Reflection coefficient of a reflectionless kink

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(Received 29 February 2024; accepted 10 April 2024; published 29 April 2024)

Classically, reflectionless kinks transmit all incident radiation. Recently, we have used an analyticity argument together with a solution of the Lippmann-Schwinger equation to write down the leading quantum correction to the reflection probability. The argument was fast, but rather indirect. In the present paper, we calculate the reflection coefficient and probability by methodically grinding through the Schrödinger picture time evolution. We find the same answer. This answer contains contributions not considered in the traditional calculation of meson-kink scattering in 1991. However, as a result of these contributions, our total result is zero in the case of the Sine-Gordon model, and so it is consistent with integrability.

DOI: 10.1103/PhysRevD.109.085019

I. INTRODUCTION

The understanding of the interactions of solitons with perturbative excitations has many potential applications, from searches for cosmic strings in the cosmic microwave [1] and gravity wave [2] backgrounds to soliton-soliton scattering, where soliton-bulk interactions play a key role [3–7].

At tree level, these interactions have long been understood [8]. However, there is reason to believe that quantum corrections qualitatively change the situation, as is thought to be the case for the oscillon [9,10] and Q-ball [11] lifetimes and dynamics [12]. This is because, in the quantum theory, the leading quantum corrections appear to make reflectionless kinks reflect perturbative mesons. The leading quantum corrections to the scattering of kinks with mesons were studied in a series of papers [13–15] culminating in Ref. [16]. Recently, in Ref. [17], we have used the Lippmann-Schwinger equations to provide a quick derivation of the one-loop quantum corrections to the elastic scattering amplitude. The result did not agree with Ref. [16]. At least some of the differences are due to the fact that some terms were explicitly dropped in Ref. [16] as they were considered to be loop corrections; however, we have shown that, in the case of the Sine-Gordon theory, these terms in fact cancel other terms of the form of those that were kept, and this cancellation is in fact a consequence of the integrability of the model.

Our derivation made several assumptions about analyticity and ignored final states that did not correspond to elastic scattering. While the Sine-Gordon theory did provide a valuable check of our results, more general models possess a cubic coupling at the minima, which yields interactions far from the kink that are not present in the Sine-Gordon model. This, together with the fact that our result disagrees with the standard result of Ref. [16], motivates an independent and robust recalculation of this scattering amplitude.

The present paper does just this. We provide a derivation of the amplitude in gory detail by considering an initial meson wave packet incident on a kink and evolving it in time, evaluating every contributing diagram up to second order in the coupling constant.

This is done using the linearized soliton perturbation theory of Refs. [18,19], reviewed in Sec. II. It is a Hamiltonian approach, which uses a decomposition of the fields in normal modes following Ref. [20]. In particular, no collective coordinate is introduced, removing many of the complications present in traditional approaches [21,22]. The transition from a Hamiltonian to a kink Hamiltonian, central to all approaches to quantum solitons since Ref. [23], takes the form of a passive unitary transformation on the regularized theory. This is in contrast with previous approaches, which regularize the vacuum and kink sectors separately and then need to introduce an arbitrary and often inconsistent matching condition for the regulators [24].

In Sec. III we calculate all contributions to the scattering amplitude not involving zero modes. The pieces of the final state containing zero modes are fixed by translation invariance [19]. However, there are contributions to the amplitude

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involving processes in which zero modes are created and then are absorbed by the free evolution of the kink center of mass. These are the hardest to calculate. In Secs. IV and V we methodically calculate the final states containing four and two zero modes. These are, as expected, determined by translation invariance. However, in Sec. VI we show that these calculations can be easily modified to generate the final states that have no zero modes, but arise from intermediate states involving zero modes. This provides the final contribution to the elastic scattering amplitude.

The contributions found here agree precisely with those of Ref. [17]. This suggests that, in the future, long calculations such as that of the present paper may be unnecessary. One may simply read the amplitudes off of the solution to the Lippmann-Schwinger equations, as was done in Ref. [17].

II. REVIEW

A. The theory

A number of efficient formalisms are available for treating quantum solitons. At one loop, as reviewed in Refs. [25,26], reliable and efficient spectral methods have long been available. Recently a classical-quantum correspondence has been introduced in Refs. [27,28] that cannot treat nonlinearities, but has been applied even well beyond the perturbative regime [29]. However, elastic scattering occurs at the next order, so these formalisms will not be suitable.

We will instead use linearized soliton perturbation theory. Linearized soliton perturbation theory was developed at one loop in Ref. [18] and beyond in Ref. [19]. So far, it has only been applied to $1+1$-dimensional models of a scalar field $\phi(x)$ and its conjugate $\pi(x)$,

$$H = \int dx {:}\mathcal{H}(x){:}_a,$$
$$\mathcal{H}(x) = \frac{\pi^2(x)}{2} + \frac{(\partial_x\phi(x))^2}{2} + \frac{V(\sqrt{\lambda}\phi(x))}{\lambda}, \quad (2.1)$$

because in these models all ultraviolet divergences are removed by the normal ordering $::_a$. However, the formalism is also compatible with a cutoff regularization and counterterms [30], and so we feel that it can be generalized to more interesting models.

The potential V is required to have degenerate minima so that there will be classical kink solutions $\phi(x,t) = f(x)$. We specialize to the case of reflectionless kinks, however, we have shown in Ref. [31] that calculations such as those that follow are effortlessly generalized to reflective kinks. In the present context, the leading quantum contribution to the reflection probability would arise from cross terms between the amplitude calculated here, adjusted as in Ref. [31], and the leading order amplitude [8,32].

We will expand perturbatively in the coupling constant λ. In Refs. [31,33] we have seen that meson multiplication and Stokes scattering occur at order $O(\sqrt{\lambda})$ in the amplitude. We will see that elastic scattering amplitudes begin at order $O(\lambda)$.

The normal ordering will be defined at mass m, which in turn is defined by

$$m^2 = V^{(2)}(\sqrt{\lambda}f(\pm\infty)),$$
$$V^{(n)}(\sqrt{\lambda}\phi(x)) = \frac{\partial^n V(\sqrt{\lambda}\phi(x))}{\partial(\sqrt{\lambda}\phi(x))^n}, \quad (2.2)$$

where the masses $V^{(2)}(\sqrt{\lambda}f(\infty))$ and $V^{(2)}(\sqrt{\lambda}f(-\infty))$ need to agree in order for a stationary kink state to exist [34].

B. States and sectors

The field $\phi(x)$ has perturbative excitations. As usual, these are created and destroyed by operators $A^\dagger$ and A that are in turn constructed by decomposing $\phi(x)$ and $\pi(x)$ into plane waves. This is to be expected, as plane waves are the solutions of the linearized classical equations of motion. We refer to such perturbative excitations as mesons. The Fock space consisting of the vacuum plus some finite number of mesons will be called the vacuum sector.

In the presence of a kink, the linearized equations of motion become the Sturm-Liouville equation

$$V^{(2)}(\sqrt{\lambda}f(x))\mathfrak{g}(x) = \omega^2\mathfrak{g}(x) + \mathfrak{g}''(x),$$
$$\phi(x,t) = f(x) + e^{-i\omega t}\mathfrak{g}(x). \quad (2.3)$$

The solutions to this equation are normal modes $\mathfrak{g}(x)$. Normal modes can be divided into three categories, depending on their frequency ω. First, there is a single zero mode

$$\mathfrak{g}_B(x) = -\frac{f'(x)}{\sqrt{Q_0}} \quad (2.4)$$

with frequency $\omega_B = 0$. Here Q_i is the order $O(\lambda^{i/2-1})$ quantum correction to the kink mass, so Q_0 is just the classical kink mass. Second, for every real number k there is a continuum mode with $\omega_k = \sqrt{m^2+k^2}$. Finally, there may be discrete, real shape modes $\mathfrak{g}_S(x)$ with $0 < \omega_S < m$. We chose the convention $\mathfrak{g}_k^* = \mathfrak{g}_{-k}$ and fix the normalizations via

$$\int dx|\mathfrak{g}_B(x)|^2 = 1, \quad \int dx\mathfrak{g}_{k_1}(x)\mathfrak{g}^*_{k_2}(x) = 2\pi\delta(k_1-k_2),$$
$$\int dx\mathfrak{g}_{S_1}(x)\mathfrak{g}^*_{S_2}(x) = \delta_{S_1S_2}. \quad (2.5)$$

Following Ref. [20], we may use the normal modes to decompose the Schrödinger picture fields

$$\phi(x) = \phi_0 \mathfrak{g}_B(x) + \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\left(B_k^{\ddagger} + \frac{B_{-k}}{2\omega_k}\right)\mathfrak{g}_k(x),$$
$$\pi(x) = \pi_0 \mathfrak{g}_B(x) + i\sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\left(\omega_k B_k^{\ddagger} - \frac{B_{-k}}{2}\right)\mathfrak{g}_k(x), \quad (2.6)$$

where we have defined the shorthand

$$B_k^{\ddagger} = \frac{B_k^{\dagger}}{2\omega_k}, \qquad B_{-S} = B_S, \qquad \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} = \int \frac{dk}{2\pi} + \sum_S. \quad (2.7)$$

The canonical commutation relations satisfied by $\phi(x)$ and $\pi(x)$ imply that ϕ_0, π_0, B, and $B^{\ddagger}$ satisfy the algebra

$$[\phi_0, \pi_0] = i, \qquad \left[B_{S_1}, B_{S_2}^{\ddagger}\right] = \delta_{S_1 S_2},$$
$$\left[B_{k_1}, B_{k_2}^{\ddagger}\right] = 2\pi\delta(k_1 - k_2). \quad (2.8)$$

The interpretation of these new operators is straightforward. In states with a kink, the operator $B_k^{\ddagger}$ creates a continuum normal mode, which we also call a meson. The operator $B_S^{\ddagger}$ excites an internal shape mode. The operators ϕ_0 and π_0 correspond to the position and momentum of the kink's center of mass.

We refer to the kink ground state plus any number of mesons and shape modes with any wave function composed of ϕ_0 as a kink sector state.

C. The kink sector

How do we construct a kink sector state? In classical field theory, vacuum sector states correspond to fields $\phi(x,t)$ that are close to a minimum of the potential, which we take be zero, while kink sector states correspond to $\phi(x,t)$ close to $f(x)$. Thus, one can turn a vacuum sector state into a kink sector state by shifting $\phi(x,t) \to \phi(x,t) + f(x)$.

In quantum field theory, one needs to be careful because such a shift may be incompatible with the regularization [35]. Instead, we will work directly in the regularized theory and will, as described below, make use of the unitary displacement operator

$$\mathcal{D}_f = \mathrm{Exp}\left[-i\int dx f(x)\pi(x)\right]. \quad (2.9)$$

In the absence of a momentum cutoff, this indeed shifts the field.

The key observation is that acting the operator $\mathcal{D}_f$ on a vacuum sector state yields a kink sector state, and all kink sector states can be constructed in this way. Indeed, this is just the old coherent state construction of soliton states [36,37]. For example, we may write the soliton ground state as $\mathcal{D}_f|0\rangle$, where $|0\rangle$ is some state in the vacuum sector, and a Hamiltonian eigenstate with one soliton and one meson as $\mathcal{D}_f|k_1\rangle$, where $|k_1\rangle$ is another vacuum sector state.

The appearance of a $\mathcal{D}_f$ factor in every state is annoying, and so we will remove it with a passive transformation. We stress that this passive transformation is a convenience, merely relabeling the coordinates on the Hilbert space. The passive transformation is defined as follows.

We define a "frame" to be an identification of Hilbert space (projective) vectors with states. The usual identification of Hilbert space vectors with states is called the "defining frame." Then we define the "kink frame" as follows. In the kink frame, the Hilbert space vector $|\psi\rangle$ is identified with the state that is identified with the Hilbert space vector $\mathcal{D}_f|\psi\rangle$ in the defining frame. In other words, $|\psi\rangle$ in the kink frame is just our old state $\mathcal{D}_f|\psi\rangle$ without bothering to write the $\mathcal{D}_f$. So in the kink frame, we write $|0\rangle$ for the kink ground state and $|k_1\rangle$ for a state with one kink and one meson.

Of course, as is always the case with passive transformations, one needs to simultaneously transform the operators that act on the states. For example, in the kink frame, time evolution and spatial translations are generated by the kink Hamiltonian and momentum

$$H' = \mathcal{D}_f^{\dagger} H \mathcal{D}_f, \qquad P' = \mathcal{D}_f^{\dagger} P \mathcal{D}_f. \quad (2.10)$$

These are easily evaluated. The kink momentum is

$$P' = \sqrt{Q_0}\pi_0 + P, \qquad P = -\int dx \pi(x)\partial_x\phi(x), \quad (2.11)$$

where the π_0 term is the momentum of the kink center of mass, while P represents the momentum in the mesons. The kink Hamiltonian is

$$H' = \sum_{n=0}^{\infty} H'_n, \qquad H'_0 = Q_0, \qquad H'_1 = 0,$$
$$H'_{n>2} = \lambda^{\frac{n}{2}-1}\int dx \frac{V^{(n)}(\sqrt{\lambda}f(x))}{n!} :\phi^n(x):_a, \quad (2.12)$$

where H'_n is of order $O(\lambda^{n/2-1})$. We will write H'_2 momentarily.

D. The perturbation theory

What have we gained by decomposing kink sector states into $\mathcal{D}_f|\psi\rangle$ and then dropping the $\mathcal{D}_f$? The main advantage of this formalism is that $|\psi\rangle$ may be found perturbatively using the eigenvalue equation for H'. This is the main advantage of linearized perturbation theory, the nonperturbative problem of finding the kink states becomes entirely perturbative. Similarly, Schrödinger picture time evolution may be performed perturbatively using $e^{-iH't}$.

The perturbation theory begins with the free part of the kink Hamiltonian,

$$H'_2 = Q_1 + H_{\rm free}, \quad H_{\rm free} = \frac{\pi_0^2}{2} + \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\omega_k B_k^\ddagger B_k. \tag{2.13}$$

Recall that Q_1 is a scalar, it is just the one-loop correction to the kink mass. The $\pi_0^2/2$ term is the kinetic energy of the kink center of mass, while the other terms are quantum harmonic oscillators for the shape and continuum modes. We will always work in the center of mass frame. The ground state $|0\rangle_0$ of the free Hamiltonian is the quantum field theory state, which is the ground state of all of these quantum mechanical models; in other words, it is the unique state that satisfies

$$\pi_0|0\rangle_0 = B_k|0\rangle_0 = B_S|0\rangle_0 = 0. \tag{2.14}$$

We can write any state in the kink sector by applying creation operators $B^\ddagger$ and zero modes ϕ_0 to this state. $B^\ddagger$ converts H'_2 eigenstates into other H'_2 eigenstates, which we will denote with a subscript 0,

$$B_{k_1}^\ddagger \cdots B_{k_n}^\ddagger |0\rangle_0 = |k_1\cdots k_n\rangle_0. \tag{2.15}$$

We are interested not in eigenstates $|k_1\cdots k_n\rangle_0$ of the free Hamiltonian H'_2, but rather in eigenstates $|k_1\cdots k_n\rangle$ of the full Hamiltonian H'. To find these, perturbatively, we decompose them in powers of the coupling

$$|k_1\cdots k_n\rangle = \sum_{i=0}^\infty |k_1\cdots k_n\rangle_i, \tag{2.16}$$

where $|k_1\cdots k_n\rangle_i$ is of order $O(\lambda^{i/2})$ when expanded in the basis that we will describe shortly. The perturbative expansion starts with the approximation $i=0$ given in Eq. (2.15).

As the Hamiltonian is translation invariant, we may specialize to states that are translation invariant. In other words, we are only interested in states annihilated by P'. Now the states are described by a wave function in the kink center of mass position ϕ_0, but translation invariance means that, if we find the part of a state near[1] $\phi_0 = 0$, then we can use translation invariance to reconstruct it elsewhere. Thus, we expand about $\phi_0 = 0$. In terms of operators, this means that we consider a polynomial expansion in ϕ_0, which is a good approximation for the part of the state near the zero eigenvalue of ϕ_0. In summary, a basis of states is given by

$$\phi_0^m B_{k_1}^\ddagger \cdots B_{k_n}^\ddagger |0\rangle_0. \tag{2.17}$$

We refer to the part of a state[2] with $m=0$ as the primary part and the $m>0$ part as the descendants. In Ref. [19] we showed that all of the descendants are determined by translation invariance $P'|\psi\rangle = 0$. Therefore, we only use perturbation theory to determine the primaries.

The last ingredient that we will need for our perturbative treatment is Wick's theorem [39], which relates the normal ordering $::_a$ to a normal ordering $::_b$, in which π_0 and B_k appear at the end,

$$:\phi^j(x):_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!}\mathcal{I}^m(x):\phi^{j-2m}(x):_b,$$
$$\mathcal{I}(x) = \int\frac{dk}{2\pi}\frac{|\mathfrak{g}_k(x)|^2-1}{2\omega_k} + \sum_S \frac{|\mathfrak{g}_S(x)|^2}{2\omega_S}. \tag{2.18}$$

The contraction factor $\mathcal{I}(x)$ will be represented pictorially below as a loop that begins and ends at the same vertex. This theorem lets us convert the formula (2.12) for the interactions in the kink Hamiltonian into the formulas that will appear in the text.

III. CONTRIBUTIONS WITH NO ZERO MODES

We are interested in the following process. Meson 1 strikes the kink from the left. An interaction occurs at order $O(\lambda)$ and meson 2 leaves the kink, again to the left. The initial and final states both contain a single unexcited kink and a single meson.

A. Generalities

1. Initial condition

More precisely, our system begins in the state

$$|t=0\rangle = \int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2 - i(k_1-k_0)x_0}|k_1\rangle, \tag{3.1}$$

where meson 1 is centered at a position $x_0 < 0$ relative to the kink in a wave packet of width σ and average momentum $k_0 > 0$. Recall that $|k_1\rangle$ is the translation-invariant H' eigenstate consisting of a single kink and a single meson with momentum k_1. It is invariant under simultaneous translations of the kink and the meson, preserving their separation. The state was constructed explicitly up to order $O(\lambda)$ in Ref. [38].

[1]This crude notation means that we decompose the state into eigenvalues of ϕ_0 and then consider components with eigenvalues close to zero. It is explained more precisely in Ref. [38].

[2]We use the letter m as both a non-negative integer index counting zero modes and as a real, positive number describing the meson mass.

We will be interested in the limits

$$\frac{x_0}{\sigma} \to -\infty, \qquad m\sigma \to \infty. \tag{3.2}$$

The first limit states that the initial meson wave packet does not overlap with the kink, while the second limit states that the wave packet is nearly monochromatic.

As $|k_1\rangle$ is a translation-invariant Hamiltonian eigenstate, $|t=0\rangle$ is also translation-invariant. However, it is not a Hamiltonian eigenstate, as each $|k_1\rangle$ has a different eigenvalue. The Hamiltonian and momentum commute $[H,P] = [H',P'] = 0$ and so, evolving in time, the state will remain translation invariant.

The details of the initial state will not be relevant to the elastic scattering amplitude. In other words, if we perform the following calculation with a different initial state, then the amplitude will be unchanged so long as the smearing of the initial kink-meson relative momentum involves Fourier modes corresponding to momenta much less than m and the initial meson is fully localized to the left of the kink. Of course, we cannot begin with the meson in a momentum eigenstate, as that would not be localized to the left of the kink. The choice of a Gaussian wave packet is convenient because it is simple and also normalizable, a property which we will use in Sec. VII when we divide by the norm squared in the definition of the elastic scattering probability.

2. Evolution operator

The Schrödinger picture evolution operator is

$$U(t) = e^{-iH't} = \sum_{n=0}^{\infty} U_n(t). \tag{3.3}$$

Here we have decomposed it into the order $O(\lambda^{n/2})$ contributions U_n. Up to order $O(\lambda)$, these are

$$U_0(t) = e^{-iH'_2 t}, \qquad U_1(t) = -i\int_0^t d\tau_1 e^{-iH'_2(t-\tau_1)} H'_3 e^{-iH'_2\tau_1},$$
$$U_2(t) = -i\int_0^t d\tau_1 e^{-iH'_2(t-\tau_1)} H'_4 e^{-iH'_2\tau_1} - \int_0^t d\tau_1 \int_0^t d\tau_2 e^{-iH'_2(t-\tau_2)} H'_3 e^{-iH'_2(\tau_2-\tau_1)} H'_3 e^{-iH'_2\tau_1}. \tag{3.4}$$

We will define x_t, which, before the collision, is the meson's position at time t, and also t_c, the collision time, by

$$x_t = x_0 + \frac{k_0}{\omega_{k_0}} t, \qquad t_c = -\frac{\omega_{k_0}}{k_0} x_0. \tag{3.5}$$

We will be interested in the limit $(t - t_c)/\sigma \to \infty$, so that by the end of the experiment meson 2 is far from the kink.

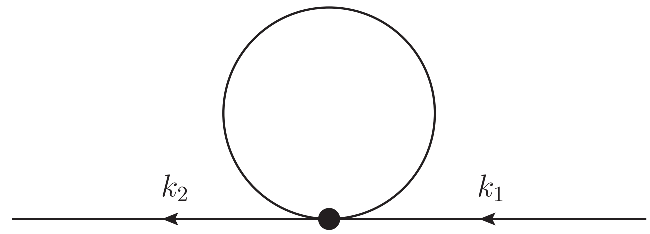

FIG. 1. Time runs to the left. This is a schematic drawing of the following process. Meson 1 travels. Then it emits and absorbs the same virtual particle and in the process becomes meson 2. This interaction is proportional to $\mathcal{I}(x)$, which falls exponentially in mx far from the kink, thus the interaction necessarily happens close to the kink. However, the kink is not drawn.

