Parameter dependence of entanglement spectra in quantum field theories

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(Received 3 March 2024; accepted 24 March 2024; published 15 April 2024)

In this paper, we explore the characteristics of reduced density matrix spectra in quantum field theories. Previous studies mainly focus on the function $\mathcal{P}(\lambda) \coloneqq \sum_i \delta(\lambda - \lambda_i)$, where λ_i denote the eigenvalues of the reduced density matrix. We introduce a series of functions designed to capture the parameter dependencies of these spectra. These functions encompass information regarding the derivatives of eigenvalues concerning the parameters, notably including the function $\mathcal{P}_{\alpha_J}(\lambda) \coloneqq \sum_i \frac{\partial \lambda_i}{\partial \alpha_J} \delta(\lambda - \lambda_i)$, where α_J denotes the specific parameter. Computation of these functions is achievable through the utilization of Rényi entropy. Intriguingly, we uncover compelling relationships among these functions and demonstrate their utility in constructing the eigenvalues of reduced density matrices for select cases. We perform computations of these functions across several illustrative examples. Especially, we conducted a detailed study of the variations of $\mathcal{P}(\lambda)$ and $\mathcal{P}_{\alpha_J}(\lambda)$ under general perturbation, elucidating their physical implications. In the context of holographic theory, we ascertain that the zero point of the function $\mathcal{P}_{\alpha_J}(\lambda)$ possesses universality, determined as $\lambda_0 = e^{-S}$, where S denotes the entanglement entropy of the reduced density matrix. Furthermore, we exhibit potential applications of these functions in analyzing the properties of entanglement entropy.

DOI: 10.1103/PhysRevD.109.086016

I. INTRODUCTION

Entanglement has emerged as a novel tool for discerning the structure of quantum field theories (QFTs) in recent years. Typically, various measures are introduced to quantify entanglement, with one of the most extensively studied being the entanglement entropy (EE). In certain QFTs, the entanglement entropy can be computed either analytically or numerically [1–6]. EE characterizes the quantum correlations between different types of spatial regions within field theory. Interestingly, within the framework of AdS/CFT [7–9], entanglement entropy has been found to be related to minimal surfaces in the dual spacetime, following a law similar to the area law observed in black holes [10,11].

By partitioning the entire system into two parts, denoted as A and its complementary $\bar{A}$, one can introduce the reduced density matrix $\rho_A \coloneqq \mathrm{tr}_{\bar{A}}\rho$, where ρ represents the density matrix of the system. EE can then be considered as a function of ρ_A, defined as the von Neumann entropy $S(\rho_A) \coloneqq -\mathrm{tr}\rho_A \log \rho_A$. In QFTs, the replica method via Euclidean path integrals is commonly employed to evaluate EE. It is necessary to initially compute the Rényi entropy, defined as

$$S^{(n)}(\rho_A) \coloneqq \frac{\log \mathrm{tr}\rho_A^n}{1-n}, \tag{1}$$

for a positive integer n. After analytically continuing n to complex numbers, the entanglement entropy is expressed as $S(\rho_A) = \lim_{n\to 1} S^{(n)}(\rho_A)$.

In QFTs, the trace in $\rho_A = \mathrm{tr}_{\bar{A}}\rho$ is typically considered a formal definition. Unlike in finite-dimensional examples, obtaining ρ_A directly through the trace operation seems unfeasible. Nevertheless, it is apparent that ρ_A encompasses the complete information of the subsystem A. Consequently, reconstructing ρ_A using entanglement measures becomes a significant area of investigation.

The spectra of ρ_A is studied in many-body system as a new topological order [12]. In two-dimensional conformal field theories (CFTs), the entanglement spectra can also be obtained using Rényi entropy [13]; see also [14]. In [15,16], the authors further investigate the entanglement spectra for the theory with holographic dual. An interesting result is that there exists an approximated state for any given states with holographic dual. Using the spectra decomposition, it is also possible to construct the so-called fixed area states in CFTs [17]. The approximated state, as

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we mentioned above, can be understood as one special fixed area state. The fixed area states are introduced in [18,19] motivated by the similarity between AdS/CFT and quantum error correction (QEC) code [20]. Hence, the entanglement spectra of ρ_A holds significance in comprehending the entanglement structure of QFTs, alongside elucidating the relationship between geometry and entanglement. There are also many studies on entanglement spectra in various directions; see, e.g., [21–26].

Generally, the Rényi entropy $S^{(n)}(\rho_A)$ encapsulates information about the spectra present within the reduced density matrix ρ_A. Numerous studies have investigated methods to obtain the density of the spectrum from Rényi entropy. For a specific theory and subsystem A, $S^{(n)}(\rho_A)$ is anticipated to depend on dimensional or dimensionless parameters, such as the subsystem's size, time, and coupling constants of the theory. It is presumed that the spectra would be related to these parameters. Nevertheless, the density of the eigenvalues may not adequately capture the parameter-dependent nature of the spectrum.

In this paper, we introduce a series of functions designed specifically to accomplish this objective; see details of the definitions in Sec. II. Roughly, the density of eigenvalues represents the probability distribution of the eigenvalues. The functions presented in this paper aim to capture the changes in eigenvalues concerning a specific parameter. For example, we introduce the function

$$\mathcal{P}_{\alpha_J}(\lambda) \coloneqq \sum_i \frac{\partial \lambda_i}{\partial \alpha_J} \delta(\lambda_i - \lambda), \tag{2}$$

where α_J is any parameter. The function $\mathcal{P}_{\alpha_J}$ can be taken as the average value of $\frac{\partial \lambda_i}{\partial \alpha_J}$ at the eigenvalue λ. These functions can be computed using the Rényi entropy, enabling an examination of the eigenvalue variations. Additionally, we uncover intriguing relationships among these functions. If the eigenvalues of ρ_A satisfy more conditions, we can demonstrate the possibility of reconstructing the form of the eigenvalues using the results obtained from these functions. This has been carried out for a single interval in the vacuum state of two-dimensional CFTs. The eigenvalues obtained by our method are consistent with the known results.

We have calculated these functions in several examples within two-dimensional CFTs, including scenarios such as the single interval in the vacuum state, short intervals in arbitrary states, and obtaining a general result for the perturbation state $\rho + \delta\rho$. Based on these results, we have discussed the inherent properties of these functions. Additionally, we have made interesting observations regarding theories with a holographic dual. In the semiclassical limit, it has been found that the zero point of $\mathcal{P}_{\alpha_J}(\lambda)$ is given by $\lambda_0 = e^{-S}$, where S is the EE for ρ_A. This particular value also appears in [16], where an approximated state for ρ_A is constructed within the semiclassical limit. However, the relationship between these two results remains unclear.

We also delve into the potential application of these functions in characterizing the phase transition of EE. It has been observed that the shape of the function $\mathcal{P}_{\alpha_J}$ does indeed mirror the variations in EE concerning the parameter. Our paper merely establishes a framework for studying the parameter dependence of the entanglement spectra in QFTs. On this basis, there exist numerous intriguing questions worthy of exploration.

The remainder of the paper is organized as follows. Section II introduces a series of functions, including $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$, which describe the entanglement spectra along with their parameter dependencies and discusses their properties. Following this, Sec. III presents the calculation of $\mathcal{P}$ and $\mathcal{P}_L$ in the vacuum state of 2D CFTs as an illustrative example. In Sec. IV, we delve into the calculation for an arbitrary state of 2D CFTs with a short interval. Notably, it reveals a shift in the zero point of $\mathcal{P}_l$ compared to the vacuum state. Section V examines the scenario where the density matrix experiences a perturbation, denoted as $\rho = \rho_0 + \delta\rho$. This section investigates the alterations in $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ subsequent to the perturbation. We also provide explanations for each term in the obtained results. In Sec. VI, the paper explores the computation of $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ in holographic theory, employing the saddle point approximation. Furthermore, it discusses the zero point of $\mathcal{P}_{\alpha_J}$ within this context and find a universal result of the zero point. Section VII extends the discussion to analyze the derivative of entanglement entropy using the function $\mathcal{P}_{\alpha_J}$. Finally, Sec. VIII presents the concluding remarks. Detailed calculations are provided in the appendixes.

II. GENERAL SETUP

Assume the spectra of ρ_A are $\{\lambda_i\}$. We can define the spectra density as

$$\mathcal{P}(\lambda) \coloneqq \sum_i \delta(\lambda_i - \lambda). \tag{3}$$

Roughly, it can be understood as the number of degenerate eigenstates for the eigenvalue λ.

By the definition, it is easy to know that it has property

$$\int_{-\infty}^{+\infty} f(\lambda)\mathcal{P}(\lambda) d\lambda = \sum_i f(\lambda_i). \tag{4}$$

For example, when $f(\lambda) = \lambda$, $\int_{-\infty}^{+\infty} \lambda \mathcal{P}(\lambda) d\lambda = \sum_i \lambda_i = 1$; when $f(\lambda) = -\lambda \log \lambda$, $\int_{-\infty}^{+\infty} -\lambda \log \lambda \mathcal{P}(\lambda) d\lambda = -\sum_i \lambda_i \times \log \lambda_i = S_{\mathcal{A}}$. From the above example, it can also be seen that, when we obtain $\mathcal{P}(\lambda)$, the entanglement entropy $S_{\mathcal{A}}$ can be easily calculated, so it can be seen that the information of the entanglement spectra is greater than the entanglement entropy.

Notice in $\{\lambda_i\}$, $\lambda_i > 0$ and there is a maximum eigenvalue λ_m, so we can rewrite (4) to

$$\int_0^{\lambda_m} f(\lambda)P(\lambda)d\lambda = \sum_i f(\lambda_i). \tag{5}$$

Generally, the eigenvalue λ_i can be taken as functions of some parameters denoted by $\{\alpha_J\}$. To characterize the relation we would like to introduce and explore a new quantity

$$\mathcal{P}_{\alpha_J}(\lambda) := \sum_i \frac{\partial\lambda_i}{\partial\alpha_J}\delta(\lambda_i - \lambda). \tag{6}$$

The function $\bar{\mathcal{P}}_{\alpha_J}(\lambda) := \frac{\mathcal{P}_{\alpha_J}(\lambda)}{\mathcal{P}(\lambda)}$ can be taken as the average value of $\frac{\partial\lambda_i}{\partial\alpha_J}$ for the eigenvalue λ. It is obvious that we should have the constraint

$$\int_0^{\lambda_m} d\lambda\mathcal{P}_{\alpha_J}(\lambda) = 0, \tag{7}$$

where λ_m is the maximal eigenvalue of ρ_A. The above constraint comes from the normalization of the reduced density matrix. We also have interest at

$$\mathcal{P}_{\alpha_{Jm}} := \sum_i \frac{\partial^m\lambda_i}{\partial\alpha_J^m}\delta(\lambda_i - \lambda), \tag{8}$$

$$\mathcal{P}_{\alpha_J^m} := \sum_i \left(\frac{\partial\lambda_i}{\partial\alpha_J}\right)^m \delta(\lambda_i - \lambda), \tag{9}$$

for the integer m.

Most generally, we can also define the following quantities:

$$\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})} := \sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\cdots \times \frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\delta(\lambda_i - \lambda), \tag{10}$$

for given integer n and $\{m_1, \ldots m_n\}$. We can define the order $\mathcal{N}$ of the functions by counting the power of the derivatives, $\mathcal{N} := \sum_{i=1}^n m_i$.

Especially, we can get

$$\mathcal{P}_{\alpha_{J_1}\ldots\alpha_{J_m}} := \mathcal{P}_{(\alpha_{J_1}\ldots\alpha_{J_m})} = \sum_i \frac{\partial^m\lambda_i}{\partial\alpha_{J_1}\ldots\partial\alpha_{J_m}}\delta(\lambda_i - \lambda), \tag{11}$$

$$\mathcal{P}_{(\alpha_{J_1})\ldots(\alpha_{J_m})} = \sum_i \frac{\partial\lambda_i}{\partial\alpha_{J_1}}\cdots\frac{\partial\lambda_i}{\partial\alpha_{J_m}}\delta(\lambda_i - \lambda), \tag{12}$$

for a given integer m. When $\alpha_{J_1} = \alpha_{J_2} = \cdots = \alpha_{J_m} = \alpha_J$, (11) becomes (8), and (12) becomes (9).

By the definition, the above functions are determined once all the eigenvalues are given. But in most cases, especially examples in QFTs, we have very limited information about the eigenvalues. Our motivation to define these functions is to gain more information on the distribution and parameter dependence of the eigenvalues. These functions appear to be independent, but we will demonstrate later that there are connections between them, which are implicit in their definitions.

A. Relations among the functions

All the functions that we defined in the last section can be evaluated by the Rényi entropy. Recall the definition of Rényi entropy

$$S^{(n)} = \frac{1}{1-n}\log \mathrm{Tr}_{\mathcal{A}}\rho_{\mathcal{A}}^n = \frac{1}{1-n}\log\sum_i \lambda_i^n. \tag{13}$$

By using the property (5), the above equation can be rewritten as

$$\begin{aligned}\sum_i \lambda_i^n &= e^{(1-n)S^{(n)}},\\ \sum_i\int_0^{\lambda_m}\lambda^n\delta(\lambda_i - \lambda)d\lambda &= e^{(1-n)S^{(n)}},\\ \int_0^{\lambda_m}\lambda^n\mathcal{P}(\lambda)d\lambda &= e^{(1-n)S^{(n)}}.\end{aligned} \tag{14}$$

Compare the form of Laplace transformation:

$$\mathcal{L}[f(t)] := \int_0^\infty e^{nt}f(t)dt. \tag{15}$$

We find that the above formula (14) is similar to the form of Laplace transformation (15). For further calculation, let $\lambda = e^{-b-t}$, where $b = -\log\lambda_m$.

Actually, from (13) we have $b = \lim_{n\to\infty}S^{(n)}$. By using the above replacement, it is easy to know $\lambda = 0$ corresponding $t = \infty$ and $\lambda = \lambda_m$ corresponding $t = 0$, so (14) becomes

$$\mathcal{L}[\mathcal{P}(e^{-b-t})e^{-b-t}] = e^{(1-n)S^{(n)}}e^{nb}. \tag{16}$$

By using inverse Laplace transformation, we can obtain the density of eigenvalue $\mathcal{P}$:

$$\mathcal{P}(e^{-b-t}) = \lambda^{-1}\mathcal{L}^{-1}[e^{nb+(1-n)S^{(n)}}], \tag{17}$$

where $\mathcal{L}^{-1}$ is defined as

$$f(t) = \mathcal{L}^{-1}[F(n)] = \frac{1}{2\pi i}\int_{\gamma_0-i\infty}^{\gamma_0+i\infty} F(n)e^{nt}dn \quad (18)$$

and γ_0 is chosen for the convergence of the integration.

So we can say that density of eigenvalue $\mathcal{P}$ and Rényi entropy $S^{(n)}$ are each other's (inverse) Laplace transformation.

If taking derivative with respect to α_J for both sides of the above equation (13), we have

$$\sum_i n\lambda_i^{n-1}\frac{\partial\lambda_i}{\partial\alpha_J} = (1-n)\frac{\partial S^{(n)}}{\partial\alpha_J}e^{(1-n)S^{(n)}}. \quad (19)$$

By using the definition (6), the above equation can be rewritten as

$$\int_0^{\lambda_m} d\lambda n\lambda^{n-1}\mathcal{P}_{\alpha_J}(\lambda) = (1-n)\frac{\partial S^{(n)}}{\partial\alpha_J}e^{(1-n)S^{(n)}}. \quad (20)$$

Let $\lambda = e^{-b-t}$, where $b = -\log\lambda_m$, and we have

$$\int_0^\infty dte^{-nt}\mathcal{P}_{\alpha_J}(e^{-b-t}) = \frac{1-n}{n}e^{nb+(1-n)S^{(n)}}\frac{\partial S^{(n)}}{\partial\alpha_J}. \quad (21)$$

Similar as the case for density of eigenvalue $\mathcal{P}$ (17), one could evaluate $\mathcal{P}_{\alpha_J}$ by using inverse Laplace transformation, that is,

$$\mathcal{P}_{\alpha_J}(e^{-b-t}) = \mathcal{L}^{-1}\left[\frac{1-n}{n}e^{nb+(1-n)S^{(n)}}\frac{\partial S^{(n)}}{\partial\alpha_J}\right]. \quad (22)$$

Note that the expression in the square brackets is a function of n. One could obtain $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ once knowing the Rényi entropy $S^{(n)}$.

