(\Delta_4 + k - 1, \Delta_5 - 1) \mathcal{F}^{(1)}_k(\{\epsilon_i, z_i, \bar{z}_i\}) \delta(x - \bar{x}). \tag{5.21}$$

Here we have replaced the $\mathcal{F}_0^{(1)}$ dependent term in (5.17) in terms of a soft graviton mode acting on the 4-point amplitude. Let us now repeat the same procedure for $\mathcal{F}_1^{(1)}$.

By taking the subleading conformal soft limit $\Delta_4 \to 0$ in (5.21), we get

$$\lim_{\Delta_4\to 0}\Delta_4\mathcal{M}_5|_{\mathcal{O}(1)} = -(\Delta_5-2)\mathcal{H}^1_{-\frac{3}{2},\frac{1}{2}}(5)\mathcal{M}_4(1^+_{\Delta_1},2^+_{\Delta_2},3^+_{\Delta_3},5^+_{\Delta_5-1}) - \frac{i}{(4\pi)^2 960}2^5\frac{\Gamma(\sum_{k=1,k\neq 4}^5\Delta_k)}{(i\mathcal{D})^{\sum_{k=1,k\neq 4}^5\Delta_k}}\prod_{i=1}^3(\epsilon_i\sigma_{i,1})^{\Delta_i-1}\mathcal{F}_1^{(1)}(\{\epsilon_i,z_i,\bar{z}_i\})\delta(x-\bar{x}). \tag{5.22}$$

Now, from subleading soft graviton theorem we know that

$$\lim_{\Delta_4\to 0}\Delta_4\mathcal{M}_5|_{\mathcal{O}(1)} = -\mathcal{H}^0_{-1,1}(5)\mathcal{H}^1_{-\frac{1}{2},-\frac{1}{2}}(5)\mathcal{M}_4(1^+_{\Delta_1},2^+_{\Delta_2},3^+_{\Delta_3},5^+_{\Delta_5-1}). \tag{5.23}$$

Again, consistency of the two equations (5.22) and (5.23) gives us the function $\mathcal{F}_1^{(1)}$ in terms of the leading and subleading soft modes. Substituting this back in (5.21) results in

$$\mathcal{M}_5|_{\mathcal{O}(1)} = \frac{\Gamma(\Delta_4+1)}{\Gamma(\Delta_4)}B(\Delta_4-1,\Delta_5-1)\mathcal{H}^1_{-\frac{3}{2},\frac{1}{2}}(5)\mathcal{M}_4(1^+_{\Delta_1},2^+_{\Delta_2},3^+_{\Delta_3},5^+_{\Delta_4+\Delta_5-1}) + B(\Delta_4,\Delta_5-1)\mathcal{H}^0_{-1,1}(5)\left(-\mathcal{H}^1_{-\frac{1}{2},-\frac{1}{2}}(5)\mathcal{M}_4(1^+_{\Delta_1},2^+_{\Delta_2},3^+_{\Delta_3},5^+_{\Delta_4+\Delta_5-1})\right) - \frac{i}{(4\pi)^2 960}2^5\frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}}\prod_{i=1}^3(\epsilon_i\sigma_{i,1})^{\Delta_i-1}\sum_{k=2}^4 B(\Delta_4+k-1,\Delta_5-1)\mathcal{F}_k^{(1)}(\{\epsilon_i,z_i,\bar{z}_i\})\delta(x-\bar{x}). \tag{5.24}$$

We continue this process till all the $\mathcal{F}_k^{(1)}$'s have been replaced by descendant correlation functions of the soft modes. From the above equation (5.24), it is clear that to replace all the $\mathcal{F}_k^{(1)}$'s by the descendant correlation functions of the soft modes, we have to go till sub$^4$leading order in the soft limits of $\Delta_4$. We only write the final result here which is given by

$$\mathcal{M}_5(1^+_{\Delta_1},2^+_{\Delta_2},3^+_{\Delta_3},4^+_{\Delta_4},5^+_{\Delta_5})|_{\mathcal{O}(1)} = \sum_{k=0}^4\frac{1}{(4-k)!}\frac{\Gamma(\Delta_4+4)}{\Gamma(\Delta_4+k)}B(\Delta_4+k-1,\Delta_5-1) \times\mathcal{H}^{1-k}_{\frac{k-3}{2},\frac{k+1}{2}}(5)\left(\mathcal{H}^1_{-\frac{1}{2},-\frac{1}{2}}(5)\right)^k\mathcal{M}_4(1^+_{\Delta_1},2^+_{\Delta_2},3^+_{\Delta_3},5^+_{\Delta_4+\Delta_5-1}). \tag{5.25}$$

Now that we have factorized the $\mathcal{O}(1)$ terms in the 5-point Mellin amplitude completely in terms of soft modes acting on the 4-point amplitude, we can easily extract the $\mathcal{O}(1)$ graviton graviton OPE from the above equation. It is given by

$$G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\mathcal{O}(1)} = \sum_{k=0}^4\frac{1}{(4-k)!}\frac{\Gamma(\Delta_4+4)}{\Gamma(\Delta_4+k)}B(\Delta_4+k-1,\Delta_5-1) \times H^{1-k}_{\frac{k-3}{2},\frac{k+1}{2}}\left(H^1_{-\frac{1}{2},-\frac{1}{2}}\right)^k G^+_{\Delta_4+\Delta_5-1}(z_5,\bar{z}_5). \tag{5.26}$$

We can rewrite (5.26) using the null states of MHV-sector. From (H1), it is clear that all the soft modes $H^{1-k}_{\frac{k-3}{2},\frac{k+1}{2}}$ with $k=1,\ldots,4$ can be replaced by the MHV null states $\{\Phi_k, k=1,\ldots,4\}$. Thus, (5.25) in terms of the $\mathcal{O}(1)$ MHV null states (H1), becomes

$$G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\mathcal{O}(1)} = B(\Delta_4-1,\Delta_5-1)H^1_{-\frac{3}{2},\frac{1}{2}}G^+_{\Delta_4+\Delta_5-1}(z_5,\bar{z}_5) + \sum_{k=1}^4\frac{1}{(4-k)!}\frac{\Gamma(\Delta_4+4)}{\Gamma(\Delta_4+k)}B(\Delta_4+k-1,\Delta_5-1)\Phi_k(\Delta_4+\Delta_5). \tag{5.27}$$

Thus, we see that, the $\mathcal{O}(1)$ terms in the self-dual OPE between two positive helicity outgoing gravitons can completely be written in terms of the $\mathcal{O}(1)$ MHV OPE and the $\mathcal{O}(1)$ null states of the MHV sector. We can also see that the $\mathcal{O}(1)$ OPE in the self-dual theory gets contribution from the descendants of the conformally soft operators up to $\Delta = -3$, i.e., $H^{-3}(z,\bar{z})$. This is consistent with what we found in [39] based on $w_{1+\infty}$ symmetry. This is somewhat surprising given that $w_{1+\infty}$ has an infinite number of soft currents. We further discuss this in Sec. VI.

Now, as discussed in Appendix H, we can define a new basis for MHV null states instead of $\Phi_k$'s. This new basis is given by (H3). For our convenience, let us write Eq. (H3) here again,

$$\Omega_k(\Delta) = \sum_{n=1}^{k} \frac{1}{(k-n)!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta+n-2)}\Phi_n(\Delta). \tag{5.28}$$

This basis has nice transformation properties under the $w$-algebra [39], reviewed in Appendix G. Represented in terms of this new $\Omega$-basis, the graviton-graviton OPE (5.27) takes a very simple form,

$$\begin{aligned} &G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\mathcal{O}(1)} \\ &= B(\Delta_4-1,\Delta_5-1)H^1_{-\frac{3}{2},\frac{1}{2}}G^+_{\Delta_4+\Delta_5-1}(z_5,\bar{z}_5) \\ &\quad + \sum_{k=1}^{4} B(\Delta_4+k-1,\Delta_5-1)\Omega_k(\Delta_4+\Delta_5). \end{aligned} \tag{5.29}$$

### 3. $\mathcal{O}(\bar{z}_{45})$ term

The soft modes that appear at order $\bar{z}_{45}$ from the $w$-algebra are given by

$$H^k_{-\frac{k+2}{2},-\frac{k}{2}}, \quad k = 1,0,-1,\ldots. \tag{5.30}$$

Now, like the $\mathcal{O}(1)$ OPE we can factorize the $\mathcal{O}(\bar{z}_{45})$ terms from the 5-point amplitude using the soft limits and $w$-modes. The crucial difference from $\mathcal{O}(1)$ is that, now we have to go one order higher in the soft limits than $\mathcal{O}(1)$. We start by writing the $\mathcal{O}(\bar{z}_{45})$ term of the 5-point Mellin amplitude given by [see (E17)]

$$\mathcal{M}_5|_{\mathcal{O}(\bar{z}_{45})} = -\frac{i}{(4\pi)^2 960}2^5\frac{\Gamma(\Delta)}{(i\mathcal{D})^\Delta}\prod_{i=1}^{3}(\epsilon_i\sigma_{i,1})^{\Delta_i-1}\sum_{k=1}^{5}B(\Delta_4+k-1,\Delta_5-1)\mathcal{F}^{(\bar{z})}_k(\{\epsilon_i,z_i,\bar{z}_i\}). \tag{5.31}$$

One can easily see from (5.31) that, to factorize the 5-point Mellin amplitude completely, i.e., to replace all the functions $\mathcal{F}^{(\bar{z})}_k(\{\epsilon_i,z_i,\bar{z}_i\})$ by the descendant correlation functions of soft modes, we have to continue taking the soft limits in $\Delta_4$ till we reach $\Delta_4 \to -4$. Thus, the highest soft modes that can appear in the OPE at $\mathcal{O}(\bar{z}_{45})$ are given by $H^{-4}_{1,2}$. We have discussed how to factorize the amplitude at $\mathcal{O}(1)$ in terms of the descendant correlators of the soft modes in the previous section in detail. One has to repeat the same procedure for $\mathcal{O}(\bar{z}_{45})$ as well. Without going into much detail we directly write the $\mathcal{O}(\bar{z}_{45})$ OPE which is given by

$$\begin{aligned} G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\mathcal{O}(\bar{z}_{45})} &= G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\text{MHV at }\mathcal{O}(\bar{z}_{45})} \\ &\quad + \sum_{k=1}^{4}\frac{1}{(n-k)!}\frac{\Gamma(\Delta_4+n+1)}{\Gamma(\Delta_4+k+1)}B(\Delta_4+k,\Delta_5-1)\Psi_k(\Delta_4+\Delta_5+1), \end{aligned} \tag{5.32}$$

where

$$G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\text{MHV at }\mathcal{O}(\bar{z}_{45})} = B(\Delta_4-1,\Delta_5-1)\left[\frac{\Delta_4-1}{\Delta_4+\Delta_5-2}H^0_{-1,0}\left(-H^1_{-\frac{1}{2},-\frac{1}{2}}\right)+\Delta_4 H^1_{-\frac{3}{2},-\frac{1}{2}}\right]G^+_{\Delta_4+\Delta_5-1} \tag{5.33}$$

and $\Psi_k(\Delta_4+\Delta_5+1)$ is given by

$$\begin{aligned} \Psi_k(\Delta_4+\Delta_5+1) = \Bigg[&H^{-k}_{\frac{k-2}{2},\frac{k}{2}}(-H^1_{-\frac{1}{2},-\frac{1}{2}})^{k+1} - \frac{(-1)^k}{k!}\frac{\Gamma(\Delta_4+\Delta_5+k-1)}{\Gamma(\Delta_4+\Delta_5-1)}H^0_{-1,0}\left(-H^1_{-\frac{1}{2},-\frac{1}{2}}\right) \\ &-(-1)^k\frac{k}{(k+1)!}\frac{\Gamma(\Delta_4+\Delta_5+k-1)}{\Gamma(\Delta_4+\Delta_5-2)}H^1_{-\frac{3}{2},-\frac{1}{2}}\Bigg]G^+_{\Delta_4+\Delta_5-1}. \end{aligned} \tag{5.34}$$

In terms of the new basis defined in (H4), the above OPE can again be written in a very nice and simple form given by

$$
\begin{aligned}
&G^+_{\Delta_4}(z_4,\bar z_4)G^+_{\Delta_5}(z_5,\bar z_5)|_{\mathcal{O}(\bar z_{45})} \\
&= G^+_{\Delta_4}(z_4,\bar z_4)G^+_{\Delta_5}(z_5,\bar z_5)|_{\mathrm{MHV\ at}\,\mathcal{O}(\bar z_{45})} \\
&\quad + \bar z_{45}\sum_{k=1}^{4} B(\Delta_4+k,\Delta_5-1)\Pi_k(\Delta_4+\Delta_5+1). \qquad (5.35)
\end{aligned}
$$

Thus we see that the $\mathcal{O}(\bar z_{45})$ terms in the OPE again truncate at $H^{-4}(z,\bar z)$.

## VI. DISCUSSION

Operator product expansion plays a very important role in any quantum field theory and therefore it is important to understand the structure of OPE in the celestial CFTs. In its current formulation, celestial CFTs differ from more conventional CFTs in many ways. The primary difference is that the spectrum of the operator dimensions in celestial CFTs is not bounded from below. Taken at face value, this implies that the number of descendants that can appear at any given order of the celestial OPE can be infinite. However, this is not a very desirable feature and warrants further study.

In this paper, we have undertaken the task of computing the celestial OPE of two positive helicity outgoing gravitons in the quantum self-dual gravity. It is known that the self-dual gravity enjoys $w$ invariance. Therefore, one should be able to express the OPE in terms of $w$ descendants of the graviton primary. This is what we have found. However, the most surprising fact which comes out of our study is that at any given order the OPE contains only a finite number of $w$ descendants. Therefore, the self-dual gravity behaves like any other CFT with a spectrum of operator dimensions bounded from below.