As $m\sigma \to \infty$, in the support of the Gaussian $e^{-\sigma^2(k_1-k_0)^2}$ we may approximate $k_1 \sim k_0$ and so linearly expand

$$\omega_{k_1} = \omega_{k_0} + \frac{k_0}{\omega_{k_0}}(k_1 - k_0). \tag{3.6}$$

B. One interaction

The simplest process that leads to elastic scattering is drawn in Fig. 1. Meson 1, with momentum k_1, interacts via the interaction

$$H_4^{(1)\prime} = \frac{\lambda}{2}\int \frac{d^2k}{(2\pi)^2} V_{\mathcal{I}-k_1k_2} B_{k_2}^{\ddagger} \frac{B_{k_1}}{2\omega_{k_1}} \tag{3.7}$$

at time τ_1. Here $H_4^{(1)\prime}$ is a term in H'_4. This interaction involves a virtual meson that it both creates and annihilates, and it leaves meson 2, with momentum k_2. Each loop at the same vertex gives a factor of the function $\mathcal{I}(x)$.

Here we have used the shorthand V to denote an n-point function defined as follows:

$$V_{k_1\cdots k_n} = \int dx V^{(n)}(\sqrt{\lambda}f(x)) \mathfrak{g}_{k_1}(x)\cdots \mathfrak{g}_{k_n}(x),$$
$$V_{\mathcal{I}k_1\cdots k_{n-2}} = \int dx V^{(n)}(\sqrt{\lambda}f(x)) \mathcal{I}(x) \mathfrak{g}_{k_1}(x)\cdots \mathfrak{g}_{k_{n-2}}(x), \tag{3.8}$$

where we remind the reader that the loop factor $\mathcal{I}(x)$ was defined in Eq. (2.18).

This interaction is proportional to λ already, and so a final state proportional to λ may only arise if one acts it on a state of order $O(\lambda^0)$. In other words, we must act it on the leading order term of the Hamiltonian eigenstate $|k_1\rangle$,

$$|k_1\rangle_0 = B_{k_1}^{\ddagger}|0\rangle_0. \tag{3.9}$$

This is not a Hamiltonian eigenstate, but it is an eigenstate of the free Hamiltonian H'_2.

Acting the interaction (3.7) on $|k_1\rangle_0$ one finds

$$H_4^{(1)\prime}|k_1\rangle_0 = \frac{\lambda}{4\omega_{k_1}}\int\frac{dk_2}{2\pi}V_{\mathcal{I}-k_1k_2}|k_2\rangle_0. \tag{3.10}$$

Our goal is to obtain $U(t)|t=0\rangle$. Now we are ready to calculate one term, the contribution from the interaction $H_4^{(1)\prime}$.

Let us write the corresponding part of the evolution operator as

$$U_2(t) = -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_4^{(1)\prime}e^{-iH_2'\tau_1}. \tag{3.11}$$

This is an abuse of our notation, as we have already defined $U_2(t)$ to be the complete evolution operator at order $O(\lambda)$ and (3.11) is just one term in $U_2(t)$; however, it would be cumbersome to give separate names to every term in the evolution operator.

Evolving the initial state, we find

$$\begin{aligned}U_2(t)|t=0\rangle &= -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_4^{(1)\prime}e^{-iH_2'\tau_1}\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_1\rangle_0\\ &= -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_4^{(1)\prime}\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0-i\omega_{k_1}\tau_1}|k_1\rangle_0\\ &= -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_4^{(1)\prime}e^{-i\omega_{k_0}\tau_1}\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_1\rangle_0.\end{aligned} \tag{3.12}$$

Using (3.10), one finds

$$U_2(t)|t=0\rangle = -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}\frac{V_{\mathcal{I}-k_1k_2}}{\omega_{k_1}}\int_0^t d\tau_1 e^{-i\omega_{k_2}(t-\tau_1)}e^{-i\omega_{k_0}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0.$$

Recall that, as $m\sigma\to\infty$, k_1 is very close to k_0 in the support of the Gaussian weight. This means that ω_{k_1} is very close to ω_{k_0}, and so we replace the ω_{k_1} in the denominator with ω_{k_0}. However, we cannot do the same with phase factors of the form k_1x_0, for example, because $|x_0|\gg\sigma$, and so this would create an error in the phase of order x_0/σ which is very large. In summary, we will make the approximations

$$\omega_{k_1} = \omega_{k_0}, \qquad \mathfrak{g}_{-k_1}(x) = \mathfrak{g}_{-k_0}(x)e^{i(k_1-k_0)x}, \tag{3.13}$$

but we will not drop the $(k_1-k_0)x$ terms. The second approximation comes from the fact that, for a reflectionless kink, $\mathfrak{g}_k(x)$ consists of e^{-ikx} times various terms that vary with respect to k with a characteristic scale of order $O(m)$, which is much greater than $1/\sigma$ and so these terms may be considered to be constant over the width of the Gaussian $e^{-\sigma^2(k_1-k_0)^2}$.

This leaves

$$\begin{aligned}U_2(t)|t=0\rangle &= -i\frac{\lambda}{4\omega_{k_0}}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1 e^{-i\omega_{k_2}(t-\tau_1)-i\omega_{k_0}\tau_1}\int dx V^{(4)}(\sqrt{\lambda}f(x))\mathcal{I}(x)\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\\ &\quad\times\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)(x_{\tau_1}-x)}|k_2\rangle_0\\ &= -i\frac{\lambda}{4\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}\int dx V^{(4)}(\sqrt{\lambda}f(x))\mathcal{I}(x)\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0.\end{aligned} \tag{3.14}$$

Now $\mathcal{I}(x)$ has its support at $x\sim O(1/m)$ and so x/σ tends to 0 in our limit. So can we drop the x/σ term in the Gaussian factor? A shift in x of order $O(1/m)$ would shift the dummy variable x_{τ_1} and so τ_1 by of order $O(1/m)$ for relativistic mesons. This would in turn shift the phase factor $e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}$ by a phase of order $(\omega_{k_0}-\omega_{k_2})/m$. However, as we will see momentarily and is anyway clear from momentum conservation, ω_{k_0} and ω_{k_2} are quite close, differing by of order $O(1/\sigma)$, and so the corresponding phase shift would be of order $O(1/(m\sigma))$, which vanishes in our limit.

In conclusion, we may safely drop the x from the Gaussian term, and so pull it out of the x integral, leaving

$$\begin{aligned} U_2(t)|t=0\rangle &= -i\frac{\lambda}{4\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}V_{\mathcal{I}k_0-k_2}e^{-i\omega_{k_2}t}\int_0^t d\tau_1 e^{-x_{\tau_1}^2/(4\sigma^2)-i(\omega_{k_0}-\omega_{k_2})\tau_1}|k_2\rangle_0 \\ &= -i\frac{\lambda}{4k_0}\int\frac{dk_2}{2\pi}V_{\mathcal{I}k_0-k_2}e^{-i\omega_{k_2}t}e^{-\sigma^2(\omega_{k_0}-\omega_{k_2})^2\omega_{k_0}^2/k_0^2-i(\omega_{k_0}-\omega_{k_2})t_c}|k_2\rangle_0. \end{aligned} \tag{3.15}$$

The expression $(\omega_{k_0}-\omega_{k_2})$ vanishes at $k_2=\pm k_0$ and so the Gaussian factor has two peaks. The $k_2=k_0$ peak corresponds to forward scattering. We are not interested in it, so we will drop it. About the other peak we may use (3.6) to rewrite the $(\omega_{k_0}-\omega_{k_2})$ terms as $(k_0+k_2)k_0/\omega_{k_0}$ and so

$$U_2(t)|t=0\rangle = -i\frac{\lambda}{4k_0}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}V_{\mathcal{I}k_0-k_2}e^{-\sigma^2(k_0+k_2)^2+i(k_0+k_2)x_0}|k_2\rangle_0. \tag{3.16}$$

C. A tadpole

All other contributions to elastic scattering involve two H_3' interactions. In this subsection, we will consider the interactions

$$\begin{aligned} H_3^{(1)\prime} &= \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\!\int\frac{dk_1}{2\pi}\sum\!\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{-k_1k_2k'}B_{k_2}^{\ddagger}\left(B_{k'}^{\ddagger}+\frac{B_{-k'}}{2\omega_{k'}}\right)\frac{B_{k_1}}{2\omega_{k_1}}, \\ H_3^{(2)\prime} &= \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{\mathcal{I}k'}\left(B_{k'}^{\ddagger}+\frac{B_{-k'}}{2\omega_{k'}}\right). \end{aligned} \tag{3.17}$$

In the interaction $H_3^{(1)\prime}$, at time τ_1 the meson k_1 changes to k_2 and a virtual meson of momentum k' is emitted or absorbed. At this point we allow both k' and also k_2 to be a continuum or a shape mode, since we do not yet know which will be the virtual meson. In the tadpole interaction $H_3^{(2)\prime}$, at time τ_2 the virtual meson is absorbed or emitted and another virtual meson travels in a loop to the same vertex. Finally, we will restrict our attention to final states in which meson 2 is a continuum excitation. This restriction is not really necessary, as it is not hard to show that if the final state consists, instead, of a kink and an excited shape mode, since this cannot be on shell, the amplitude vanishes.

As drawn in Fig. 2, the interactions may occur in either order. The evolution operator is

$$U_2^A(t) = -\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-iH_2'(t-\tau_2)}H_3^{(2)\prime}e^{-iH_2'(\tau_2-\tau_1)}H_3^{(1)\prime}e^{-iH_2'\tau_1} \tag{3.18}$$

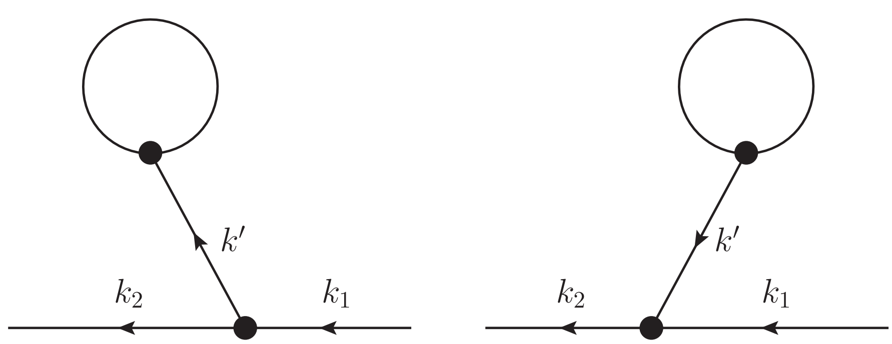

FIG. 2. Right: while meson 1 approaches, a virtual particle pair comes in and out of existence, leaving behind it a virtual particle. We will say that the virtual particle is created by a tadpole, although one might rightly note that it is emitted by the kink that is never drawn. This virtual particle merges with meson 1, leaving meson 2. Left: meson 1 emits a virtual particle, becoming meson 2. This emitted virtual particle decays via a virtual particle pair tadpole process.

if $\tau_1 < \tau_2$ and, otherwise, it is

$$U_2^B(t) = -\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 e^{-iH_2'(t-\tau_1)} H_3^{(1)\prime} e^{-iH_2'(\tau_1-\tau_2)} H_3^{(2)\prime} e^{-iH_2'\tau_2}. \tag{3.19}$$

1. The case $\tau_1 < \tau_2$

In this case, projecting out the three-meson sector and remembering the factor of 2 from the choice of contractions of $B_{k'}$, the interaction terms act as

$$H_3^{(1)\prime}|k_1\rangle_0 = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}}{2\omega_{k_1}}|k_2k'\rangle_0,$$
$$H_3^{(2)\prime}|k_2k'\rangle_0 = \frac{\sqrt{\lambda}}{2}\frac{V_{\mathcal{I}-k'}}{2\omega_{k'}}|k_2\rangle_0 + \frac{\sqrt{\lambda}}{2}\frac{V_{\mathcal{I}-k_2}}{2\omega_{k_2}}|k'\rangle_0. \tag{3.20}$$

As always, when considering the leading contribution to the initial state, one begins at time τ_1 with

$$e^{-iH_2'\tau_1}|t=0\rangle_0 = e^{-i\omega_{k_0}\tau_1}\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_1\rangle_0, \tag{3.21}$$

where x_{τ_1} is defined in the first expression in Eq. (3.5). At time t this evolves to

$$\begin{aligned}
U_2^A(t)|t=0\rangle_0 &= -\frac{\lambda}{8}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 \sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}\\
&\quad\times\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\frac{V_{-k_1k_2k'}V_{\mathcal{I}-k'}}{\omega_{k_1}\omega_{k'}}|k_2\rangle_0\\
&= -\frac{\lambda}{8\omega_{k_0}}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}\\
&\quad\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)(x_{\tau_1}-x)}|k_2\rangle_0\\
&= -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}\\
&\quad\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0.
\end{aligned} \tag{3.22}$$

2. Showing that the first interaction occurs near the kink

Unlike the previous process, the x integrand no longer obviously has compact support unless k' is a shape mode. To see that it in fact does have compact support, even if k' is not a shape mode, when integrated over k' and τ_2, let us first multiply the integrand by a normalized bump function $e^{-(x-\hat{x})^2/(4\hat{\sigma}^2)}/(2\sqrt{\pi}\hat{\sigma})$,

$$\begin{aligned}
U_2^A(\hat{x},t)|t=0\rangle_0 &= -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}\\
&\quad\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)-(x-\hat{x})^2/(4\hat{\sigma}^2)}|k_2\rangle_0,
\end{aligned} \tag{3.23}$$

where $|\hat{x}| \gg \hat{\sigma} \gg 1/m$ and $\hat{\sigma} \ll \sigma$. This will allow us to determine the contribution to the integral arising from $x \sim \hat{x}$. We will now show that it vanishes for all $\hat{x}$ satisfying $|\hat{x}| \gg \hat{\sigma} \gg 1/m$.

As the x integral now has support at $|x| \sim |\hat{x}| \gg 1/m$, we may replace

$$V^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x) \tag{3.24}$$

by its asymptotic value at $|x| \gg 1/m$. In the case of classically reflectionless kinks, this is

$$\mathfrak{g}_k(x) = \begin{cases} \mathcal{B}_k e^{-ikx} & \text{if } x \ll -1/m \\ \mathcal{D}_k e^{-ikx} & \text{if } x \gg 1/m, \end{cases}$$
$$|\mathcal{B}_k|^2 = |\mathcal{D}_k|^2 = 1, \qquad \mathcal{B}_k^* = \mathcal{B}_{-k}, \qquad \mathcal{D}_k^* = \mathcal{D}_{-k}, \tag{3.25}$$

where the phases $\mathcal{B}_k$ and $\mathcal{D}_k$ vary slowly with respect to k.

For concreteness, choose $\hat{x} < 0$, as the following argument proceeds identically with the other sign choice. Then we replace $V^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)$ with $V^{(3)L}_{-k_0k_2k'}e^{-i(-k_0+k_2+k')x}$, where

$$V^{(3)L}_{-k_0k_2k'} = V^{(3)}(\sqrt{\lambda}f(-\infty))\mathcal{B}_{-k_0}\mathcal{B}_{k_2}\mathcal{B}_{k'}. \tag{3.26}$$

The support of the state near $\hat{x}$ is then

$$U_2^A(\hat{x},t)|t=0\rangle_0 = -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\int\frac{dk'}{2\pi}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}$$
$$\times\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}V^{(3)L}_{-k_0k_2k'}\int dx e^{-i(-k_0+k_2+k')x}e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}e^{-(x-\hat{x})^2/(4\hat{\sigma}^2)}|k_2\rangle_0.$$

As $\hat{\sigma} \ll \sigma$, in the support of the bump function, we may replace $e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}$ with $e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}$ and pull it out of the x integral. Then

$$U_2^A(\hat{x},t)|t=0\rangle_0 = -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\int\frac{dk'}{2\pi}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}$$
$$\times\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}V^{(3)L}_{-k_0k_2k'}e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)-\hat{\sigma}^2(-k_0+k_2+k')^2}e^{-i(-k_0+k_2+k')\hat{x}}|k_2\rangle_0.$$

Now k' is close to $k_0 - k_2$ as a result of the Gaussian $e^{-\hat{\sigma}^2(-k_0+k_2+k')^2}$. Physically, this is because the virtual meson is created at $\hat{x}$, which is far from the kink where mesons cannot transfer momentum to the kink. This means that we may expand about $k' \sim k_0 - k_2$,

$$\omega_{k'} = \omega_{k_0-k_2} + \frac{k_0-k_2}{\omega_{k_0-k_2}}(-k_0+k_2+k'). \tag{3.27}$$

We then find

$$U_2^A(\hat{x},t)|t=0\rangle_0 = -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}V^{(3)L}_{-k_0,k_2,k_0-k_2}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-i\omega_{k_0}\tau_1-i\omega_{k_0-k_2}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}$$
$$\times e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k_2-k_0}}e^{-\hat{\sigma}^2(-k_0+k_2+k')^2}e^{-i(-k_0+k_2+k')\left(\hat{x}+\frac{k_0-k_2}{\omega_{k_0-k_2}}(\tau_2-\tau_1)\right)}|k_2\rangle_0$$
$$= -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}V^{(3)L}_{-k_0,k_2,k_0-k_2}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-i\omega_{k_0}\tau_1-i\omega_{k_0-k_2}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}$$
$$\times e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}\frac{1}{\omega_{k_2-k_0}}\int dy V^{(3)}(\sqrt{\lambda}f(y))\mathcal{I}(y)\mathfrak{g}_{k_2-k_0}(y)$$
$$\times\int\frac{dk'}{2\pi}e^{-\hat{\sigma}^2(-k_0+k_2+k')^2}e^{-i(-k_0+k_2+k')\left(\hat{x}-y+\frac{k_0-k_2}{\omega_{k_0-k_2}}(\tau_2-\tau_1)\right)}|k_2\rangle_0$$

$$= -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}V^{(3)L}_{-k_0,k_2,k_0-k_2}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-i\omega_{k_0}\tau_1-i\omega_{k_0-k_2}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}$$
$$\times e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}\frac{1}{\omega_{k_2-k_0}}\int dyV^{(3)}(\sqrt{\lambda}f(y))\mathcal{I}(y)\mathfrak{g}_{k_2-k_0}(y)e^{-\left(\hat{x}-y+\frac{k_0-k_2}{\omega_{k_0-k_2}}(\tau_2-\tau_1)\right)^2/(4\hat{\sigma}^2)}|k_2\rangle_0.$$

Now unlike x, which was the location of the first interaction, y, the location of the second interaction, must be close to the kink. This is mandated by the $\mathcal{I}(y)$ term which has support at $y\sim O(1/m)$. Therefore, $y/\hat{\sigma}$ can be set to zero, implying that the corresponding Gaussian factor is y independent and can be pulled out of the y integral,

$$U_2^A(\hat{x},t)|t=0\rangle_0 = -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}V^{(3)L}_{-k_0,k_2,k_0-k_2}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-i\omega_{k_0}\tau_1-i\omega_{k_0-k_2}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}$$
$$\times e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}e^{-\left(\hat{x}+\frac{k_0-k_2}{\omega_{k_0-k_2}}(\tau_2-\tau_1)\right)^2/(4\hat{\sigma}^2)}\frac{V_{\mathcal{I},k_2-k_0}}{\omega_{k_2-k_0}}|k_2\rangle_0.$$

Finally, consider the τ_2 Gaussian integration. Depending on the values of τ_1 and $\hat{x}$, the range of integration may or may not overlap with the support of the second Gaussian factor. If it does not overlap, this integral trivially vanishes. If it does overlap, then it overlaps for a range of $\hat{\sigma}\omega_{k_0-k_2}/(k_0-k_2)>\hat{\sigma}$. During this time, the phase $e^{-i\omega_{k_0-k_2}\tau_2}$ decreases by more than $\omega_{k_0-k_2}\hat{\sigma}>m\hat{\sigma}$ units. Thus, the integral yields a factor of less than $e^{-m^2\hat{\sigma}^2}$, which vanishes in our limit $m\hat{\sigma}\to\infty$. We thus conclude that, including a bump function near $x=\hat{x}$,

$$U_2^A(\hat{x},t)|t=0\rangle_0 = 0 \tag{3.28}$$

for $|\hat{x}|\gg 1/m$. In other words, there is no contribution to $U_2^A(t)|t=0\rangle_0$ from x near $\hat{x}$. As a result, the position x of the first interaction is necessarily inside the kink $x\sim O(1/m)$, where the mesons and kink may exchange momentum.

To make this statement more quantitative, assume for a moment that the limit $|\hat{x}|/\sigma$ is nonzero. As the limit $m\sigma$ tends to ∞, in this case $m\hat{x}$ also tends to ∞. One therefore can choose $\hat{\sigma}$ so that $|\hat{x}|\gg\hat{\sigma}\gg 1/m$. Now the results of this subsubsection imply that such a $\hat{x}$ does not contribute to the integral. Thus, contributions to the integral can only arise when the limit of $|\hat{x}|/\sigma$ tends to zero. In other words, the support of our original integral is at the limit $|x|/\sigma\to 0$, where we may drop the x/σ term in the Gaussian exponential.