By using the property of inverse Laplace transformation, one could derive the relations between the functions. Using (17) and (22), we find

$$\frac{\partial\mathcal{P}(\lambda)}{\partial\alpha_J} = -\frac{\partial\mathcal{P}_{\alpha_J}(\lambda)}{\partial\lambda}. \quad (23)$$

Further taking derivative with respect to a_J for both sides of (19), other quantities (8) and (9) would appear. One could obtain these quantities by a similar method as above. For example, taking twice derivative we would obtain

$$\sum_i n(n-1)\lambda_i^{(n-2)}\left(\frac{\partial\lambda_i}{\partial\alpha_J}\right)^2 + \sum_i n\lambda_i^{(n-1)}\frac{\partial^2\lambda_i}{\partial\alpha_J^2} = \frac{\partial^2}{\partial L^2}e^{(1-n)S^{(n)}}. \quad (24)$$

Similarly, we will have

$$\int_0^{\lambda_m} n(n-1)\lambda^{n-2}P_{\alpha_J^2}(\lambda)d\lambda + \int_0^{\lambda_m} n\lambda^{n-1}P_{\alpha_{J2}}(\lambda)d\lambda = \frac{\partial^2}{\partial\alpha_J^2}e^{(1-n)S^{(n)}}. \quad (25)$$

By using the property of inverse Laplace transformation and (22), we find

$$\frac{\partial\mathcal{P}_{\alpha_J}}{\partial\alpha_J} = \mathcal{P}_{\alpha_{J2}} - \frac{\partial\mathcal{P}_{\alpha_J^2}}{\partial\lambda}. \quad (26)$$

For higher power we can also obtain similar relations, as we will show in the next section.

B. Consistent with definition

In fact, the relations of the functions are also consistent with the definition of these functions. By the definition of $\mathcal{P}$, taking derivative with respect to α_J for $\mathcal{P}$ we have

$$\frac{\partial\mathcal{P}}{\partial\alpha_J} = \sum_i\frac{\partial}{\partial\alpha_J}\delta(\lambda_i-\lambda) = \sum_i\frac{\partial\lambda_i}{\partial\alpha_J}\delta'(\lambda_i-\lambda). \quad (27)$$

$\mathcal{P}$ depends on the parameter α_J through λ_i. One should keep in mind that λ is independent with the parameter. Similarly, $\frac{\partial\lambda_i}{\partial\alpha_J}$ is also independent with λ; thus, we find

$$\frac{\partial\mathcal{P}}{\partial\alpha_J} = -\frac{\partial}{\partial\lambda}\sum_i\frac{\partial\lambda_i}{\partial\alpha_J}\delta(\lambda_i-\lambda) = -\frac{\partial\mathcal{P}_{\alpha_J}}{\partial\lambda}. \quad (28)$$

By the same logic, we can derive the relation (E1) as

$$\begin{aligned}\frac{\partial}{\partial\alpha_J}P_{\alpha_J}(\lambda) &= \frac{\partial}{\partial\alpha_J}\sum_i\frac{\partial\lambda_i}{\partial\alpha_J}\delta(\lambda_i-\lambda)\\ &= \sum_i\frac{\partial^2\lambda_i}{\partial\alpha_J^2}\delta(\lambda_i-\lambda) + \sum_i\frac{\partial\lambda_i}{\partial\alpha_J}\frac{\partial\lambda_i}{\partial\alpha_J}\delta'(\lambda_i-\lambda)\\ &= \sum_i\frac{\partial^2\lambda_i}{\partial\alpha_J^2}\delta(\lambda_i-\lambda) + \sum_i\left(\frac{\partial\lambda_i}{\partial\alpha_J}\right)^2\\ &\quad\times\left(-\frac{\partial}{\partial\lambda}\delta(\lambda_i-\lambda)\right)\\ &= P_{\alpha_{J2}}(\lambda) - \frac{\partial}{\partial\lambda}P_{\alpha_J^2}(\lambda).\end{aligned} \quad (29)$$

Most generally, we can get that

$$\begin{aligned}
&\frac{\partial}{\partial\alpha_K}\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})}\\
&=\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}}\alpha_K)(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})}\\
&\quad+\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}}\alpha_K)\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})}+\cdots\\
&\quad+\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}}\alpha_K)}\\
&\quad-\frac{\partial}{\partial\lambda}\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})(\alpha_K)}. \qquad (30)
\end{aligned}$$

Please see Appendix A 1 for more details of the calculations.

With relation (30), it is easy to get an interesting conclusion about two parameters. Consider two unrelated parameters α_{J_1} and α_{J_2}, and it is easy to get

$$\frac{\partial}{\partial\alpha_2}\mathcal{P}_{\alpha_{J_1}}=\frac{\partial}{\partial\alpha_1}\mathcal{P}_{\alpha_{J_2}}. \qquad (31)$$

C. With further assumptions

In the above discussions, we find the functions at the order of $\mathcal{N}$ would have some relations. For example, for $\mathcal{N}=2$, $\mathcal{P}_{\alpha_J^2}$ and $\mathcal{P}_{\alpha_{J2}}$ are not independent. In fact, this means one cannot solve $\mathcal{P}_{\alpha_J^2}$ and $\mathcal{P}_{\alpha_{J2}}$ separately by only using Rényi entropy. To obtain them, we should have more assumptions.

In general, $\frac{\partial\lambda_i}{\partial\alpha_J}$ should not depend on the eigenvalue λ_i. But in some special case we find that $\frac{\partial\lambda_i}{\partial\alpha_J}$ can still be seen as a function of λ_i, that is, $\frac{\partial\lambda_i}{\partial\alpha_J}=f(\lambda_i,\alpha_J)$. We do not expect this is true for general cases. In the appendixes, we use simple examples to show this. For the special case we find all the functions can be solved.

With the assumption, we have

$$\begin{aligned}
\mathcal{P}_{\alpha_J}(\lambda)&=\sum_i f(\lambda_i,\alpha_J)\delta(\lambda_i-\lambda)\\
&=f(\lambda,\alpha_J)\sum_i\delta(\lambda_i-\lambda)=f(\lambda,\alpha_J)\mathcal{P}(\lambda). \qquad (32)
\end{aligned}$$

By using (32), we have the equation

$$\frac{\partial\lambda_i}{\partial\alpha_J}=f(\lambda_i,\alpha_J)=\frac{\mathcal{P}_{\alpha_J}(\lambda_i)}{\mathcal{P}(\lambda_i)}, \qquad (33)$$

where $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ can be obtained by $S^{(n)}$. Once knowing $S^{(n)}$, one could solve the equation with suitable conditions. With these results, one could obtain more details of the eigenvalues λ_i. By choosing more parameters α_J, one could reconstruct the eigenvalues of ρ_A. In the following, we will show some examples. On the contrary, one may assume the eigenvalues λ_i satisfy the relation $\frac{\partial\lambda_i}{\partial\alpha_J}=f(\lambda_i,\alpha_J)$. If the differential equation has no proper solutions, one can conclude this assumption is false.

The higher-order function can also be associated with $\mathcal{P}$. For example, by definition

$$\mathcal{P}_{\alpha_J^2}(\lambda)=f(\lambda,\alpha_J)^2\mathcal{P}(\lambda), \qquad (34)$$

$$\begin{aligned}
\mathcal{P}_{\alpha_{J2}}(\lambda)&=\sum_i\frac{d}{d\alpha_J}f(\lambda_i,\alpha_J)\delta(\lambda_i-\lambda)\\
&=\sum_i\left(\frac{\partial}{\partial\lambda_i}f(\lambda_i,\alpha_J)\frac{\partial\lambda_i}{\partial\alpha_J}+\frac{\partial}{\partial\alpha_J}f(\lambda_i,\alpha_J)\right)\delta(\lambda_i-\lambda)\\
&=\sum_i\left(\frac{\partial f(\lambda_i,\alpha_J)}{\partial\lambda_i}f(\lambda_i,\alpha_J)+\frac{\partial}{\partial\alpha_J}f(\lambda_i,\alpha_J)\right)\delta(\lambda_i-\lambda)\\
&=\left(\frac{\partial f(\lambda,\alpha_J)}{\partial\lambda}f(\lambda,\alpha_J)+\frac{\partial}{\partial\alpha_J}f(\lambda,\alpha_J)\right)\sum_i\delta(\lambda_i-\lambda)\\
&=\left(\frac{\partial f(\lambda,\alpha_J)}{\partial\lambda}f(\lambda,\alpha_J)+\frac{\partial}{\partial\alpha_J}f(\lambda,\alpha_J)\right)\mathcal{P}(\lambda). \qquad (35)
\end{aligned}$$

So we can write $\mathcal{P}_{\alpha_J}(\lambda)$, $\mathcal{P}_{\alpha_J^2}(\lambda)$, and $\mathcal{P}_{\alpha_{J2}}(\lambda)$ just by $f(\lambda,\alpha_J)$ and $\mathcal{P}(\lambda)$. We can test the self-consistency of (32), (34), and (35) through the relation (E1).

More generally, we want to write $\mathcal{P}_{\alpha_J^m}(\lambda)$ and $\mathcal{P}_{\alpha_{Jm}}(\lambda)$ by $f(\lambda,\alpha_J)$ and $\mathcal{P}(\lambda)$. We define a new derivation

$$\frac{D}{D\alpha_J}=f(\lambda,\alpha_J)\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\alpha_J}. \qquad (36)$$

So $\mathcal{P}_{\alpha_{J2}}(\lambda)$ can be rewritten as $\mathcal{P}_{\alpha_{J2}}(\lambda)=\frac{Df(\lambda,\alpha_J)}{D\alpha_J}P(\lambda)$.

One could show that

$$\mathcal{P}_{\alpha_J^m}(\lambda)=f(\lambda,\alpha_J)^m\mathcal{P}(\lambda), \qquad (37)$$

$$\mathcal{P}_{\alpha_{Jm}}(\lambda)=\frac{D^{m-1}f(\lambda,\alpha_J)}{D\alpha_J^{m-1}}\mathcal{P}(\lambda), \qquad (38)$$

where $m\in Z$ and $m\geq 2$. See Appendix A 2 for more details of the calculations.

III. EXAMPLES IN TWO-DIMENSIONAL CFTs

There are many known analytic results of Rényi entropy of one interval for some states in two-dimensional CFTs. Using these results, we could directly obtain the functions discussed in previous sections. We will first evaluate $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ for α_J being the size of the interval L and central charge c. With further assumption, one could obtain the eigenvalues. The eigenvalues of this example can be derived by conformal mapping method used in [27]. Our results are consistent with the ones in [27].

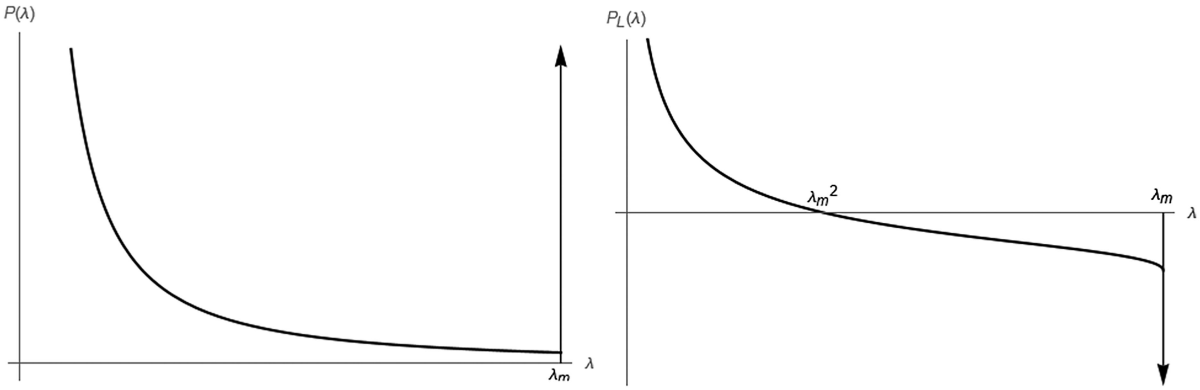

FIG. 1. The illustration of $\mathcal{P}(\lambda)$ and $\mathcal{P}_L(\lambda)$ in the case of the vacuum state of 2D CFT, where the arrows represent the Dirac delta function.

A. One interval on infinite line, vacuum state

Using the replica method, we can get the Rényi entropy $S^{(n)}(\rho_A)$ of one interval on the infinite line in the vacuum state of 2D CFTs [4,5]:

$$S^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{L}{\epsilon}, \tag{39}$$

where L is the length of system A and ϵ is the UV cutoff. We have $b = -\log\lambda_m = \frac{c}{6}\log L/\epsilon$.

Using (17) and (22), we can get

$$\mathcal{P}(\lambda) = \frac{1}{\lambda}\left[\frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} + \delta(t)\right],$$
$$\mathcal{P}_L(\lambda) = -\frac{c}{6L}\left[\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}} + \delta(t)\right],$$
$$\mathcal{P}_c(\lambda) = -\frac{\log L}{6}\left[\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}} + \delta(t)\right], \tag{40}$$

where $I_n(x)$ is the modified Bessel functions of the first kind and we have the relation $\lambda = e^{-b-t}$. It is straightforward to check that (40) satisfy the relation (23), that is,

$$\frac{\partial}{\partial L}P(\lambda) = -\frac{\partial}{\partial\lambda}P_L(\lambda). \tag{41}$$

See Appendix B for the details. We also find the relation (31):

$$\frac{\partial}{\partial c}P_L(\lambda) = \frac{\partial}{\partial L}P_c(\lambda). \tag{42}$$

See Appendix B for the details.

$\mathcal{P}_L(\lambda)$ as a function of λ can be used to reflect how the eigenvalues change with the scale of the subsystem. For the maximal eigenvalue λ_m, one could obtain $\frac{\partial\lambda_m}{\partial L}$ by using $-\log\lambda_m = \frac{c}{6}\log L/\epsilon$. One could also check that $\frac{\partial\lambda_m}{\partial L} = \tilde{\mathcal{P}}(\lambda_m)$. For $\lambda \neq \lambda_m$, there is a zero point of the function $\mathcal{P}_L$, which is given by $t = b$ or $\lambda_0 = e^{-2b} = \lambda_m^2$. For $\lambda < \lambda_0$, $\mathcal{P}_L(\lambda) > 0$, which means that, as the scale of the subsystem increases, the average eigenvalues smaller than λ_0 are increasing. While for $\lambda > \lambda_0$, $\mathcal{P}_L(\lambda) < 0$, the average eigenvalues are decreasing. The physical significance of the zero point is not very clear, but we can see that the function $\mathcal{P}_L(\lambda)$ must have at least one zero because the integral result of it should be zero. We plot the function $\mathcal{P}(\lambda)$ and $\mathcal{P}_L(\lambda)$ in Fig. 1.

B. One interval on cylinder, vacuum state

Consider the CFT is defined on a cylinder with circumference R. The interval is $A = [0, L]$ with length L. The Rényi entropy for this case is given by

$$S^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left(\frac{R}{\epsilon\pi}\sin\frac{\pi L}{R}\right). \tag{43}$$

We have $b = -\log\lambda_m = \frac{c}{6}\log\left(\frac{R}{\epsilon\pi}\sin\frac{\pi L}{R}\right)$. It is straightforward to obtain the functions

$$\mathcal{P}(\lambda) = \frac{1}{\lambda}\left[\frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}} + \delta(t)\right],$$
$$\mathcal{P}_L(\lambda) = -\frac{c}{6}\frac{\pi\cot(\frac{\pi L}{R})}{R}\left[\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}} + \delta(t)\right]. \tag{44}$$

One could check that $\int_0^{\lambda_m}\lambda\mathcal{P}(\lambda)d\lambda = 1$ and $\int_0^{\lambda_m}\mathcal{P}_L(\lambda)d\lambda = 0$.