This raises some interesting questions. For example, we know that the HSA contains an infinite tower of holomorphic currents $H^k_n(z)$ with $k$ going from 1 to $-\infty$. Our calculation shows that in the self-dual theory at $\mathcal{O}(1)$ and at $\mathcal{O}(\bar z)$ the list of $w$ descendants truncate at $k=-3$ and $k=-4$, respectively. However, this is somewhat unnatural given the fact that the currents $H^{-3}_n(z)$ and $H^{-4}_n(z)$ do not play any distinguished role in the algebra. Therefore, it is natural to wonder if there are other $w$ invariant theories where the truncation occurs at other values of $k$. The answer is *yes*. In fact in our earlier work [39] we wrote down the general structure of $w$ invariant OPEs which can be derived using the representation theory of $w$ algebra. In [39] we found that at $\mathcal{O}(z^0\bar z^0)$ the OPE can truncate at any integer value of $k=1,0,-1,-2,\ldots$. For example, in the MHV sector it truncates at $k=1$ and in the self-dual gravity theory it truncates at $k=-3$. But truncation at other values of $k$ also gives us $w$ invariant OPE. This is true also for other subleading-order terms in the OPE. Therefore, the value of $k$ at which the OPE truncates at any particular subleading order is not determined by the $w$ symmetry. The Lagrangian description of the infinite family of $w$ invariant theories remains as an outstanding problem.

Before we end, we would like to point out that truncation means that the self-dual theory in many ways behave like theories with operator dimensions bounded from below. So it is very likely that the self-dual theory and the (tree-level) MHV sector of GR can be reformulated in terms of celestial primary operators with dimensions strictly bounded from below. Interesting proposals along this line has been put forward in [67,68].[8] It will be fascinating if they can be applied to the present problem.

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## APPENDIX A: BREIF REVIEW OF CELESTIAL OR MELLIN AMPLITUDES FOR MASSLESS PARTICLES

The celestial or Mellin amplitude for massless particles in four dimensions is defined as the Mellin transformation of the $S$-matrix element, given by [2,3]

$$
\mathcal{M}_n(\{z_i,\bar z_i,h_i,\bar h_i\}) = \prod_{i=1}^{n}\int_0^\infty d\omega_i\omega_i^{\Delta_i-1}A_n(\{\omega_i,z_i,\bar z_i,\sigma_i\}), \qquad (A1)
$$

where $\sigma_i$ denotes the helicity of the $i$th particle and the on shell momenta are parametrized by (2.1). The scaling dimensions $(h_i,\bar h_i)$ are defined as

$$
h_i = \frac{\Delta_i+\sigma_i}{2}, \qquad \bar h_i = \frac{\Delta_i-\sigma_i}{2}. \qquad (A2)
$$

Under the Lorentz transformation (2.2), the Mellin amplitude $\mathcal{M}_n$ transforms as

[8]In conventional CFTs we get a finite number of descendants at every order of the OPE because the set of conformal dimensions of primary operators is bounded from below. The OPE in the self-dual gravity behaves in the same way, i.e., we get only a finite number of $w$ descendants at every order. The basis proposed in the Refs. [67,68] consists of operators whose dimensions are $\Delta=0,-1,-2,\ldots$. This is a basis where the set of operator dimensions is bounded from above. So, this is exactly the opposite of what happens in conventional CFTs but, this comes very close. We leave further study of this potential connection to future works.

$$\mathcal{M}_n(\{z_i,\bar{z}_i,h_i,\bar{h}_i\}) = \prod_{i=1}^{n}\frac{1}{(cz_i+d)^{2h_i}}\frac{1}{(\bar{c}\bar{z}_i+\bar{d})^{2\bar{h}_i}}\mathcal{M}_n\left(\frac{az_i+b}{cz_i+d},\frac{\bar{a}\bar{z}_i+\bar{b}}{\bar{c}\bar{z}_i+\bar{d}},h_i,\bar{h}_i\right). \tag{A3}$$

This is the familiar transformation law for the correlation function of primary operators of weight $(h_i,\bar{h}_i)$ in a 2D CFT under the global conformal group.

In Einstein gravity, the Mellin amplitude as defined in (A1) usually diverges. This divergence can be regulated by defining a modified Mellin amplitude as [4,65]

$$\mathcal{M}_n(\{u_i,z_i,\bar{z}_i,h_i,\bar{h}_i\}) = \prod_{i=1}^{n}\int_0^\infty d\omega_i\omega_i^{\Delta_i-1}e^{-i\sum_{i=1}^{n}\epsilon_i\omega_i u_i}A_n(\{\omega_i,z_i,\bar{z}_i,\sigma_i\}), \tag{A4}$$

where $u$ can be thought of as a time coordinate and $\epsilon_i=\pm1$ for an outgoing (incoming) particle. Under (Lorentz) conformal transformation the modified Mellin amplitude $\mathcal{M}_n$ transforms as

$$\mathcal{M}_n(\{u_i,z_i,\bar{z}_i,h_i,\bar{h}_i\}) = \prod_{i=1}^{n}\frac{1}{(cz_i+d)^{2h_i}}\frac{1}{(\bar{c}\bar{z}_i+\bar{d})^{2\bar{h}_i}}\mathcal{M}_n\left(\frac{u_i}{|cz_i+d|^2},\frac{az_i+b}{cz_i+d},\frac{\bar{a}\bar{z}_i+\bar{b}}{\bar{c}\bar{z}_i+\bar{d}},h_i,\bar{h}_i\right). \tag{A5}$$

Under global space-time translation, $u\to u+A+Bz+\bar{B}\,\bar{z}+Cz\bar{z}$, the modified Mellin amplitude is invariant, i.e.,

$$\mathcal{M}_n(\{u_i+A+Bz_i+\bar{B}\bar{z}_i+Cz_i\bar{z}_i,z_i,\bar{z}_i,h_i,\bar{h}_i\}) = \mathcal{M}_n(\{u_i,z_i,\bar{z}_i,h_i,\bar{h}_i\}). \tag{A6}$$

Now in order to make manifest the conformal nature of the dual theory living on the celestial sphere it is useful to write the (modified) Mellin amplitude as a correlation function of conformal primary operators. So let us define a generic conformal primary operator as

$$\phi^{\epsilon}_{h,\bar{h}}(z,\bar{z}) = \int_0^\infty d\omega\omega^{\Delta-1}a(\epsilon\omega,z,\bar{z},\sigma), \tag{A7}$$

where $\epsilon=\pm1$ for an annihilation (creation) operator of a massless particle of helicity $\sigma$. Under (Lorentz) conformal transformation the conformal primary transforms like a primary operator of scaling dimension $(h,\bar{h})$,

$$\phi'^{\epsilon}_{h,\bar{h}}(z,\bar{z}) = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\,\bar{z}+\bar{d})^{2\bar{h}}}\phi^{\epsilon}_{h,\bar{h}}\left(\frac{az+b}{cz+d},\frac{\bar{a}\,\bar{z}+\bar{b}}{\bar{c}\,\bar{z}+\bar{d}}\right). \tag{A8}$$

Similarly in the presence of the time coordinate $u$ we have

$$\phi^{\epsilon}_{h,\bar{h}}(u,z,\bar{z}) = \int_0^\infty d\omega\omega^{\Delta-1}e^{-i\epsilon\omega u}a(\epsilon\omega,z,\bar{z},\sigma). \tag{A9}$$

Under (Lorentz) conformal transformations,

$$\phi'^{\epsilon}_{h,\bar{h}}(u,z,\bar{z}) = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\,\bar{z}+\bar{d})^{2\bar{h}}}\times\phi^{\epsilon}_{h,\bar{h}}\left(\frac{u}{|cz+d|^2},\frac{az+b}{cz+d},\frac{\bar{a}\,\bar{z}+\bar{b}}{\bar{c}\,\bar{z}+\bar{d}}\right). \tag{A10}$$

In terms of (A7), the Mellin amplitude can be written as the correlation function of conformal primary operators,

$$\mathcal{M}_n = \left\langle\prod_{i=1}^{n}\phi^{\epsilon_i}_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle. \tag{A11}$$

Similarly using (A9), the modified Mellin amplitude can be written as

$$\mathcal{M}_n = \left\langle\prod_{i=1}^{n}\phi^{\epsilon_i}_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle. \tag{A12}$$

### 1. Comments on notation in the paper

Note that the conformal primaries carry an extra index $\epsilon$ which distinguishes between an incoming and an outgoing particle. In this paper, for notational simplicity, we omit this additional index unless this plays an important role. So in most places we simply write the (modified) Mellin amplitude as

$$\mathcal{M}_n = \left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(z_i,\bar{z}_i)\right\rangle \tag{A13}$$

or

$$\mathcal{M}_n = \left\langle\prod_{i=1}^{n}\phi_{h_i,\bar{h}_i}(u_i,z_i,\bar{z}_i)\right\rangle. \tag{A14}$$

Similarly in many places in the paper we denote a graviton primary of weight $\Delta = h + \bar{h}$ by $G^{\sigma}_{\Delta}$ where $\sigma = \pm 2$ is the helicity ($= h - \bar{h}$). Since we are considering pure gravity, we can further simplify the notation to $G^{\pm}_{\Delta}$ by omitting the 2.

## APPENDIX B: PARAMETRIZATION OF THE DELTA FUNCTIONS

In this appendix, we parametrize the 4-point and 5-point delta functions which will be convenient for our purpose of extracting the OPE.

### 1. 4-Point delta function

In (2, 2) split signature, the parametrization of the null momentum ($p_i$) for $i$th massless particle in terms of $(\omega_i, z_i, \bar{z}_i)$ is given by

$$p_i = \omega_i\{1 + z_i\bar{z}_i, z_i + \bar{z}_i, (z_i - \bar{z}_i), 1 - z_i\bar{z}_i\}, \qquad p_i^2 = 0. \tag{B1}$$

This allows us to write down the 4-point momentum conserving delta function in the following way which is more convenient for us:

$$\begin{aligned}\delta^{(4)}\left(\sum_{i=1,\neq 4}^{5}\epsilon_i p_i\right) &= \frac{1}{4}\delta\left(\sum_{i=1,\neq 4}^{5}\epsilon_i\omega_i\right)\delta\left(\sum_{i=1}^{3}\epsilon_i\omega_i z_{i5}\right) \\ &\quad\times\delta\left(\sum_{i=1}^{3}\epsilon_i\omega_i\bar{z}_{i5}\right)\delta\left(\sum_{i=1}^{3}\epsilon_i\omega_i z_{i5}\bar{z}_{i5}\right) \\ &= \epsilon_1\epsilon_2\epsilon_3\epsilon_5\frac{1}{4\omega_5}\delta(\omega_1 - \omega_1^*)\delta(\omega_2 - \omega_2^*) \\ &\quad\times\delta(\omega_3 - \omega_3^*)\delta(x - \bar{x}),\end{aligned} \tag{B2}$$

where $\epsilon_i = \pm 1$ for outgoing (incoming) particle and

$$\omega_i^* = \epsilon_5\omega_5\epsilon_i\sigma_{i,1}, \tag{B3}$$

$$\sigma_{1,1} = -\frac{z_{25}\bar{z}_{35}}{z_{12}\bar{z}_{13}}, \tag{B4}$$

$$\sigma_{2,1} = \frac{z_{15}\bar{z}_{35}}{z_{12}\bar{z}_{23}}, \tag{B5}$$

$$\sigma_{3,1} = -\frac{z_{25}\bar{z}_{15}}{z_{23}\bar{z}_{13}}, \tag{B6}$$

$$x = z_{12}z_{35}\bar{z}_{13}\bar{z}_{25}, \qquad \bar{x} = z_{13}z_{25}\bar{z}_{12}\bar{z}_{35}. \tag{B7}$$

The $\sigma_{i,1}$'s defined above satisfy the following identities with the support of $\delta(x - \bar{x})$,

$$\sigma_{1,1} + \sigma_{2,1} + \sigma_{3,1} + 1 = 0, \tag{B8}$$

$$z_{15}\sigma_{1,1} + z_{25}\sigma_{2,1} + z_{35}\sigma_{3,1} = 0, \tag{B9}$$

$$\bar{z}_{15}\sigma_{1,1} + \bar{z}_{25}\sigma_{2,1} + \bar{z}_{35}\sigma_{3,1} = 0. \tag{B10}$$

This representation for the 4-point delta function and the properties of $\sigma_{i,1}$'s will be useful in extracting the OPE. Note that in this delta function representation, we have indexed the four particles by 1, 2, 3, and 5 because to extract the OPE, we take the $4 \to 5$ OPE limit in the 5-point Mellin amplitude and then factorize it in terms of the 4-point Mellin amplitude now indexed by 1, 2, 3, 5. This is a notation that we followed throughout the paper.