3. Continuing with the computation

This long argument has been made to justify dropping the x/σ term in Eq. (3.22), as the x integral has support at $|x|\ll\sigma$,

$$\begin{aligned}
U_2^A(t)|t=0\rangle_0 &= -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_1)}e^{-x_{\tau_1}^2/(4\sigma^2)}V_{-k_0k_2k'}|k_2\rangle_0\\
&= -i\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}V_{\mathcal{I}-k'}}{\omega_{k'}^2}\int_0^t d\tau_1 e^{-x_{\tau_1}^2/(4\sigma^2)}e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}\left(e^{-i\omega_{k'}(t-\tau_1)}-1\right)|k_2\rangle_0\\
&= -i\frac{\lambda}{8k_0}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}V_{\mathcal{I}-k'}}{\omega_{k'}^2}e^{-i(\omega_{k_0}-\omega_{k_2})t_c}\\
&\quad\times\left(e^{-i\omega_{k'}(t-t_c)}e^{-\sigma^2\frac{\omega_{k_0}^2}{k_0^2}(\omega_{k_0}-\omega_{k_2}-\omega_{k'})^2}-e^{-\sigma^2\frac{\omega_{k_0}^2}{k_0^2}(\omega_{k_0}-\omega_{k_2})^2}\right)|k_2\rangle_0\\
&= A_1+A_2,
\end{aligned}\tag{3.29}$$

where A_1 and A_2 are the contributions arising from the first and second terms in the parentheses.

Consider A_1, which arose from the $\tau_2 = t$ late time limit of the τ_2 integration. This has support at $\omega_{k_0} \sim \omega_{k_2} + \omega_{k'}$, where the virtual meson is on shell. In fact, it is unrelated to elastic scattering; instead, it represents a quantum correction to meson multiplication.

Now consider the k' integral of A_1. In the support of the Gaussian, $\omega_{k'}$ may be expanded to linear order in k' as in Eq. (3.6). Recall that the linear coefficient is the group velocity. Then, the size of the support of the Gaussian factor is equal to $1/\sigma$ times the ratio of the k_0 to the k' velocities, which is of order unity. Over this range, the phase $e^{-i\omega_{k'}(t-t_c)}$ changes by of order $(t-t_c)/\sigma$. This leads to a suppression factor of less than $e^{-(t-t_c)^2/\sigma^2}$ after k' integration, and so this term vanishes. This argument, of course, does not apply if k' is a shape mode, in which case it is discrete. We will turn to that case in Appendix B.

What about A_2? This has two peaks, at $k_2 = \pm k_0$. The positive sign corresponds to forward scattering, which we are not interested in here. Therefore, we keep the negative sign,

$$A_2 = i\frac{\lambda}{8k_0}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}V_{\mathcal{I}-k'}}{\omega_{k'}^2}e^{-\sigma^2(k_0+k_2)^2+i(k_0+k_2)x_0}|k_2\rangle_0.$$

4. The case $\tau_1 > \tau_2$

If the tadpole creates the virtual meson that is then absorbed by the incoming meson, then the interaction terms act as follows:

$$H_3^{(2)\prime}|k_1\rangle_0 = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{\mathcal{I}k'}|k_1k'\rangle_0,$$
$$H_3^{(1)\prime}|k_1k'\rangle_0 = \frac{\sqrt{\lambda}}{4\omega_{k_1}\omega_{k'}}\sum\!\!\!\!\!\!\!\int\frac{dk_2}{2\pi}V_{-k_1k_2-k'}|k_2\rangle_0, \tag{3.30}$$

leading to the final state,

$$U_2^B(t)|t=0\rangle_0 = -\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}}e^{-i\omega_{k_0}\tau_1-i\omega_{k'}(\tau_1-\tau_2)-i\omega_{k_2}(t-\tau_1)}$$
$$\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0. \tag{3.31}$$

Integrating over τ_2, we obtain two terms corresponding to the two limits of integration,

$$U_2^B(t)|t=0\rangle_0 = B_1 + B_2,$$
$$B_1 = i\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}^2}e^{-i\omega_{k_0}\tau_1-i\omega_{k_2}(t-\tau_1)}$$
$$\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0,$$
$$B_2 = -i\frac{\lambda}{8\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{\mathcal{I}-k'}}{\omega_{k'}^2}e^{-i(\omega_{k_0}+\omega_{k'})\tau_1-i\omega_{k_2}(t-\tau_1)}$$
$$\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k'}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0. \tag{3.32}$$

The calculation of B_1 proceeds identically to that of A_2, leading to the same result. Summing them, yields a factor of 2,

$$A_2 + B_1 = i\frac{\lambda}{4k_0}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}e^{-\sigma^2(k_0+k_2)^2+i(k_0+k_2)x_0}$$
$$\times\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}V_{\mathcal{I}-k'}}{\omega_{k'}^2}|k_2\rangle_0. \tag{3.33}$$

We will show that A_1 and B_2 cancel other contributions, and so, when these contributions are added, this expression is in fact equal to $(U_2^A(t) + U_2^B(t))|t = 0\rangle_0$.

5. Initial state corrections

There are also two initial state corrections, corresponding intuitively to the case in which either of these interactions has occurred in the distant past. More precisely, these correspond to the evolution of the $|k_1\rangle_1$ subleading term in the $|k_1\rangle$ in Eq. (3.1).

These corrections were found in Ref. [40]. The corresponding amplitudes are again of order $O(\lambda)$, but now the initial state is suppressed by a factor of $\sqrt{\lambda}$, while the evolution operator is order $O(\sqrt{\lambda})$. In other words, the only corrections that lead to elastic scattering are those that can be transmuted into a single backward traveling meson using a single interaction H_3'.

We will not draw these, but given any diagram in this paper, one may arrive at the corresponding diagram for initial state corrections as follows. First choose a time τ. Then remove the part of the diagram at earlier times $\tau' < \tau$, corresponding to everything that appears to the right of the time τ.

In the first case, one considers a virtual meson in the meson cloud about the kink. After a time t_c, the incoming meson strikes the virtual meson and creates the final meson. The virtual meson contributes a phase factor of $e^{-i(\omega_{k'}+\omega_{k_0}-\omega_{k_2})t_c}$, which oscillates rapidly with respect to k' unless the $\omega_{k_2} = \omega_{k_0} + \omega_{k'}$, corresponding to the limit in which the virtual meson is on shell. Like the A_1 term in the parentheses in Eq. (3.29), the k' integration over a domain of order $O(1/\sigma)$ leads to interference in the $e^{-i\omega_{k'}t_c}$ phase which annihilates this correction.

The second initial state contribution arises from a quantum correction to the incoming meson, which consists of two mesons of momenta k_2 and $k_1 - k_2$, one of which interacts with a virtual meson created by the kink once they arrive at the kink, after a time $\tau_2 \sim t_c$. One needs to integrate over τ_2, and each value is weighted by a phase $e^{-i\omega_{k_0-k_2}\tau_2}$. As $\omega_{k_0-k_2} > m$, one finds of order mt_c oscillations, and so after integrating over k' this contribution is hopelessly suppressed.

What if the virtual meson is a shape mode? Then k' is discrete and cannot be integrated, so this argument fails. The shape mode contribution to the meson cloud falls exponentially with the distance from the kink, so one can ignore the second initial state contribution.

More generally, the only corrections to the initial state that do not vanish when the meson is far from the kink and can contribute to elastic scattering after an interaction H_3' consist of components of $|k_1\rangle_1$ with two mesons and no zero modes. In Ref. [40] this contribution to $|k_1\rangle_1$ was quantified,

$$|k_1\rangle_1^{02} = -\frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{V_{\mathcal{I}k'}}{\omega_{k'}}|k_1 k'\rangle_0, \tag{3.34}$$

where the 02 superscript means that we are interested in the no zero-mode, two-meson Fock space in the kink sector. We have dropped all terms that vanish when the meson and kink are well separated, but they are summarized in Eq. (B6).

The corresponding contribution to the initial state is

$$\begin{aligned}|t=0\rangle_1^{02} &= \int \frac{dk_1}{2\pi} e^{-\sigma^2(k_1-k_0)^2 - i(k_1-k_0)x_0}|k_1\rangle_1^{02}\\ &= -\frac{\sqrt{\lambda}}{2}\int \frac{dk_1}{2\pi} e^{-\sigma^2(k_1-k_0)^2 - i(k_1-k_0)x_0}\\ &\quad\times \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{V_{\mathcal{I}k'}}{\omega_{k'}}|k_1 k'\rangle_0.\end{aligned} \tag{3.35}$$

Evolving this to time t, at order $O(\sqrt{\lambda})$, using the interaction $H_3^{(1)\prime}$ from Eq. (3.17), one finds

$$\begin{aligned}&U_1(t)|t=0\rangle_1^{02} = -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_3^{(1)\prime}\\ &e^{-iH_2'\tau_1}|t=0\rangle_1^{02} = -B_2.\end{aligned} \tag{3.36}$$

We thus conclude that B_2, which arose from the early time limit of the τ_2 integration, is canceled by an initial state correction. One may then suspect that A_1, which arises from the late time limit of the τ_2 integration, is canceled by a final state correction. As we will show in Appendix B, this is partly true. The corrections to the final state are given in Eq. (B6) and two of the five are canceled by A_1.

D. A bubble

The contribution that motivates our project is drawn in Fig. 3. There are again two interactions. At time τ_1, the interaction

$$\begin{aligned}H_3^{(1)\prime} &= \frac{\sqrt{\lambda}}{4}\sum\!\!\!\!\!\!\int \frac{dk_1}{2\pi}\sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}V_{-k_1k_1'k_2'}\\ &\quad\times\left(B_{k_1'}^{\ddagger}B_{k_2'}^{\ddagger} + \frac{B_{-k_1'}B_{-k_2'}}{12\omega_{k_1'}\omega_{k_2'}}\right)\frac{B_{k_1}}{\omega_{k_1}}\end{aligned} \tag{3.37}$$

connects the incoming meson 1 with two virtual mesons $1'$ and $2'$. These might lie in the continuum, but they may also be shape modes, or perhaps one of each. In particular, if both are shape modes, this corresponds to an unstable resonance. Next, at time τ_2, the interaction

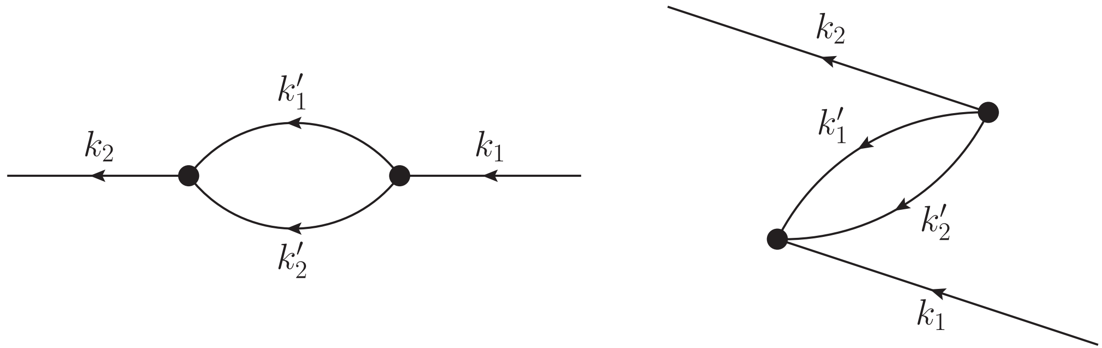

FIG. 3. Left: we see meson 1 splitting into two virtual mesons that recombine into meson 2. Right: two virtual mesons and meson 2 are created together, and later the two virtual mesons annihilate themselves together with meson 1.

$$H_3^{(2)\prime}=\frac{\sqrt{\lambda}}{6}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}V_{k_2k'_1k'_2}B^{\ddagger}_{k_2}\left(B^{\ddagger}_{k'_1}B^{\ddagger}_{k'_2}+\frac{3B_{-k'_1}B_{-k'_2}}{4\omega_{k'_1}\omega_{k'_2}}\right) \tag{3.38}$$

connects the two virtual mesons to the outgoing meson 2. We expect the amplitude to have a peak at the energy of the twice-excited shape mode.

1. The case $\tau_1<\tau_2$

In this case, projecting out five-meson final states and remembering a factor of 2 from the choice of which annihilation operator annihilates which virtual meson, the interactions act as

$$H_3^{(1)\prime}|k_1\rangle_0=\frac{\sqrt{\lambda}}{4}\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}\frac{V_{-k_1k'_1k'_2}}{\omega_{k_1}}|k'_1k'_2\rangle_0,\qquad H_3^{(2)\prime}|k'_1k'_2\rangle_0=\frac{\sqrt{\lambda}}{4}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\frac{V_{k_2-k'_1-k'_2}}{\omega_{k'_1}\omega_{k'_2}}|k_2\rangle_0. \tag{3.39}$$

The evolution operator (3.18) and the early state (3.21) then yield

$$\begin{aligned}U_2^A(t)|t=0\rangle_0&=-\frac{\lambda}{16}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}e^{-i\omega_{k_0}\tau_1-i(\omega_{k'_1}+\omega_{k'_2})(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_2)}\\&\quad\times\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\frac{V_{-k_1k'_1k'_2}V_{k_2-k'_1-k'_2}}{\omega_{k_1}\omega_{k'_1}\omega_{k'_2}}|k_2\rangle_0\\&=-\frac{\lambda}{16\omega_{k_0}}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k'_1-k'_2}}{\omega_{k'_1}\omega_{k'_2}}e^{-i\omega_{k_0}\tau_1-i(\omega_{k'_1}+\omega_{k'_2})(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_2)}\\&\quad\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k'_1}(x)\mathfrak{g}_{k'_2}(x)\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)(x_{\tau_1}-x)}|k_2\rangle_0\\&=-\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k'_1-k'_2}}{\omega_{k'_1}\omega_{k'_2}}e^{-i\omega_{k_0}\tau_1-i(\omega_{k'_1}+\omega_{k'_2})(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_2)}\\&\quad\times\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k'_1}(x)\mathfrak{g}_{k'_2}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0.\end{aligned} \tag{3.40}$$

2. Showing that the first interaction occurs near the kink

Again, we would like to drop the x/σ term when k'_1 and k'_2 are both continuum modes so that the x integrand does not have compact support. In this subsubsection we will try to argue that, even when k'_1 and k'_2 are continuum modes, after performing the other integrals, the x integral vanishes except when x/σ tends to zero. The argument will be similar to the tadpole case, but not quite the same.

As in the previous case, to see that it has compact support if integrated over k' and τ_2, we multiply the integrand by the normalized bump function $e^{-(x-\hat{x})^2/(4\hat{\sigma}^2)}/(2\sqrt{\pi}\hat{\sigma})$, where $|\hat{x}| \gg \hat{\sigma} \gg 1/m$ and $\hat{\sigma} \ll \sigma$. We also choose $\hat{x} < 0$, promising the reader that the manipulations are identical in the case $\hat{x} > 0$.

Again, this allows us to replace $V^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_1'}(x)\mathfrak{g}_{k_2'}(x)$ with $V^{(3)L}_{-k_0k_1'k_2'}e^{-i(-k_0+k_1'+k_2')x}$. The support of the state near $\hat{x}$ is then

$$\begin{aligned}U_2^A(\hat{x},t)|t=0\rangle_0 = &-\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\int\frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k_1'-k_2'}V^{(3)L}_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}}\\ &\times e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_2'})(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_2)}\int dx e^{-i(-k_0+k_1'+k_2')x}e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}e^{-(x-\hat{x})^2/(4\hat{\sigma}^2)}|k_2\rangle_0\\ = &-\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\int\frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k_1'-k_2'}V^{(3)L}_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}}\\ &\times e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_2'})(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_2)}e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)-\hat{\sigma}^2(-k_0+k_1'+k_2')^2}e^{-i(-k_0+k_1'+k_2')\hat{x}}|k_2\rangle_0. \end{aligned} \tag{3.41}$$

Now k_2' is close to k_0-k_1' as a result of the Gaussian $e^{-\hat{\sigma}^2(-k_0+k_1'+k_2')^2}$. Again, this is because the virtual mesons are created at $\hat{x}$, which is far from the kink where mesons cannot transfer momentum to the kink. Expanding k_2' about $k_2' \sim k_0 - k_1'$,

$$\omega_{k_2'} = \omega_{k_0-k_1'} + \frac{k_0-k_1'}{\omega_{k_0-k_1'}}(-k_0+k_1'+k_2'). \tag{3.42}$$

We then find

$$\begin{aligned}U_2^A(\hat{x},t)|t=0\rangle_0 = &-\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}\\ &\times\int\frac{dk_1'}{2\pi}e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_0-k_1'})(\tau_2-\tau_1)-i\omega_{k_2}(t-\tau_2)}\\ &\times\int\frac{dk_2'}{2\pi}\frac{V_{k_2-k_1'-k_2'}V^{(3)L}_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}}e^{-\hat{\sigma}^2(-k_0+k_1'+k_2')^2-i(-k_0+k_1'+k_2')\left(\hat{x}+\frac{k_0-k_1'}{\omega_{k_0-k_1'}}(\tau_2-\tau_1)\right)}|k_2\rangle_0\\ = &-\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}\\ &\times\int\frac{dk_1'}{2\pi}V^{(3)L}_{-k_0,k_1',k_0-k_1'}\frac{e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_0-k_1'})(\tau_2-\tau_1)+i\omega_{k_2}\tau_2}}{\omega_{k_1'}\omega_{k_0-k_1'}}\\ &\times\int dy V^{(3)}(\sqrt{\lambda}f(y))\mathfrak{g}_{k_2}(y)\mathfrak{g}_{-k_1'}(y)\mathfrak{g}_{k_1'-k_0}(y)\\ &\times\int\frac{dk_2'}{2\pi}e^{-\hat{\sigma}^2(-k_0+k_1'+k_2')^2-i(-k_0+k_1'+k_2')\left(\hat{x}-y+\frac{k_0-k_1'}{\omega_{k_0-k_1'}}(\tau_2-\tau_1)\right)}|k_2\rangle_0\end{aligned}$$

$$= -\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}$$
$$\times\int\frac{dk_1'}{2\pi}V^{(3)L}_{-k_0,k_1',k_0-k_1'}\frac{e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_0-k_1'})(\tau_2-\tau_1)+i\omega_{k_2}\tau_2}}{\omega_{k_1'}\omega_{k_0-k_1'}}$$
$$\times\int dy V^{(3)}(\sqrt{\lambda}f(y))\mathfrak{g}_{k_2}(y)\mathfrak{g}_{-k_1'}(y)\mathfrak{g}_{-k_0+k_1'}(y)e^{-\left(\hat{x}-y+\frac{k_0-k_1'}{\omega_{k_0-k_1'}}(\tau_2-\tau_1)\right)^2/(4\hat{\sigma}^2)}|k_2\rangle_0. \tag{3.43}$$

First, we studied the one vertex interaction, in which we found that x must be close to the kink because of the $\mathcal{I}(x)$ loop factor. Then we turned to a tadpole interaction, in which x was not obviously close, but y was close because of an $\mathcal{I}(y)$ term, which allowed us to show that x is close. However, in the case of the present interaction, even y is not obviously small.

To show that the y integral has support at small y, after integration over τ_2, we will insert another normalized bump function $e^{-(y-\hat{y})^2/(4\hat{\sigma}^2)}/(2\hat{\sigma}\sqrt{\pi})$ into the y integral, which satisfies the same limits as the x bump function, in particular, $m|\hat{y}|\gg 1$. Again, for concreteness, we will make the irrelevant choice $\hat{y}<0$. Then we may replace $V^{(3)}(\sqrt{\lambda}f(y))\mathfrak{g}_{k_2}(y)\mathfrak{g}_{-k_1'}(y)\mathfrak{g}_{-k_0+k_1'}(y)$ with $V^{(3)L}_{k_2,-k_1',-k_0+k_1'}e^{-i(k_2-k_0)y}$ and the localized final state is

$$U_2^A(\hat{x},\hat{y},t)|t=0\rangle_0 = -\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}$$
$$\times\int\frac{dk_1'}{2\pi}V^{(3)L}_{-k_0,k_1',k_0-k_1'}V^{(3)L}_{k_2,-k_1',-k_0+k_1'}\frac{e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_0-k_1'})(\tau_2-\tau_1)+i\omega_{k_2}\tau_2}}{\omega_{k_1'}\omega_{k_0-k_1'}}$$
$$\times\int dy e^{-i(k_2-k_0)y}e^{-\left(\hat{x}-y+\frac{k_0-k_1'}{\omega_{k_0-k_1'}}(\tau_2-\tau_1)\right)^2/(4\hat{\sigma}^2)}|k_2\rangle_0$$
$$= -\frac{\lambda}{16\omega_{k_0}}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}$$
$$\times\int\frac{dk_1'}{2\pi}V^{(3)L}_{-k_0,k_1',k_0-k_1'}V^{(3)L}_{k_2,-k_1',-k_0+k_1'}\frac{e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_0-k_1'})(\tau_2-\tau_1)+i\omega_{k_2}\tau_2}}{\omega_{k_1'}\omega_{k_0-k_1'}}$$
$$\times e^{-\hat{\sigma}^2(k_2-k_0)^2-i(k_2-k_0)\left(\hat{x}+\frac{k_0-k_1'}{\omega_{k_0-k_1'}}(\tau_2-\tau_1)\right)}|k_2\rangle_0. \tag{3.44}$$

The term $e^{-\hat{\sigma}^2(k_2-k_0)^2}$ ensures that the outgoing meson 2 has the same momentum as the incoming meson 1. Thus, this process describes forward scattering, which we are not interested in. The reason, of course, is that we chose both $|x|$ and $|y|$ to be greater than $O(1/m)$, so that both interactions occurred far from the kink. Thus, no momentum could be exchanged between the kink and the mesons.