For $P_L(\lambda)$ we also have a zero point at $t_0 = b$ or $\lambda_0 = e^{-2b} = \lambda_m^2$. The figure of $\mathcal{P}_L(\lambda)$ is slightly different from the case on an infinite line. If $L < R/2$, we have $\mathcal{P}_L > 0$ for $0 < \lambda < \lambda_0$, while $\mathcal{P}_L < 0$ for $\lambda_0 < \lambda < \lambda_m$. The result is similar as the case on an infinite line. But for $L > R/2$, the figure is flipped. We have $\mathcal{P}_L < 0$ for $0 < \lambda < \lambda_0$, while $\mathcal{P}_L > 0$ for $\lambda_0 < \lambda < \lambda_m$. There exists a critical point $L = R/2$; the function $\mathcal{P}_L$ is vanishing at this point. We show the results in Fig. 2.

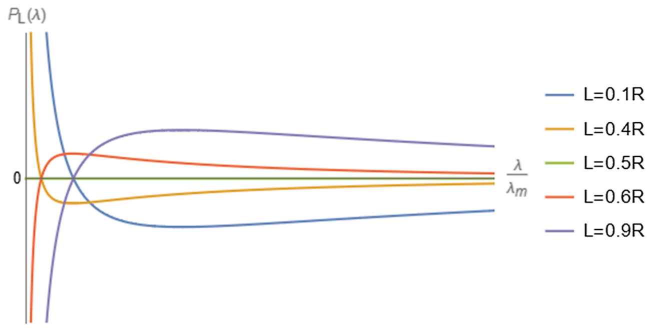

FIG. 2. The plots of $\mathcal{P}_L(\lambda)$ with various parameters in the scenario of a single interval on a cylinder. We omit the representation of the term $\delta(\lambda_m - \lambda)$ in the plot, as it is not important for our current discussion.

C. Reconstruction of the eigenvalues with further assumption

Without further assumption, one cannot obtain higher-order functions, as we have shown in the previous section. For the present example, we find the maximal eigenvalue satisfies $\frac{\partial \lambda_m}{\partial L} = \bar{\mathcal{P}}(\lambda_m)$. Let us assume for other eigenvalues we also have $\frac{\partial \lambda_i}{\partial L} = f(\lambda_i, L)$. From the discussions in Sec. II C, using (40), we have

$$\frac{\partial \lambda_i}{\partial L} = \frac{-\log \lambda_i - 2b}{b} \frac{c}{6L} \lambda_i. \tag{45}$$

Solving the above differential equation, we can get

$$\lambda_i = e^{-\frac{C_i}{\log L/\epsilon} - b}, \tag{46}$$

where C_i are constants unrelated to L.

D. Eigenvalues of modular Hamiltonian by conformal mapping

For the one interval example in the last sections, one could reconstruct the eigenvalues of the modular Hamiltonian by using the functions $\mathcal{P}_L$ and $\mathcal{P}$. For the simple example, one could obtain the eigenvalues by the methods explored in [27]. As the Rényi entropy and entanglement entropy are UV divergence, we should introduce some regulator to obtain the results. Let us focus on the two-dimensional CFTs. In [27], the authors show one could consider only the states that are projected out the basis in a small spatial region of thickness ϵ around the common boundary of A and $\bar{A}$. In the Euclidean path integral representation of the reduced density matrix ρ_A, this is to introduce a hole around the boundary point of A. Some suitable boundary conditions should be imposed on the boundary of the hole. For the one interval example, the topology of the manifold is an annulus.

Suppose $A = [0, L]$. The system is in the vacuum state on the infinite line. The corresponding state is associated with the Euclidean spacetime with a disk of radius ϵ removed at end points of A, which can be mapped to the annulus by the conformal mapping:

$$w = \log \frac{z}{L - z}, \tag{47}$$

where w is the coordinate of the annulus. The width of the annulus is $W = f(L - \epsilon) - f(\epsilon) \simeq 2 \log \frac{L}{\epsilon} = \frac{12b}{c}$. The modular Hamiltonian K_A is locally a generator of rotation around the end points of A. Under the conformal map (47), K_A is mapped to the time evolution operator $H_w \coloneqq \int dv T_{vv}$ along the direction $v \coloneqq \mathrm{Im}(w)$ up to some constants, and K_A and H_w are unitarily equivalent. Thus, the eigenvalues of K_A should be same as H_w. By using $T_{vv} = T(w) + \bar{T}(\bar{w})$, we have

$$H_w = \int dw T(w) + \int d\bar{w}\, \bar{T}(\bar{w}). \tag{48}$$

Under the conformal transformation (47), we obtain

$$H_w = K_A + \frac{c}{12} \int_\epsilon^{L-\epsilon} dx \frac{L}{x(L - x)} = K_A + b, \tag{49}$$

where the constant term is from the Schwartzian term and we define

$$K_A \coloneqq \int_\epsilon^{L-\epsilon} dz \frac{z(L - z)}{L} T(z) + \int_\epsilon^{L-\epsilon} d\bar{z} \frac{\bar{z}(L - \bar{z})}{L} \bar{T}(\bar{z}), \tag{50}$$

which is the regularized modular Hamiltonian of the single interval on an infinite line.

For CFTs on the annulus with width W, the eigenvalues of H_w are given by $\frac{\Delta_i - \frac{c}{24}}{W}$. Thus, by using (49) the eigenvalues of K_A are given by $\frac{\Delta_i - \frac{c}{24}}{W} - b$. In Ref. [16], the author shows the reduced density matrix $\rho_A = e^{-K_A - 2b}$ by normalization. Therefore, we expect the eigenvalues of ρ_A should be $e^{-\frac{\Delta_i - \frac{c}{24}}{W} - b}$, which is just the form as (46).

IV. SHORT INTERVAL IN ARBITRARY STATE

Computing the Rényi entropy of an arbitrary state is usually very challenging, but in some cases we can obtain the result perturbatively using the operator product expansion of the twistor operator [28–32]. In this section, we will focus on the two-dimensional CFT with a short interval.

A. Rényi entropy for arbitrary state

Assume the length of the interval is l and the state is ρ. For simplicity, we list only the contributions from the operators in the vacuum conformal family, such as T, $\bar{T}$, and $\mathcal{A}$, and assume the state is translationally invariant.

Up to $O(l^4)$ the Rényi entropy can be expanded in terms of l as[1]

$$S^{(n)} = \frac{c}{6}\frac{n+1}{n}\log\frac{l}{a} + \frac{n+1}{n}k_2 l^2 + \left[\frac{(n+1)(n^2-1)}{n^3}k_4 + \frac{(n+1)(n^2+11)}{n^3}k_4'\right]l^4 + O(l^6), \tag{51}$$

with

$$\begin{aligned} k_2 &= -\frac{1}{12}\left(\langle T\rangle_\rho + \langle \bar{T}\rangle_\rho\right), \\ k_4 &= -\frac{1}{288}\left(\langle A\rangle_\rho + \langle \bar{A}\rangle_\rho\right) + \frac{1}{288}\left(\langle T\rangle_\rho^2 + \langle \bar{T}\rangle_\rho^2\right), \\ k_4' &= -\frac{1}{720c}\left(\langle T\rangle_\rho^2 + \langle \bar{T}\rangle_\rho^2\right), \end{aligned} \tag{52}$$

where $\langle\chi\rangle_\rho \coloneqq \mathrm{tr}(\rho\chi)$ for $\chi = T, \bar{T}, \mathcal{A}$. Note that k_2, k_4, and k_4' are independent with n.

B. Perturbative results of the functions $\mathcal{P}$ and $\mathcal{P}_l$

If we retain only up to the second order, the result is similar with the vacuum case, requiring only the replacement of $b = \frac{c}{6}\log\frac{l}{\epsilon}$ with $b' = \frac{c}{6}\log\frac{l}{\epsilon} + k_2 l^2$, since their dependence on n is the same:

$$S^{(n)} = \frac{n+1}{n}\left(\frac{c}{6}\log\frac{l}{a} + k_2 l^2\right) + O(l^4). \tag{53}$$

At the order of $O(l^4)$, we will obtain more intriguing results. By definition, we have

$$b \coloneqq -\log\lambda_m = \lim_{n\to\infty} S^{(n)} = \frac{c}{6}\log\frac{l}{a} + k_2 l^2 + k_4 l^4 + k_4' l^4, \tag{54}$$

where λ_m is the maximal eigenvalue in this case. Let us also define $b_0 \coloneqq \frac{c}{6}\log\frac{l}{a}$.

By using (17) and (51), we obtain

$$\mathcal{P}(\lambda) = \lambda^{-1}\left[\frac{\sqrt{b_0}}{\sqrt{t}}I_1 + \delta(t) + k_2 I_0 l^2 + \left[\left(2k_4 - 10k_4'\right)I_0 + \frac{1}{2}k_2^2 t^{\frac{1}{2}} b_0^{-\frac{1}{2}} I_1 + \left(-k_4 + 11k_4'\right)t b_0^{-1} I_2\right]l^4\right], \tag{55}$$

where the argument of I_n is $2\sqrt{b_0 t}$ and we have used

$$\mathcal{L}^{-1}\left[e^{\frac{b_0}{n}}n^k\right] = b_0^{(1+k)/2} t^{-(1+k)/2} I_{-(1+k)}\left(2\sqrt{b_0 t}\right). \tag{56}$$

Furthermore, by using (22) and (51), we have

$$\begin{aligned} \mathcal{P}_l(\lambda) = -\frac{\partial b}{\partial l}\Bigg[&\frac{c}{6}b_0^{-\frac{1}{2}} t^{\frac{1}{2}} I_1 l^{-1} + \frac{\sqrt{b_0}}{\sqrt{t}}I_1 + \delta(t) + \left[\frac{c}{6}k_2 I_0 + 2k_2 t^{\frac{1}{2}} b_0^{-\frac{1}{2}} I_1 + \frac{c}{6}k_2 t b_0^{-1} I_2\right]l + \Bigg[\left(-2k_2^2 - \frac{c}{6}(2k_4 - 10k_4')\right)I_0 \\ &+ \left(-\frac{c}{6}\frac{1}{2}k_2^2 + 4(2k_4 - 10k_4')\right)t^{\frac{1}{2}} b_0^{-\frac{1}{2}} I_1 + \left(2k_2^2 + \frac{c}{6}(3k_4 - 21k_4')\right)t b_0^{-1} I_2 + \left(\frac{c}{6}\frac{1}{2}k_2^2 - 4(k_4 - 11k_4')\right)t^{\frac{3}{2}} b_0^{-\frac{3}{2}} I_3 \\ &- \left(\frac{c}{6}(k_4 - 11k_4')\right)t^2 b_0^{-2} I_4\Bigg]l^3\Bigg], \end{aligned} \tag{57}$$

where the argument of I_n is $2\sqrt{b_0 t}$.

It can be seen that the functions $\mathcal{P}$ and $\mathcal{P}_l$ depend on the expectation value $\langle\chi\rangle_\rho$. One could check that they satisfy the relation $\frac{\partial\mathcal{P}}{\partial l} = -\frac{\partial\mathcal{P}_l}{\lambda}$.

One could also check the above results by the example of the thermal state. Consider the thermal state with $\rho = e^{-\beta H}/Z(\beta)$. The Rényi entropy is given by

$$S^{(n)} = \frac{c}{6}\left(1 + \frac{1}{n}\right)\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi l}{\beta}\right)\right], \tag{58}$$

where l represents the length of the interval, which is not necessarily assumed to be small. With this result, one could obtain $\mathcal{P}(\lambda)$ and $\mathcal{P}_l(\lambda)$ for the thermal state. The results are similar to the vacuum cases in Sec. III A. Then one could expand the function $\mathcal{P}(\lambda)$ and $\mathcal{P}_l(\lambda)$ in terms of $\frac{l}{\beta}$ in the region $l/\beta \ll 1$. The results should be the same with (11) and (57) up to $O(l^4)$ by using the expectation values of $\langle\chi\rangle_\beta \coloneqq \mathrm{tr}(e^{-\beta H}\chi)/Z(\beta)$. The details of the calculations can be found in Appendix. D.

C. Zero point of $\mathcal{P}_l$

For the vacuum cases, we find the function $\mathcal{P}_L$ has one zero point which is given by $\lambda_m^2 = e^{-2b_0}$. Here, we would like to study the zero point of $\mathcal{P}_l$ for the short interval case. We assume the zero point is given by the form $\tilde{\lambda}_0 = e^{-\tilde{t}_0 - b}$,

[1]The extension of the calculation to arbitrary situations and higher order is straightforward. In the following, we will use this result for holographic CFTs, in which the contributions from the vacuum conformal family are dominant.

with $\tilde{t}_0 = b_0 + t_2 l^2 + t_4 l^4 + O(l^6)$, where b_0 is the zero point for the vacuum case. t_2 and t_4 should satisfy the following equations:

$$t_2 = k_2,$$
$$t_4 = 2k_4 - 10k_4' - \frac{48}{c}k_4\frac{I_2}{I_1} + \frac{528}{c}k_4'\frac{I_2}{I_1} - 3k_4\frac{I_3}{I_1} + 33k_4'\frac{I_3}{I_1}, \tag{59}$$

where the argument of I_n is $2b_0$. For general theory, the result is complicated. But for CFTs with holographic dual, we have a large central charge c. For the function $I_n(x)$, since $\lim_{x\to\infty} I_n(x) = \frac{1}{\sqrt{2\pi x}}e^x$, in the limit $x \to \infty$ one would have $\lim_{x\to\infty}\frac{I_n(x)}{I_m(x)} = 1$. Thus, for large $c \gg 1$, we have $b_0 \gg 1$ and

$$\tilde{t}_0 = b_0 + k_2 l^2 - k_4 l^4 + 23k_4' l^4 + O(1/c). \tag{60}$$

By using (51), the EE is given by

$$S = \lim_{n\to1} S^{(n)} = 2b_0 + 2k_2 l^2 + 24k_4' l^4 + O(l^6). \tag{61}$$

It is remarkable that the zero point $\tilde{t}_0$ is associated with the EE S and $b = S^\infty$:

$$\tilde{t}_0 = S - b + O(1/c, l^6). \tag{62}$$

For the vacuum case, we have $S = 2b$, and the zero point is given by $t_0 = b$, which is consistent with the above results. One could also check the above relation for higher order of the short interval expansion. In the following section, we will discuss the holographic CFTs. One would find the above relation is actually correct for arbitrary states that are dual to a bulk geometry.

V. PERTURBATION STATES

Consider the density matrix

$$\rho = \rho_0 + \delta\rho, \tag{63}$$

with the condition $\mathrm{tr}\delta\rho = 0$. One could obtain the Rényi entropy

$$\begin{aligned} S^{(n)} &= \frac{1}{1-n}\log(\mathrm{tr}\rho^n) \\ &= S^{(n)}(\rho_0) + \frac{n\,\mathrm{tr}(\rho_0^{n-1}\delta\rho)}{(1-n)\mathrm{tr}(\rho_0^n)} + O(\delta\rho^2) \\ &= S^{(n)}(\rho_0) + \delta S^{(n)} + O(\delta\rho^2), \end{aligned} \tag{64}$$

where we define

$$\delta S^{(n)} \coloneqq \frac{n\mathrm{tr}(\rho_0^{n-1}\delta\rho)}{(1-n)\mathrm{tr}(\rho_0^n)}. \tag{65}$$

In the following, we will keep only the leading order of the perturbation.