### 2. 5-point delta function

We now write down the representation for the delta function for five particles. For concreteness, we take $\epsilon_4 = \epsilon_5 = +1$. Since we are interested in the OPE limit $4 \to 5$, it is convenient to use the following parametrization:

$$\omega_4 = t\omega_P, \qquad \omega_5 = (1 - t)\omega_P, \tag{B11}$$

in representing the 5-point delta function. For the case of $n = 5$ particles in four spacetime dimensions we have four constraint equations coming from the four components of the energy momentum conserving equations. We can solve these four constraint equations for three energy variables $\{\omega_1, \omega_2, \omega_3\}$ in terms of $\omega_4$ and $\omega_5$. Thus, the representation of the 5-point delta function which is better suited for our purposes of performing the OPE decomposition of the Mellin amplitude in the (4, 5) channel, is given by [19][9]

$$\begin{aligned}\delta^{(4)}\left(\sum_{i=1}^{5}\epsilon_i\omega_i q_i\right) &= \frac{1}{4\omega_P}\delta(\omega_1 - \omega_1^*)\delta(\omega_2 - \omega_2^*)\delta(\omega_3 - \omega_3^*) \\ &\quad\times\delta\left(x - \bar{x} - tz_{45}\left(\frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}}\right)\right. \\ &\quad - t\bar{z}_{45}\left(\frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}}\right) \\ &\quad\left. + tz_{45}\bar{z}_{45}\left(\frac{x}{z_{35}\bar{z}_{25}} - \frac{\bar{x}}{z_{25}\bar{z}_{35}}\right)\right),\end{aligned} \tag{B12}$$

where for $i = \{1, 2, 3\}$ we have

$$\begin{aligned}\omega_i^* &= \omega_P\tilde{\omega}_i^*, \\ \tilde{\omega}_i^* &= \epsilon_i(\sigma_{i,1} + tz_{45}\sigma_{i,2} + t\bar{z}_{45}\sigma_{i,3} + tz_{45}\bar{z}_{45}\sigma_{i,4}),\end{aligned} \tag{B13}$$

[9] Please note that in [19] the OPE factorization has been done starting from the 6-point Mellin amplitude whereas in this paper it is done starting from the 5-point amplitude. Thus, in parametrizing the 5-point delta function in this paper, we have used the same methodology which was used for 6-point delta function in [19].

and the $\sigma_{i,1}, x, \bar{x}$ are given by (B3)–(B7). We also have

$$\sigma_{i,2} = \frac{\partial \sigma_{i,1}}{\partial z_5}, \qquad \sigma_{i,3} = \frac{\partial \sigma_{i,1}}{\partial \bar{z}_5}, \qquad \sigma_{i,4} = \frac{\partial \sigma_{i,1}}{\partial z_5 \partial \bar{z}_5}, \qquad \forall \;\; i = 1, 2, 3. \tag{B14}$$

## APPENDIX C: SIMPLIFICATION OF THE 4-POINT AMPLITUDE

In this appendix, we simplify the 4-point self-dual one-loop amplitude in momentum space which is used in Sec. V. We start with Eq. (5.2) for the 4-point amplitude,

$$\begin{aligned} B_4 &= \sum_{\substack{1 \le a < b \le 4 \\ M,N}} h(a, M, b) h(b, N, a) \mathrm{tr}^3[aMbN] \\ &= h(1,3,2)h(2,4,1)\mathrm{tr}^3[1324] + h(1,2,3)h(3,4,1)\mathrm{tr}^3[1234] + h(1,2,4)h(4,3,1)\mathrm{tr}^3[1243] \\ &\quad + h(2,1,3)h(3,4,2)\mathrm{tr}^3[2134] + h(2,1,4)h(4,3,2)\mathrm{tr}^3[2143] + h(3,1,4)h(4,2,3)\mathrm{tr}^3[3142]. \end{aligned}$$

The trace function is given by

$$\mathrm{tr}[aMbN] = \langle a|K_M|b] \langle b|K_N|a] + [a|K_M|b\rangle [b|K_N|a\rangle. \tag{C1}$$

For $M = \{i\}, N = \{l\}$ we have

$$\begin{aligned} \mathrm{tr}[aibl] &= \langle a|k_i|b] \langle b|k_l|a] + [a|k_i|b\rangle [b|k_l|a\rangle \\ &= \langle ai\rangle [ib] \langle bl\rangle [la] + \langle bi\rangle [ia] \langle al\rangle [lb]. \end{aligned} \tag{C2}$$

From the above equation we can see that $\mathrm{tr}[aibl] = \mathrm{tr}[ialb]$. Using this property of the trace function and the expression for the half-soft function,

$$h(a, i, b) = \frac{1}{\langle ai\rangle^2 \langle ib\rangle^2}, \tag{C3}$$

(C1) can be simplified as

$$\begin{aligned} B_4 = 2(&h(1,3,2)h(2,4,1)\mathrm{tr}^3[1324] + h(1,2,3) \\ &\times h(3,4,1)\mathrm{tr}^3[1234] + h(1,2,4)h(4,3,1)\mathrm{tr}^3[1243]). \end{aligned} \tag{C4}$$

Now, using the momentum conservation for four particles in the trace functions (C2) and the explicit expressions of the half-soft functions (C3), (C4) finally gives

$$B_4 = -2^4 \left( \frac{\langle 13\rangle \langle 23\rangle ([13][23])^3}{\langle 14\rangle^2 \langle 24\rangle^2} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right). \tag{C5}$$

As mentioned earlier, since we will index the four particles as 1, 2, 3, 5, relabeling 4 as 5 in the above expression gives the following form of the 4-point amplitude in momentum space,

$$B_4 = -2^4 \left( \frac{\langle 13\rangle \langle 23\rangle ([13][23])^3}{\langle 15\rangle^2 \langle 25\rangle^2} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right). \tag{C6}$$

## APPENDIX D: SIMPLIFICATION OF THE 5-POINT AMPLITUDE

Similar to what was done for the 4-point case, we will now simplify the 5-point self-dual one loop amplitude in momentum space which is used in Sec. V C by considering Eq. (5.8),

$$\begin{aligned} B_5 &= \sum_{\substack{1 \le a < b \le 5 \\ M,N}} h(a, M, b) h(b, N, a) \mathrm{tr}^3[aMbN] \\ &= h(1,M,2)h(2,N,1)\mathrm{tr}^3[1M2N] + h(1,M,3)h(3,N,1)\mathrm{tr}^3[1M3N] + h(1,M,4)h(4,N,1)\mathrm{tr}^3[1M4N] \\ &\quad + h(1,M,5)h(5,N,1)\mathrm{tr}^3[1M5N] + h(2,M,3)h(3,N,2)\mathrm{tr}^3[2M3N] + h(2,M,4)h(4,N,2)\mathrm{tr}^3[2M4N] \\ &\quad + h(2,M,5)h(5,N,2)\mathrm{tr}^3[2M5N] + h(3,M,4)h(4,N,3)\mathrm{tr}^3[3M4N] + h(3,M,5)h(5,N,3)\mathrm{tr}^3[3M5N] \\ &\quad + h(4,M,5)h(5,N,4)\mathrm{tr}^3[4M5N]. \end{aligned} \tag{D1}$$

The two sets $M$ and $N$ are such that $M \cup N = 1, \ldots a-1, a+1, \ldots b-1, b+1, \ldots n$, and $M \cap N = \phi$ and the sum is over all possible $a$, $b$ and sets $(M, N)$, where $(M, N)$ and $(N, M)$ are not distinguished. For 5-point amplitudes, with $M = \{i, j\}, N = \{l\}$, the trace function given by

$$\mathrm{tr}[aMbN] = \langle a|K_M|b] \langle b|K_N|a] + [a|K_M|b\rangle [b|K_N|a\rangle \tag{D2}$$

becomes

$$\begin{aligned}\mathrm{tr}[a\{i+j\}b\{l\}] &= \langle a|k_i+k_j|b]\langle b|k_l|a] + [a|k_i+k_j|b\rangle[b|k_l|a\rangle \\ &= \mathrm{tr}[aibl] + \mathrm{tr}[ajbl] \\ &= (\langle ai\rangle[ib] + \langle aj\rangle[jb])\langle bl\rangle[la] + (\langle bi\rangle[ia] + \langle bj\rangle[ja])\langle al\rangle[lb]. \end{aligned} \tag{D3}$$

Now, using momentum conservation in the spinor notation,

$$\begin{aligned}&\langle ai(\neq\{a,b,j,k\})\rangle[i(\neq\{a,b,j,k\})b] + \langle aj(\neq\{a,b,i,k\})\rangle[j(\neq\{a,b,j,k\})b] \\ &\quad + \langle ak(\neq\{a,b,i,j\})\rangle[k(\neq\{a,b,i,j\})b] = 0\end{aligned} \tag{D4}$$

one can show that,

$$\mathrm{tr}[a\{i+j\}b\{l\}] = -2\langle al\rangle[al]\langle bl\rangle[bl],$$

where each label is different. Thus we see that $\mathrm{tr}[a\{i+j\}b\{l\}]$ is independent of $\{i,j\}$. The half soft functions needed for the simplification of the 5-point amplitude are given by

$$h(a,\{i,j\},b) = \frac{[ij]}{\langle ij\rangle\langle ai\rangle\langle aj\rangle\langle ib\rangle\langle jb\rangle}, \tag{D5}$$

$$h(a,\{i\},b) = \frac{1}{\langle ai\rangle^2\langle ib\rangle^2}. \tag{D6}$$

Thus, we see that $h(a,\{i,j\},b) = h(a,\{j,i\},b)$.

Now, using the explicit form of the trace and half-soft functions in terms of spinor helicity brackets, we can write (D1) as

$$\begin{aligned}B_5 = -8\Bigg[&\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle} + \frac{[24]\langle 13\rangle\langle 35\rangle([13][35])^3}{\langle 24\rangle\langle 12\rangle\langle 14\rangle\langle 25\rangle\langle 45\rangle} + \frac{[15]\langle 23\rangle\langle 34\rangle([23][34])^3}{\langle 15\rangle\langle 21\rangle\langle 25\rangle\langle 14\rangle\langle 54\rangle} \\ &+ \frac{[14]\langle 23\rangle\langle 35\rangle([23][35])^3}{\langle 14\rangle\langle 21\rangle\langle 24\rangle\langle 15\rangle\langle 45\rangle} + \frac{[45]\langle 13\rangle\langle 23\rangle([13][23])^3}{\langle 45\rangle\langle 14\rangle\langle 15\rangle\langle 42\rangle\langle 52\rangle} + \frac{[34]\langle 25\rangle\langle 15\rangle([15][25])^3}{\langle 34\rangle\langle 13\rangle\langle 14\rangle\langle 32\rangle\langle 42\rangle} \\ &+ \frac{[35]\langle 14\rangle\langle 24\rangle([14][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 32\rangle\langle 52\rangle} + \frac{[12]\langle 34\rangle\langle 35\rangle([34][35])^3}{\langle 12\rangle\langle 41\rangle\langle 42\rangle\langle 15\rangle\langle 25\rangle} + \frac{[12]\langle 35\rangle\langle 45\rangle([35][45])^3}{\langle 12\rangle\langle 31\rangle\langle 32\rangle\langle 14\rangle\langle 24\rangle} \\ &+ \frac{[12]\langle 34\rangle\langle 45\rangle([34][45])^3}{\langle 12\rangle\langle 31\rangle\langle 32\rangle\langle 15\rangle\langle 25\rangle}\Bigg] + (1\leftrightarrow 3) + (2\leftrightarrow 3).\end{aligned} \tag{D7}$$

Before simplifying this, first note that the first four terms (and hence a total of 12 terms) in the above expression have the apparent form that seems to go like $\sim\frac{1}{\langle 45\rangle}$. However, it cannot be true that the 5-point amplitude has a leading behaviour of $\sim\frac{1}{\langle 45\rangle}$. We will show that these terms add up to contribute to the leading order $[\mathcal{O}(\frac{[45]}{\langle 45\rangle})]$, $\mathcal{O}(1)$, and higher orders as expected. Hence, to simplify further, let us first write down these 12 terms explicitly,

$$\begin{aligned}-\frac{B_5^{\mathrm{S}}}{8} = &\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle} + \frac{[35]\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 34\rangle\langle 54\rangle} + \frac{[24]\langle 13\rangle\langle 35\rangle([13][35])^3}{\langle 24\rangle\langle 12\rangle\langle 14\rangle\langle 25\rangle\langle 45\rangle} \\ &+ \frac{[34]\langle 12\rangle\langle 25\rangle([12][25])^3}{\langle 34\rangle\langle 13\rangle\langle 14\rangle\langle 35\rangle\langle 45\rangle} + \frac{[15]\langle 23\rangle\langle 34\rangle([23][34])^3}{\langle 15\rangle\langle 21\rangle\langle 25\rangle\langle 14\rangle\langle 54\rangle} + \frac{[35]\langle 12\rangle\langle 14\rangle([12][14])^3}{\langle 35\rangle\langle 23\rangle\langle 25\rangle\langle 34\rangle\langle 54\rangle} \\ &+ \frac{[14]\langle 23\rangle\langle 35\rangle([23][35])^3}{\langle 14\rangle\langle 21\rangle\langle 24\rangle\langle 15\rangle\langle 45\rangle} + \frac{[34]\langle 12\rangle\langle 15\rangle([12][15])^3}{\langle 34\rangle\langle 23\rangle\langle 24\rangle\langle 35\rangle\langle 45\rangle} + \frac{[15]\langle 23\rangle\langle 24\rangle([23][24])^3}{\langle 15\rangle\langle 31\rangle\langle 35\rangle\langle 14\rangle\langle 54\rangle} \\ &+ \frac{[25]\langle 13\rangle\langle 14\rangle([13][14])^3}{\langle 25\rangle\langle 32\rangle\langle 35\rangle\langle 24\rangle\langle 54\rangle} + \frac{[14]\langle 23\rangle\langle 25\rangle([23][25])^3}{\langle 14\rangle\langle 31\rangle\langle 34\rangle\langle 15\rangle\langle 45\rangle} + \frac{[24]\langle 13\rangle\langle 15\rangle([13][15])^3}{\langle 24\rangle\langle 32\rangle\langle 34\rangle\langle 25\rangle\langle 45\rangle}.\end{aligned} \tag{D8}$$

Keeping terms only up to $\mathcal{O}(\bar{z}_{45})$, the first term above can be rewritten as

$$\begin{aligned}\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle} = &-\frac{1}{\langle 45\rangle}\frac{[25]\langle 12\rangle^3\langle 24\rangle^3[12]^3[24]^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 13\rangle^2\langle 34\rangle^2} + 3\frac{[45]}{\langle 45\rangle}\frac{[25]\langle 12\rangle^2\langle 15\rangle\langle 24\rangle^3[12]^3[24]^2}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 13\rangle^2\langle 34\rangle^2}\\ &+ 3\frac{[25]\langle 12\rangle^3\langle 24\rangle^2[12]^2[15][24]^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 13\rangle^2\langle 34\rangle^2} - 3\langle 45\rangle\frac{[25]\langle 12\rangle^2[12][15]^2[24]^3}{\langle 25\rangle\langle 15\rangle\langle 13\rangle^2\langle 34\rangle^2}\\ &- 9[45]\frac{[25]\langle 12\rangle\langle 24\rangle[12]^2[15][24]^2}{\langle 25\rangle\langle 13\rangle^2\langle 34\rangle^2}. \end{aligned} \tag{D9}$$

Now we use a little trick to explicitly show that the terms in (D8) add up to give $[\mathcal{O}(\frac{[45]}{\langle 45\rangle})]$, $\mathcal{O}(1)$ and higher orders contributions. It involves appropriately combining terms in the equation. To see this, note that the first term in rhs of (D9) and second term in rhs of (D8) can be combined to get

$$\begin{aligned}&-\frac{1}{\langle 45\rangle}\frac{[25]\langle 12\rangle^3\langle 24\rangle^3[12]^3[24]^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 13\rangle^2\langle 34\rangle^2} + \frac{[35]\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 34\rangle\langle 54\rangle}\\ &= -\frac{1}{\langle 45\rangle}\frac{\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 13\rangle^2\langle 15\rangle\langle 25\rangle\langle 34\rangle^2\langle 35\rangle}(\langle 12\rangle\langle 24\rangle\langle 35\rangle[25] + \langle 13\rangle\langle 34\rangle\langle 25\rangle[35]). \end{aligned} \tag{D10}$$