We therefore conclude that only $y\sim O(1/m)$ can contribute to elastic scattering if $|x|\gg O(1/m)$. In particular, $|y/\hat{\sigma}|$ limits to zero and so may be dropped in Eq. (3.43), leading to

$$U_2^A(\hat{x},t)|t=0\rangle_0 = -\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\frac{\sqrt{\pi}}{2\pi\hat{\sigma}}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-(x_{\tau_1}-\hat{x})^2/(4\sigma^2)}$$
$$\times\int\frac{dk_1'}{2\pi}\frac{e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_0-k_1'})(\tau_2-\tau_1)+i\omega_{k_2}\tau_2}}{\omega_{k_1'}\omega_{k_0-k_1'}}e^{-\left(\hat{x}+\frac{k_0-k_1'}{\omega_{k_0-k_1'}}(\tau_2-\tau_1)\right)^2/(4\hat{\sigma}^2)}$$
$$\times V^{(3)L}_{-k_0,k_1',k_0-k_1'}V_{k_2,-k_1',-k_0+k_1'}|k_2\rangle_0. \tag{3.45}$$

Finally, we turn to the integrals of the interaction times. The τ_2 integral yields a Gaussian whose exponential is equal to $-\hat{\sigma}^2(\omega_{k_1'}+\omega_{k_0-k_1'}-\omega_{k_2})^2$ divided by a velocity squared, while the τ_1 integral yields a Gaussian whose exponential is

$-\sigma^2(k_0+k_2)^2$, where we have chosen the sign of k_2 to yield elastic scattering and not forward scattering. In the support of this later Gaussian, we may replace ω_{k_2} by ω_{k_0} in the former Gaussian, so that its exponential is $-\hat{\sigma}^2(\omega_{k'_1}+\omega_{k_0-k'_1}-\omega_{k_0})^2$. This is of order $-\hat{\sigma}^2m^2$ for all values of k_0 and k'_1, as two-body decay to two particles of the same mass as the original particle cannot simultaneously conserve momentum and energy. Therefore, the first exponential vanishes, and we find that $U_2^A(\hat{x},t)|t=0\rangle_0$ vanishes when the first interaction is localized near any $\hat{x}$ that is not of order $O(1/m)$, as was the case for the previous two interactions.

3. Continuing with the computation

Finally, we are justified in dropping the x/σ factor in Eq. (3.40), which leaves

$$\begin{aligned} U_2^A(t)|t=0\rangle_0 &= -\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 \sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k'_1-k'_2}V_{-k_0k'_1k'_2}}{\omega_{k'_1}\omega_{k'_2}} \\ &\quad\times e^{-x_{\tau_1}^2/(4\sigma^2)-i(\omega_{k_0}-\omega_{k'_1}-\omega_{k'_2})\tau_1-i(\omega_{k'_1}+\omega_{k'_2}-\omega_{k_2})\tau_2}|k_2\rangle_0 \\ &= i\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}\int_0^t d\tau_1 \sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k'_1-k'_2}V_{-k_0k'_1k'_2}}{\omega_{k'_1}\omega_{k'_2}(\omega_{k_2}-\omega_{k'_1}-\omega_{k'_2})} \\ &\quad\times e^{-x_{\tau_1}^2/(4\sigma^2)-i(\omega_{k_0}-\omega_{k'_1}-\omega_{k'_2})\tau_1}\left(e^{-i(\omega_{k'_1}+\omega_{k'_2}-\omega_{k_2})t}-e^{-i(\omega_{k'_1}+\omega_{k'_2}-\omega_{k_2})\tau_1}\right)|k_2\rangle_0 \\ &= i\frac{\lambda}{16k_0}e^{-i\omega_{k_0}t_c}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}(t-t_c)} \sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k'_1-k'_2}V_{-k_0k'_1k'_2}}{\omega_{k'_1}\omega_{k'_2}(\omega_{k_2}-\omega_{k'_1}-\omega_{k'_2})} \\ &\quad\times\left(e^{-\sigma^2\frac{\omega_{k_0}^2}{k_0^2}(\omega_{k_0}-\omega_{k'_1}-\omega_{k'_2})^2-i(\omega_{k'_1}+\omega_{k'_2}-\omega_{k_2})(t-t_c)}-e^{-\sigma^2\frac{\omega_{k_0}^2}{k_0^2}(\omega_{k_0}-\omega_{k_2})^2}\right)|k_2\rangle_0. \end{aligned} \tag{3.46}$$

Note that there is no pole at $\omega_{k_2}=\omega_{k'_1}+\omega_{k'_2}$, as the sum of the two terms in the parentheses has a simple zero there, leaving a term proportional to $t-t_c$. Of course, this does nonetheless diverge if one naively takes a $t\to\infty$ limit before integrating over the meson momenta.

As in the tadpole case, the first term in the parentheses corresponds not to elastic scattering, but rather to meson multiplication. One may again note that over the support of the Gaussian its phase varies many times, and so it should not contribute once the virtual meson momenta have been integrated. This argument applies here as it did there, away from $\omega_{k_2}=\omega_{k'_1}+\omega_{k'_2}$. What about at $\omega_{k_2}=\omega_{k'_1}+\omega_{k'_2}$, where the momenta cannot be freely varied as the surface is constrained?

Since the integrand is in fact everywhere finite, there is a vanishingly small contribution from any vanishingly small neighborhood of $\omega_{k_2}=\omega_{k'_1}+\omega_{k'_2}$. One may therefore remove such a neighborhood from the domain of integration; in other words, one may evaluate the integral close to $\omega_{k_2}=\omega_{k'_1}+\omega_{k'_2}$ using a principal value prescription without changing the value of the integral

$$\begin{aligned} U_2^A(t)|t=0\rangle_0 &= i\frac{\lambda}{16k_0}e^{-i\omega_{k_0}t_c}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}(t-t_c)} \sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k'_1-k'_2}V_{-k_0k'_1k'_2}}{\omega_{k'_1}\omega_{k'_2}} \\ &\quad\times e^{-\sigma^2\frac{\omega_{k_0}^2}{k_0^2}(\omega_{k_0}-\omega_{k_2})^2}\mathrm{PV}\left[\frac{e^{-\sigma^2\frac{\omega_{k_0}^2}{k_0^2}[(\omega_{k_0}-\omega_{k'_1}-\omega_{k'_2})^2-(\omega_{k_0}-\omega_{k_2})^2]-i(\omega_{k'_1}+\omega_{k'_2}-\omega_{k_2})(t-t_c)}-1}{\omega_{k_2}-\omega_{k'_1}-\omega_{k'_2}}\right]|k_2\rangle_0. \end{aligned} \tag{3.47}$$

The principal value is additive, so the two terms in the numerator may be separated, yielding the sum of two principal values.

In the support of the overall Gaussian, we may replace $V_{k_2-k'_1-k'_2}$ with $V_{k_0-k'_1-k'_2}$. We do not replace the k_2 in the phase, as it is multiplied by a group velocity factor times t, which is the scale at which the naive divergence is cut off.

Now consider the k_2' integral of the first term,

$$\frac{e^{-\sigma^2(\omega_{k_0}^2/k_0^2)(\omega_{k_0}-\omega_{k_1'}-\omega_{k_2'})^2}e^{-i(\omega_{k_1'}+\omega_{k_2'}-\omega_{k_2})(t-t_c)}}{\omega_{k_2}-\omega_{k_1'}-\omega_{k_2'}}. \tag{3.48}$$

In the limit $m(t-t_c)\to\infty$, the phase rotates so quickly that the k_2' integral is exponentially suppressed, being roughly of order $\exp(-(t-t_c)^2/\sigma^2)$. This vanishes as we take $(t-t_c)/\sigma\to\infty$ so that the final wave packet has no overlap with the kink. However, when the denominator is less than this exponentially suppressed factor, as occurs near the poles, this argument fails. The poles lie at

$$k_2' = \pm\sqrt{(\omega_{k_2}-\omega_{k_1'})^2-m^2} = \pm k_I, \tag{3.49}$$

where we have introduced the positive momentum notation k_I. Therefore, we must evaluate the contribution from a neighborhood of order $O(1/(t-t_c))$ of the poles.

Near each of these poles, the contribution to the principal value is nonzero as a result of the phase factor. Near each pole, the phase decreases as $\omega_{k_2'}$ increases and so as $|k_2'|$ increases. As a result, near the $k_2'=-k_I$ pole, the phase increases with k_2' and near the $k_2'=k_I$ pole it decreases. This implies that the principal value is $\mp i\pi$ times the residue at the $k_2'=\pm k_I$ pole. The residue is $-\omega_{k_2'}/k_2'$, times the various coefficients of the square brackets evaluated at the pole, at both poles. Summing the contributions at the two poles, one finds

$$i\pi\frac{\omega_{k_2'}}{|k_2'|}(\delta(k_2'-k_I)+\delta(k_2'+k_I)) = i\pi\delta(\omega_{k_2'}+\omega_{k_1'}-\omega_{k_2}). \tag{3.50}$$

We have argued that we may replace the first term in square brackets with $i\pi\delta(-\omega_{k_2}+\omega_{k_1'}+\omega_{k_2'})$. This may in turn be absorbed into the other principal value term using the Sokhotski–Plemelj theorem,

$$\begin{aligned} &i\pi\delta(-\omega_{k_2}+\omega_{k_1'}+\omega_{k_2'}) + \mathrm{PV}\left[\frac{1}{-\omega_{k_2}+\omega_{k_1'}+\omega_{k_2'}}\right] \\ &= \frac{1}{-\omega_{k_2}+\omega_{k_1'}+\omega_{k_2'}-i\epsilon}, \end{aligned} \tag{3.51}$$

where the limit $\epsilon\to 0^+$ is implicit.

In conclusion, we may replace the first term in the parentheses with an ϵ shift. Now, we are interested in elastic, not forward, scattering, so we will choose the sign of k_2 in the Gaussian peak considered, removing the forward scattering part, yielding

$$\begin{aligned} U_2^A(t)|t=0\rangle_0 &= -i\frac{\lambda}{16k_0}e^{-i\omega_{k_0}t_c}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}(t-t_c)}e^{-\sigma^2(k_0+k_2)^2} \\ &\quad\times \sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}\frac{V_{k_0-k_1'-k_2'}V_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}(\omega_{k_0}-\omega_{k_1'}-\omega_{k_2'}+i\epsilon)}|k_2\rangle_0. \end{aligned} \tag{3.52}$$

In the denominator, we have replaced ω_{k_2} with ω_{k_0}, using the fact that they are equal in the support of the Gaussian in our $m\sigma\to\infty$ limit. We recognize the $+i\epsilon$ in the final state as the usual one appearing in the in states in the Lippmann-Schwinger equation.

4. The case $\tau_1 > \tau_2$

This case is identical, except that the virtual mesons exchange their creation and annihilation operators. This leads to the final state

$$\begin{aligned} U_2^B(t)|t=0\rangle_0 = &-\frac{\lambda}{16\omega_{k_0}}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}\frac{V_{k_2-k_1'-k_2'}}{\omega_{k_1'}\omega_{k_2'}}e^{-i\omega_{k_0}\tau_1-i(\omega_{k_1'}+\omega_{k_2'})(\tau_1-\tau_2)} \\ &\times e^{-i\omega_{k_2}(t-\tau_2)}\int dxV^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_{-k_0}(x)\mathfrak{g}_{k_1'}(x)\mathfrak{g}_{k_2'}(x)e^{-(x_{\tau_1}-x)^2/(4\sigma^2)}|k_2\rangle_0. \end{aligned} \tag{3.53}$$

Therefore, an identical derivation to the one above follows. The τ_2 integral leads to a $(\omega_{k_2}+\omega_{k_1'}+\omega_{k_2'})$ in the denominator so there is not even superficially a pole, and no $i\epsilon$ is required. The τ_1 integral again gives two terms, and this time it is the second term that corresponds to an on shell k' and vanishes upon integration. As these two terms differ by a sign, and as it is the first and not the second term that remains, one obtains an overall sign flip with respect to the $\tau_1<\tau_2$ case, yielding

$$U_2^B(t)|t=0\rangle_0 = i\frac{\lambda}{16k_0}e^{-i\omega_{k_0}t_c}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}(t-t_c)}e^{-\sigma^2(k_0+k_2)^2} \times \sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}\frac{V_{k_0-k_1'-k_2'}V_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}(\omega_{k_0}+\omega_{k_1'}+\omega_{k_2'})}|k_2\rangle_0. \quad (3.54)$$

Adding these two contributions, we find

$$(U_2^A(t)+U_2^B(t))|t=0\rangle_0 = -i\frac{\lambda}{8k_0}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}e^{-\sigma^2(k_0+k_2)^2+i(k_0+k_2)x_0} \times \sum\!\!\!\!\!\!\int \frac{d^2k'}{(2\pi)^2}\frac{(\omega_{k_1'}+\omega_{k_2'})V_{k_0-k_1'-k_2'}V_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}\left(\omega_{k_0}^2-\left(\omega_{k_1'}+\omega_{k_2'}\right)^2+i\epsilon\right)}|k_2\rangle_0. \quad (3.55)$$

IV. ϕ_0^4 TERMS

Recall that translation invariance dictates all terms with zero modes [19]. These terms have two contributions. First, there is the cloud of mesons around the incoming or outcoming meson. Next, there is the cloud of mesons around the kink. In both cases, the quantum corrections contain more mesons than the leading order kets or, more precisely, more $B^\ddagger$ operators, except when the incoming or outgoing meson is close to the kink, in which case the incoming or outgoing meson may be absorbed by the kink [41]. In particular, in the asymptotic past and future, when the incoming and outgoing meson are far from the kink, these quantum corrections to components with zero modes ϕ_0 will all have at least two mesons.

This argument implies that there should not be any terms with zero modes and only one meson or, more precisely, terms of the form $\phi_0^m B^\ddagger|0\rangle_0$ with $m>0$, at times t late enough that the meson has traveled far from the kink. In the current section, we will verify that this is indeed the case for terms with ϕ_0^4 in the final state $U(t)|t=0\rangle$ at order $O(\lambda)$, which is the leading order at which ϕ_0^4 may arise.

A. The main contribution

Let us begin with the case in which e^{-iHt} is evaluated at order $O(\lambda)$ and $|t=0\rangle$ at order $O(1)$.

We will consider the interactions

$$H_3^{(1)\prime} = \frac{\sqrt{\lambda}}{2}\int\frac{dk_1}{2\pi}V_{BB-k_1}\frac{B_{k_1}}{2\omega_{k_1}}\phi_0^2,$$
$$H_3^{(2)\prime} = \frac{\sqrt{\lambda}}{2}\int\frac{dk_2}{2\pi}V_{BBk_2}B_{k_2}^\ddagger\phi_0^2. \quad (4.1)$$

In this case, meson 1 is annihilated by $H_3^{(1)\prime}$ at time τ_1, while meson 2 is created by $H_3^{(2)\prime}$ at time τ_2. This is drawn in Fig. 4.

There are two cases to consider, corresponding to the sign of $\tau_1-\tau_2$.

1. $\tau_1<\tau_2$

First, consider the case $\tau_1<\tau_2$, in which meson 1 is absorbed by the kink before meson 2 is emitted. Now the interactions act as

$$H_3^{(1)\prime}|k_1\rangle_0 = \sqrt{\lambda}\frac{V_{BB-k_1}}{4\omega_{k_1}}\phi_0^2|0\rangle_0,$$
$$H_3^{(2)\prime}\phi_0^2|0\rangle_0 = \frac{\sqrt{\lambda}}{2}\int\frac{dk_2}{2\pi}V_{BBk_2}\phi_0^4|k_2\rangle_0. \quad (4.2)$$

The corresponding contribution to the final state is

$$U_2^A(t)|t=0\rangle_0 = -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}} \times e^{-i\omega_{k_2}(t-\tau_2)-i\omega_{k_1}\tau_1} \times e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^4|k_2\rangle_0. \quad (4.3)$$

2. $\tau_1>\tau_2$

Next we turn to the case in which meson 2 is emitted before meson 1 is absorbed. Now the interactions act as

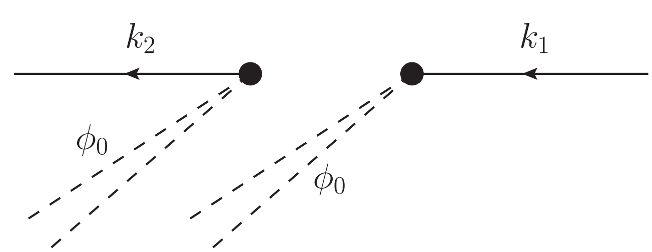

FIG. 4. Meson 1 is absorbed by the kink, leaving two zero modes. The kink also emits meson 2, together with two more zero modes.

$$H_3^{(2)\prime}|k_1\rangle_0 = \frac{\sqrt{\lambda}}{2}\int\frac{dk_2}{2\pi}V_{BBk_2}\phi_0^2|k_1k_2\rangle_0,$$
$$H_3^{(1)\prime}\phi_0^2|k_1k_2\rangle_0 = \sqrt{\lambda}\frac{V_{BB-k_2}}{4\omega_{k_2}}\phi_0^4|k_1\rangle_0 + \sqrt{\lambda}\frac{V_{BB-k_1}}{4\omega_{k_1}}\phi_0^4|k_2\rangle_0, \tag{4.4}$$

leading to the contribution

$$U_2^B(t)|t=0\rangle_0 = -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}$$
$$\times e^{-i\omega_{k_2}(t-\tau_2)-i\omega_{k_1}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^4|k_2\rangle_0, \tag{4.5}$$

where we have removed the forward scattering part, proportional to $|k_1\rangle_0$.

The integrand is equal to the previous case, and so these contributions are easily added,

$$\begin{aligned}(U_2^A(t)+U_2^B(t))|t=0\rangle_0 &= -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}e^{-i\omega_{k_2}t}\\ &\quad\times\int_0^t d\tau_1\int_0^t d\tau_2 e^{i\omega_{k_2}\tau_2-i\omega_{k_1}\tau_1}\phi_0^4|k_2\rangle_0\\ &= \frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}^2\omega_{k_2}}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}(1-e^{-i\omega_{k_2}t})(1-e^{-i\omega_{k_1}t})\phi_0^4|k_2\rangle_0\\ &= \frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}^2\omega_{k_2}}e^{-\sigma^2(k_1-k_0)^2}(1-e^{-i\omega_{k_2}t})\\ &\quad\times(e^{-i(k_1-k_0)x_0}-e^{-i\omega_{k_0}t}e^{-i(k_1-k_0)x_t})\phi_0^4|k_2\rangle_0.\end{aligned} \tag{4.6}$$

The Gaussian factor implies that k_1 has its support in a domain of width of order $O(1/\sigma)$. The phase changes rapidly in this domain, x_0/σ times and x_t/σ times in the first and the second terms of the last parentheses. This leads to an exponential suppression, after integrating over k_1, of order $O(e^{-x_0^2/(4\sigma^2)})$ and $O(e^{-x_t^2/(4\sigma^2)})$, respectively. These both converge rapidly to 0 in our limit in which σ/t and σ/x_0 tend to zero. We thus conclude that there is no ϕ_0^4 contribution.

B. Initial state contributions

Contributions may also arise from subleading terms in the initial state $|t=0\rangle$. Were $|t=0\rangle$ an eigenstate of the full Hamiltonian H', there would be three contributions, arising from terms of form $\phi_0^2|k_1k_2\rangle_0$, $\phi_0^2|0\rangle_0$, and $\phi_0^4|k_2\rangle_0$, with $k_2\neq k_1$, in the initial state. However, $|t=0\rangle$ is not a Hamiltonian eigenstate, it is an asymptotic state. As shown in Ref. [41], where the asymptotic states are evaluated explicitly, the second and third terms are therefore not present. This fact can be derived directly by considering the Hamiltonian eigenstate and integrating over the wave packet (3.1). Terms in which the k_1 meson has been annihilated contain an integral over k_1 that vanishes similarly.