On the other hand, by using $\rho_0 = \sum_i \lambda_i^0 |\lambda_i^0\rangle\langle\lambda_i^0|$, we can rewrite (64) as

$$\delta S^{(n)} \coloneqq \frac{n\sum_i(\lambda_i^0)^{n-1}\delta\lambda_i}{(1-n)\sum_i(\lambda_i^0)^n}, \tag{66}$$

where we define

$$\delta\lambda_i \coloneqq \langle\lambda_i^0|\delta\rho|\lambda_i^0\rangle. \tag{67}$$

A. Density of eigenstates

Define $\lambda_m \coloneqq e^{-b}$ and $\lambda_m^0 \coloneqq e^{-b_0}$, where λ_m and λ_m^0 are maximal eigenvalues of ρ and ρ_0, respectively. By definition, we have $b = \lim_{n\to\infty} S^{(n)}(\rho)$ and $b_0 = \lim_{n\to\infty} S^{(n)}(\rho_0)$. Thus, by using (66), we have

$$\begin{aligned} \lim_{n\to\infty} S^{(n)}(\rho) &= \lim_{n\to\infty}(S^{(n)}(\rho_0) + \delta S^{(n)}), \\ b &= b_0 + \delta b, \end{aligned} \tag{68}$$

with

$$\begin{aligned} \delta b &\coloneqq -\frac{\delta\lambda_m}{\lambda_m^0}, \\ \delta\lambda_m &\coloneqq \langle\lambda_m^0|\delta\rho|\lambda_m^0\rangle, \end{aligned} \tag{69}$$

where $|\lambda_m^0\rangle$ denotes the eigenstate for the maximal eigenvalue λ_m^0. A useful form that we will utilize hereafter is $e^{\delta b} = 1 + \delta b = \frac{\lambda_m^0}{\lambda_m}$.

Now we are ready to evaluate the function $\mathcal{P}$. By using Eq. (17), we have

$$\begin{aligned} \mathcal{P}(e^{-b-t}) &= \lambda^{-1}\mathcal{L}^{-1}[e^{nb+(1-n)S^{(n)}(\rho)}](t) \\ &= \left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)^{-1} e^{\delta b}\mathcal{L}^{-1}[e^{nb_0+(1-n)S^{(n)}(\rho_0)}](t) \\ &\quad + \lambda^{-1}\delta b\mathcal{L}^{-1}[ne^{nb_0+(1-n)S^{(n)}(\rho_0)}](t) \\ &\quad + \lambda^{-1}\mathcal{L}^{-1}[ne^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)](t). \end{aligned} \tag{70}$$

Because of the complexity of the above calculations, we will discuss the inverse Laplace transform in the above expression term by term. First, let us consider the first term. Using $\lambda = e^{-b-t}$, since $e^{-b_0-t} = e^{-b-t}e^{\delta b} = \frac{\lambda_m^0}{\lambda_m}\lambda$, we have

$$\begin{aligned} &\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)^{-1} e^{\delta b}\mathcal{L}^{-1}[e^{nb_0+(1-n)S^{(n)}(\rho_0)}](t) \\ &\quad = (1+\delta b)(e^{-b_0-t})^{-1}\mathcal{L}^{-1}[e^{nb_0+(1-n)S^{(n)}(\rho_0)}](t) \\ &\quad = \mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + \delta b\mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right), \end{aligned} \tag{71}$$

where $\mathcal{P}_0$ denotes the density of eigenstate for the state ρ_0. Note that, in the last, we substituted the variable $\mathcal{P}_0$ with $\frac{\lambda_m^0}{\lambda_m}\lambda$ to adjust for the range of values. The reason is easy to see in the above mathematical calculations, since the inverse Laplace transform of $\mathcal{L}^{-1}[e^{nb_0+(1-n)S^{(n)}(\rho_0)}](t)$ is $e^{-b_0-t}\mathcal{P}_0(e^{-b_0-t})$.

On the other hand, this adjustment can be also explained as follows: λ in $\mathcal{P}(\lambda)=\sum_i\delta(\lambda_i-\lambda)$ falls within the range $(0,\lambda_m]$, while λ' in $\mathcal{P}_0(\lambda')=\sum_i\delta(\lambda_i^0-\lambda')$ falls within the range $(0,\lambda_m^0]$. Hence, the purpose of this transformation is to ensure that both sides of the equation share the same variable range.

Let us go on discussing the remaining two inverse Laplace transformation terms in (71). By using the formula

$$\mathcal{L}^{-1}\{s\mathcal{L}[f](s)\}(t)=\delta(t)f(0)+f'(t), \tag{72}$$

we have

$$\begin{aligned}
&\lambda^{-1}\delta b\mathcal{L}^{-1}[ne^{nb_0+(1-n)S^{(n)}(\rho_0)}](t)\\
&=\lambda^{-1}\delta b\mathcal{L}^{-1}\left[n\mathcal{L}[e^{-b_0-t}\mathcal{P}_0(e^{-b_0-t})]\right](t)\\
&=\lambda^{-1}\delta b\left[\delta(t)e^{-b_0}\mathcal{P}_0(e^{-b_0})+e^{-b_0-t}\frac{d\mathcal{P}_0(e^{-b_0-t})}{dt}\right.\\
&\quad\left.-e^{-b_0-t}\mathcal{P}_0(e^{-b_0-t})\right]\\
&=\delta b\delta(t)\mathcal{P}_0(e^{-b_0})-\delta b\lambda\mathcal{P}_0'\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)-\delta b\mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right),
\end{aligned} \tag{73}$$

where $\mathcal{P}_0'(\lambda):=\frac{\partial\mathcal{P}_0(\lambda)}{\partial\lambda}$. It is worth noting that the final term in the expression above cancels out with the second term in formula (70).

To calculate the last term of (70), we define

$$\mathcal{P}_\delta(\lambda'):=\sum_i\delta\lambda_i\delta(\lambda_i^0-\lambda'), \tag{74}$$

which can be taken as the average expectation value of the perturbation $\delta\rho$ in the eigenstates with eigenvalue $\lambda=e^{-b-t}$. It can be related to $\mathrm{tr}(\rho_0^{n-1}\delta\rho)$ by

$$\begin{aligned}
\mathrm{tr}(\rho_0^{n-1}\delta\rho)&=\sum_i(\lambda_i^0)^{n-1}\delta\lambda_i\\
&=\int_0^{\lambda_m^0}d\lambda'\lambda'^{n-1}\sum_i\delta\lambda_i\delta(\lambda_i^0-\lambda')\\
&=\int_0^\infty dte^{-n(b_0+t)}\mathcal{P}_\delta(e^{-b_0-t}),
\end{aligned} \tag{75}$$

where we use $\lambda':=e^{-b_0-t}$. One could obtain $\mathcal{P}_\delta(e^{-b_0-t})$ by inverse Laplace transformation once $\mathrm{tr}(\rho_0^{n-1}\delta\rho)$ is known. By using (65) and (66), we have

$$\mathcal{P}_\delta(e^{-b_0-t})=\mathcal{L}^{-1}\left[\frac{1-n}{n}e^{nb_0+(1-n)S^{(n)}(\rho_0)}\delta S^n\right], \tag{76}$$

the form of which is similar to $\mathcal{P}_{\alpha_J}$ (22). Again using the formula (72), we obtain

$$\begin{aligned}
&\lambda^{-1}\mathcal{L}^{-1}[ne^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)](t)\\
&=\lambda^{-1}\left[\delta(t)\mathcal{P}_\delta(e^{-b_0})+\frac{d\mathcal{P}_\delta(e^{-b_0-t})}{dt}\right]\\
&=\lambda^{-1}\delta(t)\mathcal{P}_\delta(e^{-b_0})-\mathcal{P}_\delta'\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right),
\end{aligned} \tag{77}$$

where $\mathcal{P}_\delta'(\lambda):=\frac{\partial\mathcal{P}_\delta}{\partial\lambda}$. We compare the terms in (73) and (77) that contain $\delta(t)$ and find

$$\lambda^{-1}\delta(t)\mathcal{P}_\delta(e^{-b_0})=-\delta b\delta(t)\mathcal{P}_0(e^{-b_0}), \tag{78}$$

in the leading order of the perturbation.

Combining all the aforementioned results, we derive the final expression. In summary, the function $\mathcal{P}(\lambda)$ (70) can be structured in the following form:

$$\mathcal{P}(\lambda)=\mathcal{P}(e^{-b-t})=\mathcal{P}_0(e^{-b_0-t})+\delta\mathcal{P}(e^{-b_0-t}), \tag{79}$$

with

$$\begin{aligned}
&\mathcal{P}_0(e^{-b_0-t})=\mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right),\\
&\delta\mathcal{P}(e^{-b_0-t})=-\delta b\lambda\mathcal{P}_0'\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)-\mathcal{P}_\delta'\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right).
\end{aligned} \tag{80}$$

B. Further discussion on the perturbation result

1. Normalization

One could promptly verify that (79) complies with the normalization condition. Through direct calculations, we have

$$\begin{aligned}
\int_0^{\lambda_m}\lambda\mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)d\lambda&=1-2\delta b,\\
-\int_0^{\lambda_m}\lambda\delta b\lambda\mathcal{P}_0'\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)d\lambda&=-\delta b\frac{\lambda_m^3}{\lambda_m^0}\mathcal{P}_0(\lambda_m^0)+2\delta b,\\
-\int_0^{\lambda_m}\lambda\mathcal{P}_\delta'\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)d\lambda&=-\frac{\lambda_m^2}{\lambda_m^0}\mathcal{P}_\delta(\lambda_m^0)\\
&\quad+\frac{\lambda_m}{\lambda_m^0}\int_0^{\lambda_m^0}\mathcal{P}_\delta(\lambda')d\lambda'\\
&=-\frac{\lambda_m^2}{\lambda_m^0}\mathcal{P}_\delta(\lambda_m^0),
\end{aligned}$$

where in the last step we use the fact that $\mathrm{tr}\delta\rho=\int_0^{\lambda_m^0}\mathcal{P}_\delta(\lambda')d\lambda'=0$. Summing over the above results and

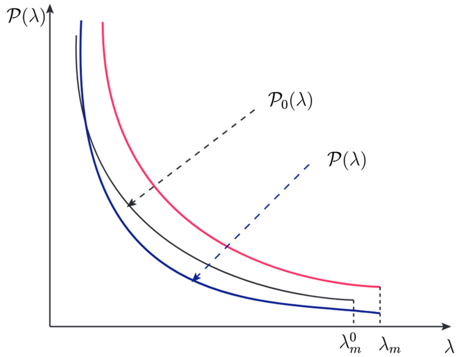

FIG. 3. Illustration of the function $\mathcal{P}(\lambda)$ in the perturbation state. The black line is the unperturbed function $\mathcal{P}_0$. The range of the eigenvalue would change under perturbation. The red line illustrates the perturbation of the function $\mathcal{P}$ due to the alteration in the range of the variable λ. But it does not satisfy the normalization condition $\int_0^{\lambda_m} d\lambda \mathcal{P}(\lambda)\lambda = 1$. The blue one includes the adjustment to satisfy the normalization.

using the relation (78), we arrive at the normalization condition

$$\int_0^{\lambda_m} \lambda \mathcal{P}(\lambda) d\lambda = 1. \tag{81}$$

Let us briefly analyze the implications of each term in the above expression (79). $\mathcal{P}_0$ represents the unperturbed result of the density of eigenstates. As perturbation affects the range of values of eigenvalues, i.e., the maximal eigenvalue changes from λ_m^0 to λ_m. The first term $\mathcal{P}_0$ can be viewed as the change in the density distribution with the alteration in the range of the variable λ. As we can see from (81), it does not satisfy the normalization. The second term in $\delta\mathcal{P}(e^{-b_0-t})$ can be seen as the adjustment of the density distribution function itself to satisfy the normalization requirement. We illustrate the above explanation in Fig. 3.

2. *The number of eigenvalues*

Since the function $\mathcal{P}$ can be seen as the density of eigenstates, we can define the number of eigenstates larger than λ as

$$n(\lambda) = \int_\lambda^{\lambda_m} \mathcal{P}(\lambda') d\lambda'. \tag{82}$$

Taking (79) into the integration (82), we have

$$n(\lambda) = n_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + \delta b \lambda \mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + \mathcal{P}_\delta\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right), \tag{83}$$

where n_0 is the number of the unperturbed density matrix defined by

$$n_0(\lambda) = \int_\lambda^{\lambda_m^0} \mathcal{P}_0(\lambda') d\lambda'. \tag{84}$$

It is important to note that integration typically leads to divergence. For instance, in Sec. III A, one can readily verify that the number of eigenstates is infinite. In essence, $N \coloneqq n(\lambda = 0)$ can be regarded as an approximation to the dimension of the density matrix ρ_A, which tends to be infinite in QFTs. Nevertheless, formally, we observe that the variation in dimension due to perturbations is linked to the function $\mathcal{P}_\delta$, that is,

$$\Delta N \coloneqq n(0) - n_0(0) = \mathcal{P}_\delta(0). \tag{85}$$

Here, the number of the maximal eigenvalue is well defined and generally finite. Taking $\lambda = \lambda$ into (83) and using (78), we obtain

$$n(\lambda_m) = n_0(\lambda_m^0), \tag{86}$$

which means the number of the maximal eigenvalue would be invariant at the leading-order perturbation.

C. The function $\mathcal{P}_{\alpha_J}$

It is straightforward to calculate the function $\mathcal{P}_\alpha$ as we have done in the previous section. By using formula (22), we have

$$\begin{aligned}
\mathcal{P}_{\alpha_J}(e^{-b-t}) &= \mathcal{L}^{-1}\left[\frac{1-n}{n} e^{nb+(1-n)S^{(n)}(\rho)} \frac{\partial S^{(n)}(\rho)}{\partial \alpha_J}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1-n}{n} e^{nb_0+(1-n)S^{(n)}(\rho_0)} (1 + n\delta b + (1-n)\delta S^{(n)}) \frac{\partial (S^{(n)}(\rho_0) + \delta S^{(n)})}{\partial \alpha_J}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1-n}{n} e^{nb_0+(1-n)S^{(n)}(\rho_0)} \frac{\partial S^{(n)}(\rho_0)}{\partial \alpha_J}\right] + \mathcal{L}^{-1}\left[n \frac{1-n}{n} e^{nb_0+(1-n)S^{(n)}(\rho_0)} \frac{\partial S^{(n)}(\rho_0)}{\partial \alpha_J} \delta b\right] \\
&\quad + \mathcal{L}^{-1}\left[\frac{(1-n)^2}{n} e^{nb_0+(1-n)S^{(n)}(\rho_0)} \delta S^{(n)} \frac{\partial S^{(n)}(\rho_0)}{\partial \alpha_J}\right] + \mathcal{L}^{-1}\left[\frac{1-n}{n} e^{nb_0+(1-n)S^{(n)}(\rho_0)} \frac{\partial \delta S^{(n)}}{\partial \alpha_J}\right] + \mathcal{L}^{-1}[O(\delta\rho^2)].
\end{aligned} \tag{87}$$

Let us discuss the four terms in the above results separately.