Note that although we are writing equalities everywhere, one should keep in mind that there are higher-order terms as well. However, here, and throughout this paper, we will always write expressions keeping terms only up to $\mathcal{O}(\bar{z}_{45})$. Now, using the Shouten identity $\langle 24\rangle\langle 35\rangle = \langle 25\rangle\langle 34\rangle + \langle 23\rangle\langle 45\rangle$ and momentum conservation equation, we can write the above equation as

$$\begin{aligned}&-\frac{1}{\langle 45\rangle}\frac{[25]\langle 12\rangle^3\langle 24\rangle^3[12]^3[24]^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 13\rangle^2\langle 34\rangle^2} + \frac{[35]\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 34\rangle\langle 54\rangle}\\ &= -\frac{1}{\langle 45\rangle}\frac{\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 13\rangle^2\langle 15\rangle\langle 25\rangle\langle 34\rangle^2\langle 35\rangle}(-\langle 14\rangle\langle 25\rangle\langle 34\rangle[45] + \langle 12\rangle\langle 23\rangle\langle 45\rangle[25]). \end{aligned} \tag{D11}$$

Hence the first two terms in (D8) give

$$\begin{aligned}\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle} + \frac{[35]\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 34\rangle\langle 54\rangle} &= 3\frac{[45]}{\langle 45\rangle}\frac{[25]\langle 12\rangle\langle 24\rangle^2[12]^3[24]^2}{\langle 25\rangle\langle 13\rangle^2\langle 34\rangle^2}\\ + 3\frac{[25]\langle 12\rangle^2\langle 24\rangle[12]^2[15][24]^3}{\langle 25\rangle\langle 15\rangle\langle 13\rangle^2\langle 34\rangle^2} &+ \frac{[45]}{\langle 45\rangle}\frac{\langle 12\rangle\langle 14\rangle\langle 24\rangle[12]^3[24]^3}{\langle 13\rangle^2\langle 15\rangle\langle 34\rangle\langle 35\rangle} - \frac{\langle 12\rangle^2\langle 23\rangle\langle 24\rangle[25][12]^3[24]^3}{\langle 13\rangle^2\langle 15\rangle\langle 25\rangle\langle 34\rangle^2\langle 35\rangle}\\ - 3\langle 45\rangle\frac{[25]\langle 12\rangle^2[12][15]^2[24]^3}{\langle 25\rangle\langle 15\rangle\langle 13\rangle^2\langle 34\rangle^2} &- 9[45]\frac{[25]\langle 12\rangle\langle 24\rangle[12]^2[15][24]^2}{\langle 25\rangle\langle 13\rangle^2\langle 34\rangle^2}. \end{aligned} \tag{D12}$$

Using momentum conservation again in the fourth term in the rhs of the above equation, we finally get

$$\begin{aligned}\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle} + \frac{[35]\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 34\rangle\langle 54\rangle} &= 3\frac{[45]}{\langle 45\rangle}\frac{[25]\langle 12\rangle\langle 24\rangle^2[12]^3[24]^2}{\langle 25\rangle\langle 13\rangle^2\langle 34\rangle^2}\\ + \frac{[45]}{\langle 45\rangle}\frac{\langle 12\rangle\langle 14\rangle\langle 24\rangle[12]^3[24]^3}{\langle 13\rangle^2\langle 15\rangle\langle 34\rangle\langle 35\rangle} &+ 2\frac{[25]\langle 12\rangle^2\langle 24\rangle[12]^2[15][24]^3}{\langle 13\rangle^2\langle 25\rangle\langle 15\rangle\langle 34\rangle^2} - \frac{[25]\langle 12\rangle^2\langle 24\rangle[12]^2[14][24]^3}{\langle 13\rangle^2\langle 15\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle}\\ - 3\langle 45\rangle\frac{[25]\langle 12\rangle^2[12][15]^2[24]^3}{\langle 25\rangle\langle 15\rangle\langle 13\rangle^2\langle 34\rangle^2} &- 9[45]\frac{[25]\langle 12\rangle\langle 24\rangle[12]^2[15][24]^2}{\langle 25\rangle\langle 13\rangle^2\langle 34\rangle^2}. \end{aligned} \tag{D13}$$

Similarly, the third and fourth terms in rhs of (D8) give

$$
\begin{aligned}
&\frac{[24]\langle 13\rangle\langle 35\rangle([13][35])^3}{\langle 24\rangle\langle 12\rangle\langle 14\rangle\langle 25\rangle\langle 45\rangle}+\frac{[34]\langle 12\rangle\langle 25\rangle([12][25])^3}{\langle 34\rangle\langle 13\rangle\langle 14\rangle\langle 35\rangle\langle 45\rangle}\\
&=3\frac{[45]}{\langle 45\rangle}\frac{[24]\langle 12\rangle\langle 25\rangle^2[12]^3[25]^2}{\langle 24\rangle\langle 13\rangle^2\langle 35\rangle^2}+\frac{[45]}{\langle 45\rangle}\frac{\langle 12\rangle\langle 15\rangle\langle 25\rangle[12]^3[25]^3}{\langle 13\rangle^2\langle 14\rangle\langle 34\rangle\langle 35\rangle}+2\frac{[24]\langle 12\rangle^2\langle 25\rangle[12]^2[14][25]^3}{\langle 24\rangle\langle 14\rangle\langle 13\rangle^2\langle 35\rangle^2}\\
&\quad-\frac{[24]\langle 12\rangle^2\langle 25\rangle[12]^2[15][25]^3}{\langle 13\rangle^2\langle 14\rangle\langle 24\rangle\langle 34\rangle\langle 35\rangle}+3\langle 45\rangle\frac{[24]\langle 12\rangle^2[12][14]^2[25]^3}{\langle 24\rangle\langle 14\rangle\langle 13\rangle^2\langle 35\rangle^2}+9[45]\frac{[24]\langle 12\rangle\langle 25\rangle[12]^2[14][25]^2}{\langle 24\rangle\langle 13\rangle^2\langle 35\rangle^2}.
\end{aligned}
\tag{D14}
$$

As is clear from the above equations, we can combine the 12 terms of (D8) in groups of two as shown above to see that the leading-order contribution coming from (D8) is indeed $\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})$ instead of the apparent $\mathcal{O}(\frac{1}{z_{45}})$.

Now, we rewrite the first four terms in (D8) in terms of $\{\omega_i, z_i, \bar{z}_i\}$, and then expand around $z_{45}=\bar{z}_{45}=0$. As mentioned earlier, we only keep terms up to $\mathcal{O}(\bar{z}_{45})$ to get

$$
\begin{aligned}
&\frac{[25]\langle 13\rangle\langle 34\rangle([13][34])^3}{\langle 25\rangle\langle 12\rangle\langle 15\rangle\langle 24\rangle\langle 54\rangle}+\frac{[35]\langle 12\rangle\langle 24\rangle([12][24])^3}{\langle 35\rangle\langle 13\rangle\langle 15\rangle\langle 34\rangle\langle 54\rangle}+\frac{[24]\langle 13\rangle\langle 35\rangle([13][35])^3}{\langle 24\rangle\langle 12\rangle\langle 14\rangle\langle 25\rangle\langle 45\rangle}+\frac{[34]\langle 12\rangle\langle 25\rangle([12][25])^3}{\langle 34\rangle\langle 13\rangle\langle 14\rangle\langle 35\rangle\langle 45\rangle}\\
&=2^4\epsilon_1\frac{\omega_1\omega_2^4}{\omega_3^2\omega_4\omega_5}(\omega_4+\omega_5)^3\frac{\bar{z}_{45}}{z_{45}}\frac{z_{12}z_{25}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2z_{35}^2}-2^4\epsilon_1\frac{\omega_1\omega_2^4}{\omega_3^2\omega_4\omega_5}[(\omega_4+\omega_5)^3-5\omega_4\omega_5(\omega_4+\omega_5)]\frac{z_{12}^2\bar{z}_{12}^2\bar{z}_{15}\bar{z}_{25}^4}{z_{13}^2z_{15}z_{35}^2}\\
&\quad+2^4\epsilon_1\bar{z}_{45}\frac{z_{12}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2z_{15}z_{35}^3}\frac{\omega_1\omega_2^4}{\omega_3^2\omega_4\omega_5}[-z_{15}z_{35}\omega_4(\omega_4^2+6\omega_4\omega_5-3\omega_5^2)+z_{25}z_{35}(-\omega_4^3+\omega_5^3)\\
&\quad+z_{15}z_{25}(\omega_4^3+6\omega_4^2\omega_5+\omega_5^3)]+2^4\epsilon_1\bar{z}_{45}\frac{z_{12}^2\bar{z}_{12}^2\bar{z}_{25}^3}{z_{13}^2z_{15}z_{35}^2}\frac{\omega_1\omega_2^4}{\omega_3^2\omega_4\omega_5}[\bar{z}_{25}\omega_4(\omega_4^2-2\omega_5^2)\\
&\quad+\bar{z}_{15}(3\omega_4^3-6\omega_4^2\omega_5-2\omega_4\omega_5^2+\omega_5^3)]+\cdots.
\end{aligned}
\tag{D15}
$$

The contribution from the other eight terms in (D8) is simply obtained by taking different permutations of 1, 2, and 3 in the above expression. Setting $\omega_4=t\omega_P, \omega_5=(1-t)\omega_P$ and collecting all the singular terms we finally get

$$
B_5^{\mathrm{S}}=-2^7\frac{\omega_P}{t(1-t)}\left(\frac{\bar{z}_{45}}{z_{45}}T_L^{\mathrm{S}}+T_{\mathcal{O}(1)}^{\mathrm{S}}+\bar{z}_{45}T_{\bar{z}}^{\mathrm{S}}\right),
\tag{D16}
$$

where

$$
T_L^{\mathrm{S}}=\left[\epsilon_1\frac{z_{12}z_{25}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2z_{35}^2}\frac{\omega_1\omega_2^4}{\omega_3^2}+\epsilon_2\frac{z_{12}z_{15}\bar{z}_{12}^3\bar{z}_{15}^3}{z_{23}^2z_{35}^2}\frac{\omega_1^4\omega_2}{\omega_3^2}+\epsilon_3\frac{z_{13}z_{15}\bar{z}_{13}^3\bar{z}_{15}^3}{z_{23}^2z_{25}^2}\frac{\omega_1^4\omega_3}{\omega_2^2}\right],
\tag{D17}
$$

$$
T_{\mathcal{O}(1)}^{\mathrm{S}}=-\left[\epsilon_1\frac{z_{12}^2\bar{z}_{12}^2\bar{z}_{15}\bar{z}_{25}^4}{z_{13}^2z_{15}z_{35}^2}\frac{\omega_1\omega_2^4}{\omega_3^2}+\epsilon_2\frac{z_{12}^2\bar{z}_{12}^2\bar{z}_{25}\bar{z}_{15}^4}{z_{23}^2z_{25}z_{35}^2}\frac{\omega_1^4\omega_2}{\omega_3^2}+\epsilon_3\frac{z_{13}^2\bar{z}_{13}^2\bar{z}_{35}\bar{z}_{15}^4}{z_{23}^2z_{35}z_{25}^2}\frac{\omega_1^4\omega_3}{\omega_2^2}\right][1-5t(1-t)],
\tag{D18}
$$

and

$$
\begin{aligned}
T_{\bar{z}}^{\mathrm{S}}&=\epsilon_1\frac{z_{12}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2z_{15}z_{35}^3}\frac{\omega_1\omega_2^4}{\omega_3^2}[-z_{15}z_{35}t(t^2+6t(1-t)-3(1-t)^2)+z_{25}z_{35}(-t^3+(1-t)^3)\\
&\quad+z_{15}z_{25}(t^3+6t^2(1-t)+(1-t)^3)]+\epsilon_1\frac{z_{12}^2\bar{z}_{12}^2\bar{z}_{25}^3}{z_{13}^2z_{15}z_{35}^2}\frac{\omega_1\omega_2^4}{\omega_3^2}[\bar{z}_{25}t(t^2-2(1-t)^2)\\
&\quad+\bar{z}_{15}(3t^3-6t^2(1-t)-2t(1-t)^2+(1-t)^3)]\\
&\quad+\epsilon_2\frac{z_{12}\bar{z}_{12}^3\bar{z}_{15}^3}{z_{23}^2z_{25}z_{35}^3}\frac{\omega_1^4\omega_2}{\omega_3^2}[-z_{25}z_{35}t(t^2+6t(1-t)-3(1-t)^2)+z_{15}z_{35}(-t^3+(1-t)^3)
\end{aligned}
$$

$$
\begin{aligned}
&+z_{25}z_{15}(t^3+6t^2(1-t)+(1-t)^3)]+\epsilon_2\frac{z_{12}^2\bar{z}_{12}^2\bar{z}_{15}^3}{z_{23}^2z_{25}z_{35}^2}\frac{\omega_1^4\omega_2}{\omega_3^2}[\bar{z}_{15}t(t^2-2(1-t)^2)\\
&+\bar{z}_{25}(3t^3-6t^2(1-t)-2t(1-t)^2+(1-t)^3)]\\
&+\epsilon_3\frac{z_{13}\bar{z}_{13}^3\bar{z}_{15}^3}{z_{23}^2z_{35}z_{25}^3}\frac{\omega_1^4\omega_3}{\omega_2^2}[-z_{35}z_{25}t(t^2+6t(1-t)-3(1-t)^2)+z_{15}z_{25}(-t^3+(1-t)^3)\\
&+z_{35}z_{15}(t^3+6t^2(1-t)+(1-t)^3)]+\epsilon_3\frac{z_{13}^2\bar{z}_{13}^2\bar{z}_{15}^3}{z_{23}^2z_{35}z_{25}^2}\frac{\omega_1^4\omega_3}{\omega_2^2}[\bar{z}_{15}t(t^2-2(1-t)^2)\\
&+\bar{z}_{35}(3t^3-6t^2(1-t)-2t(1-t)^2+(1-t)^3)].
\end{aligned}
\tag{D19}
$$