This leaves terms of the first form. There is only one such quantum correction [41],

$$|k_1\rangle_1|_{\phi_0^2} = -\frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\frac{V_{BBk_2}}{\omega_{k_2}}\phi_0^2|k_1k_2\rangle_0. \tag{4.7}$$

This yields a quantum correction to the initial wave packet $|t=0\rangle$,

$$\begin{aligned}|t=0\rangle_1 &= \int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_1\rangle_1\\ &= -\frac{\sqrt{\lambda}}{2}\int\frac{dk_1}{2\pi}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\\ &\quad\times\frac{V_{BBk_2}}{\omega_{k_2}}\phi_0^2|k_1k_2\rangle_0.\end{aligned} \tag{4.8}$$

We evolve this with

$$U_1(t) = -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_3^{(1)\prime}e^{-iH_2'\tau_1} \tag{4.9}$$

to produce the contribution

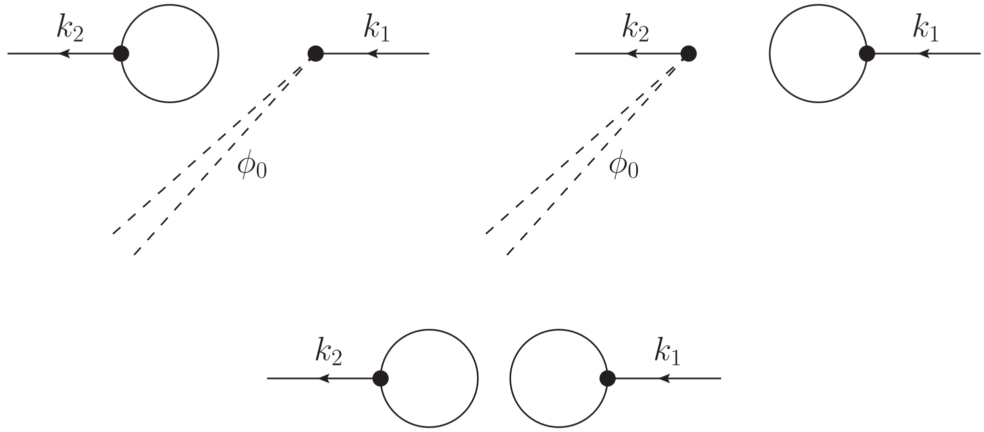

FIG. 5. Meson 1 is destroyed by a tadpole (right and bottom) or converted into two zero modes (left) and meson 2 is created by a tadpole (left and bottom) or together with two zero modes (right).

$$
\begin{aligned}
U_1(t)|t=0\rangle_1 &= \frac{i\lambda}{8}\int_0^t d\tau_1 \int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}\omega_{k_2}} e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0-i\omega_{k_1}\tau_1-i\omega_{k_2}t}\phi_0^4|k_2\rangle_0 \\
&= \frac{\lambda}{8}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}^2\omega_{k_2}}(1-e^{-i\omega_{k_1}t})e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0-i\omega_{k_2}t}\phi_0^4|k_2\rangle_0
\end{aligned} \tag{4.10}
$$

to the final state, where we removed the forward scattering part in the first line. We also removed the contribution from final states in which there is an excited shape mode and no continuum mesons, as these terms do not correspond to elastic scattering and anyway vanish as they can never conserve energy on shell.

The contributions arising from the continuum k_2 integral cancel the second term in the first parentheses in the last expression in Eq. (4.6). We have already argued that these terms each vanish at large t, but for completeness if we add the present contribution to (4.6) we obtain

$$
\begin{aligned}
U(t)|t=0\rangle = &\frac{\lambda}{8}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}^2\omega_{k_2}} \\
&\times e^{-\sigma^2(k_1-k_0)^2}(e^{-i(k_1-k_0)x_0} \\
&- e^{-i\omega_{k_0}t}e^{-i(k_1-k_0)x_t})\phi_0^4|k_2\rangle_0.
\end{aligned} \tag{4.11}
$$

As argued above, this vanishes upon performing the k_1 integration. It would not vanish were x_t close to zero, reflecting the fact that, during the meson-kink collision, there are indeed nonvanishing ϕ_0^4 terms with a single meson. We will see below that these terms are important, as they lead to ϕ_0^2 terms that are necessary to maintain translation invariance.

Equation (4.10) also includes contributions in which k_2 is a shape mode. In this case, the final state is not a kink and a meson, but instead an excited kink. It therefore does not correspond to elastic scattering. In the case of this process, the final energy is necessarily less than that of the initial state and so this can never be on shell, so one can show that after k_1 integration the amplitude vanishes exponentially in $t-t_c$.

C. A generalization

We have just shown that the interaction terms (4.1) in H_3', those that are proportional to ϕ_0^2, do not lead to any contribution proportional to ϕ_0^4 at any time t except within of order $O(\sigma)$ of t_c. In particular, such contributions vanish at large times, when the experiment ends. The argument relied on the fact that this term is proportional to $\mathfrak{g}_B^2(x)$, which is localized at $|x|\sim 1/m \ll \sigma$, which let us drop x/σ terms.

The interaction $H_3'\Big|_{\mathcal{I}}$ possesses a similar term,

$$
H_3'\Big|_{\mathcal{I}} = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}V_{\mathcal{I}k'}\left(B_{k'}^{\ddagger}+\frac{B_{-k'}}{2\omega_{k'}}\right). \tag{4.12}
$$

The same arguments may then be applied to calculate the final state of the process shown in the bottom panel of Fig. 5 to show that there is no contribution to the state $U(t)|t=0\rangle$ proportional to $\mathcal{I}^2(x)$.

What about the initial state contribution? Again from Ref. [41] the leading correction to the $|k_1\rangle$ asymptotic state is

$$
|k_1\rangle_1\Big|_{\mathcal{I}} = -\frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk_2}{2\pi}\frac{V_{\mathcal{I}k_2}}{\omega_{k_2}}|k_1k_2\rangle_0, \tag{4.13}
$$

which is identical to (4.7) except the ϕ_0^2 is missing and the $\mathfrak{g}_B^2(x)$ has been replaced by $\mathcal{I}(x)$, which again is supported at $|x| \sim 1/m$. Thus, even this contribution can be calculated identically.

In fact, one can do better. One can repeat the argument with the sum of these two contributions,

$$H_3'\Big|_{\mathcal{I},\phi_0^2} = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}(V_{\mathcal{I}k'} + V_{BBk'}\phi_0^2)\left(B_{k'}^{\ddagger} + \frac{B_{-k'}}{2\omega_{k'}}\right). \tag{4.14}$$

The argument again proceeds identically, but now one can see that even terms with one $\mathcal{I}$ and one ϕ_0^2, seen in the top of Fig. 5, vanish at all times t.

V. ϕ_0^2 TERMS

In this section, we systematically study the components of the state at a time t that have two zero modes or, more precisely, a factor of $\phi_0^2|k_2\rangle_0$. Contributions to such states can be decomposed into four categories, to each of which we dedicate a subsection. First, we consider contributions with a single four-point interaction. The other three categories each contain two three-point interactions. Of these, in the first, both zero modes arise from the same interaction. In the second, one zero mode arises from each interaction. In the last, each interaction generates two zero modes, as in Sec. IV, but two of these zero modes are eliminated by the kinetic term for the kink center of mass.

A. A single interaction

The simplest contribution to final states of the form $\phi_0^2|k_2\rangle_0$ arises from a single interaction,

$$H_4^{(1)\prime} = \frac{\lambda}{2}\int\frac{d^2k}{(2\pi)^2}V_{BB-k_1k_2}B_{k_2}^{\ddagger}\frac{B_{k_1}}{2\omega_{k_1}}\phi_0^2. \tag{5.1}$$

Acting on an initial meson $|k_1\rangle_0$, it yields

$$H_4^{(1)\prime}|k_1\rangle_0 = \frac{\lambda}{4\omega_{k_1}}\int\frac{dk_2}{2\pi}V_{BB-k_1k_2}\phi_0^2|k_2\rangle_0. \tag{5.2}$$

This leads to the final state,

$$U_2(t)|t=0\rangle_0 = -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1k_2}}{\omega_{k_1}}e^{-i\omega_{k_2}t} \times\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1} \times e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\phi_0^2|k_2\rangle_0. \tag{5.3}$$

The corresponding process is drawn in Fig. 6.

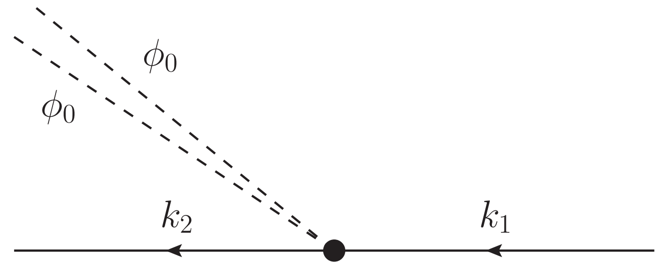

FIG. 6. Meson 1 converts into meson 2, emitting two zero modes in the process.

B. A virtual meson that decays to two zero modes

Next let us consider the contribution with two H_3' interactions drawn in Fig. 7. In the first, at time τ_1 meson 1 changes to meson 2 and a virtual meson of momentum k' is emitted or absorbed. In the second, at time τ_2 the virtual meson is absorbed or emitted and two zero modes are created.

The two relevant interactions are

$$H_3^{(1)\prime} = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{-k_1k_2k'}B_{k_2}^{\ddagger}\left(B_{k'}^{\ddagger} + \frac{B_{-k'}}{2\omega_{k'}}\right)\frac{B_{k_1}}{2\omega_{k_1}},$$
$$H_3^{(2)\prime} = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{BBk'}\left(B_{k'}^{\ddagger} + \frac{B_{-k'}}{2\omega_{k'}}\right)\phi_0^2. \tag{5.4}$$

1. The case $\tau_1 < \tau_2$

In this case, the virtual meson is emitted by meson 1,

$$H_3^{(1)\prime}|k_1\rangle_0 = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}}{2\omega_{k_1}}|k_2k'\rangle_0 \tag{5.5}$$

and it is then absorbed by the kink,

$$H_3^{(2)\prime}|k_2k'\rangle_0 = \frac{\sqrt{\lambda}}{2}\frac{V_{BB-k'}}{2\omega_{k'}}\phi_0^2|k_2\rangle_0 + \frac{\sqrt{\lambda}}{2}\frac{V_{BB-k_2}}{2\omega_{k_2}}\phi_0^2|k'\rangle_0. \tag{5.6}$$

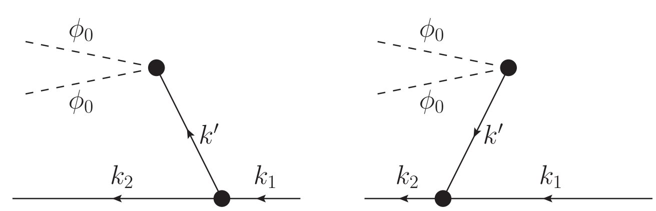

FIG. 7. Right: a virtual meson is created together with two zero modes. The virtual meson strikes meson 1 and turns it into meson 2. Left: meson 1 nucleates a virtual meson, which decays into two zero modes.

The resulting final state is

$$U_2^A(t)|t=0\rangle_0 = -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}}$$
$$\times e^{-i\omega_{k_2}(t-\tau_1)-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_0}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\phi_0^2|k_2\rangle_0$$
$$= i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}^2}e^{-i\omega_{k_2}t}$$
$$\times\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\phi_0^2|k_2\rangle_0, \tag{5.7}$$

where, in the τ_2 integration, we have dropped the boundary term at $\tau_2=t$ as it corresponds to the limit in which the virtual meson goes on shell. Like the two-process cases above, this term vanishes after k' is integrated, as its phase oscillates rapidly.

2. The case $\tau_1 > \tau_2$

In this case, the virtual meson is first emitted by the kink

$$H_3^{(2)\prime}|k_1\rangle_0 = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{BBk'}\phi_0^2|k_1k'\rangle_0 \tag{5.8}$$

and then it is absorbed by meson 1,

$$H_3^{(1)\prime}|k_1k'\rangle_0 = \frac{\sqrt{\lambda}}{4}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\frac{V_{-k_1k_2-k'}}{\omega_{k_1}\omega_{k'}}|k_2\rangle_0, \tag{5.9}$$

leading to the final state

$$U_2^B(t)|t=0\rangle_0 = -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}}$$
$$\times e^{-i\omega_{k_2}(t-\tau_1)-i\omega_{k'}(\tau_1-\tau_2)-i\omega_{k_0}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\phi_0^2|k_2\rangle_0$$
$$= i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}^2}e^{-i\omega_{k_2}t}$$
$$\times\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\phi_0^2|k_2\rangle_0. \tag{5.10}$$

This time, when performing the τ_2 integral, we have dropped the contribution from $\tau_2=0$. This term is in fact exactly canceled by an initial state contribution, but anyway corresponds to the on-shell limit of our virtual meson in which the k' integration yields zero.

This contribution to the final state is equal to that of Eq. (5.7) with the other ordering. Adding them then yields a factor of 2. Using the Ward identity (A8), this can be summarized,

$$(U_2^A(t)+U_2^B(t))|t=0\rangle_0 = i\frac{\lambda}{4\sqrt{\lambda Q_0}}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}\Delta_{-k'B}}{\omega_{k_1}}e^{-i\omega_{k_2}t}$$
$$\times\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}\phi_0^2|k_2\rangle_0. \tag{5.11}$$

Here we have used the shorthand

$$\Delta_{ij} = \int dx\,\mathfrak{g}_i(x)\mathfrak{g}_j'(x), \tag{5.12}$$

where i and j run over the normal mode indices B, S, and k. Intuitively, the matrix Δ represents the momentum operator acting on the mesons.

C. One zero mode at each vertex

Next we turn to the case in which there is a single zero mode created at each interaction of the form $(\sqrt{\lambda}/2)\int dx\mathfrak{g}_B(x)\phi_0 :\phi^2(x):_b$. At times τ_1 and τ_2, we place the interactions

$$H_3^{(1)\prime} = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk_1}{2\pi}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi} V_{B-k_1k'}\left(2B_{k'}^\ddagger + \frac{B_{-k'}}{2\omega_{k'}}\right)\frac{B_{k_1}}{2\omega_{k_1}}\phi_0,$$
$$H_3^{(2)\prime} = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\int \frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi} V_{Bk'k_2}B_{k_2}^\ddagger\left(B_{k'}^\ddagger + \frac{B_{-k'}}{\omega_{k'}}\right)\phi_0, \tag{5.13}$$

respectively, bearing in mind that we are interested in the components of the final state with a single meson. This is drawn in Fig. 8.

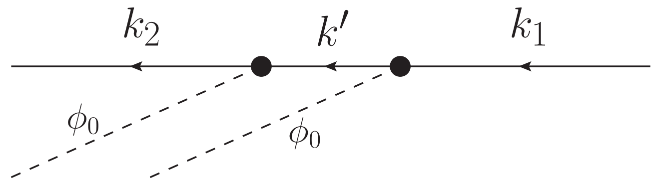

FIG. 8. Meson 1 turns into a virtual meson, emitting a zero mode. The virtual meson emits yet another zero mode, converting into meson 2.

1. The case $\tau_1 < \tau_2$

At each interaction, the meson interacts with the kink, exciting a single zero mode,

$$H_3^{(1)\prime}|k_1\rangle_0 = \sqrt{\lambda}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{V_{B-k_1k'}}{2\omega_{k_1}}\phi_0|k'\rangle_0,$$
$$H_3^{(2)\prime}\phi_0|k'\rangle_0 = \sqrt{\lambda}\sum\!\!\!\!\!\!\int \frac{dk_2}{2\pi}\frac{V_{B-k'k_2}}{2\omega_{k'}}\phi_0^2|k_2\rangle_0. \tag{5.14}$$

The corresponding contribution to the final state is

$$U_2^A(t)|t=0\rangle_0 = -\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}}$$
$$\times e^{-i\omega_{k_2}(t-\tau_2)-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_1}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^2|k_2\rangle_0. \tag{5.15}$$

If we first integrate τ_1 from 0 to τ_2, dropping the vanishing contribution from $\tau_1 = 0$, we obtain

$$U_2^A(t)|t=0\rangle_0 = -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}(\omega_{k_1}-\omega_{k'})}$$
$$\times e^{-i(\omega_{k_1}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^2|k_2\rangle_0$$
$$= -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}(\omega_{k_1}-\omega_{k'})}$$
$$\times e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \tag{5.16}$$

If instead we first integrate τ_2 from τ_1 to t, and drop the vanishing contribution at $\tau_2 = t$, then we obtain

$$U_2^A(t)|t=0\rangle_0 = -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\int_0^t d\tau_1\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}(\omega_{k_2}-\omega_{k'})}$$
$$\times e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^2|k_2\rangle_0$$
$$= -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}(\omega_{k_2}-\omega_{k'})}$$
$$\times e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \tag{5.17}$$

Of course, this must equal (5.16), as the finite τ_i integrals commute. In particular, both must equal their average, which will be more convenient below,

$$U_2^A(t)|t=0\rangle_0 = i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}}\times\left[\frac{1}{\omega_{k'}-\omega_{k_2}}+\frac{1}{\omega_{k'}-\omega_{k_1}}\right]e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \tag{5.18}$$

2. The case $\tau_1 > \tau_2$

Now the first interaction creates two new mesons,

$$H_3^{(2)\prime}|k_1\rangle_0 = \frac{\sqrt{\lambda}}{2}\sum\!\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}V_{Bk'k_2}\phi_0|k_1k_2k'\rangle_0, \tag{5.19}$$

while the second destroys one of these together with meson 1,

$$H_3^{(1)\prime}\phi_0|k_1k_2k'\rangle_0 = \frac{\sqrt{\lambda}}{4}\frac{V_{B-k_1-k'}}{\omega_{k_1}\omega_{k'}}\phi_0^2|k_2\rangle_0 + \frac{\sqrt{\lambda}}{4}\frac{V_{B-k_1-k_2}}{\omega_{k_1}\omega_{k_2}}\phi_0^2|k'\rangle_0 + \frac{\sqrt{\lambda}}{4}\frac{V_{B-k_2-k'}}{\omega_{k_2}\omega_{k'}}\phi_0^2|k_1\rangle_0, \tag{5.20}$$

where the last term will correspond to forward scattering and we will remove it when calculating the final state. As k' and k_2 are both dummy variables, in the case of the $|k'\rangle$ term, we can and will exchange their names, so that the final state is proportional to $|k_2\rangle$ and the first two terms on the right-hand side are equal.

Evolving to time t, we find the state

$$U_2^B(t)|t=0\rangle_0 = -\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}}\times e^{-i\omega_{k_2}(t-\tau_2)-i\omega_{k'}(\tau_1-\tau_2)-i\omega_{k_1}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^2|k_2\rangle_0. \tag{5.21}$$

Integration over τ_1 from τ_2 to t, dropping $\tau_1 = t$, yields

$$U_2^B(t)|t=0\rangle_0 = i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}(\omega_{k_1}+\omega_{k'})}\times e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0,$$

whereas integration over τ_2, dropping $\tau_2 = 0$, would instead yield

$$U_2^B(t)|t=0\rangle_0 = i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}(\omega_{k_2}+\omega_{k'})}\times e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0.$$

Averaging, one finds

$$U_2^B(t)|t=0\rangle_0 = i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_2\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}}\times\left[\frac{1}{\omega_{k_2}+\omega_{k'}}+\frac{1}{\omega_{k_1}+\omega_{k'}}\right]e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \tag{5.22}$$

3. Conclusions

Finally, we add the contribution (5.18) from the case $\tau_1 < \tau_2$ to obtain

$$(U_2^A(t) + U_2^B(t))|t=0\rangle_0 = i\frac{\lambda}{4}\int \frac{d^2k}{(2\pi)^2} e^{-i\omega_{k_2}t} \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\int_0^t d\tau_2 \frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}} \times \left[\frac{1}{\omega_{k'}^2-\omega_{k_2}^2}+\frac{1}{\omega_{k'}^2-\omega_{k_1}^2}\right] e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2} e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \tag{5.23}$$

Using the Ward identity (A7) this can be simplified somewhat,

$$(U_2^A(t) + U_2^B(t))|t=0\rangle_0 = i\frac{\lambda}{4\sqrt{\lambda Q_0}}\int \frac{d^2k}{(2\pi)^2} e^{-i\omega_{k_2}t} \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{(V_{B-k_1k'}\Delta_{-k'k_2}+\Delta_{k'-k_1}V_{B-k'k_2})}{\omega_{k_1}} \times \int_0^t d\tau_2 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2} e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \tag{5.24}$$

Now we can see the reason that we chose the complicated prescription of averaging over the two orders of time integration. Although of course these integrals commute, we see that the average prescription used here leads to the combination $V\Delta + \Delta V$ in round brackets in (5.24) which is the same as that in the Ward identity (A16), even without setting $k_0 = -k_2$.

Could we have simply set $k_0 = -k_2$ and just chose one ordering for the time integrals? Well, the uncertainty principle says that $k_0 + k_2$ will be of order $O(1/t)$, which indeed tends to zero at large t, although it is dimensionful and so one needs to be more careful. The problem, as we will see below, is that the $e^{-i\pi_0^2 t/2}$ term in the evolution operator contains, at first order, $-i\pi_0^2 t/2$, which leads to a zero-mode-free term proportional to t. In all, this contribution would be proportional to $t(k_0 + k_2)$, which is indeed dimensionless and does not tend to zero at large t. Therefore, in terms with zero modes we need to be careful about factors of $k_0 + k_2$ or, equivalently, $\omega_{k_2} - \omega_{k_0}$ or, even worse, $\omega_{k_2} - \omega_{k_1}$.

We note that there are neither initial nor final state corrections, as they would consist of a single meson and a Δ_{kB} term that vanishes when folded into the initial or final wave packet, which is far from the kink or, more precisely, the support of $\mathfrak{g}_B(x)$.

D. Two zero modes from four zero modes

The final contribution to the two zero-mode sector of the final state arises from interactions in which four zero modes are created, two by each of two H_3' terms in (4.1), and then two of these four zero modes are destroyed by the $\pi_0^2/2$ in the free Hamiltonian H_2'. This process is depicted in Fig. 9.