The first term is the function $\mathcal{P}_{\alpha_J}$ for the unperturbed density matrix ρ_0; denote it by $\mathcal{P}^0_{\alpha_J}(e^{-b_0-t})$. By using the formula (72), the second term is given by

$$\mathcal{L}^{-1}\left[n\frac{1-n}{n}e^{nb_0+(1-n)S^{(n)}(\rho_0)}\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\delta b = \mathcal{L}^{-1}[n\mathcal{L}[\mathcal{P}^0_{\alpha_J}(e^{-b_0-t})]]\delta b$$
$$= \delta b\left(\delta(t)\mathcal{P}^0_{\alpha_J}(e^{-b_0}) + \frac{d\mathcal{P}^0_{\alpha_J}(e^{-b_0-t})}{dt}\right)$$
$$= \delta b\delta(t)\mathcal{P}^0_{\alpha_J}(e^{-b_0}) - \delta b\lambda\mathcal{P}^{0\prime}_{\alpha_J}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right), \tag{88}$$

where $\mathcal{P}^{0\prime}_{\alpha_J} \coloneqq \frac{\partial\mathcal{P}^0_{\alpha_J}}{\partial\lambda}$. The third term can be simplified as follows:

$$\mathcal{L}^{-1}\left[\frac{(1-n)^2}{n}e^{nb_0+(1-n)S^{(n)}(\rho_0)}\delta S^{(n)}(\rho_0)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right] = \mathcal{L}^{-1}\left[\frac{(1-n)^2}{n}e^{nb_0+(1-n)S^{(n)}(\rho_0)}\frac{n\mathrm{tr}(\rho_0^{n-1}\delta\rho)}{(1-n)\mathrm{tr}(\rho_0^n)}\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]$$
$$= \mathcal{L}^{-1}\left[(1-n)e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]. \tag{89}$$

One could further simplify the above term if the Rényi entropy $S^{(n)}(\rho_0)$ is given. Similar terms will also appear in the fourth term, albeit with opposite signs, allowing them to cancel each other out. The fourth term is considerably more intricate. In order to articulate the outcomes, we need to introduce the following quantities:

$$\mathcal{P}_{(\delta\alpha_J)}(\lambda') \coloneqq \sum_i\frac{\partial\delta\lambda_i}{\partial\alpha_J}\delta(\lambda^0_i-\lambda') \tag{90}$$

$$\mathcal{P}_{(\delta)(\alpha_J)}(\lambda') \coloneqq \sum_i\delta\lambda_i\frac{\partial\lambda^0_i}{\partial\alpha_J}\delta(\lambda^0_i-\lambda'). \tag{91}$$

These two functions illustrate the relationship between the variation $\delta\lambda_i$ and the parameter α_J. They can be computed once $\frac{\partial\mathrm{tr}(\rho_0^{n-1}\delta\rho)}{\partial\alpha_J}$ is known. With these definitions, we have

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{1-n}{n}e^{nb_0+(1-n)S^{(n)}(\rho_0)}\frac{\partial\delta S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\\
&= \mathcal{L}^{-1}\left[\frac{1-n}{n}e^{nb_0+(1-n)S^{(n)}(\rho_0)}\frac{\partial}{\partial\alpha_J}\frac{n\mathrm{tr}(\rho_0^{n-1}\delta\rho)}{(1-n)\mathrm{tr}(\rho_0^n)}\right]\\
&= \mathcal{L}^{-1}\left[e^{nb_0}\frac{\partial}{\partial\alpha_J}[\mathrm{tr}(\rho_0^{n-1}\delta\rho)]\right] - \mathcal{L}^{-1}\left[e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)(1-n)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\\
&= \mathcal{L}^{-1}\left[e^{nb_0}\frac{\partial}{\partial\alpha_J}\left[\sum_i(\lambda^0_i)^{n-1}\langle i|\delta\rho|i\rangle\right]\right] - \mathcal{L}^{-1}\left[(1-n)e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\\
&= \mathcal{L}^{-1}\left[e^{nb_0}\sum_i(\lambda^0_i)^{n-1}\frac{\partial\langle i|\delta\rho|i\rangle}{\partial\alpha_J}\right] + \mathcal{L}^{-1}\left[e^{nb_0}\sum_i(n-1)(\lambda^0_i)^{n-2}\frac{\partial\lambda^0_i}{\partial\alpha_J}\langle i|\delta\rho|i\rangle\right] - \mathcal{L}^{-1}\left[(1-n)e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\\
&= \mathcal{L}^{-1}[\mathcal{L}[\mathcal{P}_{(\delta\alpha_J)}(e^{-b_0-t})]] + \mathcal{L}^{-1}[(n-1)\mathcal{L}[\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0-t})e^{b_0+t}]] - \mathcal{L}^{-1}\left[(1-n)e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\\
&= \mathcal{P}_{(\delta\alpha_J)}(e^{-b_0-t}) + \delta(t)e^{b_0}\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0}) + \left[e^{b_0+t}\frac{d}{dt}\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0-t}) + e^{b_0+t}\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0-t})\right] - e^{b_0+t}\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0-t})\\
&\quad - \mathcal{L}^{-1}\left[(1-n)e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right]\\
&= \mathcal{P}_{(\delta\alpha_J)}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) + \delta(t)e^{b_0}\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0}) - \mathcal{P}'_{(\delta)(\alpha_J)}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) - \mathcal{L}^{-1}\left[(1-n)e^{nb_0}\mathrm{tr}(\rho_0^{n-1}\delta\rho)\frac{\partial S^{(n)}(\rho_0)}{\partial\alpha_J}\right],
\end{aligned} \tag{92}$$

where in the derivation we use the formula (72) again and define $\mathcal{P}'_{(\delta)(\alpha_J)}(\lambda) \coloneqq \frac{\partial \mathcal{P}_{(\delta)(\alpha_J)}}{\partial \lambda}$. As mentioned earlier, the final term in the above results exactly cancels out with the result from the third term (89).

We also notice that

$$\delta(t)e^{b_0}\mathcal{P}_{(\delta)(\alpha_J)}(e^{-b_0}) = -\delta(t)\delta b\mathcal{P}^0_{\alpha_J}(e^{-b_0}). \tag{93}$$

In summary, the final result can be expressed in the following concise form:

$$\mathcal{P}_{\alpha_J}(e^{-b-t}) = \mathcal{P}^0_{\alpha_J}(e^{-b_0-t}) + \delta\mathcal{P}_{\alpha_J}(e^{-b_0-t}) \tag{94}$$

with

$$\mathcal{P}^0_{\alpha_J}(e^{-b_0-t}) = \mathcal{P}^0_{\alpha_J}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) \tag{95}$$

and

$$\delta\mathcal{P}_{\alpha_J}(e^{-b_0-t}) = -\delta b\lambda\mathcal{P}^{0\prime}_{\alpha_J}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + \mathcal{P}_{(\delta\alpha_J)}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \mathcal{P}'_{(\delta)(\alpha_J)}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right). \tag{96}$$

The aforementioned result for $\mathcal{P}_{\alpha_J}(\lambda)$ is very similar to $\mathcal{P}(\lambda)$. Similar explanations can be provided for each term in (94), akin to what we have done in the previous section.

We can also check that $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ satisfy the consistent relation (28). See Appendix E for the details.

D. A simple example of perturbation states

Let us consider the example studied in Sec. III A. Suppose the density matrix ρ_0 corresponds to the interval with length l_0, while the density matrix $\rho = \rho_0 + \delta\rho$ corresponds to the interval $l = l_0 + \delta l$ with $\delta \ll l$. Thus, we can take $\delta\rho$ as perturbation and obtain

$$\delta S^{(n)} = \frac{c}{6}\left(1 + \frac{1}{n}\right)\frac{\delta l}{l_0}, \tag{97}$$

$$\delta b = b - b_0 = \frac{c}{6}\frac{\delta l}{l_0}. \tag{98}$$

Now it is straightforward to obtain the function $\mathcal{P}_\delta$:

$$\mathcal{P}_\delta(e^{-b_0-t}) = \mathcal{L}^{-1}\left[\frac{(1-n)}{n}e^{nb_0}e^{(1-n)S^{(n)}(\rho_0)}\delta S^{(n)}\right](t) = -\frac{c}{6}\frac{\delta l}{l_0}\left[\frac{(b_0-t)I_1(2\sqrt{b_0 t})}{\sqrt{b_0 t}} + \delta(t)\right]. \tag{99}$$

Further using (79), we have

$$\mathcal{P}(\lambda) = \mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \delta b\lambda\mathcal{P}'_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \mathcal{P}'_\delta\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) = \frac{1}{\lambda}\frac{\sqrt{b_0}I_1}{\sqrt{-b_0-\log\lambda'}} + \delta(\lambda_m - \lambda) + \frac{c}{6}\frac{\delta l}{l_0}\left[\frac{I_1}{\lambda'\sqrt{(-b_0-\log\lambda')b_0}} + \frac{I_2}{\lambda'}\right], \tag{100}$$

where $\lambda' \coloneqq \frac{\lambda_m^0}{\lambda_m}\lambda$. On the other hand, we can also expand $\mathcal{P}(\lambda)$ in (40) and keep the first order $O(\frac{\delta l}{l_0})$. The result is the same as (100). This can be seen as a consistent check of our general result (79).

In this example, we can also check the number of eigenstates (83). With some calculations, we have

$$n(e^{-b-t}) = n_0(e^{-b_0-t}) + \delta b\lambda\mathcal{P}_0(e^{-b_0-t}) + \mathcal{P}_\delta(e^{-b_0-t}) = n_0(e^{-b_0-t}) + \frac{\sqrt{t}I_1(2\sqrt{b_0 t})}{\sqrt{b_0}}\delta b, \tag{101}$$

where $n_0(e^{-b_0-t}) = I_0(2\sqrt{b_0 t})$. Let us consider the two limits $\lambda \to 0$ and $\lambda \to \lambda_m$. We care about $n(0)$ and $n(\lambda_m)$. As expected, $n(0)$ is divergent. Formally, we have

$$N = n(0) - n_0(0) = \mathcal{P}_\delta(0) = \lim_{t\to\infty}\frac{\sqrt{t}I_1(2\sqrt{b_0 t})}{\sqrt{b_0}}\delta b. \tag{102}$$

On the other hand, we also find

$$n(\lambda_m) = n_0(\lambda_m^0) = I_0(0) = 1, \tag{103}$$

which is consistent with our previous discussion; the number of maximal eigenvalues does not change at the first-order perturbation. We anticipate that this outcome holds true beyond the leading-order perturbation, owing to the presence of a Dirac delta function $\delta(\lambda - \lambda_m)$ in the density of eigenstates $\mathcal{P}(\lambda)$. In this example, the maximal eigenstate corresponds to the vacuum state on the annulus following the conformal transformation (47). It is natural for the vacuum state to be nondegenerate in this context. Our results indicate that this nondegeneracy exhibits robustness under first-order perturbations.

VI. GEOMETRIC STATES IN HOLOGRAPHIC THEORY

In this section, our focus will be on holographic theory, a framework wherein certain special states can be effectively described by classical geometry in the semiclassical limit $G \to 0$. These states are referred to as geometric states. In the semiclassical limit $G \to 0$, quantum fluctuations are suppressed. Certain nonlocal observables, such as entanglement entropy, may have a bulk geometric dual. Entanglement can be utilized as a probe to determine whether a given state can be dual to bulk geometry. In fact, constructing states that cannot be dual to bulk geometry is not a difficult task; refer to [33] for details.

A. The functions $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$

The density of eigenstates $\mathcal{P}$ also has some interesting features. By the formula (17), one has

$$\mathcal{P}(e^{-b-t})e^{-b-t} = \mathcal{L}^{-1}[e^{nb+(1-n)S^{(n)}}] = \frac{1}{2\pi i}\int_{\gamma_0-i\infty}^{\gamma_0+i\infty} dne^{s_n}, \tag{104}$$

where

$$s_n \coloneqq n(t+b) + (1-n)S^{(n)}. \tag{105}$$

One important feature of the geometric state is that the Rényi entropy has a gravity dual and follows the area law formula [34]:

$$n^2\partial_n\left(\frac{n-1}{n}S^{(n)}\right) = \frac{\mathcal{B}_n}{4G}, \tag{106}$$

where $\mathcal{B}_n$ denotes the area of the bulk codimension-2 brane. The tension of the brane is related to the index n by $T_n = \frac{n-1}{4nG}$. In the limit $n \to 1$, we would obtain the Ryu-Takayanagi formula.

Note that, by the above holographic formula, one can see that the Rényi entropy for the geometric state should be of $O(1/G)$. This permits us to use saddle point approximation to evaluate the functions $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$. By using saddle point approximation and the holographic Rényi entropy formula, one could derive

$$\mathcal{P}(e^{-b-t})e^{-b-t} = \frac{1}{2\pi}\sqrt{\frac{2\pi}{\frac{\partial^2 s_n}{\partial n^2}|_{n=n*}}}e^{\frac{\mathcal{B}_{n^*}}{4G}}, \tag{107}$$

where $\mathcal{B}_{n^*}$ is the area of the cosmic brane with tension $\mu = \frac{n^*-1}{4Gn^*}$ and n^* is determined by the saddle point condition

$$\partial_n s_n|_{n=n^*} = 0. \tag{108}$$

We can proceed with the evaluation of the function $\mathcal{P}_{\alpha_J}$ for the geometric state. It needs to evaluate the inverse Laplace transformation (22), that is,

$$\mathcal{P}_{\alpha_J}(e^{-b-t}) = \frac{1}{2\pi i}\int_{\gamma_0-i\infty}^{\gamma_0+i\infty} dne^{s_{\alpha_J,n}}, \tag{109}$$

where

$$s_{\alpha_J,n} \coloneqq nt + nb + (1-n)S^{(n)} + \log\frac{\partial S^{(n)}}{\partial\alpha_J} + \log\frac{1-n}{n}. \tag{110}$$

For geometric states, we expect the Rényi entropy should be of the order of $O(1/G)$. Thus, in the semiclassical limit, the last two logarithmic terms can be ignored. Therefore, the saddle point approximation equation

$$\partial_n s_{\alpha_J,n} \simeq \partial_n s_n = 0 \tag{111}$$

holds at the leading order of $O(1/G)$. The solution is given by n^*. Taking n^* back into $s_{\alpha_J,n}$, the function $\mathcal{P}_{\alpha_J}(e^{-b-t})$ can be approximated by

$$\mathcal{P}_{\alpha_J}(e^{-b-t}) \simeq \frac{1}{2\pi}\sqrt{\frac{2\pi}{\frac{\partial^2 s_{\alpha_J,n}}{\partial n^2}|_{n=n*}}}e^{s_{\alpha_J,n^*}} \propto e^{\frac{\mathcal{B}_{n^*}}{4G}}\frac{\partial S^{(n^*)}}{\partial\alpha_J}\frac{1-n^*}{n^*}. \tag{112}$$

Let us analyze the zero point of the function $\mathcal{P}_{\alpha_J}$. The function $\frac{\partial S^{(n)}}{\partial\alpha_J}$ generally does not vanish except at certain special values of α_J. For instance, consider α_J as the size of the subsystem denoted by L. It can be demonstrated that, for a pure state, $\frac{\partial S^{(n)}}{\partial L}$ would vanish if L corresponds to half of the entire system size; refer to the discussions in the next section. Here, we assume that it is not at this critical point. Therefore, the zero point of the function $\mathcal{P}_{\alpha_J}$ is given by $n^* = 1$. The equation

$$\partial_n s_n|_{n=n^*=1} = 0 \tag{113}$$

gives the solution $t = S - b$ or, equally, $t = S - S^\infty$, where S is entanglement entropy.

In Sec. IV B, we evaluate the function $\mathcal{P}_L$ and discover that its zero point is also determined by $t = S - b$ in the large c limit. In fact, for a holographic theory, the central charge $c \sim O(1/G)$. Hence, the large c limit precisely corresponds to the semiclassical limit. The zero point of $\mathcal{P}_L$ in the large c limit discussed in Sec. IV B serves as a nontrivial validation of our general conclusion. It is important to note that our findings in this section hold true for arbitrary parameters α_J, except for the parameters c or G. In the Refs. [16,17], the authors construct the so-called fixed area state in CFTs. It is also noteworthy that the fixed area state with $t = S - b$ is indeed quite special, as it can be considered as an approximate state for the reduced density matrix ρ_A in the large c limit. There might exist profound connections between these intriguing results.

B. Higher dimension example

In [14], the authors consider the holographic Rényi entropy for a sphere region in d-dimensional spacetime. For the dual bulk theory being Einstein gravity, the holographic Rényi entropy is

$$S_d^{(n)} = \frac{n}{n-1}\pi V_\Sigma\left(\frac{\tilde{L}}{l_P}\right)^{d-1}\left(2 - x_n^{d-2}(1+x_n^2)\right), \tag{114}$$

where $x_n = \frac{1}{dn}(1+\sqrt{1-2dn^2+d^2n^2})$, V_Σ denote the "coordinate" volume of the hyperbolic plane, $l_P^{d-1} = 8\pi G$, and $\tilde{L}^2$ gives the AdS curvature scale. Thus, we have

$$S = \lim_{n\to 1} S_d^{(n)} = 2\pi V_\Sigma \left(\frac{\tilde{L}}{l_P}\right)^{d-1},$$
$$b = \lim_{n\to\infty} S_d^{(n)} = 2\pi V_\Sigma \left(\frac{\tilde{L}}{l_P}\right)^{d-1}\left(1-\frac{d-1}{d}\left(\frac{d-2}{d}\right)^{\frac{d-2}{2}}\right). \tag{115}$$

$S_d^{(n)}$ can be rewritten as

$$S_d^{(n)} = \frac{n}{n-1}\left(1-\frac{1}{2}x_n^{d-2}(1+x_n^2)\right)S. \tag{116}$$

We care about the zero point of the function $\mathcal{P}_{\alpha_J}$. In principle, one could obtain the expression of $\mathcal{P}_{\alpha_J}$ by directly computing the inverse Laplace transformation (22). However, we cannot obtain an analytical result. Since we are considering the holographic theory, one could use the saddle point approximation. It can be found that the zero point is given by $t = S - b$. And further, by using (115), the zero point can be written as

$$t = \left[\left(1-\frac{d-1}{d}\left(\frac{d-2}{d}\right)^{\frac{d-2}{2}}\right)^{-1} - 1\right]b. \tag{117}$$

The corresponding zero point λ_0 is given by

$$\lambda_0 = \lambda_m^{(1-\frac{d-1}{d}(\frac{d-2}{d})^{\frac{d-2}{2}})^{-1}}, \tag{118}$$

where $\lambda_m = e^{-b}$. If $d = 2$, $\lambda_0 = \lambda_m^2$, which is consistent with the result in Sec. III A. In the limit $d \to \infty$, the zero point $\lambda_0 \to \lambda_m \lambda_m^{\frac{1}{e-1}}$. It can be shown λ_0 is a monotonically increasing function of d. We plot the zero point λ_0 as a function of the dimension d in Fig. 4.