Taking into account the other 18 terms [although note that at $\mathcal{O}(\bar{z}_{45})$, only 12 of these contribute and the ninth and tenth term in (D7) and the $(1\leftrightarrow 3)$ and $(2\leftrightarrow 3)$ permutation of those do not contribute at this order] in (D7) we finally get $B_5$ as

$$
B_5=-2^7\frac{\omega_P}{t(1-t)}\left(\frac{\bar{z}_{45}}{z_{45}}T_L+T_{\mathcal{O}(1)}+\bar{z}_{45}T_{\bar{z}}\right)+\cdots. \tag{D20}
$$

where we have neglected the higher-order terms in the expansion of the rhs of (D7) around $z_{45}=\bar{z}_{45}=0$ in $(\omega,z,\bar{z})$ space and

$$
\begin{aligned}
T_L&=T_L^{\mathrm{S}}+\left[\epsilon_1\epsilon_2\frac{\omega_1\omega_2\omega_3^4}{\omega_P^3}\frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2z_{25}^2}+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right],\\
T_{\mathcal{O}(1)}&=T_{\mathcal{O}(1)}^{\mathrm{S}}+\left[\epsilon_1\epsilon_2\epsilon_3\frac{\omega_1\omega_2\omega_P^2}{\omega_3}\frac{\bar{z}_{35}(\bar{z}_{15}\bar{z}_{25})^3}{z_{35}z_{13}z_{23}}\{(1-t)^5+t^5\}+\epsilon_1\epsilon_2\frac{\omega_P\omega_3^4}{\omega_1\omega_2}\frac{\bar{z}_{12}z_{35}^2\bar{z}_{35}^6}{z_{12}z_{15}^2z_{25}^2}t^2(1-t)^2+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right],\\
T_{\bar{z}}&=T_{\bar{z}}^{\mathrm{S}}+\left[\epsilon_1\epsilon_2\frac{\omega_1\omega_2\omega_3^4}{\omega_P^3}\frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2z_{25}^2}\left(\frac{1}{z_{15}}+\frac{1}{z_{25}}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right]\\
&\quad-\left[\epsilon_1\epsilon_2\epsilon_3\frac{\omega_1\omega_2\omega_P^2}{\omega_3}\frac{(\bar{z}_{15}\bar{z}_{25})^3}{z_{13}z_{23}z_{35}}(1-t)^5+3\epsilon_1\epsilon_2\epsilon_3\frac{\omega_1\omega_2\omega_P^2}{\omega_3}\frac{\bar{z}_{35}(\bar{z}_{15}\bar{z}_{25})^2(\bar{z}_{15}+\bar{z}_{25})}{z_{13}z_{23}z_{35}}t^5\right.\\
&\quad\left.+\epsilon_1\epsilon_2\frac{\omega_3^4\omega_P}{\omega_1\omega_2}\frac{\bar{z}_{12}z_{35}^2\bar{z}_{35}^5}{z_{12}z_{15}^2z_{25}^2}t^2(1-t)^2+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right].
\end{aligned}
\tag{D21}
$$

We will now Mellin transform (D20) and take the OPE limit $4\to 5$. We want to emphasize here that after Mellin transformation, the higher-order terms in the OPE expansion of the Mellin amplitude may receive contribution from the lower-order terms. This is because of the fact that, after Mellin transformation the Mellin amplitude will depend on $\tilde{\omega}_i^*$'s as well as the delta function involving cross ratios coming from 5-point momentum conserving delta function as parametrized in (B12). In the next subsection we analyze this in detail and calculate the order-by-order terms in the OPE expansion $4\to 5$ of the 5-point Mellin amplitude.

## APPENDIX E: MELLIN TRANSFORMATION OF THE 5-POINT AMPLITUDE

For the discussion of this appendix, the prefactor $\frac{i}{(4\pi)^2 960}$ in (5.7) is not important. Thus we only Mellin transform $B_5$ and keep terms only up to $\mathcal{O}(\bar{z}_{45})$. Substituting (D20) in (5.12) we get

$$
\begin{aligned}
\tilde{B}_5=-2^7\int_0^\infty\prod_{i=1}^5 d\omega_i\omega_i^{\Delta_i-1}e^{-i\sum_{i=1}^4\epsilon_i\omega_i u_{i5}}\frac{\omega_P}{t(1-t)}&\left(\frac{\bar{z}_{45}}{z_{45}}T_L(\omega_1,\omega_2,\omega_3,\omega_P)+T_{\mathcal{O}(1)}(\omega_1,\omega_2,\omega_3,\omega_P)\right.\\
&\left.+\bar{z}_{45}T_{\bar{z}}(\omega_1,\omega_2,\omega_3,\omega_P)\right)\delta^{(4)}\left(\sum_{i=1}^5\epsilon_i\omega_i q_i\right),
\end{aligned}
\tag{E1}
$$

where $T_L, T_{\mathcal{O}(1)}$, and $T_{\bar{z}}$ are given by (D21) and we have kept their $\{\omega\}$ dependence explicit for our convenience. Also we have used momentum conservation in the exponential. Now using the parametrization (B12), we can perform the $(\omega_1,\omega_2,\omega_3)$ integrals to obtain,

$$\tilde{B}_5 = -2^5 \int_0^1 dt t^{\Delta_4 - 2} (1-t)^{\Delta_5 - 2} \int_0^\infty d\omega_P \omega_P^{\Delta_4 + \Delta_5 - 1} \prod_{i=1}^3 (\omega_i^*)^{\Delta_i - 1} e^{-i \sum_{i=1}^3 \epsilon_i \omega_i^* u_{i5} - i\omega_P t u_{45}}$$
$$\times \left( \frac{\bar{z}_{45}}{z_{45}} T_L(\omega_1^*, \omega_2^*, \omega_3^*, \omega_P) + T_{\mathcal{O}(1)}(\omega_1^*, \omega_2^*, \omega_3^*, \omega_P) + \bar{z}_{45} T_{\bar{z}}(\omega_1^*, \omega_2^*, \omega_3^*, \omega_P) \right)$$
$$\times \delta\left( x - \bar{x} - t z_{45} \left( \frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}} \right) - t \bar{z}_{45} \left( \frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}} \right) + t z_{45} \bar{z}_{45} \left( \frac{x}{z_{35}\bar{z}_{25}} - \frac{\bar{x}}{z_{25}\bar{z}_{35}} \right) \right). \tag{E2}$$

Now from (B13) and the explicit expressions of $T_L, T_{\mathcal{O}(1)}$, and $T_{\bar{z}}$ given by (D21) one can see that

$$T_L(\omega_1^*, \omega_2^*, \omega_3^*, \omega_P) = \omega_P^3 \mathcal{T}_L(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*),$$
$$T_{\mathcal{O}(1)}(\omega_1^*, \omega_2^*, \omega_3^*, \omega_P) = \omega_P^3 \mathcal{T}_{\mathcal{O}(1)}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*),$$
$$T_{\bar{z}}(\omega_1^*, \omega_2^*, \omega_3^*, \omega_P) = \omega_P^3 \mathcal{T}_{\bar{z}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*), \tag{E3}$$

where

$$\mathcal{T}_L(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) = \mathcal{T}_L^{\mathrm{S}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) + \left[ \epsilon_1 \epsilon_2 \tilde{\omega}_1^* \tilde{\omega}_2^* (\tilde{\omega}_3^*)^4 \frac{z_{13} z_{23} (\bar{z}_{13} \bar{z}_{23})^3}{z_{15}^2 z_{25}^2} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right],$$
$$\mathcal{T}_{\mathcal{O}(1)}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) = \mathcal{T}_{\mathcal{O}(1)}^{\mathrm{S}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) + \left[ \epsilon_1 \epsilon_2 \epsilon_3 \frac{\tilde{\omega}_1^* \tilde{\omega}_2^*}{\tilde{\omega}_3^*} \frac{\bar{z}_{35} (\bar{z}_{15} \bar{z}_{25})^3}{z_{35} z_{13} z_{23}} \{ (1-t)^5 + t^5 \} \right.$$
$$\left. + \epsilon_1 \epsilon_2 \frac{(\tilde{\omega}_3^*)^4}{\tilde{\omega}_1^* \tilde{\omega}_2^*} \frac{\bar{z}_{12} z_{35}^2 \bar{z}_{35}^6}{z_{12} z_{15}^2 z_{25}^2} t^2 (1-t)^2 + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right],$$
$$\mathcal{T}_{\bar{z}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) = \mathcal{T}_{\bar{z}}^{\mathrm{S}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) + \left[ \epsilon_1 \epsilon_2 \tilde{\omega}_1^* \tilde{\omega}_2^* (\tilde{\omega}_3^*)^4 \frac{z_{13} z_{23} (\bar{z}_{13} \bar{z}_{23})^3}{z_{15}^2 z_{25}^2} \left( \frac{1}{z_{15}} + \frac{1}{z_{25}} \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right]$$
$$- \left[ \epsilon_1 \epsilon_2 \epsilon_3 \frac{\tilde{\omega}_1^* \tilde{\omega}_2^*}{\tilde{\omega}_3^*} \frac{(\bar{z}_{15} \bar{z}_{25})^3}{z_{13} z_{23} z_{35}} (1-t)^5 + 3 \epsilon_1 \epsilon_2 \epsilon_3 \frac{\tilde{\omega}_1^* \tilde{\omega}_2^*}{\tilde{\omega}_3^*} \frac{\bar{z}_{35} (\bar{z}_{15} \bar{z}_{25})^2 (\bar{z}_{15} + \bar{z}_{25})}{z_{13} z_{23} z_{35}} t^5 \right.$$
$$\left. + \epsilon_1 \epsilon_2 \frac{(\tilde{\omega}_3^*)^4}{\tilde{\omega}_1^* \tilde{\omega}_2^*} \frac{\bar{z}_{12} z_{35}^2 \bar{z}_{35}^5}{z_{12} z_{15}^2 z_{25}^2} t^2 (1-t)^2 + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right], \tag{E4}$$

and $\mathcal{T}_L^{\mathrm{S}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*), \mathcal{T}_{\mathcal{O}(1)}^{\mathrm{S}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*)$, and $\mathcal{T}_{\bar{z}}^{\mathrm{S}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*)$ are given by (D17), (D18), and (D19) respectively with $\{\omega_1, \omega_2, \omega_3\}$ replaced by $\{\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*\}$. Now we can perform the $\omega_P$ integral in (E2) and obtain,

$$\tilde{B}_5 = -2^5 \Gamma(\Delta) \int_0^1 dt t^{\Delta_4 - 2} (1-t)^{\Delta_5 - 2} \frac{\prod_{i=1}^3 (\tilde{\omega}_i^*)^{\Delta_i - 1}}{[i(\sum_{i=1}^3 \epsilon_i \tilde{\omega}_i^* u_{i5} + t u_{45})]^\Delta}$$
$$\times \left( \frac{\bar{z}_{45}}{z_{45}} \mathcal{T}_L(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) + \mathcal{T}_{\mathcal{O}(1)}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) + \bar{z}_{45} \mathcal{T}_{\bar{z}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) \right)$$
$$\times \delta\left( x - \bar{x} - t z_{45} \left( \frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}} \right) - t \bar{z}_{45} \left( \frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}} \right) + t z_{45} \bar{z}_{45} \left( \frac{x}{z_{35}\bar{z}_{25}} - \frac{\bar{x}}{z_{25}\bar{z}_{35}} \right) \right), \tag{E5}$$

where $\Delta = \sum_{i=1}^5 \Delta_i$. We now expand the above equation around $z_{45} = \bar{z}_{45} = u_{45} = 0$.

### 1. Evaluating the leading-order contribution

It is clear from (E5) that the leading order term goes as $\sim \frac{\bar{z}_{45}}{z_{45}}$ and the contribution to the leading order can come only from the term containing $\mathcal{T}_L(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*)$. At leading order we have $\tilde{\omega}_i^* = \epsilon_i \sigma_{i,1}$. Thus, the leading-order term of $\tilde{B}_5$ is given by

$$\tilde{B}_5|_{\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})} = -2^5 \frac{\bar{z}_{45}}{z_{45}} \frac{\Gamma(\Delta)}{(i\mathcal{D})^\Delta} \prod_{i=1}^3 (\epsilon_i \sigma_{i,1})^{\Delta_i - 1} \int_0^1 dt t^{\Delta_4 - 2} (1-t)^{\Delta_5 - 2} \mathcal{T}_L(\epsilon_1 \sigma_{1,1}, \epsilon_2 \sigma_{2,1}, \epsilon_3 \sigma_{3,1}) \delta(x - \bar{x}), \tag{E6}$$

where $\mathcal{D} = (\sum_{i=1}^{3} \sigma_{i,1} u_{i5})$. From (D17) and the first equation of (D21) we have

$$\mathcal{T}_L(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1}) = \left[\frac{z_{12}z_{25}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2 z_{35}^2}\frac{\sigma_{1,1}\sigma_{2,1}^4}{\sigma_{3,1}^2} + \frac{z_{12}z_{15}\bar{z}_{12}^3\bar{z}_{15}^3}{z_{23}^2 z_{35}^2}\frac{\sigma_{1,1}^4\sigma_{2,1}}{\sigma_{3,1}^2} + \frac{z_{13}z_{15}\bar{z}_{13}^3\bar{z}_{15}^3}{z_{23}^2 z_{25}^2}\frac{\sigma_{1,1}^4\sigma_{3,1}}{\sigma_{2,1}^2}\right] + \left[\sigma_{1,1}\sigma_{2,1}\sigma_{3,1}^4 \frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2 z_{25}^2} + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]. \tag{E7}$$

Now, using (B3)–(B7), one can show that

$$\begin{aligned} \frac{z_{12}z_{25}\bar{z}_{12}^3\bar{z}_{25}^3}{z_{13}^2 z_{35}^2}\frac{\sigma_{1,1}\sigma_{2,1}^4}{\sigma_{3,1}^2} &= \frac{z_{12}z_{13}(\bar{z}_{12}\bar{z}_{13})^3}{z_{25}^2 z_{35}^2}\sigma_{1,1}^4\sigma_{2,1}\sigma_{3,1},\\ \frac{z_{12}z_{15}\bar{z}_{12}^3\bar{z}_{15}^3}{z_{23}^2 z_{35}^2}\frac{\sigma_{1,1}^4\sigma_{2,1}}{\sigma_{3,1}^2} &= \frac{z_{12}z_{23}(\bar{z}_{12}\bar{z}_{23})^3}{z_{15}^2 z_{35}^2}\sigma_{1,1}\sigma_{2,1}^4\sigma_{3,1},\\ \frac{z_{13}z_{15}\bar{z}_{13}^3\bar{z}_{15}^3}{z_{23}^2 z_{25}^2}\frac{\sigma_{1,1}^4\sigma_{3,1}}{\sigma_{2,1}^2} &= \frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2 z_{25}^2}\sigma_{1,1}\sigma_{2,1}\sigma_{3,1}^4. \end{aligned} \tag{E8}$$

Using the above relations, we can simplify (E7) to get

$$\begin{aligned} \mathcal{T}_L(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1}) &= 2\left[\sigma_{1,1}\sigma_{2,1}\sigma_{3,1}^4 \frac{z_{13}z_{23}(\bar{z}_{13}\bar{z}_{23})^3}{z_{15}^2 z_{25}^2} + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]\\ &= 2[\mathcal{N}_4 + \mathcal{N}_4(1\leftrightarrow 3) + \mathcal{N}_4(2\leftrightarrow 3)], \end{aligned} \tag{E9}$$

where the second equality follows from (5.6). Since this is independent of $t$, we can easily carry out the $t$-integral in (E6) to get

$$\tilde{B}_5|_{\mathcal{O}(\frac{\bar{z}_{45}}{z_{45}})} = -2^6\frac{\bar{z}_{45}}{z_{45}}B(\Delta_4 - 1, \Delta_5 - 1)\frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}}\delta(x - \bar{x})\prod_{i=1}^{3}(\epsilon_i\sigma_{i,1})^{\Delta_i - 1}[\mathcal{N}_4 + \mathcal{N}_4(1\leftrightarrow 3) + \mathcal{N}_4(2\leftrightarrow 3)]. \tag{E10}$$

This precisely gives Eq. (5.13).