The free propagator H_2' consists of a $\pi_0^2/2$ term, as well as harmonic oscillator terms for the normal modes. These all commute, and so the respective parts of the free propagator may be factorized. Concretely, consider a basis element of the kink sector $\phi_0^m|k_1\cdots k_n\rangle_0$. Then the free propagator acts as

$$e^{-iH_2'T}\phi_0^m|k_1\cdots k_n\rangle_0 = e^{-i\omega T}e^{-i\pi_0^2 T/2}\phi_0^m|k_1\cdots k_n\rangle_0, \qquad \omega = \sum_{i=1}^n \omega_{k_n}. \tag{5.25}$$

The contribution of interest in this subsection uses a single π_0^2 to reduce the number of zero modes from 4 to 2 and so corresponds to the term

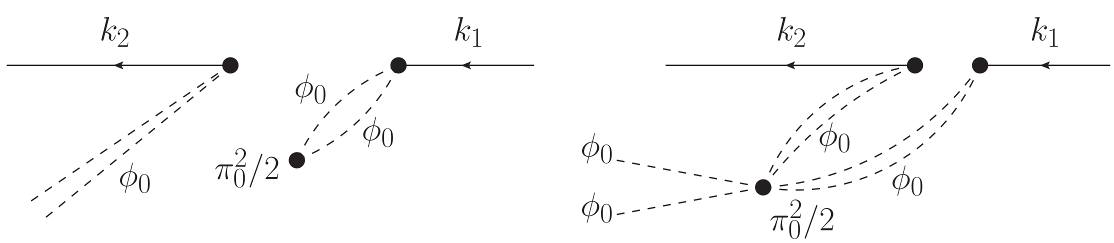

FIG. 9. This process is as in Fig. 4. However, two of the four zero modes are annihilated by the $\pi_0^2/2$ term in the free Hamiltonian H_2'.

$$e^{-i\omega T}\left(-i\frac{\pi_0^2}{2}T\right)\phi_0^m|k_1\cdots k_n\rangle_0 = i\frac{m(m-1)T}{2}e^{-i\omega T}\phi_0^{m-2}|k_1\cdots k_n\rangle_0. \tag{5.26}$$

Now observe that $e^{-i\omega T}\phi_0^m|k_1\cdots k_n\rangle_0$ is the result of the free evolution in which no zero modes are annihilated. And so, once one has calculated the m zero-mode sector at an arbitrary time τ as an integral over the various interaction times, one need only include a factor of $im(m-1)T/2$ in the integrand to obtain the contribution to the $m-2$ zero-mode sector. This needs to be done during the free evolution between each pair of interactions, as two zero modes may, in principle, be annihilated between any pair of interactions. Here T is the time that passes between the pair of interactions.

In kink-meson elastic scattering at order $O(\lambda)$, the only pair of interactions that creates four zero modes is written as an integral of interaction times in Eqs. (4.3) and (4.5). Consider first the case $\tau_1 < \tau_2$. Then, including the factors of $im(m-1)T/2$, where $m=2$ between the interactions and $m=4$ after both, one obtains the final state contribution,

$$U_2^A(t)|t=0\rangle_0 = -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}e^{-i\omega_{k_2}t}I_A e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^2|k_2\rangle_0, \tag{5.27}$$

where

$$I_A = i\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{i\omega_{k_2}\tau_2-i\omega_{k_1}\tau_1}((\tau_2-\tau_1)+6(t-\tau_2)). \tag{5.28}$$

Despite the linear growth in t, the arguments above show that the $\tau_2=t$ contribution vanishes exponentially and so we may drop it,

$$\begin{aligned}
I_A &= i\int_0^t d\tau_1 e^{-i\omega_{k_1}\tau_1}\left(6t-\tau_1+5i\frac{\partial}{\partial\omega_{k_2}}\right)\int_{\tau_1}^t d\tau_2 e^{i\omega_{k_2}\tau_2}\\
&= -\int_0^t d\tau_1 e^{-i\omega_{k_1}\tau_1}\left(6t-\tau_1+5i\frac{\partial}{\partial\omega_{k_2}}\right)\frac{e^{i\omega_{k_2}\tau_1}}{\omega_{k_2}}\\
&= \int_0^t d\tau_1\frac{e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}}{\omega_{k_2}}\left(-6t+6\tau_1+\frac{5i}{\omega_{k_2}}\right).
\end{aligned} \tag{5.29}$$

Integrating τ_1 first and dropping $\tau_1=0$ would instead yield

$$\begin{aligned}
I_A &= i\int_0^t d\tau_2 e^{i\omega_{k_2}\tau_2}\left(6t-5\tau_2-i\frac{\partial}{\partial\omega_{k_1}}\right)\int_0^{\tau_2}d\tau_1 e^{-i\omega_{k_1}\tau_1}\\
&= \int_0^t d\tau_1\frac{e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}}{\omega_{k_1}}\left(-6t+6\tau_1-\frac{i}{\omega_{k_1}}\right).
\end{aligned} \tag{5.30}$$

In the case $\tau_1 > \tau_2$, one finds

$$U_2^B(t)|t=0\rangle_0 = -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}e^{-i\omega_{k_2}t}I_B e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}\phi_0^2|k_2\rangle_0, \tag{5.31}$$

where

$$I_B = i\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2 e^{i\omega_{k_2}\tau_2-i\omega_{k_1}\tau_1}((\tau_1-\tau_2)+6(t-\tau_1)). \tag{5.32}$$

Now we drop the vanishing $\tau_2=0$ contribution to arrive at

$$I_B = i\int_0^t d\tau_1 e^{-i\omega_{k_1}\tau_1}\left(6t - 5\tau_1 + i\frac{\partial}{\partial\omega_{k_2}}\right)\int_0^{\tau_1} d\tau_2 e^{i\omega_{k_2}\tau_2}$$
$$= \int_0^t d\tau_1 \frac{e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}}{\omega_{k_2}}\left(6t - 6\tau_1 - \frac{i}{\omega_{k_2}}\right), \quad (5.33)$$

while integrating τ_1 first and then renaming τ_2 would give

$$I_B = \int_0^t d\tau_1 \frac{e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}}{\omega_{k_1}}\left(6t - 6\tau_1 + \frac{5i}{\omega_{k_1}}\right). \quad (5.34)$$

We see that the naively divergent $(t-\tau_1)$ terms cancel in $I_A + I_B$. This linear divergence would be caused by the fact that the constant ϕ_0^4 term, created at time τ_1 or τ_2, would create ϕ_0^2 at a constant rate as a result of the $\pi_0^2/2$ in H_2'. The cancellation occurs because, as we have shown, the ϕ_0^4 term itself vanishes at late times.

Summing the two cases, and again replacing I_A and I_B by the average of the expressions obtained from the two integration orders, one finds the contribution to the final state to be

$$(U_2^A(t) + U_2^B(t))|t=0\rangle_0 = -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}\left(\frac{1}{\omega_{k_1}^2}+\frac{1}{\omega_{k_2}^2}\right)$$
$$\times\int_0^t d\tau_2 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \quad (5.35)$$

Again, it will be convenient to rewrite this using a Ward identity,

$$(U_2^A(t) + U_2^B(t))|t=0\rangle_0 = -i\frac{\lambda}{4\sqrt{\lambda Q_0}}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\frac{(V_{BB-k_1}\Delta_{k_2B}+V_{BBk_2}\Delta_{-k_1B})}{\omega_{k_1}}$$
$$\times\int_0^t d\tau_2 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0. \quad (5.36)$$

E. The total

Finally, we are ready to add the two zero-mode, one-meson contributions to the elastic scattering of the final state given in Eqs. (5.3), (5.11), (5.24), and (5.36),

$$U(t)|t=0\rangle = i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\frac{S_2}{\omega_{k_1}}\int_0^t d\tau_2 e^{-i(\omega_{k_0}-\omega_{k_2})\tau_2}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_2}}\phi_0^2|k_2\rangle_0,$$

where

$$S_2 = -V_{BB-k_1k_2} + \frac{1}{\sqrt{\lambda Q_0}}\Big[-V_{BB-k_1}\Delta_{k_2B} - V_{BBk_2}\Delta_{-k_1B}$$
$$+ \sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}(V_{-k_1k_2k'}\Delta_{-k'B} + V_{B-k_1k'}\Delta_{-k'k_2} + \Delta_{k'-k_1}V_{B-k'k_2})\Big] = 0. \quad (5.37)$$

The last equality is a result of the Ward identity (A16) for translation invariance. This implies that no ϕ_0^2 terms appear at first order in the one-meson sector, as is demanded by translation invariance.

VI. FROM ZERO MODES TO NO ZERO MODES

Recall that any translation-invariant state in the kink sector is entirely determined by its primary components, those with no zero modes. Furthermore, the reduced inner product of Ref. [38] allows one to compute amplitudes using only the no zero-mode sector of the final state. Therefore, the computation of any initial value problem reduces to the computation of the no zero-mode sector of the final state.

So then why have we wasted so much space calculating the sector of the final state with zero modes? Because, following the strategy of Sec. V D, we can easily modify those computations to yield the zero-mode free parts of the final state resulting from interactions that create zero modes, which are later destroyed by the free H_2' evolution.

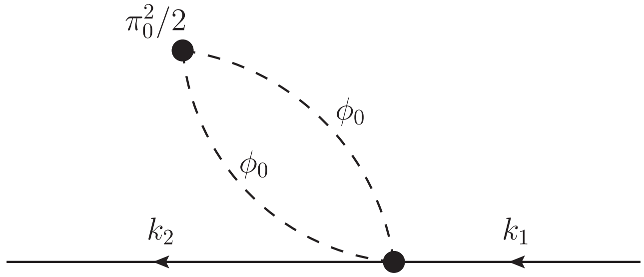

FIG. 10. This interaction is as in Fig. 6. However, the two zero modes are absorbed by the $\pi_0^2/2$ kinetic term for the kink center of mass.

A. A single interaction

As always, the simplest case is that with a single interaction; in this case, that of Eq. (5.1). This creates $m = 2$ zero modes, and so we must insert a factor of

$$im(m-1)T/2 = i(t-\tau_1), \tag{6.1}$$

where $T = t - \tau_1$ is the time after the creation of the zero modes. This changes the ϕ_0^2 part of the final state, given in Eq. (5.3), into the ϕ_0^0 part

$$U_2(t)|t=0\rangle_0 = \frac{\lambda}{4}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1k_2}}{\omega_{k_1}}e^{-i\omega_{k_2}t}\int_0^t d\tau_1(t-\tau_1)e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0. \tag{6.2}$$

This process in drawn in Fig. 10.

B. A virtual meson that decays to two zero modes

Next we turn to the interactions (5.4) in which a virtual meson is emitted by meson 1 at time τ_1 and it is absorbed by the kink, creating two zero modes, at time τ_2. The process in which these two zero modes are removed by the free evolution, drawn in Fig. 11, contains a factor of

$$im(m-1)T/2 = i(t-\tau_2) \tag{6.3}$$

with respect to the ϕ_0^2 contributions calculated in Sec. V B.

Including this factor in Eq. (5.7), one finds that the contribution from the case $\tau_1 < \tau_2$ is

$$\begin{aligned}
U_2^A(t)|t=0\rangle_0 &= -i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}}(t-\tau_2)\\
&\quad\times e^{-i\omega_{k_2}(t-\tau_1)-i\omega_{k'}(\tau_2-\tau_1)-i\omega_{k_0}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0\\
&= -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}^2}e^{-i\omega_{k_2}t}\\
&\quad\times\int_0^t d\tau_1\left(t-\tau_1+\frac{i}{\omega_{k'}}\right)e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0,
\end{aligned} \tag{6.4}$$

where again we have dropped the contribution at $\tau_2 = t$,

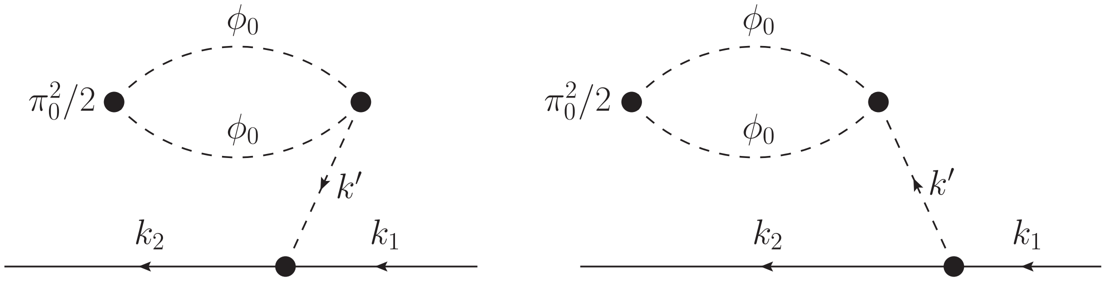

FIG. 11. This interaction is as in Fig. 7. However, the two zero modes are annihilated by the $\pi_0^2/2$ term in the free evolution.

$$
\begin{aligned}
& i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}^3}e^{-i(\omega_{k_2}+\omega_{k'})t} \\
&\quad\times\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2}-\omega_{k'})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0 \\
&= i\frac{\lambda}{8}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}V_{BB-k'}}{\omega_{k_0}\omega_{k'}^3}e^{-i(\omega_{k_2}+\omega_{k'})t}\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2}-\omega_{k'})\tau_1}e^{-x_{\tau_1}^2/(4\sigma^2)}|k_2\rangle_0 \\
&= i\frac{\lambda}{8\sqrt{\lambda Q_0}}e^{-i\omega_{k_0}t_c}\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}\Delta_{-k'B}}{k_0\omega_{k'}}e^{-\sigma^2(\omega_{k_0}/k_0)^2(\omega_{k_0}-\omega_{k_2}-\omega_{k'})^2-i(\omega_{k_2}+\omega_{k'})(t-t_c)}|k_2\rangle_0. \qquad (6.5)
\end{aligned}
$$

The k' integration causes this term to vanish, as the integrand oscillates quickly. This argument fails if k' is a discrete shape mode, and so we will handle this case separately in Appendix B.

Similarly, in the case $\tau_1 > \tau_2$, we include the factor in Eq. (5.10),

$$
\begin{aligned}
U_2^B(t)|t=0\rangle_0 &= -i\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\int_0^t d\tau_1\int_0^{\tau_1}d\tau_2\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}}(t-\tau_2) \\
&\quad\times e^{-i\omega_{k_2}(t-\tau_1)-i\omega_{k'}(\tau_1-\tau_2)-i\omega_{k_0}\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0 \\
&= -\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}V_{BB-k'}}{\omega_{k_1}\omega_{k'}^2}e^{-i\omega_{k_2}t} \\
&\quad\times\int_0^t d\tau_1\left(t-\tau_1-\frac{i}{\omega_{k'}}\right)e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0. \qquad (6.6)
\end{aligned}
$$

Adding the two contributions, one finds

$$
\begin{aligned}
(U_2^A(t)+U_2^B(t))|t=0\rangle_0 &= -\frac{\lambda}{4\sqrt{\lambda Q_0}}\int\frac{d^2k}{(2\pi)^2}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_1k_2k'}\Delta_{-k'B}}{\omega_{k_1}}e^{-i\omega_{k_2}t} \\
&\quad\times\int_0^t d\tau_1(t-\tau_1)e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0. \qquad (6.7)
\end{aligned}
$$

C. One zero mode at each vertex

Now consider the case in which each vertex creates a single zero mode ϕ_0. Since the only operator in the free Hamiltonian that annihilates zero modes is $\pi_0^2/2$, no zero modes can be annihilated until both are created. The time T will therefore be equal to t minus whichever of τ_1 and τ_2 is greater. This process is drawn in Fig. 12.

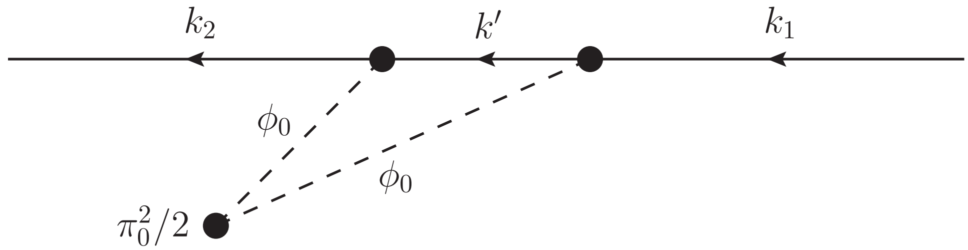

FIG. 12. Most of this paper is about the computation of this term, which is the only contribution to the scattering amplitude resulting from zero modes. The process is as in Fig. 8, except that the two zero modes are annihilated by the $\pi_0^2/2$ in the free evolution operator.

If $\tau_1 < \tau_2$, then Eq. (5.15) is modified to

$$\begin{aligned} U_2^A(t)|t=0\rangle_0 &= -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}I_A\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}} \\ &\quad\times e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_2\rangle_0, \\ I_A &= \int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{-i(\omega_{k'}-\omega_{k_2})\tau_2-i(\omega_{k_1}-\omega_{k'})\tau_1}(t-\tau_2). \end{aligned} \tag{6.8}$$

Integrating τ_1 first yields a factor of

$$\begin{aligned} I_A &= \int_0^t d\tau_2 e^{-i(\omega_{k'}-\omega_{k_2})\tau_2}(t-\tau_2)\int_0^{\tau_2}d\tau_1 e^{-i(\omega_{k_1}-\omega_{k'})\tau_1} \\ &= \frac{i}{\omega_{k_1}-\omega_{k'}}\int_0^t d\tau_2 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_2}(t-\tau_2), \end{aligned} \tag{6.9}$$

whereas integrating τ_2 first would yield

$$\begin{aligned} I_A &= \int_0^t d\tau_1 e^{-i(\omega_{k_1}-\omega_{k'})\tau_1}\left(t-i\frac{\partial}{\partial\omega_{k'}}\right)\int_{\tau_1}^t d\tau_2 e^{-i(\omega_{k'}-\omega_{k_2})\tau_2} \\ &= \frac{i}{\omega_{k_2}-\omega_{k'}}\int_0^t d\tau_1 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}\left(t-\tau_1-\frac{i}{\omega_{k_2}-\omega_{k'}}\right). \end{aligned} \tag{6.10}$$

Again, the integrals commute and so these expressions are equal. It will be convenient to use the average.

If $\tau_1 > \tau_2$, then Eq. (5.21) is modified to

$$\begin{aligned} U_2^B(t)|t=0\rangle_0 &= -i\frac{\lambda}{4}\int\frac{d^2k}{(2\pi)^2}e^{-i\omega_{k_2}t}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}I_B\frac{V_{B-k_1k'}V_{B-k'k_2}}{\omega_{k_1}\omega_{k'}} \\ &\quad\times e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_2\rangle_0, \\ I_B &= \int_0^t d\tau_1\int_0^{\tau_1}d\tau_2 e^{i(\omega_{k'}+\omega_{k_2})\tau_2-i(\omega_{k_1}+\omega_{k'})\tau_1}(t-\tau_1). \end{aligned} \tag{6.11}$$

Integrating τ_1 first,

$$\begin{aligned} I_B &= \int_0^t d\tau_2 e^{i(\omega_{k'}+\omega_{k_2})\tau_2}\left(t-i\frac{\partial}{\partial\omega_{k'}}\right)\int_{\tau_2}^t d\tau_1 e^{-i(\omega_{k_1}+\omega_{k'})\tau_1} \\ &= -\frac{i}{\omega_{k'}+\omega_{k_1}}\int_0^t d\tau_2 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_2}\left(t-\tau_2+\frac{i}{\omega_{k'}+\omega_{k_1}}\right), \end{aligned} \tag{6.12}$$

while integrating τ_2 first,

$$I_B = -\frac{i}{\omega_{k'} + \omega_{k_2}} \int_0^t d\tau_1 e^{-i(\omega_{k_1} - \omega_{k_2})\tau_1} (t - \tau_1). \tag{6.13}$$

Now, replacing all dummy variables τ_2 with τ_1 and averaging over the integral orderings, one finds

$$I_A + I_B = \int_0^t d\tau_1 e^{-i(\omega_{k_1} - \omega_{k_2})\tau_1} \left[i \left(\frac{\omega_{k'}}{\omega_{k_1}^2 - \omega_{k'}^2} + \frac{\omega_{k'}}{\omega_{k_2}^2 - \omega_{k'}^2} \right) (t - \tau_1) + \frac{1}{2(\omega_{k_1} + \omega_{k'})^2} + \frac{1}{2(\omega_{k_2} - \omega_{k'})^2} \right]. \tag{6.14}$$

Reinserting these integrals in the equations for the final states, one finds

$$\begin{aligned}(U_2^A(t) + U_2^B(t))|t = 0\rangle_0 = {}& \frac{\lambda}{4} \int \frac{d^2k}{(2\pi)^2} \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi} \frac{V_{B-k_1 k'} V_{B-k' k_2}}{\omega_{k_1}} e^{-i\omega_{k_2} t} \\ & \times \int_0^t d\tau_1 \left[(t - \tau_1) \left(\frac{1}{\omega_{k_1}^2 - \omega_{k'}^2} + \frac{1}{\omega_{k_2}^2 - \omega_{k'}^2} \right) - \frac{i}{2\omega_{k'}(\omega_{k_1} + \omega_{k'})^2} - \frac{i}{2\omega_{k'}(\omega_{k_2} - \omega_{k'})^2} \right] \\ & \times e^{-i(\omega_{k_0} - \omega_{k_2})\tau_1} e^{-\sigma^2(k_1 - k_0)^2 - i(k_1 - k_0)x_{\tau_1}} |k_2\rangle_0 = A + B,\end{aligned}$$

where

$$\begin{aligned}A = {}& \frac{\lambda}{4\sqrt{\lambda Q_0}} \int \frac{d^2k}{(2\pi)^2} \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi} \frac{(V_{B-k_1 k'} \Delta_{k_2 - k'} + V_{B-k' k_2} \Delta_{-k_1 k'})}{\omega_{k_1}} e^{-i\omega_{k_2} t} \\ & \times \int_0^t d\tau_1 (t - \tau_1) e^{-i(\omega_{k_0} - \omega_{k_2})\tau_1} e^{-\sigma^2(k_1 - k_0)^2 - i(k_1 - k_0)x_{\tau_1}} |k_2\rangle_0\end{aligned} \tag{6.15}$$

and

$$\begin{aligned}B = {}& -i\frac{\lambda}{8} \int \frac{d^2k}{(2\pi)^2} \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi} \frac{V_{B-k_1 k'} V_{B-k' k_2}}{\omega_{k_1} \omega_{k'}} e^{-i\omega_{k_2} t} \\ & \times \int_0^t d\tau_1 \left[\frac{1}{(\omega_{k_1} + \omega_{k'})^2} + \frac{1}{(\omega_{k_2} - \omega_{k'})^2} \right] e^{-i(\omega_{k_0} - \omega_{k_2})\tau_1} e^{-\sigma^2(k_1 - k_0)^2 - i(k_1 - k_0)x_{\tau_1}} |k_2\rangle_0.\end{aligned} \tag{6.16}$$

While the term A looks like that seen in the previous processes, the term B is different, in that it does not contain a $t - \tau$ factor. We will see that it is the only term in this section that contributes to elastic scattering.