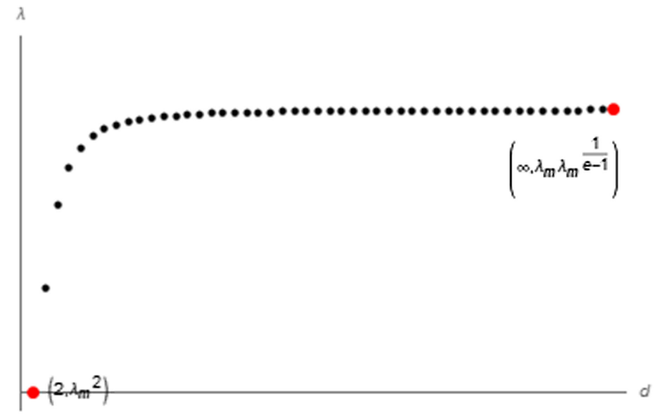

FIG. 4. Illustration of the zero point λ varies with dimension d.

VII. ENTANGLEMENT ENTROPY AND THE FUNCTIONS

In the preceding sections, we explored the properties of functions like $\mathcal{P}$, $\mathcal{P}_{\alpha_J}$, and others. These functions inherently encompass more information than just the entanglement measure, such as the entanglement entropy. In this section, our aim is to demonstrate how the properties of these functions directly correlate with certain aspects of entanglement entropy.

A. First derivative of entanglement entropy with respect to the subsystem size

Recall the definition of entanglement entropy

$$S = -\sum_i \lambda_i \log \lambda_i. \tag{119}$$

The dependence of entanglement entropy on certain parameters is directly related to those functions we have previously studied. Let us focus on the size of the subsystem L. We have

$$\frac{\partial S}{\partial L} = -\sum_i \frac{\partial \lambda_i}{\partial L}\log \lambda_i - \sum_i \frac{\partial \lambda_i}{\partial L},$$
$$= -\int_0^{\lambda_m} d\lambda \mathcal{P}_L(\lambda) \log \lambda, \tag{120}$$

where we have used the fact $\sum_i \frac{\partial \lambda_i}{\partial L} = 0$. Support the function $\mathcal{P}_L$ is given by the black line shown in Fig. 5, which is similar to the example of a single interval in the vacuum state on an infinite line (see Fig. 1). There is a zero point λ_0, $\mathcal{P}_L > 0$ for $\lambda < \lambda_0$ and $\mathcal{P}_L < 0$ for $\lambda_0 < \lambda \le \lambda_m$. It can be shown that

$$\begin{aligned}\frac{\partial S}{\partial L} &= -\int_0^{\lambda_0} d\lambda \mathcal{P}_L(\lambda)\log\lambda - \int_{\lambda_0}^{\lambda_m} d\lambda \mathcal{P}_L(\lambda)\log\lambda \\ &\ge -\int_0^{\lambda_0} d\lambda \mathcal{P}_L(\lambda)\log\lambda_0 - \int_{\lambda_0}^{\lambda_m} d\lambda \mathcal{P}_L(\lambda)\log\lambda \\ &= \int_{\lambda_0}^{\lambda_m} d\lambda \mathcal{P}_L(\lambda)\log\lambda_0 - \int_{\lambda_0}^{\lambda_m} d\lambda \mathcal{P}_L(\lambda)\log\lambda \\ &= \int_{\lambda_0}^{\lambda_m} d\lambda \mathcal{P}_L(\lambda)\log\frac{\lambda_0}{\lambda} \ge 0,\end{aligned} \tag{121}$$

where in the first step we use $-\log\lambda \ge -\log\lambda_0$ for $0 < \lambda < \lambda_0$, in the second step we use $\sum_i \frac{\partial \lambda_i}{\partial L} = \int_0^{\lambda_m} d\lambda \mathcal{P}_L(\lambda) = 0$, and in the last step $\mathcal{P}_L \le 0$ and $\log\frac{\lambda_0}{\lambda} \le 0$ for $\lambda_0 \le \lambda \le \lambda_m$.

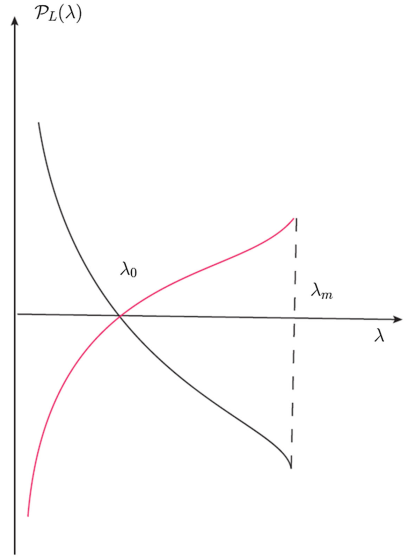

FIG. 5. Illustration of the function $\mathcal{P}_L$. The black line and red line are two typical functions for $\mathcal{P}_L$. λ_0 is the zero point of $\mathcal{P}_L$, and λ_m is the maximal eigenvalue.

While if the function $\mathcal{P}_L$ is like the form of the red line in Fig. 5, we can demonstrate, as previously done, that $\frac{\partial S}{\partial L} \leq 0$.

The above discussion shows that whether the entanglement entropy S increases or decreases with the increase of L depends on the characteristics of function $\mathcal{P}_L$. In Sec. III B, we obtain the function $\mathcal{P}_L$ for one interval with length L on cylinder with circumference R. $\mathcal{P}_L$ is taken as the form of the black line in Fig. 5 for $L < \frac{R}{2}$. This shows that S is a monotonically increasing function of L in this region, while S is monotonically decreasing function of L in the region $\frac{R}{2} < L < R$. There is a critical point $L = \frac{R}{2}$, where $\frac{\partial S}{\partial L} = 0$. At this point we also have $\mathcal{P}_L = 0$, since $\cot\frac{\pi}{2} = 0$.

For one interval in arbitrary pure state, say, $|\psi\rangle$, on a cylinder, we have $S(R - L) = S(L)$, which leads to

$$-S'(R - L) = S'(L). \tag{122}$$

Thus, one could obtain $\frac{\partial S}{\partial L}|_{L=\frac{R}{2}} = 0$. At the point $L = \frac{R}{2}$ we expect the function $\mathcal{P}_L = 0$. By utilizing (122), one can observe that the sign of $\frac{\partial S}{\partial L}$ differs between the two cases: $L < \frac{R}{2}$ and $L > \frac{R}{2}$. Hence, we anticipate that the function $\mathcal{P}_L$ would resemble the black and red lines depicted in Fig. 5 for $L < \frac{R}{2}$ and $L > \frac{R}{2}$, respectively. The above assertions can be verified through specific explicit examples.

One more interesting example is one interval in thermal state with β on cylinder with circumference R. For high-temperature limit R/β, the gravity dual is described by a Banãdos-Teitelboim-Zanelli black hole. By using the RT formula, one could directly evaluate the holographic entanglement entropy by choosing the global minimal surface. It has been demonstrated that the holographic entanglement entropy undergoes a phase transition at a critical point $L = L_c$. These two phases correspond to distinct types of minimal surfaces, illustrated in Fig. 6. For $L < L_c$, it is observed that S is a monotonically increasing function of L, whereas for $L > L_c$, it behaves as a monotonically decreasing function. Based on our earlier discussions, we can conclude that, at the critical point L_c, $\mathcal{P}_{L_c} = 0$. Consequently, the function $\mathcal{P}_L$ serves as a means to identify the phase transition of entanglement entropy.

B. Second derivative of entanglement entropy with respect to the subsystem size

In [35], the author introduce the so-called entropy c function $c(L) \coloneqq L\frac{\partial S}{\partial L}$. By the combination of the Lorentz

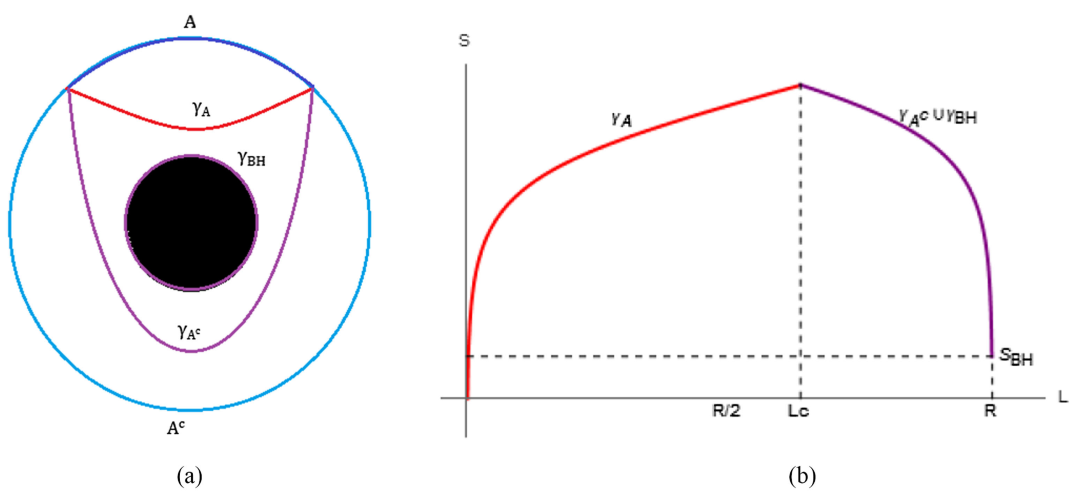

FIG. 6. (a) Shows two types of minimal surfaces γ_A and $\gamma_{A^c} \cup \gamma_{\mathrm{BH}}$, which are shown in red and purple lines, respectively. We notice that γ_A is homotopy with $\gamma_{A^c} \cup \gamma_{\mathrm{BH}}$ instead of γ_{A^c} in this case. Since the result should take the minimum value in the extreme surfaces, for $L < L_c$ the S is given by γ_A, whereas for $L > L_c$ the S is given by $\gamma_{A^c} \cup \gamma_{\mathrm{BH}}$, which are shown in red and purple, respectively, in (b).

symmetry and the strong subadditivity of entropy, it can be shown that $c'(L) \leq 0$. This prompts us to consider the quantity $c'(L) = L\frac{\partial}{\partial L}(L\frac{\partial S}{\partial L})$ and its relation to the functions $\mathcal{P}_L$, $\mathcal{P}_{L^2}$, and $\mathcal{P}_{L_2}$. By definition, we have

$$\begin{aligned} c'(L) &= -L\sum_i \frac{\partial\lambda_i}{\partial L}\log\lambda_i \\ &\quad - L^2\left(\sum_i \frac{\partial^2\lambda_i}{\partial L^2}\log\lambda_i + \sum_i\left(\frac{\partial\lambda_i}{\partial L}\right)^2\lambda_i^{-1}\right) \\ &= -L\int_0^{\lambda_m} d\lambda\mathcal{P}_L\log\lambda \\ &\quad - L^2\left(\int_0^{\lambda_m} d\lambda\mathcal{P}_{L_2}\log\lambda + \int_0^{\lambda_m} d\lambda\mathcal{P}_{L^2}\lambda^{-1}\right). \end{aligned}$$

First, let us consider the contribution from the maximal eigenvalue λ_m. Suppose the maximal eigenvalue is nondegenerate. Its contribution is given by

$$\begin{aligned} &-L\frac{\partial\lambda_m}{\partial L}\log\lambda_m - L^2\left(\frac{\partial^2\lambda_m}{\partial L^2}\log\lambda_m + \left(\frac{\partial\lambda_m}{\partial L}\right)^2\lambda_m^{-1}\right) \\ &\quad = -L\left(1+L\frac{\partial}{\partial L}\right)\left(\frac{\partial\lambda_m}{\partial L}\log\lambda_m\right). \end{aligned} \tag{123}$$

These terms come from the Dirac delta $\delta(\lambda-\lambda_m)$ in the functions $\mathcal{P}_L$, $\mathcal{P}_{L^2}$, and $\mathcal{P}_{L_2}$. Let us define the functions without the Dirac delta terms by $\tilde{\mathcal{P}}_L$, $\tilde{\mathcal{P}}_{L^2}$, and $\tilde{\mathcal{P}}_{L_2}$. Generally, the functions can be written as

$$\begin{aligned} \mathcal{P}_L &= \tilde{\mathcal{P}}_L + \frac{\partial\lambda_m}{\partial L}\delta(\lambda_m-\lambda), \\ \mathcal{P}_{L^2} &= \tilde{\mathcal{P}}_{L^2} + \left(\frac{\partial\lambda_m}{\partial L}\right)^2\delta(\lambda_m-\lambda), \\ \mathcal{P}_{L_2} &= \tilde{\mathcal{P}}_{L_2} + \frac{\partial^2\lambda_m}{\partial L^2}\delta(\lambda_m-\lambda). \end{aligned} \tag{124}$$

By using (E1) with $\alpha_J = L$, the contribution from other eigenvalues is given by

$$\begin{aligned} &-L\int_0^{\lambda_m} d\lambda\tilde{\mathcal{P}}_L\log\lambda - L^2\int_0^{\lambda_m} d\lambda\frac{\partial\tilde{\mathcal{P}}_L}{\partial L}\log\lambda \\ &\quad - L^2\tilde{\mathcal{P}}_{L^2}\log\lambda|_0^{\lambda_m}, \end{aligned} \tag{125}$$

where the last term is the boundary term at the eigenvalues λ_m and 0. In summary, $c'(L)$ can be expressed as

$$\begin{aligned} c'(L) &= -L\int_0^{\lambda_m} d\lambda\left(1+L\frac{\partial}{\partial L}\right)\mathcal{P}_L\log\lambda \\ &\quad - L^2\tilde{\mathcal{P}}_{L^2}\log\lambda|_0^{\lambda_m}. \end{aligned} \tag{126}$$

Let us consider the boundary term $\lim_{\lambda\to 0}L^2\tilde{\mathcal{P}}_{L^2}\log\lambda$, which can be determined by studying the behavior of $\mathcal{P}_{L^2}$ as λ approaches zero. However, our knowledge about the properties of this function is limited. An explicit example might be found in the case of a single interval in the vacuum state [see (III A)]. Using Eq. (45), we find that $\mathcal{P}_{L^2}(\lambda) \sim \lambda(\log\lambda)^2 e^{2\sqrt{-b\log\lambda}}$ as λ approaches zero. For this particular example, we observe that $\lim_{\lambda\to 0}L^2\tilde{\mathcal{P}}_{L^2}\log\lambda = 0$. Starting from (123), $c'(L)$ can be expressed as an integration involving functions $\mathcal{P}_{L^2}$. Convergence of this integration is expected, given that $c'(L)$ is generally finite. This expectation implies that $\int_0^\epsilon d\lambda\mathcal{P}_{L^2}\lambda^{-1}$ should yield a constant for any positive ϵ, ensuring convergence. Consequently, we obtain $\mathcal{P}_{L^2}(\epsilon)\epsilon^{-1} \to C$, where C is a constant. Consequently, we derive

$$L^2\lim_{\lambda\to 0}\tilde{\mathcal{P}}_{L^2}(\lambda)\log\lambda = L^2\lim_{\lambda\to 0}\tilde{\mathcal{P}}_{L^2}(\lambda)\lambda^{-1}\lambda\log\lambda \to 0. \tag{127}$$

The other boundary term $-L^2\tilde{\mathcal{P}}_{L^2}(\lambda_m)\log\lambda_m$ is typically nonzero. Since we have $0 \leq \mathcal{P}_{L^2}$ and $0 < \lambda_m < 1$, this term is positive.