### 2. Evaluating the $\mathcal{O}(1)$ contribution

From (E5), we can see that the $\mathcal{O}(1)$ contribution to the 5-point amplitude essentially comes only from the term containing $\mathcal{T}_{\mathcal{O}(1)}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*)$ when $\tilde{\omega}_i^*$'s take their leading-order value given by $\epsilon_i\sigma_{i,1}$. Let us write the Mellin integral at order one,

$$\tilde{B}_5|_{\mathcal{O}(1)} = -2^5\frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}}\prod_{i=1}^{3}(\epsilon_i\sigma_{i,1})^{\Delta_i - 1}\int_0^1 dt t^{\Delta_4 - 2}(1-t)^{\Delta_5 - 2}\mathcal{T}_{\mathcal{O}(1)}(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1})\delta(x - \bar{x}). \tag{E11}$$

We will not attempt to take the explicit expressions of $\mathcal{T}_{\mathcal{O}(1)}(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1})$ and Mellin integrate it. Rather we will take a different approach which is more helpful for our purpose of the OPE factorization. Firstly, from the second equation of (D21) we observe that $\mathcal{T}_{\mathcal{O}(1)}(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1})$ is a polynomial in $t$ with the highest power being 4. We use this fact and write $\mathcal{T}_{\mathcal{O}(1)}(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1})$ as

$$\mathcal{T}_{\mathcal{O}(1)}(\epsilon_1\sigma_{1,1}, \epsilon_2\sigma_{2,1}, \epsilon_3\sigma_{3,1}) = \sum_{k=0}^{4} t^k \mathcal{F}_k^{(1)}(\{\epsilon_i, z_i, \bar{z}_i\}). \tag{E12}$$

The explicit expressions for the functions $\mathcal{F}_k^{(1)}(\{\epsilon_i, z_i, \bar{z}_i\})$ can be read out from the second equation of (D21). However, they are not relevant for our discussions and hence we will not write them explicitly. Using (E12), we can easily evaluate the integral (E11) to get,

$$\tilde{B}_5|_{\mathcal{O}(1)} = -2^5 \frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}} \prod_{i=1}^{3} (\epsilon_i \sigma_{i,1})^{\Delta_i - 1} \sum_{k=0}^{4} B(\Delta_4 + k - 1, \Delta_5 - 1) \mathcal{F}_k^{(1)}(\{\epsilon_i, z_i, \bar{z}_i\}) \delta(x - \bar{x}). \tag{E13}$$

This is the expression we have used in Sec. V E 2.

### 3. Evaluating the order $\mathcal{O}(\bar{z}_{45})$ contribution

We apply the same strategy as the previous section here. However, we have to be careful now as there will be contributions at $\mathcal{O}(\bar{z}_{45})$ from the lower-order terms. Like $\mathcal{O}(1)$ terms, here also we are only concerned about the $t$-dependence. Before proceeding further let us first write down the expansion of different components in (E5) around $z_{45} = \bar{z}_{45} = u_{45} = 0$. Keeping terms only up to $\mathcal{O}(\bar{z}_{45})$ we have

$$\begin{aligned}
\tilde{\omega}_i^* &= \epsilon_i(\sigma_{i,1} + t z_{45} \sigma_{i,2} + t \bar{z}_{45} \sigma_{i,3}), \\
\mathcal{T}_L(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) &= \mathcal{T}_L(\epsilon_1 \sigma_{1,1}, \epsilon_2 \sigma_{2,1}, \epsilon_3 \sigma_{3,1}) + z_{45} \mathcal{T}_L^{(z)}(\{\epsilon_i, z_i, \bar{z}_i\}) + \bar{z}_{45} \mathcal{T}_L^{(\bar{z})}(\{\epsilon_i, z_i, \bar{z}_i\}), \\
\mathcal{T}_{\mathcal{O}(1)}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) &= \mathcal{T}_{\mathcal{O}(1)}(\epsilon_1 \sigma_{1,1}, \epsilon \sigma_{2,1}, \epsilon_3 \sigma_{3,1}) + z_{45} \mathcal{T}_{\mathcal{O}(1)}^{(z)}(\{\epsilon_i, z_i, \bar{z}_i\}) + \bar{z}_{45} \mathcal{T}_{\mathcal{O}(1)}^{(\bar{z})}(\{\epsilon_i, z_i, \bar{z}_i\}), \\
\mathcal{T}_{\bar{z}}(\tilde{\omega}_1^*, \tilde{\omega}_2^*, \tilde{\omega}_3^*) &= \mathcal{T}_{\bar{z}}(\epsilon_1 \sigma_{1,1}, \epsilon_2 \sigma_{2,1}, \epsilon_3 \sigma_{3,1}) + z_{45} \mathcal{T}_{\bar{z}}^{(z)}(\{\epsilon_i, z_i, \bar{z}_i\}) + \bar{z}_{45} \mathcal{T}_{\bar{z}}^{(\bar{z})}(\{\epsilon_i, z_i, \bar{z}_i\}).
\end{aligned} \tag{E14}$$

The explicit expressions for different $\mathcal{T}$'s are not required for our discussions. For notational convenience, we will not write the arguments of different $\mathcal{T}$'s and replace $\mathcal{T}_{L,\mathcal{O}(1),\bar{z}}(\epsilon_1 \sigma_{1,1}, \epsilon_2 \sigma_{2,1}, \epsilon_3 \sigma_{3,1})$ by $\mathcal{T}^{(0)}_{L,\mathcal{O}(1),\bar{z}}$. Let us first write down all possible contributions to $\tilde{B}_5$ at $\mathcal{O}(\bar{z}_{45})$. From (E5) we have

$$\begin{aligned}
\tilde{B}_5|_{\mathcal{O}(\bar{z}_{45})} = -2^5 \frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}} \prod_{i=1}^{3} (\epsilon_i \sigma_{i,1})^{\Delta_i - 1} \int_0^1 dt t^{\Delta_4 - 2} (1-t)^{\Delta_5 - 2} \Bigg[ & (\mathcal{T}_L^{(z)} + \mathcal{T}_{\mathcal{O}(1)}^{(\bar{z})} + \mathcal{T}_{\bar{z}}^{(0)}) \delta(x - \bar{x}) \\
& - t \left\{ \left( \frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}} \right) \mathcal{T}_L^{(0)} + \mathcal{T}_{\mathcal{O}(1)}^{(0)} \left( \frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}} \right) \right\} \delta'(x - \bar{x}) \\
& - t \left\{ (\Delta_1 - 1) \frac{\sigma_{1,2}}{\sigma_{1,1}} + (\Delta_2 - 1) \frac{\sigma_{2,2}}{\sigma_{2,1}} + (\Delta_3 - 1) \frac{\sigma_{3,2}}{\sigma_{3,1}} - \Delta \frac{\sum_{i=1}^{3} \sigma_{i,2} u_{i5}}{\mathcal{D}} \right\} \mathcal{T}_L^{(0)} \delta(x - \bar{x}) \\
& - t \left\{ (\Delta_1 - 1) \frac{\sigma_{1,3}}{\sigma_{1,1}} + (\Delta_2 - 1) \frac{\sigma_{2,3}}{\sigma_{2,1}} + (\Delta_3 - 1) \frac{\sigma_{3,3}}{\sigma_{3,1}} - \Delta \frac{\sum_{i=1}^{3} \sigma_{i,3} u_{i5}}{\mathcal{D}} \right\} \mathcal{T}_{\mathcal{O}(1)}^{(0)} \delta(x - \bar{x}) \Bigg].
\end{aligned} \tag{E15}$$

Now, by expanding the $\mathcal{T}$'s in (D21) around $z_{45} = \bar{z}_{45} = 0$ and keeping terms only up to $\mathcal{O}(\bar{z}_{45})$, one can check that all the terms at different orders in the expansion are polynomial of $t$. The highest degree of polynomial is 5 and appears in $\mathcal{T}_{\bar{z}}^{(0)}$ only. All the other $\mathcal{T}$'s have less power of $t$. Thus we conclude that the terms in the parenthesis $[\cdots]$ in (E15) can be written as a polynomial of $t$ in the following way:

$$\begin{aligned}
& \Bigg[ (\mathcal{T}_L^{(z)} + \mathcal{T}_{\mathcal{O}(1)}^{\bar{z}} + \mathcal{T}_{\bar{z}}^{(0)}) \delta(x - \bar{x}) - t \left\{ \left( \frac{x}{z_{35}} - \frac{\bar{x}}{z_{25}} \right) \mathcal{T}_L^{(0)} + \mathcal{T}_{\mathcal{O}(1)}^{(0)} \left( \frac{x}{\bar{z}_{25}} - \frac{\bar{x}}{\bar{z}_{35}} \right) \right\} \delta'(x - \bar{x}) \\
& \quad - t \left\{ (\Delta_1 - 1) \frac{\sigma_{1,2}}{\sigma_{1,1}} + (\Delta_2 - 1) \frac{\sigma_{2,2}}{\sigma_{2,1}} + (\Delta_3 - 1) \frac{\sigma_{3,2}}{\sigma_{3,1}} - \Delta \frac{\sum_{i=1}^{3} \sigma_{i,2} u_{i5}}{\mathcal{D}} \right\} \mathcal{T}_L^{(0)} \delta(x - \bar{x}) \\
& \quad - t \left\{ (\Delta_1 - 1) \frac{\sigma_{1,3}}{\sigma_{1,1}} + (\Delta_2 - 1) \frac{\sigma_{2,3}}{\sigma_{2,1}} + (\Delta_3 - 1) \frac{\sigma_{3,3}}{\sigma_{3,1}} - \Delta \frac{\sum_{i=1}^{3} \sigma_{i,3} u_{i5}}{\mathcal{D}} \right\} \mathcal{T}_{\mathcal{O}(1)}^{(0)} \delta(x - \bar{x}) \Bigg] \\
& = \sum_{k=1}^{5} t^k \mathcal{F}_k^{(\bar{z})}(\{\epsilon_i, z_i, \bar{z}_i\}),
\end{aligned} \tag{E16}$$

where once again the explicit expressions of $\mathcal{F}_k^{(\bar{z})}(\{\epsilon_i, z_i, \bar{z}_i\})$ are not relevant for our discussions. Substituting (E16) in (E15) and performing the $t$-integral, we finally get

$$\tilde{B}_5|_{\mathcal{O}(\bar{z}_{45})} = -2^5 \frac{\Gamma(\Delta)}{(i\mathcal{D})^{\Delta}} \prod_{i=1}^{3} (\epsilon_i \sigma_{i,1})^{\Delta_i - 1} \sum_{k=1}^{5} B(\Delta_4 + k - 1, \Delta_5 - 1) \mathcal{F}_k^{(\bar{z})}(\{\epsilon_i, z_i, \bar{z}_i\}). \tag{E17}$$

This is the form for the $\mathcal{O}(\bar{z}_{45})$ 5-point amplitude which we use in the main text of this paper.