D. No zero modes from four zero modes

The last process that leads to a single meson creates two zero modes in each of two interactions in Eq. (4.1) and lets them both be destroyed by the $e^{-i\pi_0^2 T/2}$ in the free evolution operator. It is drawn in Fig. 13.

Consider first $\tau_1 < \tau_2$. Now, as always, two zero modes are created at τ_1 and two more at τ_2. There are two ways in which the zero modes may be destroyed. First, the linear $-i\pi_0^2(\tau_2 - \tau_1)/2$ term in the evolution operator may destroy two zero

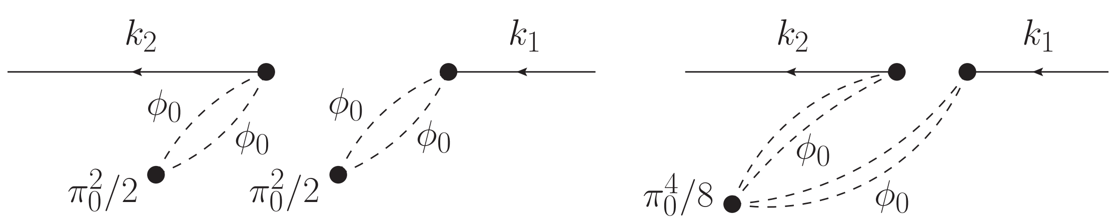

FIG. 13. This is as in Fig. 4 except that all four zero modes are annihilated by the kink center of mass kinetic term in the free evolution operator.

modes between times τ_1 and τ_2, and then the linear $-i\pi_0^2(t-\tau_2)/2$ term in the evolution operator may destroy two zero modes between times τ_2 and t. This contributes a factor of

$$[-i\pi_0^2(\tau_2-\tau_1)/2,\phi_0^2][-i\pi_0^2(t-\tau_2)/2,\phi_0^2] \\ =(\tau_1-\tau_2)(t-\tau_2) \tag{6.17}$$

to the ϕ_0^0 term in the final state with respect to the ϕ_0^4 term calculated in Sec. IV.

But it may also be that all four zero modes survive until τ_2 and so are annihilated by the $-\pi_0^4(t-\tau_2)^2/8$ quadratic term in the free evolution operator between times τ_2 and t. This possibility contributes a factor of

$$[-\pi_0^4(t-\tau_2)^2/8,\phi_0^4]=-3(t-\tau_2)^2. \tag{6.18}$$

Of course, these processes, having the same final state, add coherently and so lead to a total weight that is the sum of these factors,

$$(t-\tau_2)(-3t+\tau_1+2\tau_2). \tag{6.19}$$

The contribution (4.3) to the ϕ_0^4 sector of the final state then becomes

$$U_2^A(t)|t=0\rangle_0=-\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}I_A e^{-i\omega_{k_2}t}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_2\rangle_0, \\ I_A=\int_0^t d\tau_1\int_{\tau_1}^t d\tau_2 e^{i\omega_{k_2}\tau_2-i\omega_{k_1}\tau_1}(t-\tau_2)(-3t+\tau_1+2\tau_2). \tag{6.20}$$

Let us first integrate τ_2, as usual dropping $\tau_2=t$ as its contribution vanishes after the other integrals have been performed,

$$I_A=\int_0^t d\tau_1 e^{-i\omega_{k_1}\tau_1}\left(t+i\frac{\partial}{\partial\omega_{k_2}}\right)\left(-3t+\tau_1-2i\frac{\partial}{\partial\omega_{k_2}}\right)\int_{\tau_1}^t d\tau_2 e^{i\omega_{k_2}\tau_2} \\ =i\int_0^t d\tau_1 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}\left(t-\tau_1+i\frac{\partial}{\partial\omega_{k_2}}\right)\left(-3(t-\tau_1)-2i\frac{\partial}{\partial\omega_{k_2}}\right)\frac{1}{\omega_{k_2}} \\ =\frac{i}{\omega_{k_2}}\int_0^t d\tau_1 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}\left[-3(t-\tau_1)^2+\frac{5i(t-\tau_1)}{\omega_{k_2}}+\frac{4}{\omega_{k_2}^2}\right]. \tag{6.21}$$

On the other hand, performing the τ_1 integration first leads to

$$I_A=\int_0^t d\tau_2 e^{i\omega_{k_2}\tau_2}(t-\tau_2)\left(-3t+i\frac{\partial}{\partial\omega_{k_1}}+2\tau_2\right)\int_0^{\tau_2} d\tau_1 e^{-i\omega_{k_1}\tau_1} \\ =\frac{i}{\omega_{k_1}}\int_0^t d\tau_2 e^{i(\omega_{k_2}-\omega_{k_1})\tau_2}(t-\tau_2)\left(-3(t-\tau_2)-\frac{i}{\omega_{k_1}}\right). \tag{6.22}$$

Consider now $\tau_1>\tau_2$. The factor that one must now include is obtained by exchanging τ_1 and τ_2 in Eq. (6.19),

$$(t-\tau_1)(-3t+\tau_2+2\tau_1). \tag{6.23}$$

This modifies the contribution (4.5) to

$$U_2^B(t)|t=0\rangle_0=-\frac{\lambda}{8}\int\frac{d^2k}{(2\pi)^2}I_B e^{-i\omega_{k_2}t}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_2\rangle_0, \\ I_B=\int_0^t d\tau_1\int_0^{\tau_1} d\tau_2 e^{i\omega_{k_2}\tau_2-i\omega_{k_1}\tau_1}(t-\tau_1)(-3t+\tau_2+2\tau_1). \tag{6.24}$$

Now we first integrate τ_1, dropping $\tau_1=t$,

$$\begin{aligned} I_B &= \int_0^t d\tau_2 e^{i\omega_{k_2}\tau_2}\left(t - i\frac{\partial}{\partial\omega_{k_1}}\right)\left(-3t + \tau_2 + 2i\frac{\partial}{\partial\omega_{k_1}}\right)\int_{\tau_2}^t d\tau_1 e^{-i\omega_{k_1}\tau_1} \\ &= -i\int_0^t d\tau_2 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_2}\left(t - \tau_2 - i\frac{\partial}{\partial\omega_{k_1}}\right)\left(-3(t-\tau_2) + 2i\frac{\partial}{\partial\omega_{k_1}}\right)\frac{1}{\omega_{k_1}} \\ &= -\frac{i}{\omega_{k_1}}\int_0^t d\tau_2 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_2}\left[-3(t-\tau_2)^2 - \frac{5i(t-\tau_2)}{\omega_{k_1}} + \frac{4}{\omega_{k_1}^2}\right]. \end{aligned} \tag{6.25}$$

On the other hand, integrating τ_2 first,

$$\begin{aligned} I_B &= \int_0^t d\tau_1 e^{-i\omega_{k_1}\tau_1}(t-\tau_1)\left(-3t - i\frac{\partial}{\partial\omega_{k_2}} + 2\tau_1\right)\int_0^{\tau_1} d\tau_2 e^{i\omega_{k_2}\tau_2} \\ &= -\frac{i}{\omega_{k_2}}\int_0^t d\tau_1 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}(t-\tau_1)\left(-3(t-\tau_1) + \frac{i}{\omega_{k_2}}\right). \end{aligned} \tag{6.26}$$

Again, we replace the dummy variables τ_2 with τ_1 and average over integration orders to obtain

$$I_A + I_B = \int_0^t d\tau_1 e^{-i(\omega_{k_1}-\omega_{k_2})\tau_1}\left[-2(t-\tau_1)\left(\frac{1}{\omega_{k_2}^2} + \frac{1}{\omega_{k_1}^2}\right) + \frac{2i}{\omega_{k_2}^3} - \frac{2i}{\omega_{k_1}^3}\right]. \tag{6.27}$$

We note that, at $k_1 = \pm k_2$, corresponding to the average value in elastic scattering, the terms that are linearly divergent in $t - \tau_1$ are nonzero, but the constant piece vanishes. As a result, these terms will not contribute to our final amplitude. The contribution to the final state is

$$\begin{aligned} (U_2^A(t) + U_2^B(t))|t=0\rangle_0 = \frac{\lambda}{4}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}} e^{-i\omega_{k_2}t} \\ \times \int_0^t d\tau_1\left[(t-\tau_1)\left(\frac{1}{\omega_{k_2}^2} + \frac{1}{\omega_{k_1}^2}\right) - \frac{i}{\omega_{k_2}^3} + \frac{i}{\omega_{k_1}^3}\right] \\ \times e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1} e^{-\sigma^2(k_1-k_0)^2 - i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0 = C + D, \end{aligned}$$

where

$$\begin{aligned} C = \frac{\lambda}{4\sqrt{\lambda Q_0}}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}\Delta_{k_2B} + V_{BBk_2}\Delta_{-k_1B}}{\omega_{k_1}} e^{-i\omega_{k_2}t} \\ \times \int_0^t d\tau_1(t-\tau_1)e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1} e^{-\sigma^2(k_1-k_0)^2 - i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0 \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} D = \frac{i\lambda}{4}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BB-k_1}V_{BBk_2}}{\omega_{k_1}} e^{-i\omega_{k_2}t} \\ \times \int_0^t d\tau_1\left[\frac{1}{\omega_{k_1}^3} - \frac{1}{\omega_{k_2}^3}\right] e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1} e^{-\sigma^2(k_1-k_0)^2 - i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0. \end{aligned} \tag{6.29}$$

In this section, like the two before it, we have been careful to distinguish k_0, k_1, and k_2, even though they only differ by of order $O(1/\sigma)$. Our care has paid off, because these differences were multiplied by factors of $t - \tau$ and even $(t-\tau)^2$ in terms where zero modes were canceled. These factors resulted from the fact that the free evolution leads to a constant rate of demotion from ϕ_0^m to ϕ_0^{m-2}.

However, now we have already calculated these factors, and they are not present in D. Therefore, in D, one can safely take our limit $m\sigma \to \infty$, which implies that, in the support of our $e^{-\sigma^2(k_1-k_0)^2}$ weight, k_1 may be replaced with k_0. Thus, with the usual argument that x/σ is negligible when multiplied by $\mathfrak{g}_B(x)$, we may write

$$
\begin{aligned}
D &= \frac{i\lambda}{4}\int \frac{d^2k}{(2\pi)^2}\frac{V_{BBk_2}}{\omega_{k_0}} e^{-i\omega_{k_2}t}\int_0^t d\tau_1 \left[\frac{1}{\omega_{k_0}^3}-\frac{1}{\omega_{k_2}^3}\right] e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1} \\
&\quad\times \int dx V^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_B^2(x)\mathfrak{g}_{-k_0}(x) e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)(x_{\tau_1}-x)}|k_2\rangle_0 \\
&= \frac{i\lambda}{4}\frac{\sqrt{\pi}}{2\pi\sigma}\int \frac{dk_2}{2\pi}\frac{V_{BB-k_0}V_{BBk_2}}{\omega_{k_0}} e^{-i\omega_{k_2}t}\left[\frac{1}{\omega_{k_0}^3}-\frac{1}{\omega_{k_2}^3}\right] \\
&\quad\times \int_0^t d\tau_1 e^{-x_{\tau_1}^2/(4\sigma^2)-i(\omega_{k_0}-\omega_{k_2})\tau_1}|k_2\rangle_0 \\
&= \frac{i\lambda}{4}\int \frac{dk_2}{2\pi}\frac{V_{BB-k_0}V_{BBk_2}}{k_0} e^{-i\omega_{k_2}t} e^{-i(\omega_{k_0}-\omega_{k_2})t_c}\left[\frac{1}{\omega_{k_0}^3}-\frac{1}{\omega_{k_2}^3}\right] e^{-\sigma^2(k_0+k_2)^2}|k_2\rangle_0.
\end{aligned}
\tag{6.30}
$$

In the support of the $e^{-\sigma^2(k_0+k_2)^2}$, so that $k_2=-k_0+O(1/\sigma)$, we can see that the term in square brackets is $1/\omega_{k_0}^3$ times a factor of order $O(1/(m\sigma))$ and so vanishes as $m\sigma\to\infty$. Thus, D will not contribute to the amplitude and we will not consider it further.

E. The total

Finally, we are ready to add the contributions in Eqs. (6.2), (6.7), (6.15), (6.16), and (6.28) to the one-meson, no zero-mode part of the final state. Recall that these are the contributions arising from interactions that created zero modes, which were later annihilated. The sum is

$$
\begin{aligned}
U_2(t)|t=0\rangle_0 &= B - \frac{\lambda}{4}\int \frac{d^2k}{(2\pi)^2}\frac{S_2}{\omega_{k_1}} e^{-i\omega_{k_2}t} \\
&\quad\times \int_0^t d\tau_1 (t-\tau_1) e^{-i(\omega_{k_0}-\omega_{k_2})\tau_1} e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2\rangle_0.
\end{aligned}
\tag{6.31}
$$

The quantity S_2 was defined in Eq. (5.37) where it was noted that $S_2=0$ as a result of the Ward identity (A16). This leaves B.

The quantity B was defined in Eq. (6.16). It is

$$
\begin{aligned}
B &= -i\frac{\lambda}{8}\frac{\sqrt{\pi}}{2\pi\sigma}\int \frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{V_{B-k_0k'}V_{B-k'k_2}}{\omega_{k_0}\omega_{k'}} e^{-i\omega_{k_2}t} \\
&\quad\times \left[\frac{1}{(\omega_{k_0}+\omega_{k'})^2}+\frac{1}{(\omega_{k_2}-\omega_{k'})^2}\right]\int_0^t d\tau_1 e^{-x_{\tau_1}^2/(4\sigma^2)-i(\omega_{k_0}-\omega_{k_2})\tau_1}|k_2\rangle_0 \\
&= -i\frac{\lambda}{8}\int \frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\frac{V_{B-k_0k'}V_{B-k'k_2}}{k_0\omega_{k'}} e^{-i\omega_{k_2}t} \\
&\quad\times \left[\frac{1}{(\omega_{k_0}+\omega_{k'})^2}+\frac{1}{(\omega_{k_2}-\omega_{k'})^2}\right] e^{-\sigma^2(k_0+k_2)^2-i(\omega_{k_0}-\omega_{k_2})t_c}|k_2\rangle_0.
\end{aligned}
\tag{6.32}
$$

In the support of $e^{-\sigma^2(k_0+k_2)^2}$ we may set $\omega_{k_0}=\omega_{k_2}$ and so manipulate

$$
\begin{aligned}
V_{B-k_0k'}V_{B-k'k_2}\left[\frac{1}{(\omega_{k_0}+\omega_{k'})^2}+\frac{1}{(\omega_{k_2}-\omega_{k'})^2}\right] &= \frac{\Delta_{-k_0k'}\Delta_{-k_0-k'}}{\lambda Q_0}[(\omega_{k_0}-\omega_{k'})^2+(\omega_{k_0}+\omega_{k'})^2] \\
&= \frac{2\Delta_{-k_0k'}\Delta_{-k_0-k'}}{\lambda Q_0}(\omega_{k_0}^2+\omega_{k'}^2).
\end{aligned}
\tag{6.33}
$$

Here we replaced $V_{B-k'k_2}$ with $V_{B-k'-k_0}$, which yields a phase $e^{-i(k_2+k_0)x}$. However, the $\mathfrak{g}_B(x)$ is supported at $x\sim O(1/m)$ and k_2+k_0 is of order $O(1/\sigma)$ so the argument of the phase is of order $1/(\sigma m)$ which tends to zero, so the phase factor tends to unity.

Therefore, we conclude

$$U_2(t)|t=0\rangle_0 = -i\frac{1}{4Q_0k_0}\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}e^{-\sigma^2(k_0+k_2)^2+i(k_0+k_2)x_0} \times \sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}(\omega_{k_0}^2+\omega_{k'}^2)\frac{\Delta_{-k_0k'}\Delta_{-k_0-k'}}{\omega_{k'}}|k_2\rangle_0. \tag{6.34}$$

This, together with the terms found in Sec. III, is the part of the final state corresponding to meson with no zero modes that is not forward scattered.

VII. THE ELASTIC SCATTERING PROBABILITY

Adding together the contributions to the final state from Eqs. (3.16), (3.33), (3.55), and (6.32), finally, we find

$$U_2(t)|t=0\rangle = -i\int\frac{dk_2}{2\pi}e^{-i\omega_{k_2}t}R(k_2)e^{-\sigma^2(k_0+k_2)^2+i(k_0+k_2)x_0}|k_2\rangle_0, \tag{7.1}$$

where the reflection coefficient is

$$R(k_2) = \lambda(A(k_2)+B(k_2)+C(k_2)+D(k_2)) \tag{7.2}$$

and

$$\begin{aligned}
A(k_2) &= \frac{1}{8k_0}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\left(\frac{1}{(\omega_{k_0}+\omega_{k'})^2}+\frac{1}{(\omega_{k_2}-\omega_{k'})^2}\right)\frac{V_{B-k_0k'}V_{B-k'k_2}}{\omega_{k'}},\\
B(k_2) &= \frac{V_{\mathcal{I}k_0-k_2}}{4k_0},\\
C(k_2) &= -\frac{1}{4k_0}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}V_{\mathcal{I}-k'}}{\omega_{k'}^2},\\
D(k_2) &= \frac{1}{8k_0}\sum\!\!\!\!\!\!\int\frac{d^2k'}{(2\pi)^2}\frac{(\omega_{k_1'}+\omega_{k_2'})V_{k_0-k_1'-k_2'}V_{-k_0k_1'k_2'}}{\omega_{k_1'}\omega_{k_2'}\left(\omega_{k_0}^2-\left(\omega_{k_1'}+\omega_{k_2'}\right)^2+i\epsilon\right)}.
\end{aligned} \tag{7.3}$$

For example, $A(k_2)$ is just the coefficient in B in Eq. (6.32) divided by λ. We remind the reader that U_2 is not unitary, as we have defined it to be just the part of the evolution operator that leads to one nonforward meson and no zero modes. Note that at $k_2=-k_0$ one may simplify

$$A(-k_0) = \frac{1}{4k_0\lambda Q_0}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\left(\frac{\omega_{k_0}^2+\omega_{k'}^2}{\omega_{k'}}\right)\Delta_{-k_0k'}\Delta_{-k_0-k'}. \tag{7.4}$$

Following Ref. [17], it is easy to see that the probability of elastic scattering is $|R(-k_0)|^2$. This calculation is done using the reduced inner product of [38], which carefully removes the divergences arising from the infinite moduli space. Using

$$|k_1\rangle_0 = B_{k_1}^{\ddagger}|0\rangle_0, \quad B_{k_1}^{\ddagger\dagger} = \frac{B_{k_1}}{2\omega_{k_1}}, \quad \langle 0|0\rangle_{\rm red} = \sqrt{Q_0}, \tag{7.5}$$

one finds that at leading order the reduced inner product of $|k_1\rangle$ and $|k_2\rangle$ is $\sqrt{Q_0}2\pi\delta(k_1-k_2)/(2\omega_{k_1})$. Subleading corrections are computed in Ref. [38] and it is argued that they vanish in the present case in Ref. [17].

The reduced norm squared of the elastic scattered part of the final state (7.1) is then

$$\langle t=0|U_2^\dagger(t)U_2(t)|t=0\rangle_{\rm red} = \sqrt{Q_0}|R(-k_0)|^2\frac{\sqrt{\pi}}{4\sqrt{2}\pi\sigma\omega_{k_0}}. \tag{7.6}$$

Here we have used the fact that $\sigma m\to\infty$ to approximate R to be independent of k_2 over the support of the Gaussian, so that it could be pulled out of the integral, evaluated at $-k_0$.

On the other hand, the reduced norm squared of the total final state is equal to the reduced norm squared of the initial state $|t=0\rangle$, as a result of the unitarity of the evolution, which is

$$\langle t=0|t=0\rangle_{\rm red}=\sqrt{Q_0}\frac{\sqrt{\pi}}{4\sqrt{2}\pi\sigma\omega_{k_0}}. \tag{7.7}$$

The probability of elastic scattering is just the ratio of these two reduced norms,

$$P=\frac{\langle t=0|U_2^\dagger(t)U_2(t)|t=0\rangle_{\rm red}}{\langle t=0|t=0\rangle_{\rm red}}=|R(-k_0)|^2. \tag{7.8}$$

Indeed, the reduced norm was developed just to solve this problem.