The integration part in (126) closely resembles (120) when replacing $\mathcal{P}_L$ with $L(1+L\frac{\partial}{\partial L})\mathcal{P}_L$. Similar to our approach in the previous section, the nature of $L(1+L\frac{\partial}{\partial L})\mathcal{P}_L$ is intricately connected to the sign of the integration result.

VIII. CONCLUSION AND DISCUSSION

In this paper, we introduce a series of functions designed to characterize the dependence of the entanglement spectrum on parameters. These functions bear resemblance to the density of eigenstate $\mathcal{P}$ extensively discussed in prior literature. Our novel functions, such as $\mathcal{P}\alpha_J$ and $\mathcal{P}\alpha_{J_1}\alpha_{J_2}$, encapsulate crucial information regarding $\frac{\partial\lambda_i}{\partial\alpha_J}$ and $\frac{\partial^2\lambda_i}{\partial\alpha_{J_1}\partial\alpha_{J_2}}$. The evaluation of these functions can be accomplished through the utilization of Rényi entropy. Notably, functions of the same order exhibit intriguing relationships, e.g., Eq. (E1). Furthermore, we demonstrate that these relationships can be derived from their definitions in Sec. II B. However, our study reveals limitations in obtaining all these functions solely through the inverse Laplace transformation method employed in this paper. It appears that alternative methodologies or additional information beyond Rényi entropy may be necessary to obtain a complete set of these functions. We will delve into exploring these avenues in the near future.

If we make the additional assumption that the derivative of a given eigenvalue λ_i with respect to α_J remains a function of λ_i, i.e., $\frac{\partial\lambda_i}{\partial\alpha_J} = f(\lambda_i, \alpha_J)$, an intriguing differential equation (33) governing λ_i can be derived. Solving this differential equation enables the reconstruction of the form of λ_i. In our examination of a single interval within a vacuum state, we explicitly demonstrate how to derive the dependence of λ_i on the subsystem size L using the functions $\mathcal{P}$ and $\mathcal{P}_L$. Remarkably, this outcome aligns with the methodology involving the mapping of the modular Hamiltonian to a cylinder, presenting an interesting application of these functions. However, it is crucial to highlight a significant limitation in this process. The assumption that $\frac{\partial\lambda_i}{\partial\alpha_J} = f(\lambda_i, \alpha_J)$ may not hold universally across all scenarios. Care must be exercised when employing this assumption. An intriguing avenue for exploration involves relaxing this assumption, such as considering whether $\frac{\partial\lambda_i}{\partial\alpha_J}$ depends on all the eigenvalues. Yet, pursuing this route ultimately leads to a series of complex partial differential equations that prove challenging to solve. Obtaining the exact form of eigenvalues of entanglement Hamiltonian in QFTs remains an extremely challenging problem. While our current findings are constrained, we anticipate that our framework serves as a potential method to reconstruct the eigenvalues of the entanglement Hamiltonian using Rényi entropy.

In several instances, the Rényi entropy can be obtained using replica methods. Our paper showcases various examples illustrating how to derive the functions introduced in our study. These instances encompass scenarios such as a single interval in a vacuum state, arbitrary states for a short interval in two-dimensional CFTs, perturbation states in the general case, and holographic QFTs. Our primary focus lies on understanding the functions $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ within these examples.

Calculations in two-dimensional CFTs yield straightforward results. For perturbation states, we obtain exact expressions at the leading order of the perturbation. The final forms of $\mathcal{P}$ and $\mathcal{P}_{\alpha_J}$ offer insightful explanations. These results become applicable when the Rényi entropy of the perturbation state is known.

In the context of holographic theory, a fascinating finding arises for the function $\mathcal{P}\alpha_J$. In the semiclassical limit $G \to 0$, where G represents the gravitational constant, the zero point of this function is identified as $\lambda_0 = e^{-S}$ or $t_0 = S - S^\infty$. Here, S denotes the EE, and S^∞ signifies the minimal entropy, defined as $S^\infty := \lim_{n\to\infty} S^{(n)}$. Intriguingly, the value of λ_0 or t_0 also emerges in the approximated state for ρ_A constructed in [16]. In that work, the author observes the density of eigenstates approaching a Dirac delta function at the value $t_0 = S - S^\infty$. While the relationship between these two findings remains elusive, they signify distinct features of the geometric states in holographic theory. Specifically, the entanglement spectra of these geometric states exhibit peculiar properties near the value t_0. Further exploration into this phenomenon is planned for our future investigations.

The functions introduced in our study are intricately connected to the Rényi entropy and its derivatives through Laplace transformations. In principle, they should equate to the Rényi entropy since the Laplace transformation is reversible. One might question the necessity of investigating these functions. This parallels the field of signal processing, where Fourier or Laplace transformations are employed to convert signals into the dual space. Occasionally, the signal in the dual space offers more intuitive insights. Similarly, while the Rényi entropy encapsulates rich information regarding entanglement spectra, the functions we introduced serve as a method to extract this entanglement spectrum information.

The functions associated with λ or t can be viewed as the dual space counterparts of the Rényi index n. Specifically, in the context of holographic theory, these functions have proven useful in comprehending fixed area states and QEC codes for AdS/CFT [17]. Particularly, these functions in the dual space are anticipated to hold significant applications in elucidating the properties of geometric states. They offer an alternative perspective to understand and explore the intricacies of entanglement spectra that might not be readily apparent from the Rényi entropy alone.

ACKNOWLEDGMENTS

W.-Z. G. is supposed by the National Natural Science Foundation of China under Grant No. 12005070 and the Fundamental Research Funds for the Central Universities under Grant No. 2020kfyXJJS041.

APPENDIX A: DETAILS OF THE CALCULATIONS FOR GENERAL SETUP

1. The proof of general relation (30)

The proof is straightforward by using the definition and property of delta function, which is shown as follows:

$$\begin{aligned}
\frac{\partial}{\partial\alpha_K}\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})} &= \frac{\partial}{\partial\alpha_K}\sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\delta(\lambda_i-\lambda) \\
&= \sum_i \frac{\partial^{m_1+1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}\partial\alpha_K}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\delta(\lambda_i-\lambda) \\
&\quad + \sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2+1}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}\partial\alpha_K}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\delta(\lambda_i-\lambda) + \cdots
\end{aligned}$$

$$
\begin{aligned}
&+\sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n+1}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}\partial\alpha_K}\delta(\lambda_i-\lambda)\\
&+\sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\frac{\partial}{\partial\alpha_K}\delta(\lambda_i-\lambda)\\
=&\sum_i \frac{\partial^{m_1+1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}\partial\alpha_K}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\delta(\lambda_i-\lambda)\\
&+\sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2+1}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}\partial\alpha_K}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\delta(\lambda_i-\lambda)+\cdots\\
&+\sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n+1}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}\partial\alpha_K}\delta(\lambda_i-\lambda)\\
&+\sum_i \frac{\partial^{m_1}\lambda_i}{\partial\alpha_{J_{11}}\ldots\partial\alpha_{J_{1m_1}}}\frac{\partial^{m_2}\lambda_i}{\partial\alpha_{J_{21}}\ldots\partial\alpha_{J_{2m_2}}}\cdots\frac{\partial^{m_n}\lambda_i}{\partial\alpha_{J_{n1}}\ldots\partial\alpha_{J_{nm_n}}}\frac{\partial\lambda_i}{\partial\alpha_K}*\left[-\frac{\partial}{\partial\lambda}\delta(\lambda_i-\lambda)\right]\\
=&\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}}\alpha_K)(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})}+\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}}\alpha_K)\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})}+\cdots\\
&+\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}}\alpha_K)}-\frac{\partial}{\partial\lambda}\mathcal{P}_{(\alpha_{J_{11}}\ldots\alpha_{J_{1m_1}})(\alpha_{J_{21}}\ldots\alpha_{J_{2m_2}})\ldots(\alpha_{J_{n1}}\ldots\alpha_{J_{nm_n}})(\alpha_K)}.
\end{aligned}
\tag{A1}
$$

2. The formula derivation of (37) $\mathcal{P}_{\alpha_J^m}$ and (38) $\mathcal{P}_{\alpha_{Jm}}$

The derivation of $\mathcal{P}_{\alpha_J^m}$ is trivial, with our assumption $\frac{\partial\lambda_i}{\partial\alpha_J}=f(\lambda_i,\alpha_J)$:

$$
\begin{aligned}
\mathcal{P}_{\alpha_J^m}(\lambda)&=\sum_i\left(\frac{\partial\lambda_i}{\partial\alpha_J}\right)^m\delta(\lambda_i-\lambda)\\
&=f(\lambda,\alpha_J)^m\mathcal{P}(\lambda).
\end{aligned}
\tag{A2}
$$

Since the case where $m=2$ for $\mathcal{P}_{\alpha_{Jm}}$ we have already deduced, the case of $m>2$ can be proved by mathematical induction. Since we assume that the formula holds for $m-1$, we have

$$
\begin{aligned}
\mathcal{P}_{\alpha_{Jm-1}}(\lambda)&=\frac{D^{m-2}f(\lambda,\alpha_J)}{D\alpha_J^{m-2}}\mathcal{P}(\lambda),\\
\mathcal{P}_{\alpha_{Jm-1}}(\lambda)&=g(\lambda,\alpha_J)\mathcal{P}(\lambda),\\
\sum_i\frac{d^{m-1}}{d\alpha_J^{m-1}}f(\lambda_i,\alpha_J)\delta(\lambda_i-\lambda)&=\sum_i g(\lambda_i,\alpha_J)\delta(\lambda_i-\lambda),\\
\frac{d^{m-1}}{d\alpha_J^{m-1}}f(\lambda_i,\alpha_J)&=g(\lambda_i,\alpha_J),
\end{aligned}
\tag{A3}
$$

where we define

$$
g(\lambda,\alpha_J)\coloneqq\frac{D^{m-2}f(\lambda,\alpha_J)}{D\alpha_J^{m-2}}.
\tag{A4}
$$

So, in the case of m, we have

$$
\begin{aligned}
\mathcal{P}_{\alpha_{Jm}}(\lambda)&=\sum_i\frac{d}{d\alpha_J}\frac{d^{m-1}}{d\alpha_J^{m-1}}f(\lambda_i,\alpha_J)\delta(\lambda_i-\lambda),\\
\mathcal{P}_{\alpha_{Jm}}(\lambda)&=\sum_i\frac{d}{d\alpha_J}g(\lambda_i,\alpha_J)\delta(\lambda_i-\lambda)\\
&=\sum_i\left(\frac{\partial g(\lambda_i,\alpha_J)}{\partial\lambda_i}\frac{\partial\lambda_i}{\partial\alpha_J}+\frac{\partial g(\lambda_i,\alpha_J)}{\partial\alpha_J}\right)\delta(\lambda_i-\lambda)\\
&=\sum_i\left(\frac{\partial g(\lambda_i,\alpha_J)}{\partial\lambda_i}f(\lambda_i,\alpha_J)+\frac{\partial g(\lambda_i,\alpha_J)}{\partial\alpha_J}\right)\delta(\lambda_i-\lambda)\\
&=\left(\frac{\partial g(\lambda,\alpha_J)}{\partial\lambda}f(\lambda,\alpha_J)+\frac{\partial g(\lambda,\alpha_J)}{\partial\alpha_J}\right)\sum_i\delta(\lambda_i-\lambda)\\
&=\frac{Dg(\lambda,\alpha_J)}{D\alpha_J}\mathcal{P}(\lambda)\\
&=\frac{D^{m-1}f(\lambda,\alpha_J)}{D\alpha_J^{m-1}}\mathcal{P}(\lambda).
\end{aligned}
\tag{A5}
$$

APPENDIX B: CONSISTENT CHECK OF THE FUNCTIONS FOR VACUUM STATE

In the main text, we obtain the functions $\mathcal{P}$ and $\mathcal{P}_l$ for the one interval in the vacuum state of CFTs. In this section, we would like to check the consistent relation (28). Since $t=-b-\log\lambda$, we have $\frac{\partial t}{\partial L}=-\frac{\partial b}{\partial L}$, so

$$\begin{aligned}\frac{\partial}{\partial L}P(\lambda) &= \frac{1}{\lambda}\frac{\partial}{\partial L}\left(\frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}}+\delta(t)\right)\\ &= \frac{1}{\lambda}\left[\frac{\partial}{\partial b}\left(\frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}}\right)\frac{\partial b}{\partial L}+\frac{\partial}{\partial t}\left(\frac{\sqrt{b}I_1(2\sqrt{bt})}{\sqrt{t}}\right)\frac{\partial t}{\partial L}\right]+\frac{1}{\lambda}\frac{\partial t}{\partial L}\delta'(t)\\ &= \frac{1}{\lambda}\left[\frac{c}{6L}\left(\frac{I_1(2\sqrt{bt})}{2\sqrt{bt}}+\frac{1}{2}(I_0(2\sqrt{bt})+I_2(2\sqrt{bt}))\right)-\frac{c}{6L}\left(\frac{b(I_0(2\sqrt{bt})+I_2(2\sqrt{bt}))}{2t}-\frac{\sqrt{b}I_1(2\sqrt{bt})}{2t^{3/2}}\right)\right]-\frac{c}{6L}\frac{1}{\lambda}\delta'(t)\\ &= \frac{1}{\lambda}\frac{c}{6L}\frac{I_1(2\sqrt{bt})}{2\sqrt{bt}}+\frac{1}{\lambda}\frac{c}{6L}\frac{\sqrt{b}I_1(2\sqrt{bt})}{2t^{3/2}}+\frac{1}{\lambda}\frac{c}{6L}\frac{1}{2}\frac{t-b}{t}(I_0(2\sqrt{bt})+I_2(2\sqrt{bt}))-\frac{1}{\lambda}\frac{c}{6L}\delta'(t),\end{aligned} \tag{B1}$$

$$\begin{aligned}-\frac{\partial}{\partial\lambda}P_L(\lambda) &= \frac{\partial}{\partial\lambda}\times\frac{c}{6L}\left(\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}}+\delta(t)\right)\\ &= \frac{c}{6L}\frac{\partial}{\partial t}\left(\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}}\right)\frac{\partial t}{\partial\lambda}+\frac{c}{6L}\frac{\partial t}{\partial\lambda}\delta'(t)\\ &= \frac{1}{\lambda}\frac{c}{6L}\frac{I_1(2\sqrt{bt})}{2\sqrt{bt}}+\frac{1}{\lambda}\frac{c}{6L}\frac{\sqrt{b}I_1(2\sqrt{bt})}{2t^{3/2}}+\frac{1}{\lambda}\frac{c}{6L}\\ &\quad\times\frac{1}{2}\frac{t-b}{t}\left(I_0(2\sqrt{bt})+I_2(2\sqrt{bt})\right)-\frac{1}{\lambda}\frac{c}{6L}\delta'(t)\\ &= left.\end{aligned} \tag{B2}$$

On the other hand, since we have (40), let us rewrite

$$\begin{aligned}\mathcal{P}_L(\lambda) &= -\frac{c}{6L}\left[\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}}+\delta(t)\right] = -\frac{c}{6L}\mathcal{P}_{\alpha_J}(\lambda),\\ \mathcal{P}_c(\lambda) &= -\frac{\log L}{6}\left[\frac{(b-t)I_1(2\sqrt{bt})}{\sqrt{bt}}+\delta(t)\right]\\ &= -\frac{\log L}{6}\mathcal{P}_{\alpha_J}(\lambda),\end{aligned} \tag{B3}$$

so we have

$$\begin{aligned}\frac{\partial}{\partial c}\mathcal{P}_L(\lambda) &= -\frac{1}{6L}\mathcal{P}_{\alpha_J}(\lambda)-\frac{c}{6L}\frac{\log L}{6}\frac{\partial}{\partial b}\mathcal{P}_{\alpha_J}(\lambda),\\ \frac{\partial}{\partial L}\mathcal{P}_c(\lambda) &= -\frac{1}{6L}\mathcal{P}_{\alpha_J}(\lambda)-\frac{\log L}{6}\frac{c}{6L}\frac{\partial}{\partial b}\mathcal{P}_{\alpha_J}(\lambda)\\ &= \frac{\partial}{\partial c}\mathcal{P}_L(\lambda).\end{aligned} \tag{B4}$$