## APPENDIX F: *w*-ALGEBRA PRIMARIES

Let us start with the universal term in the OPE between two positive helicity hard gravitons given by

$$G^+_{\Delta_1}(z,\bar{z}) G^+_{\Delta}(0,0) = -\frac{1}{z} \sum_{n=0}^{\infty} B(\Delta_1 - 1 + n, \Delta - 1) \frac{\bar{z}^{n+1}}{n!} \bar{\partial}^n G^+_{\Delta+\Delta_1}(0,0). \tag{F1}$$

We now take the conformal soft limit, first by setting $\Delta_1 = k + \varepsilon$ and then taking $\varepsilon \to 0$ to get

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon G^+_{k+\varepsilon}(z,\bar{z}) G^+_{\Delta}(0,0) &= -\frac{1}{z} \sum_{n=0}^{\infty} \left[\lim_{\varepsilon\to 0} \varepsilon B(k-1+n+\varepsilon, \Delta-1)\right] \frac{\bar{z}^{n+1}}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0) \\ \Rightarrow H^k(z,\bar{z}) G^+_{\Delta}(0,0) &= -\frac{1}{z} \sum_{n=0}^{\infty} \left[\lim_{\varepsilon\to 0} \varepsilon B(k-1+n+\varepsilon, \Delta-1)\right] \frac{\bar{z}^{n+1}}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0). \end{aligned} \tag{F2}$$

Next, we mode expand the soft graviton operator $H^k(z,\bar{z})$ on the lhs of (F2) according to (3.3) and get

$$\sum_{m=\frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_m(z)}{\bar{z}^{m+\frac{k-2}{2}}} G^+_{\Delta}(0,0) = -\frac{1}{z} \sum_{n=0}^{\infty} \left[\lim_{\varepsilon\to 0} \varepsilon B(k-1+n+\varepsilon, \Delta-1)\right] \frac{\bar{z}^{n+1}}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0). \tag{F3}$$

By comparing the terms at order $\bar{z}^{n+1}$ on both the sides of (F3) for $0 \le n \le 1-k$, we get

$$H^k_{\frac{2-k}{2}-n-1}(z) G^+_{\Delta}(0,0) = -\frac{1}{z} \left[\lim_{\varepsilon\to 0} \varepsilon B(k-1+n+\varepsilon, \Delta-1)\right] \frac{1}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0). \tag{F4}$$

Now we use the holomorphic mode expansion (3.4) of the currents $H^k_{\frac{2-k}{2}-n-1}(z)$ in the above equation and obtain,

$$\sum_{\alpha} z^{-\alpha-\frac{k+2}{2}} H^k_{\alpha,\frac{2-k}{2}-n-1} G^+_{\Delta}(0,0) = -\frac{1}{z} \left[\lim_{\varepsilon\to 0} \varepsilon B(k-1+n+\varepsilon, \Delta-1)\right] \frac{1}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0). \tag{F5}$$

We can see from the above equation, that there is only a simple pole at $z = 0$ on the rhs. Thus, the holomorphic singularity structure of the above Eq. (F5) tells us that the following conditions should hold:

$$H^k_{-\frac{k+2}{2}+m,\frac{2-k}{2}-n-1} G^+_{\Delta}(0,0) = -\left[\lim_{\varepsilon\to 0} \varepsilon B(k-1+n+\varepsilon, \Delta-1)\right] \frac{1}{n!} \bar{\partial}^n G^+_{\Delta+k}(0,0) \tag{F6}$$

for $m = 1$ and

$$H^k_{-\frac{k+2}{2}+m,\frac{2-k}{2}-n-1} G^+_{\Delta}(0,0) = 0 \tag{F7}$$

for $m > 1$ and $0 \le n \le 1-k$ with $k = 1, 0, -1, \ldots$.

Moreover, from (F3), one can see that there is no term on the RHS that goes like $\bar{z}^0$. Thus, on the lhs, the coefficients of the $\bar{z}^0$ term should also vanish which gives the following condition:

$$H^k_{\frac{2-k}{2}}(z) G^+_{\Delta}(0,0) = 0. \tag{F8}$$

This equation implies

$$H^k_{-\frac{k+2}{2}+m,\frac{2-k}{2}} G^+_{\Delta}(0,0) = 0, \qquad m \ge 1. \tag{F9}$$

## APPENDIX G: TRANSFRORMATION OF THE MHV NULL STATES UNDER $sl_2(R)_V$ AND $\overline{sl_2(R)}$ ALGEBRAS

In this section of the appendix, we list the transformation properties of all the MHV null states appearing at different orders of the OPE between two positive helicity outgoing gravitons under $sl_2(R)_V$ and $\overline{sl_2(R)}$ algebras. Let us first write down their explicit expressions in terms of the descendants of the $w$-algebra. We first write down the actions of the $H^{-1}_{\frac{1}{2},\frac{1}{2}}$ on the null states $\Phi_k(\Delta)$ given by (H1) and $\Psi_k(\Delta)$ given by (H2). They are given by

$$\begin{aligned}
H^{-1}_{\frac{1}{2},\frac{1}{2}}\Phi_k(\Delta) &= -\frac{1}{2}(k+1)(k+2)\Phi_{k+1}(\Delta-1) - \frac{1}{2}(\Delta+k-3)(\Delta+k-4)\Phi_k(\Delta-1) + \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)}\Phi_1(\Delta-1), \\
H^{-1}_{\frac{1}{2},\frac{1}{2}}\Psi_k(\Delta) &= -\frac{1}{2}(k+2)(k-1)\Psi_{k+1}(\Delta-1) - \frac{1}{2}(\Delta+k-3)(\Delta+k-4)\Psi_k(\Delta-1) - \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)}\Psi_1(\Delta-1), \\
H^{-1}_{\frac{1}{2},\frac{1}{2}}\Omega_k(\Delta) &= \frac{1}{2}(k+1)(k+2)\Omega_k(\Delta-1) - \frac{1}{2}(\Delta-4)(\Delta-5)\Omega_k(\Delta-1) - \frac{1}{2}(k+1)(k+2)\Omega_{k+1}(\Delta-1), \\
H^{-1}_{\frac{1}{2},\frac{1}{2}}\Pi_k(\Delta) &= \frac{1}{2}k(k+1)\Pi_k(\Delta-1) - \frac{1}{2}(\Delta-4)(\Delta-5)\Pi_k(\Delta-1) - \frac{1}{2}(k-1)(k+2)\Pi_{k+1}(\Delta-1).
\end{aligned} \tag{G1}$$

The actions of $H^1_{-\frac{1}{2},-\frac{1}{2}}$ on the MHV null states are given by

$$\begin{aligned}
H^1_{-\frac{1}{2},-\frac{1}{2}}\Phi_k(\Delta) &= -\Phi_k(\Delta+1) - \Phi_{k-1}(\Delta+1), \\
H^1_{-\frac{1}{2},-\frac{1}{2}}\Psi_k(\Delta) &= -\Psi_k(\Delta+1) - \Psi_{k-1}(\Delta+1), \\
H^1_{-\frac{1}{2},-\frac{1}{2}}\Omega_k(\Delta) &= -\Omega_k(\Delta+1), \\
H^1_{-\frac{1}{2},-\frac{1}{2}}\Pi_k(\Delta) &= -\Pi_{k+1}(\Delta+1).
\end{aligned} \tag{G2}$$

The actions of $H^0_{0,1}$ on the MHV null states are given by

$$\begin{aligned}
H^0_{0,1}\Phi_k(\Delta) &= 0, \\
H^0_{0,1}\Psi_k(\Delta) &= (k+2)\Phi_{k+1}(\Delta-1) - 2\frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)}\Phi_1(\Delta-1), \\
H^0_{0,1}\Omega_k(\Delta) &= 0, \\
H^0_{0,1}\Pi_k(\Delta) &= -(\Delta+k-3)\Omega_k(\Delta-1) + (k+2)\Omega_{k+1}(\Delta-1).
\end{aligned} \tag{G3}$$

In deriving the above transformation properties, we have used the algebra (3.5) and the action of different operators on the primaries given in Appendix F.

## APPENDIX H: REVIEW OF GENERAL STRUCTURE OF $w$-INVARIANT OPE

It was shown in [39], that the OPE between two positive helicity outgoing graviton primaries of any $w$-invariant theory can always be written in terms of the MHV OPE's and its null states. The MHV null states that can appear at $\mathcal{O}(z^0\bar{z}^0)$ and $\mathcal{O}(z^0\bar{z})$ are given by [19,20]

$$\Phi_k(\Delta) = \left[H^{1-k}_{\frac{k-3}{2},\frac{k+1}{2}}(-H^1_{-\frac{1}{2},-\frac{1}{2}})^k - \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)}H^1_{-\frac{3}{2},\frac{1}{2}}\right]G^+_{\Delta-1} \tag{H1}$$

and

$$\Psi_k(\Delta) = \Bigg[H^{-k}_{\frac{k-2}{2},\frac{k}{2}}(-H^1_{-\frac{1}{2},-\frac{1}{2}})^{k+1} - \frac{(-1)^k}{k!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-2)} \times H^0_{-1,0}(-H^1_{-\frac{1}{2},-\frac{1}{2}}) - \frac{(-1)^k k}{(k+1)!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta-3)}H^1_{-\frac{3}{2},-\frac{1}{2}}\Bigg]G^+_{\Delta-2}, \quad \text{(H2)}$$

respectively, where $k = 1, 2, 3, \ldots, \infty$. However, it is more convenient to work with the new basis defined by

$$\Omega_k(\Delta) = \sum_{n=1}^{k} \frac{1}{(k-n)!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta+n-2)}\Phi_n(\Delta) \quad \text{(H3)}$$

for the $\mathcal{O}(z^0\bar{z}^0)$ null states and similarly for the $\mathcal{O}(z^0\bar{z})$ null states the new basis is defined by

$$\Pi_k(\Delta) = \sum_{n=1}^{k} \frac{1}{(k-n)!}\frac{\Gamma(\Delta+k-2)}{\Gamma(\Delta+n-2)}\Psi_n(\Delta). \quad \text{(H4)}$$

There is another set of null states, which are of the Knizhnik-Zamolodchikov-type and decoupling of these null states give rise to differential equations for the scattering amplitudes [19,40,42,45,69–71]. We will discuss about these null states in the context of self-dual gravity in Sec. 2. Then, using these new basis (H3) and (H4) the OPE between two positive helicity outgoing graviton primaries with dimensions $\Delta_1$ and $\Delta_2$ of any $w$-invariant theory can always be written as

$$\begin{aligned} G^+_{\Delta_1}(z,\bar{z})G^+_{\Delta_2}(0,0) = {} & -\frac{\bar{z}}{z}B(\Delta_1-1,\Delta_2-1)G^+_{\Delta_1+\Delta_2}(0,0) \\ & + G^+_{\Delta_1}(z,\bar{z})G^+_{\Delta_2}(0,0)|_{\text{MHV at }\mathcal{O}(z^0\bar{z}^0)} + \sum_{p=1}^{n} B(\Delta_1-1+p,\Delta_2-1)\Omega_p(\Delta_1+\Delta_2) \\ & + G^+_{\Delta_1}(z,\bar{z})G^+_{\Delta_2}(0,0)|_{\text{MHV at }\mathcal{O}(z^0\bar{z}^1)} + \bar{z}\sum_{p=1}^{n} B(\Delta_1+p,\Delta_2-1)\Pi_p(\Delta_1+\Delta_2+1) + \cdots, \end{aligned} \quad \text{(H5)}$$

where $G^+_{\Delta_1}(z,\bar{z})G^+_{\Delta_2}(0,0)|_{\text{MHV at }\mathcal{O}(z^0\bar{z}^0)}$ and $G^+_{\Delta_1}(z,\bar{z})\times G^+_{\Delta_2}(0,0)|_{\text{MHV at }\mathcal{O}(z^0\bar{z}^1)}$ are the MHV OPEs at $\mathcal{O}(z^0\bar{z}^0)$ and $\mathcal{O}(z^0\bar{z})$, respectively. It has been shown in [66] that the leading term in $\bar{z}$ is uniquely determined by the $sl_2(R)_V$ invariance. Once the leading term is known, the subleading terms in $\bar{z}$ of $\mathcal{O}(\frac{\bar{z}^q}{z})$, $q \geq 2$ are determined by the $\overline{sl_2(R)}$ invariance.

It was shown in [39], that both the MHV null states $\Omega_k(\Delta)$ and $\Pi_k(\Delta)$ form representations of $sl_2(R)_V$. However, these representations are reducible because for any integer $n \geq 0$, the subspaces spanned by $\{\Omega_{n+1}(\Delta), \Omega_{n+2}(\Delta), \cdots\}$ and $\{\Pi_{n+1}(\Delta), \Pi_{n+2}(\Delta), \cdots\}$ form a representation of $sl_2(R)_V$. Hence we can get smaller representations spanned by the states $\{\Omega_1(\Delta), \Omega_2(\Delta), \ldots, \Omega_n(\Delta)\}$ and $\{\Pi_1(\Delta), \Pi_2(\Delta), \ldots, \Pi_n(\Delta)\}$ if we set

$$\begin{aligned} \Omega_{k+1}(\Delta) &= 0, \qquad k \geq n \geq 0, \\ \Pi_{k+1}(\Delta) &= 0, \qquad k \geq n \geq 0. \end{aligned} \quad \text{(H6)}$$

Using the algebra (3.5), one can also check that the null states $\Omega_k(\Delta)$ and $\Pi_k(\Delta)$ are primaries under $\overline{sl_2(R)}$. Thus, the conditions (H6) are invariant under $\overline{sl_2(R)}$, hence under whole $w$-algebra.

We have showed in Sec. III that, the whole tower of $w$-currents can be generated using two subalgebras given by $\overline{sl_2(R)}$ and $sl_2(R)_V$. Moreover, the conditions (H6) are also invariant under $\overline{sl_2(R)}$ and $sl_2(R)_V$, and hence under the full $w$-algebra. Now, using these facts and the algebra (3.5), it is not hard to show the OPE (H5) is invariant under $w$-algebra. The important point we want to emphasize about the OPE (H5) is that the integer $n$ can take any arbitrary value without breaking the $w$-invariance. Hence, there exists a discrete infinite family of $w$-invariant OPEs. From (H5) it is already clear that $n = 0$ gives the MHV sector. In this paper, we have shown that $n = 4$ gives the OPE of the quantum self-dual gravity theory which is known to be $w$-invariant.

Now, the last thing we want to discuss in this section is that, the null states $\{\Omega_1(\Delta), \Omega_2(\Delta), \ldots, \Omega_n(\Delta)\}$ are not completely independent. For a given $n$, there is another set of $\lceil\frac{n}{2}\rceil$[10] nontrivial[11] states $\{\chi^1_n(\Delta), \ldots, \chi^{\lceil n/2\rceil}_n(\Delta)\}$ defined as

$$\begin{aligned} \chi^1_n(\Delta) &= \sum_{p=1}^{n}\Omega_p(\Delta), \\ \chi^i_n(\Delta) &= \sum_{p=i}^{n}\prod_{q=i}^{2i-2}(p-q)\Omega_p(\Delta), \quad i = 2, 3, \ldots, \lceil\frac{n}{2}\rceil, \end{aligned} \quad \text{(H7)}$$

[10] $\lceil\frac{n}{2}\rceil$ = Smallest integer $\geq \frac{n}{2}$.

[11] There are of course the $n$ states $\{\Omega_1(\Delta), \ldots, \Omega_n(\Delta)\}$ which transform in a representation of $sl_2(R)_V$ but, we cannot set them to zero because that will lead us again to the MHV sector.

which transform in a representation of the $sl_2(R)_V$ as a consequence of (H6). We can also set these states to zero

$$\chi_n^i(\Delta) = 0 \tag{H8}$$

without violating the $sl_2(R)_V$ or $\overline{sl_2(R)}$ symmetry.