APPENDIX C: RECONSTRUCTION OF THE EIGENVALUE

In Sec. III C, we use a further assumption that $\frac{\partial\lambda_i}{\partial L}=f(\lambda_i)$ and the functions $\mathcal{P}$ and $\mathcal{P}_L$ to reconstruct the eigenvalues of ρ_A. In principle, one could also choose other parameters, such as c. But one would obtain the wrong results as we will show below. Using (40) and the assumption $\frac{\partial\lambda_i}{\partial c}=f(\lambda_i)$, we have

$$\frac{\partial\lambda_i}{\partial c} = \frac{-\log\lambda_i-2b}{b}\frac{\log L}{6}\lambda_i, \tag{C1}$$

which can be solved as

$$\lambda_i = e^{-\frac{\tilde{C}_i}{c}-b}, \tag{C2}$$

where $\tilde{C}_i$ are constants unrelated to c. This is inconsistent with the form $e^{-\frac{\Delta_i-\frac{c}{24}}{W}-b}$ that is derived in Sec. III D. By the normalization of ρ_A, we have

$$\sum_i e^{-\frac{\Delta_i-\frac{c}{24}}{W}-b} = 1. \tag{C3}$$

Using this, one could find that $\frac{\partial\Delta_i}{\partial c}$ should satisfy the constraint

$$\sum_i\frac{\partial\Delta_i}{\partial c}e^{-\frac{\Delta_i}{W}} = -We^{b-\frac{c}{24W}}\left(\frac{\partial b}{\partial c}-\frac{1}{24W}\right), \tag{C4}$$

which means $\frac{\partial\Delta_i}{\partial c}$ is not only a function of Δ_i but depends on other Δ_j ($j\neq i$). Therefore, the assumption $\frac{\partial\lambda_i}{\partial c}=f(\lambda_i)$ is incorrect.

APPENDIX D: DETAILS OF THE CALCULATIONS FOR SHORT INTERVAL

For the thermal state, we have

$$\langle T\rangle_\beta = \langle\bar{T}\rangle_\beta = -\frac{\pi^2 c}{6\beta^2}, \qquad \langle A\rangle_\beta = \langle\bar{A}\rangle\beta = \frac{\pi^4 c(5c+22)}{180\beta^4}. \tag{D1}$$

Thus, we obtain

$$k_2 = \frac{\pi^2 c}{36\beta^2}, \qquad k_4 = -\frac{11\pi^4 c}{12960\beta^4}, \qquad k_4' = -\frac{\pi^4 c}{12960\beta^4}. \tag{D2}$$

Taking the above results into (55), we have

$$\mathcal{P} = \lambda^{-1}\left[\frac{\sqrt{b_0}}{\sqrt{t}}I_1 + \delta(t) + (k_2 l^2 + (2k_4 - 10k_4')l^4)I_0 + \frac{1}{2}k_2^2 l^4\frac{\sqrt{t}}{\sqrt{b_0}}I_1\right] = \lambda^{-1}\left[\frac{\sqrt{b_0}}{\sqrt{t}}I_1 + \delta(t) + \left(\frac{\pi^2 c}{36\beta^2}l^2 - \frac{\pi^4 c}{1080\beta^4}l^4\right)I_0 + \frac{\pi^4 c^2}{2596\beta^4}l^4\frac{\sqrt{t}}{\sqrt{b_0}}I_1\right], \tag{D3}$$

where the argument of I_n is $2\sqrt{b_0 t}$, the same as below. The Rényi entropy of the thermal state is given by

$$S^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi l}{\beta}\right)\right], \tag{D4}$$

so we can get $\mathcal{P}$ by

$$\mathcal{P}' = \lambda^{-1}\mathcal{L}^{-1}[e^{nb_T+(1-n)S_T^{(n)}}] = \frac{1}{\lambda}\left(\frac{\sqrt{b_T}I_1(2\sqrt{b_T t})}{\sqrt{t}} + \delta(t)\right) \tag{D5}$$

with

$$b_T = \frac{c}{6}\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi l}{\beta}\right)\right]. \tag{D6}$$

Expanding $\mathcal{P}'$ up to the order $(\frac{l}{\beta})^4$, we find

$$\mathcal{P}' = \lambda^{-1}\left[\frac{\sqrt{b_0}}{\sqrt{t}}I_1 + \delta(t) + \frac{c\pi^2(I_1+\sqrt{b_0 t}I_2)l^2}{36\beta^2\sqrt{b_0 t}} + \frac{[5\pi^4c^2 t I_1 - 12\pi^4 c(I_1+\sqrt{b_0 t}I_2)]l^4}{12960\beta^4\sqrt{b_0 t}}\right] + O(l^5). \tag{D7}$$

By using the relation $I_{n-1}(x) - I_{n+1}(x) = \frac{2nI_n(x)}{x}$, we have

$$I_0(2\sqrt{b_0 t}) - I_2(2\sqrt{b_0 t}) = \frac{2I_1(2\sqrt{b_0 t})}{2\sqrt{b_0 t}},$$
$$I_1(2\sqrt{b_0 t}) + \sqrt{b_0 t}I_2(2\sqrt{b_0 t}) = \sqrt{b_0 t}I_0(2\sqrt{b_0 t}). \tag{D8}$$

Thus, we find

$$\mathcal{P}' = \lambda^{-1}\left[\frac{\sqrt{b_0}}{\sqrt{t}}I_1 + \delta(t) + \left(\frac{\pi^2 c}{36\beta^2}l^2 - \frac{\pi^4 c}{1080\beta^4}l^4\right)I_0 + \frac{\pi^4 c^2}{2596\beta^4}l^4\frac{\sqrt{t}}{\sqrt{b_0}}I_1\right] = \mathcal{P}. \tag{D9}$$

One could also check the results for $\mathcal{P}_l$ by the same method.

APPENDIX E: CONSISTENT CHECK OF THE FUNCTIONS FOR PERTURBATION STATES

Before checking the relation (23) of (79) and (94), we want to find the relation between $\mathcal{P}_\delta$ and $\mathcal{P}_{(\delta\alpha_J)}$ and $\mathcal{P}_{(\delta)(\alpha_J)}$ first. By the definitions, we have

$$\frac{\partial}{\partial\alpha_J}\mathcal{P}_\delta(\lambda) = \frac{\partial}{\partial\alpha_J}\sum_i \delta\lambda_i\delta(\lambda_i^0-\lambda) = \sum_i\frac{\partial\delta\lambda_i}{\partial\alpha_J}\delta(\lambda_i^0-\lambda) + \sum_i\delta\lambda_i\frac{\partial\lambda_i^0}{\partial\alpha_J}\delta'(\lambda_i^0-\lambda) = \mathcal{P}_{(\delta\alpha_J)}(\lambda) - \frac{\partial}{\partial\lambda}\mathcal{P}_{(\delta)(\alpha_J)}(\lambda). \tag{E1}$$

By using (79), (94), and (E1),

$$\begin{aligned}\frac{\partial}{\partial\alpha_J}\mathcal{P}(\lambda) &= \frac{\partial}{\partial\alpha_J}\left[\mathcal{P}_0\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \delta b\lambda\frac{\partial\mathcal{P}_0}{\partial\lambda}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \frac{\partial\mathcal{P}_\delta}{\partial\lambda}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)\right]\\ &= \frac{\partial\mathcal{P}_0}{\partial\alpha_J}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + \frac{\partial\mathcal{P}_0(\frac{\lambda_m^0}{\lambda_m}\lambda)}{\partial\frac{\lambda_m^0}{\lambda_m}\lambda}\frac{\partial\frac{\lambda_m^0}{\lambda_m}\lambda}{\partial\alpha_J} - \frac{\partial\delta b}{\partial\alpha_J}\lambda\frac{\partial\mathcal{P}_0}{\partial\lambda}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \delta b\lambda\frac{\partial^2\mathcal{P}_0}{\partial\lambda\partial\alpha_J}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)\\ &\quad + O\left(\delta\rho^2\right) - \frac{\partial^2\mathcal{P}_\delta}{\partial\lambda\partial\alpha_J}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + O(\delta\rho^2)\\ &= -\frac{\partial\mathcal{P}_{\alpha_J}^0}{\partial\lambda}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) + \delta b\lambda\frac{\partial^2\mathcal{P}_{\alpha_J}^0}{\partial\lambda^2}\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right) - \left[\frac{\partial\mathcal{P}_{(\delta\alpha_J)}}{\partial\lambda} - \frac{\partial^2\mathcal{P}_{(\delta)(\alpha_J)}}{\partial\lambda^2}\right]\left(\frac{\lambda_m^0}{\lambda_m}\lambda\right)\end{aligned} \tag{E2}$$

and

$$\begin{aligned}-\frac{\partial \mathcal{P}_{\alpha_J}}{\partial \lambda}(\lambda) &= -\frac{\partial}{\partial \lambda}\left[\mathcal{P}^0_{\alpha_J}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) - \delta b \lambda \frac{\partial \mathcal{P}^0_{\alpha_J}}{\partial \lambda}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) + \mathcal{P}_{(\delta\alpha_J)}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) - \frac{\partial \mathcal{P}_{(\delta)(\alpha_J)}}{\partial \lambda}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right)\right] \\ &= -(1+\delta b)\frac{\partial \mathcal{P}^0_{\alpha_J}}{\partial \lambda}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) + \delta b \frac{\partial \mathcal{P}^0_{\alpha_J}}{\partial \lambda}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) + \delta b(1+\delta b)\lambda \frac{\partial^2 \mathcal{P}^0_{\alpha_J}}{\partial \lambda^2}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) \\ &\quad - \left[(1+\delta b)\frac{\partial \mathcal{P}_{(\delta\alpha_J)}}{\partial \lambda}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) - (1+\delta b)\frac{\partial^2 \mathcal{P}_{(\delta)(\alpha_J)}}{\partial \lambda^2}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right)\right] \\ &= -\frac{\partial \mathcal{P}^0_{\alpha_J}}{\partial \lambda}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) + \delta b \lambda \frac{\partial^2 \mathcal{P}^0_{\alpha_J}}{\partial \lambda^2}\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) - \left[\frac{\partial \mathcal{P}_{(\delta\alpha_J)}}{\partial \lambda} - \frac{\partial^2 \mathcal{P}_{(\delta)(\alpha_J)}}{\partial \lambda^2}\right]\left(\frac{\lambda^0_m}{\lambda_m}\lambda\right) + O(\delta\rho^2) \\ &= \frac{\partial}{\partial \alpha_J}\mathcal{P}(\lambda). \end{aligned} \tag{E3}$$

[1] C. Holzhey, F. Larsen, and F. Wilczek, Geometric and renormalized entropy in conformal field theory, Nucl. Phys. **B424**, 443 (1994).
[2] M. Srednicki, Entropy and area, Phys. Rev. Lett. **71**, 666 (1993).
[3] G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Entanglement in quantum critical phenomena, Phys. Rev. Lett. **90**, 227902 (2003).
[4] P. Calabrese and J. L. Cardy, Entanglement entropy and quantum field theory, J. Stat. Mech. (2004) P06002.
[5] P. Calabrese and J. Cardy, Entanglement entropy and conformal field theory, J. Phys. A **42**, 504005 (2009).
[6] H. Casini and M. Huerta, Entanglement entropy in free quantum field theory, J. Phys. A **42**, 504007 (2009).
[7] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. **2**, 231 (1998).
[8] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[9] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[10] S. Ryu and T. Takayanagi, Holographic derivation of entanglement entropy from AdS/CFT, Phys. Rev. Lett. **96**, 181602 (2006).
[11] V. E. Hubeny, M. Rangamani, and T. Takayanagi, A covariant holographic entanglement entropy proposal, J. High Energy Phys. 07 (2007) 062.
[12] H. Li and F. Haldane, Entanglement spectrum as a generalization of entanglement entropy: Identification of topological order in non-Abelian fractional quantum Hall effect states, Phys. Rev. Lett. **101**, 010504 (2008).
[13] P. Calabrese and A. Lefevre, Entanglement spectrum in one-dimensional systems, Phys. Rev. A **78**, 032329 (2008).
[14] L. Y. Hung, R. C. Myers, M. Smolkin, and A. Yale, Holographic calculations of Renyi entropy, J. High Energy Phys. 12 (2011) 047.
[15] W. z. Guo, Correlations in geometric states, J. High Energy Phys. 08 (2020) 125.
[16] W. z. Guo, Entanglement spectrum of geometric states, J. High Energy Phys. 02 (2021) 085.
[17] W. z. Guo, Area operator and fixed area states in conformal field theories, Phys. Rev. D **106**, L061903 (2022).
[18] C. Akers and P. Rath, Holographic Renyi entropy from quantum error correction, J. High Energy Phys. 05 (2019) 052.
[19] X. Dong, D. Harlow, and D. Marolf, Flat entanglement spectra in fixed-area states of quantum gravity, J. High Energy Phys. 10 (2019) 240.
[20] A. Almheiri, X. Dong, and D. Harlow, Bulk locality and quantum error correction in AdS/CFT, J. High Energy Phys. 04 (2015) 163.
[21] Z. C. Yang, C. Chamon, A. Hamma, and E. R. Mucciolo, Two-component structure in the entanglement spectrum of highly excited states, Phys. Rev. Lett. **115**, 267206 (2015).
[22] P. Ruggiero, V. Alba, and P. Calabrese, Negativity spectrum of one-dimensional conformal field theories, Phys. Rev. B **94**, 195121 (2016).
[23] G. Cho, A. Ludwig, and S. Ryu, Universal entanglement spectra of gapped one-dimensional field theories, Phys. Rev. B **95**, 115122 (2017).
[24] J. Kudler-Flam, V. Narovlansky, and S. Ryu, Negativity spectra in random tensor networks and holography, J. High Energy Phys. 02 (2022) 076.
[25] Z. Yan and Z. Y. Meng, Unlocking the general relationship between energy and entanglement spectra via the wormhole effect, Nat. Commun. **14**, 2360 (2023).
[26] X. Bai and J. Ren, Holographic Rényi entropies from hyperbolic black holes with scalar hair, J. High Energy Phys. 12 (2022) 038.
[27] J. Cardy and E. Tonni, Entanglement Hamiltonians in two-dimensional conformal field theory, J. Stat. Mech. (2016) 123103.
[28] J. L. Cardy, O. A. Castro-Alvaredo, and B. Doyon, Form factors of branch-point twist fields in quantum integrable models and entanglement entropy, J. Stat. Phys. **130**, 129 (2008).
[29] M. Headrick, Entanglement Rényi entropies in holographic theories, Phys. Rev. D **82**, 126010 (2010).

[30] P. Calabrese, J. Cardy, and E. Tonni, Entanglement entropy of two disjoint intervals in conformal field theory II, J. Stat. Mech. (2011) P01021.

[31] M. A. Rajabpour and F. Gliozzi, Entanglement entropy of two disjoint intervals from fusion algebra of twist fields, J. Stat. Mech. (2012) P02016.

[32] B. Chen and J.-j. Zhang, On short interval expansion of Rényi entropy, J. High Energy Phys. 11 (2013) 164.

[33] W. Z. Guo, F. L. Lin, and J. Zhang, Nongeometric states in a holographic conformal field theory, Phys. Rev. D **99**, 106001 (2019).

[34] X. Dong, The gravity dual of Renyi entropy, Nat. Commun. **7**, 12472 (2016).

[35] H. Casini and M. Huerta, A finite entanglement entropy and the c-theorem, Phys. Lett. B **600**, 142 (2004).