## APPENDIX I: NULL STATES IN SELF-DUAL GRAVITY

In this appendix, we will derive the null states of the self-dual gravity appearing at different orders of the OPE. We will first start with the OPE between two positive helicity outgoing gravitons in the self-dual gravity derived in Sec. V E. It is given by

$$\begin{aligned} G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5) = &-\frac{\bar{z}_{45}}{z_{45}}B(\Delta_4-1,\Delta_5-1)G^+_{\Delta_4+\Delta_5}(z_5,\bar{z}_5) + B(\Delta_4-1,\Delta_5-1)H^1_{-\frac{3}{2},\frac{1}{2}}G^+_{\Delta_4+\Delta_5-1}(z_5,\bar{z}_5) \\ &+\sum_{k=1}^{4}B(\Delta_4+k-1,\Delta_5-1)\Omega_k(\Delta_4+\Delta_5) + \bar{z}_{45}\Big[B(\Delta_4-1,\Delta_5-1)G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\mathrm{MHV\ at\ }\mathcal{O}(\bar{z}_{45})} \\ &+\sum_{k=1}^{4}B(\Delta_4+k,\Delta_5-1)\Pi_k(\Delta_4+\Delta_5+1)\Big] + \cdots, \end{aligned} \tag{I1}$$

where $G^+_{\Delta_4}(z_4,\bar{z}_4)G^+_{\Delta_5}(z_5,\bar{z}_5)|_{\mathrm{MHV\ at\ }\mathcal{O}(\bar{z}_{45})}$ is given by (5.33). We now derive the null states appearing at $\mathcal{O}(1)$ and $\mathcal{O}(\bar{z}_{45})$.

### 1. Null states at $\mathcal{O}(1)$

We can see from (I1) that at $\mathcal{O}(1)$ the OPE truncates at $k=4$. Now we take the conformal soft limit $\Delta_4 \to -4$ in (I1). In this limit, the soft descendant that appear at $\mathcal{O}(1)$ on the lhs of (I1) is given by $H^{-4}_{1,3}G^+_{\Delta_5}(z_5,\bar{z}_5)$. After taking the same conformal soft limits on the rhs and comparing the results we get

$$\Omega_5(\Delta) = \sum_{j=1}^{5}\frac{1}{(5-j)!}\frac{\Gamma(\Delta+3)}{\Gamma(\Delta+j-2)}\Phi_j(\Delta) = 0, \tag{I2}$$

where $\Phi_j(\Delta)$ are given by (H1). Thus, we see that $\Omega_5(\Delta)$ is a null state of the self-dual gravity. Now we will show the consistency of (I2) under $w$-algebra. Under $sl_2(R)_V$, $\Omega_5(\Delta)$ transforms as (G1), (G2),

$$\begin{aligned} H^1_{-\frac{1}{2},-\frac{1}{2}}\Omega_5(\Delta) &= -\Omega_5(\Delta+1), \\ H^{-1}_{\frac{1}{2},\frac{1}{2}}\Omega_5(\Delta) &= 21\Omega_5(\Delta-1) - \frac{1}{2}(\Delta-4)(\Delta-5)\Omega_5(\Delta-1) \\ &\quad -21\Omega_6(\Delta-1), \end{aligned} \tag{I3}$$

and $H^0_{0,0} = 2\bar{L}_0$ is diagonal on these states. However, $\Omega_6(\Delta-1)$ is also a null state of the theory and thus (I2) is invariant under $sl_2(R)_V$. One can also check that

$$H^0_{0,1}\Omega_5(\Delta) = 0. \tag{I4}$$

Thus we see that (I2) is also invariant under $\overline{sl_2(R)}$. Hence we conclude that (I2) is invariant under $w$-algebra.

There is another set of null states (H7) at $\mathcal{O}(1)$ which can be found using the commutativity property of the OPE together with the conformal soft limits. In case of self-dual gravity, they are explicitly given by

$$\begin{aligned} \chi_4^1(\Delta) &= \sum_{p=1}^{4}\Omega_p(\Delta), \\ \chi_4^2(\Delta) &= \sum_{p=3}^{4}(p-2)\Omega_p(\Delta), \end{aligned} \tag{I5}$$

These null states also transform under the representation of $sl_2(R)_V$ and $\overline{sl_2(R)}$ algebra and as a consequence one can set them to 0 without violating the $w$-symmetry. The null states (I5) play an important role in showing the invariance of the Knizhnik-Zamolodchikov-type null state under $w$-algebra which will be discussed in the next subsection.

### 2. Null states at $\mathcal{O}(\bar{z}_{45})$: Knizhnik-Zamolodchikov-type null state

Knizhnik-Zamolodchikov (KZ)-type null states occur at $\mathcal{O}(z^0\bar{z}^1)$ of the OPE. The easiest way to derive it is to use the commutativity property of the OPE and conformal soft limits together. So we start with the commutativity property of the OPE given by

$$G^+_{\Delta_1}(z_1,\bar{z}_1)G^+_{\Delta_2}(z_2,\bar{z}_2) = G^+_{\Delta_2}(z_2,\bar{z}_2)G^+_{\Delta_1}(z_1,\bar{z}_1). \tag{I6}$$

Now we use the OPE (I1) in (I6), and take the leading conformal soft limits $\Delta_1 \to 1$. Then by comparing the terms at $\mathcal{O}(\bar{z}_{45})$ we get the following Knizhnik-Zamolodchikov-type equation:

$$\Xi_4(\Delta) = \xi(\Delta) + \sum_{k=1}^{4} \Pi_k(\Delta+1) = 0, \tag{I7}$$

where $\xi(\Delta)$ is the KZ-type null state in the MHV sector given by [19]

$$\xi(\Delta) = \boxed{L_{-1}} G^{+}_{\Delta} + H^{0}_{0,-1} H^{1}_{-\frac{3}{2},\frac{1}{2}} G^{+}_{\Delta-1} + H^{0}_{-1,0} G^{+}_{\Delta} + (\Delta-1) H^{1}_{-\frac{3}{2},-\frac{1}{2}} G^{+}_{\Delta-1}. \tag{I8}$$

We have used that $\chi^1_4(\Delta)$ is a null state in this theory to arrive at the form (I7). One can check that (I7) is consistent under the actions of $\overline{sl_2(R)}$ and $sl_2(R)_V$ generators. For example,

$$H^{0}_{0,1}\Xi_4(\Delta) = 6\Omega_5(\Delta) - (\Delta-3)\chi^1_4(\Delta). \tag{I9}$$

We have already shown that $\Omega_5(\Delta)$ and $\chi^1_4(\Delta)$ are both null states in this theory, so we get

$$H^{0}_{0,1}\Xi_4(\Delta) = 0. \tag{I10}$$

Therefore, $\Xi_4(\Delta)$ is an $\overline{sl_2(R)}$ primary.

Similarly, we have

$$H^{-1}_{\frac{1}{2},\frac{1}{2}}\Xi_4(\Delta) = -\frac{1}{2}(\Delta-2)(\Delta-3)\Xi_4(\Delta-1) - 9\Pi_5(\Delta) - H^{-1}_{\frac{1}{2},-\frac{1}{2}}\chi^1_4(\Delta) - H^{0}_{0,-1}((\Delta-1)\chi^1_4(\Delta-1) + \chi^2_4(\Delta-1)). \tag{I11}$$

However, since $\Pi_5(\Delta), \chi^1_4(\Delta)$, and $\chi^2_4(\Delta)$ are null states in the theory, we get

$$H^{-1}_{\frac{1}{2},\frac{1}{2}}\Xi_4(\Delta) = -\frac{1}{2}(\Delta-2)(\Delta-3)\Xi_4(\Delta-1). \tag{I12}$$

Therefore, $\Xi_4(\Delta)$ transforms under a representation of the $sl_2(R)_V$ and we can consistently set it to zero without violating the $sl_2(R)_V$ symmetry. Hence, we conclude that (I7) is indeed $w$ invariant. Decoupling of null states gives rise to differential equations which the graviton scattering amplitudes in this theory have to satisfy.

## APPENDIX J: INVARIANCE OF THE SELF-DUAL OPE UNDER *w*-ALGEBRA

In [39], it was shown that the OPE (H5) is invariant under $w$-algebra for any arbitrary truncation in $n$, which has been reviewed in Appendix H. We have shown in Sec. V E that self-dual OPE truncates at $n = 4$ of the general OPE (H5). Thus, we can say that the invariance of the self-dual OPE under $w$-algebra is guaranteed. However, for the sake of completeness of this paper and for the better readability, we will repeat the same analysis here with focusing on the self-dual OPE. As discussed in Sec. III, the whole $w$-algebra can be derived by the combined action of $sl_2(R)_V$ and $\overline{sl_2(R)}$. Thus, it is enough to show the invariance of the OPE under these two subalgebras.

### 1. *w*-invariance at $\mathcal{O}(1)$

Let us start with the $\mathcal{O}(1)$ OPE. We write it here again for the readers convenience,

$$G^{+}_{\Delta_1}(z,\bar z)G^{+}_{\Delta_2}(0,0)|_{\mathcal{O}(1)} = B(\Delta_1-1,\Delta_2-1)H^{1}_{-\frac{3}{2},\frac{1}{2}}G^{+}_{\Delta_1+\Delta_2-1}(0,0) + \sum_{k=1}^{4} B(\Delta_1+k-1,\Delta_2-1)\Omega_k(\Delta_1+\Delta_2). \tag{J1}$$

We now show that it is invariant under the two subalgebras $sl_2(R)_V$ and $\overline{sl_2(R)}$.

#### a. $sl_2(R)_V$ invariance

To show the invariance of the OPE, we need the action of the $sl_2(R)_V$ on the MHV null states $\Omega_k(\Delta)$ that can appear at $\mathcal{O}(1)$. These actions were computed in [39] and reviewed in Appendix G. We also need the commutator algebra (3.5) along with the action of these generators on the graviton primaries given by (see Appendix F),

$$H^{1}_{-\frac{1}{2},-\frac{1}{2}}G^{+}_{\Delta}(z,\bar z) = -G^{+}_{\Delta+1}(z,\bar z),$$

$$H^{-1}_{\frac{1}{2},\frac{1}{2}}G^{+}_{\Delta}(z,\bar z) = -\frac{1}{2}[(\Delta-2)(\Delta-3) + 4(\Delta-2)\bar z\partial_{\bar z} + 3\bar z^2\partial^2_{\bar z}]G^{+}_{\Delta-1}(z,\bar z). \tag{J2}$$

Using Appendix G, (3.5), and (J2), it is not difficult to show that the $\mathcal{O}(1)$ OPE (J1) is invariant under $H^{1}_{-\frac{1}{2},-\frac{1}{2}}$ whereas the action of $H^{-1}_{\frac{1}{2},\frac{1}{2}}$ on both the sides of the OPE (J1) gives

$$H^{-1}_{\frac{1}{2},\frac{1}{2}}(\text{rhs} - \text{lhs}) \text{ of (J1)} = -12B(\Delta_1+3,\Delta_2-1) \times \Omega_5(\Delta_1+\Delta_2-1) \tag{J3}$$

However, we have already shown in Appendix 1, that $\Omega_5(\Delta)$ is a null state of the self-dual gravity appearing at $\mathcal{O}(1)$ of the OPE and as a consequence we can set it to 0. Hence, we conclude that the $\mathcal{O}(1)$ self-dual OPE (J1) is invariant under the $sl_2(R)_V$ algebra.

#### b. $\overline{sl_2(R)}$ invariance

It was shown in [19], that the OPE in the MHV sector is invariant under the action of $H^{0}_{0,1}$.[12] Also from (G3), we can see that the null states $\Omega_k(\Delta)$ are annihilated by $H^{0}_{0,1}$.

[12] $H^{0}_{0,1} \sim \bar L_1$.

Therefore, we can say that $\mathcal{O}(1)$ self-dual OPE (J1) is invariant under the $\overline{sl_2(R)}$ algebra.

### 2. $w$-invariance at $\mathcal{O}(\bar{z})$

We now move on to showing the $w$-invariance of the self-dual OPE at $\mathcal{O}(\bar{z})$. Let us first write down the $\mathcal{O}(\bar{z})$ OPE (5.35) again,

$$\begin{aligned} G^{+}_{\Delta_1}(z,\bar{z})G^{+}_{\Delta_2}(0,0)|_{\mathcal{O}(\bar{z})} &= B(\Delta_1-1,\Delta_2-1)G^{+}_{\Delta_1}(z,\bar{z}) \\ &\times G^{+}_{\Delta_2}(0,0)|_{\text{MHV at}\,\mathcal{O}(\bar{z}_{45})} \\ &+\sum_{k=1}^{4}B(\Delta_1+k,\Delta_2-1) \\ &\times \Pi_k(\Delta_1+\Delta_2+1). \end{aligned} \tag{J4}$$

From the previous subsection, it is clear that the $w$-invariance of the OPE at $\mathcal{O}(\bar{z})$ is guaranteed to follow if we can show that it is invariant under the two subalgebras $sl_2(R)_V$ and $\overline{sl_2(R)}$. Among the generators of these two subalgebras, we only show the invariance of the OPE (J4) under the actions of $H^{-1}_{\frac{1}{2},\frac{1}{2}}$ and $H^{0}_{0,1}$. This is mainly because the invariance of the OPE (J4) under the rest of the generators are fairly easy to show. By applying $H^{-1}_{\frac{1}{2},\frac{1}{2}}$ on both sides of the OPE (J4) we get

$$\begin{aligned} H^{-1}_{\frac{1}{2},\frac{1}{2}}(\text{rhs}-\text{lhs})\text{ of (J4)} &= -9B(\Delta_1+4,\Delta_2-1) \\ &\times \Pi_5(\Delta_1+\Delta_2) \end{aligned} \tag{J5}$$

and for $H^{0}_{0,1}$ we have

$$\begin{aligned} H^{(0)}_{0,1}(\text{rhs}-\text{lhs})\text{ of (J4)} &= 6B(\Delta_4+4,\Delta_5-1) \\ &\times \Omega_5(\Delta_1+\Delta_2). \end{aligned} \tag{J6}$$

However, from Appendix I, we know that both $\Pi_5(\Delta)$ and $\Omega_5(\Delta)$ are the null states of the self-dual gravity appearing at $\mathcal{O}(\bar{z})$ and $\mathcal{O}(1)$, respectively. Thus, we conclude that the $\mathcal{O}(\bar{z})$ OPE in self-dual gravity is also invariant under $sl_2(R)_V$ and $\overline{sl_2(R)}$, and hence under the whole $w$-algebra.


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