VIII. APPLICATIONS

After a long calculation, we have recovered the results of Ref. [17]. What have we gained?

We have drawn diagrams corresponding to each process. Yet, no Feynman rules have been given that would derive the corresponding contribution to the amplitude from the diagrams. We intend to use this collection of examples to guide the derivation of such Feynman rules for kink sector perturbation theory. With this, we hope that such calculations in the future may be much faster. Indeed, the fact that the derivation of the elastic scattering amplitude in Ref. [17] was so short gives us hope that such a simplification is possible.

A more streamlined framework will allow for higher order computations. These have several potential applications. First, by summing bubble diagrams, one may see a complex shift in the location of the pole corresponding to the twice-excited shape mode resonance. The width of this resonance should correspond to the lifetime of this unstable state calculated in Ref. [42], which agrees with the classical field theory calculation of Ref. [43]. One can test to see whether, like in the vacuum sector, also in the kink sector the lifetimes of unstable states may be read off of the imaginary parts of the self-energies.

The situation potentially differs qualitatively from the familiar vacuum sector case when one goes beyond leading order. Here zero modes created at one bubble may annihilate those created at another. It remains to be seen whether this simply leads to subleading corrections corresponding to larger bubbles or else a qualitative change in the structures of these resonances. Either way, we hope to calculate these subleading corrections, as they may yield, for the first time, the correction to the lifetime of an unstable excited soliton state.

Finally, we would like to study higher order diagrams to search for a kink sector Lehmann-Symanzik-Zimmermann reduction theorem. In this paper and in Ref. [31], we have observed that initial and final state corrections always seem to cancel, by a number of different mechanisms. This leads one to wonder just how generic this result is and whether bubbles on external legs can be easily summed.

If one considers initial conditions with multiple mesons, one may also study meson fusion. Using coherent states to create a classical limit as in [42], this should allow a study of the negative radiation pressure observed in Refs. [44–46].

Of course, kinks themselves have limited phenomenological interest. In general, $1+1d$ scalar models with kinks are instead used as toy models either for QCD [23] or for quantum gravity [47,48]. In the near future, we hope to generalize linearized soliton perturbation theory to solitons in more dimensions, and so the answers to the above questions may have more relevant applications.

ACKNOWLEDGMENTS

J. E. is supported by NSFC MianShang Grants No. 11875296 and No. 11675223. H. L. thanks B. Y. Zhang for the helpful discussion.

APPENDIX A: WARD IDENTITIES

Consider the n-point functions

$$V_{A_1\cdots A_n}=\int dx V^{(n)}(\sqrt{\lambda}f(x))\mathfrak{g}_{A_1}(x)\cdots\mathfrak{g}_{A_n}(x), \tag{A1}$$

where A_i runs over continuum modes k, shape modes S, and the zero mode B. These correspond to n-point functions with external legs A_i corresponding to various zero and normal modes. The generalization containing factors of $\mathcal{I}(x)$ is obvious.

Now consider an n-point function containing at least one zero mode. The n-point function is symmetric, so let us put the zero mode in the last index $A_n=B$. Then $V_{A_1\cdots A_{n-1}B}$ satisfies a Ward identity corresponding to translation invariance. Schematically, the Ward identity is

$$V_{A_1\cdots A_{n-1}B}=\frac{1}{\sqrt{\lambda Q_0}}\sum_{i=1}^{n-1}\sum\!\!\!\!\!\!\int\frac{dA'}{2\pi}\Delta_{A_i-A'}V_{A_1\cdots A_{i-1}A'A_{i+1}\cdots A_{n-1}}. \tag{A2}$$

The matrix Δ, defined in Eq. (5.12), plays the role of the momentum operator. Here, breaking from our usual notation, the symbol $\sum\!\!\!\!\!\!\int$ includes not only shape modes but also the zero mode. The constant factor of $\sqrt{\lambda Q_0}$ is the result of various conventions.

These are derived by noting that

$$V^{(n)}(\sqrt{\lambda}f(x))\mathfrak{g}_B(x)=-\frac{\partial_x V^{(n-1)}(\sqrt{\lambda}f(x))}{\sqrt{\lambda Q_0}} \tag{A3}$$

and integrating by parts to move the derivative onto the other factors $\mathfrak{g}_A(x)$. The identity, in the form of the normal mode completeness relation (A10), which is a standard result in Sturm-Liouville theory, is inserted to turn $\mathfrak{g}'_A(x)$ into a $\Delta_{A_iA_j}$ matrix, which represents translations on the normal modes. In this appendix, some such Ward identities will be derived.

Note that the only continuous global symmetry in our model is translation invariance. However, in more general models in which global symmetries are explicitly broken by classical solutions, we expect the same results to hold for the corresponding zero modes. Note that this is true even if, as in the present case, the ground state in the soliton sector preserves the classically broken symmetry as a result of the Coleman-Mermin-Wagner theorem.

1. Warm up

In the special case $N = 3$ we can use the fact that $V^{(2)}(\sqrt{\lambda}f(x))$ satisfies the Sturm-Liouville equations of motion of the normal modes to simplify the Ward identities further. This will be the first approach below.

Using

$$\mathfrak{g}_B(x) = -\frac{f'(x)}{\sqrt{Q_0}}, \tag{A4}$$

one can expand

$$\begin{aligned}V_{Bk_2k_1} &= \int dx V^{(3)}(\sqrt{\lambda}f(x))\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x)\\ &= -\frac{1}{\sqrt{Q_0}}\int dx\Big(V^{(3)}(\sqrt{\lambda}f(x))f'(x)\Big)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x)\\ &= -\frac{1}{\sqrt{\lambda Q_0}}\int dx\partial_x\Big(V^{(2)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)\\ &\quad\times \mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x),\end{aligned} \tag{A5}$$

where $C_{k_1k_2}$ is independent of x but otherwise arbitrary.

a. Approach one

As $C_{k_1k_2}$ is x independent, its derivative vanishes and we may drop it. Now, cut off the integration at $\pm\hat{x}$, such that $|\hat{x}| \gg 1/m$ and integrate by parts,

$$\begin{aligned}V_{Bk_2k_1} &= -\frac{m^2}{\sqrt{\lambda Q_0}}(\mathfrak{g}_{k_2}(\hat{x})\mathfrak{g}_{k_1}(\hat{x}) - \mathfrak{g}_{k_2}(-\hat{x})\mathfrak{g}_{k_1}(-\hat{x}))\\ &\quad + \frac{1}{\sqrt{\lambda Q_0}}\int_{-\hat{x}}^{\hat{x}} dx V^{(2)}(\sqrt{\lambda}f(x))(\mathfrak{g}_{k_2}(x)\mathfrak{g}'_{k_1}(x) + \mathfrak{g}'_{k_2}(x)\mathfrak{g}_{k_1}(x))\\ &= -\frac{m^2}{\sqrt{\lambda Q_0}}(\mathfrak{g}_{k_2}(\hat{x})\mathfrak{g}_{k_1}(\hat{x}) - \mathfrak{g}_{k_2}(-\hat{x})\mathfrak{g}_{k_1}(-\hat{x}))\\ &\quad + \frac{1}{\sqrt{\lambda Q_0}}\int_{-\hat{x}}^{\hat{x}} dx[\mathfrak{g}'_{k_1}(x)(\omega_{k_2}^2 + \partial_x^2)\mathfrak{g}_{k_2}(x) + \mathfrak{g}'_{k_2}(x)(\omega_{k_1}^2 + \partial_x^2)\mathfrak{g}_{k_1}(x)]\\ &= -\frac{m^2}{\sqrt{\lambda Q_0}}(\mathfrak{g}_{k_2}(\hat{x})\mathfrak{g}_{k_1}(\hat{x}) - \mathfrak{g}_{k_2}(-\hat{x})\mathfrak{g}_{k_1}(-\hat{x})) + \frac{1}{\sqrt{\lambda Q_0}}(\mathfrak{g}'_{k_2}(\hat{x})\mathfrak{g}'_{k_1}(\hat{x}) - \mathfrak{g}'_{k_2}(-\hat{x})\mathfrak{g}'_{k_1}(-\hat{x}))\\ &\quad + \frac{(\omega_{k_1}^2 - \omega_{k_2}^2)\Delta_{k_1k_2}}{\sqrt{\lambda Q_0}}\\ &= -\frac{m^2 + k_1k_2}{\sqrt{\lambda Q_0}}(\mathfrak{g}_{k_2}(\hat{x})\mathfrak{g}_{k_1}(\hat{x}) - \mathfrak{g}_{k_2}(-\hat{x})\mathfrak{g}_{k_1}(-\hat{x})) + \frac{(\omega_{k_1}^2 - \omega_{k_2}^2)\Delta_{k_1k_2}}{\sqrt{\lambda Q_0}}.\end{aligned} \tag{A6}$$

Note that the term in the first parentheses is proportional to $e^{\pm i\hat{x}(k_1+k_2)}$ which, by the Riemann-Lebesgue lemma, will vanish when folded into any integrable function of $k_1 + k_2$, such as our normalizable wave functions. One might have expected it to be proportional to $\delta(k_1 + k_2)$, however, it remains finite when $k_1 + k_2 = 0$ and so the constant of proportionality is zero. The left side also remains finite at $k_1 + k_2 = 0$ because $\mathfrak{g}_B(x)$ has compact support. Thus, taking the limit $\hat{x} \to \infty$, one arrives at

$$V_{Bk_2k_1} = \frac{(\omega_{k_1}^2 - \omega_{k_2}^2)\Delta_{k_1k_2}}{\sqrt{\lambda Q_0}}. \tag{A7}$$

In the above derivation, k_1 and k_2 could be continuum modes or shape modes. However, the derivation also applies to the case in which k_1 or k_2 is the zero mode. In this case, the corresponding frequency in the Sturm-Liouville equation satisfied by $\mathfrak{g}(x)$ vanishes and so one obtains

$$V_{BBk} = \frac{\omega_k^2\Delta_{kB}}{\sqrt{\lambda Q_0}}, \qquad V_{BBB} = 0. \tag{A8}$$

b. Approach two

If we keep the $C_{k_1k_2}$ when integrating by parts, one arrives at

$$V_{Bk_2k_1} = -\frac{(m^2+C_{k_1k_2})}{\sqrt{\lambda Q_0}}(\mathfrak{g}_{k_2}(\hat{x})\mathfrak{g}_{k_1}(\hat{x}) - \mathfrak{g}_{k_2}(-\hat{x})\mathfrak{g}_{k_1}(-\hat{x})),$$
$$+\frac{1}{\sqrt{\lambda Q_0}}\int dx\Big(V^{(2)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)(\mathfrak{g}_{k_2}(x)\mathfrak{g}'_{k_1}(x) + \mathfrak{g}'_{k_2}(x)\mathfrak{g}_{k_1}(x)), \tag{A9}$$

where $\hat{x}$ is a spatial cutoff, which should be taken to infinity. The boundary term on the first line oscillates rapidly and so vanishes as a distribution unless $k_1+k_2=0$. It therefore may only contribute a divergent term at $k_1+k_2=0$, but $V_{Bk_1k_2}$ has no such large x divergence as $\mathfrak{g}_B(x)$ has compact support. Therefore, the boundary term always vanishes and we will drop it. Now insert the completeness relation

$$\delta(x-y) = \mathfrak{g}_B(x)\mathfrak{g}_B(y) + \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}\mathfrak{g}_{k'}(x)\mathfrak{g}_{-k'}(y) \tag{A10}$$

to change the $\mathfrak{g}'(x)$ terms to $\mathfrak{g}'(y)$, leaving the boundary terms implicit,

$$\begin{aligned}V_{Bk_2k_1} &= \frac{1}{\sqrt{\lambda Q_0}}\int dx\int dy\delta(x-y)\Big(V^{(2)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)(\mathfrak{g}_{k_2}(x)\mathfrak{g}'_{k_1}(y) + \mathfrak{g}_{k_1}(x)\mathfrak{g}'_{k_2}(y))\\
&= \frac{1}{\sqrt{\lambda Q_0}}\int dx\Big(V^{(2)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)\Big[\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\Delta_{Bk_1} + \mathfrak{g}_B(x)\mathfrak{g}_{k_1}(x)\Delta_{Bk_2}\\
&\quad + \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}(\mathfrak{g}_{k_2}(x)\mathfrak{g}_{-k'}(x)\Delta_{k'k_1} + \mathfrak{g}_{k_1}(x)\mathfrak{g}_{-k'}(x)\Delta_{k'k_2})\Big]\\
&= \frac{1}{\sqrt{\lambda Q_0}}\Big[V_{Bk_2}\Delta_{Bk_1} + V_{Bk_1}\Delta_{Bk_2} + \sum\!\!\!\!\!\!\int \frac{dk'}{2\pi}(V_{k_2-k'}\Delta_{k'k_1} + V_{k_1-k'}\Delta_{k'k_2})\Big].\end{aligned} \tag{A11}$$

Here the C terms each vanish as a result of the orthonormality of the normal modes $\mathfrak{g}$ as well the antisymmetry of Δ.

This Ward identity relates three-point functions with contractions of two-point functions with Δ. We will see below that it can be generalized to an expression relating any n-point function with a contraction of $(n-1)$-point functions with Δ.

2. Infrared divergences

One needs to be aware of the infrared divergences that arise when some subset of the k_i sum to zero. These result from the fact that the e^{-ik_ix} factors in the corresponding $\mathfrak{g}_{k_i}(x)$ at large $|x|$ have a product that does not oscillate, and so some integrals diverge. For example, consider the manipulation

$$\begin{aligned}\Delta_{k_1k_2} &= \int dx\mathfrak{g}_{k_1}(x)\partial_x\mathfrak{g}_{k_2}(x)\\
&= \mathfrak{g}_{k_1}(x)\mathfrak{g}_{k_2}(x)|_{-\infty}^{\infty} - \int dx\mathfrak{g}_{k_2}(x)\partial_x\mathfrak{g}_{k_1}(x)\\
&= \mathfrak{g}_{k_1}(\hat{x})\mathfrak{g}_{k_2}(\hat{x})|_{-\infty}^{\infty} - \Delta_{k_2k_1}.\end{aligned} \tag{A12}$$

Generally, we drop the boundary term and summarize the result by stating that $\Delta_{k_1k_2}$ is antisymmetric. This is justified because, if we take the limit $|\hat{x}|\to\infty$ of the boundary term at the end, it oscillates rapidly in $|\hat{x}|$ and so vanishes as a distribution. However, this argument fails if $k_1=-k_2$. Thus, the antisymmetry is only up to a correction with support at $k_1=-k_2$, such as a Dirac δ function. In practice, in the case of kinks in gapped theories considered here, this term is proportional to $k_1\delta(k_1+k_2)$, which in fact is antisymmetric. In principle, such contributions may lead to finite effects in quantities of interest, and one must always be aware of them and must determine when they may contribute. For example, in the case of the one-loop mass correction, the general formula of Ref. [20] is proportional to $(\omega_k-\omega_p)^2$ and so it vanishes even when the coefficient contains a $\delta(k-p)$.

In general, we expect such divergences in our $(n-1)$-point functions on the right-hand side of the Ward identities, but we do not expect them in the n-point functions on the left-hand side because these include a $\mathfrak{g}_B(x)$ which has compact support. Let us now show that this expectation is fulfilled in the case at hand, and a divergence on the right-hand side of the Ward identity does not lead to one on the left-hand side.

The three-point function $V_{Bk_1k_2}$ plays an important role, as the vertex factor connecting a zero mode to two mesons. We now ask whether it is sensitive to δ function terms in

$V_{k_1k_2}$. In the last line of (A11), one can see that the shift $V_{k_1k_2} \to V_{k_1k_2} + \delta(k_1+k_2)$ leads to the shift $V_{Bk_1k_2} \to V_{Bk_1k_2} + (\Delta_{k_2k_1} + \Delta_{k_1k_2})/\sqrt{\lambda Q_0}$. This of course would vanish were $\Delta_{k_1k_2}$ truly antisymmetric, but as we reviewed above this argument fails at $k_1 = -k_2$. However, a finite contribution on a codimension one surface like $k_1 = -k_2$ also vanishes in the sense of a distribution, while we recall that an infinite distribution is excluded by the fact that $\mathfrak{g}_B(x)$ has compact support. Therefore, again we are not interested in such contributions, and so we conclude that a $\delta(k_1+k_2)$ contribution to the two-point function does not affect the three-point function.

3. Computation

This time, expand

$$\begin{aligned} V_{BBk_2k_1} &= \int dx V^{(4)}(\sqrt{\lambda}f(x))\mathfrak{g}_B^2(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x) \\ &= -\frac{1}{\sqrt{Q_0}}\int dx\Big(V^{(4)}(\sqrt{\lambda}f(x))f'(x)\Big)\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x) \\ &= -\frac{1}{\sqrt{\lambda Q_0}}\int dx\partial_x\Big(V^{(3)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x), \end{aligned} \tag{A13}$$

where $C_{k_1k_2}$ is independent of x but otherwise arbitrary.

As $\mathfrak{g}_B(x)$ vanishes asymptotically, no boundary term is introduced when we integrate by parts,

$$\begin{aligned} V_{BBk_2k_1} &= \frac{1}{\sqrt{\lambda Q_0}}\int dx\Big(V^{(3)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)\partial_x(\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{k_1}(x)) \\ &= \frac{1}{\sqrt{\lambda Q_0}}\int dx\Big(V^{(3)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)(\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}'_{k_1}(x) \\ &\quad + \mathfrak{g}_B(x)\mathfrak{g}_{k_1}(x)\mathfrak{g}'_{k_2}(x) + \mathfrak{g}_{k_1}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}'_B(x)). \end{aligned} \tag{A14}$$

Note that the $C_{k_1k_2}$ terms vanish as they are the integral of a total derivative of a bounded function. Of course, this is obvious because $C_{k_1k_2}$ is arbitrary.

Now insert the completeness relation

$$\delta(x-y) = \mathfrak{g}_B(x)\mathfrak{g}_B(y) + \sum\!\!\!\!\!\!\!\int \frac{dk'}{2\pi}\mathfrak{g}_{k'}(x)\mathfrak{g}_{-k'}(y) \tag{A15}$$

to change the $\mathfrak{g}'(x)$ terms to $\mathfrak{g}'(y)$,

$$\begin{aligned} V_{BBk_2k_1} &= \frac{1}{\sqrt{\lambda Q_0}}\int dx\int dy\delta(x-y)\Big(V^{(3)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)(\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}'_{k_1}(y) \\ &\quad + \mathfrak{g}_B(x)\mathfrak{g}_{k_1}(x)\mathfrak{g}'_{k_2}(y) + \mathfrak{g}_{k_1}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}'_B(y)) \\ &= \frac{1}{\sqrt{\lambda Q_0}}\int dx\Big(V^{(3)}(\sqrt{\lambda}f(x)) + C_{k_1k_2}\Big)\Big[\mathfrak{g}_B^2(x)\mathfrak{g}_{k_2}(x)\Delta_{Bk_1} + \mathfrak{g}_B^2(x)\mathfrak{g}_{k_1}(x)\Delta_{Bk_2} \\ &\quad + \sum\!\!\!\!\!\!\!\int \frac{dk'}{2\pi}(\mathfrak{g}_B(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{-k'}(x)\Delta_{k'k_1} + \mathfrak{g}_B(x)\mathfrak{g}_{k_1}(x)\mathfrak{g}_{-k'}(x)\Delta_{k'k_2} + \mathfrak{g}_{k_1}(x)\mathfrak{g}_{k_2}(x)\mathfrak{g}_{-k'}(x)\Delta_{k'B})\Big] \\ &= \frac{1}{\sqrt{\lambda Q_0}}\Big[V_{BBk_2}\Delta_{Bk_1} + V_{BBk_1}\Delta_{Bk_2} + \sum\!\!\!\!\!\!\!\int \frac{dk'}{2\pi}(V_{Bk_2-k'}\Delta_{k'k_1} + V_{Bk_1-k'}\Delta_{k'k_2} + V_{k_1k_2-k'}\Delta_{k'B})\Big]. \end{aligned} \tag{A16}$$

Now shifting $V_{k_1k_2-k'}$ by $\delta(k_1+k_2-k')$ changes the answer, with a shift proportional to Δ_{B,k_1+k_2}.

APPENDIX B: SUBLEADING CORRECTIONS TO STOKES SCATTERING

1. Shape modes

In some models, the kink possesses shape modes. In that case, the virtual meson above could be a shape mode. That invalidates two of the arguments made above.

First of all, several times above we stated that the k' integrand oscillates so rapidly that, once k' is integrated out, the contribution to the amplitude will be exponentially suppressed by interference. If k' is discrete, this argument does not work.

Second, we used the reduced inner product. The kink has an infinite moduli space of classical solutions, related by translation invariance. By choosing one kink solution, we have fixed the translation symmetry. This can be done consistently in the ratio of matrix elements, like in our formula for the probability. However, when one fixes a symmetry, a determinant term must be included.

This determinant was calculated in Ref. [38]. Including it in the inner product, we found that the inner product is nonvanishing not only when all mesons have the same momenta, but also the inner product is nonvanishing between two states that differ by one meson with momentum k'. However, in this case, there is a suppression factor that is schematically $\sqrt{\lambda}\Delta_{k'B}$.

As the zero mode B is localized close to the kink, if the virtual meson has traveled far, then it will be disjoint from $\mathfrak{g}_B(x)$ and this term will cancel. However, a shape mode is bound to the kink and so cannot travel far. Therefore, one can expect a contribution to the inner product arising from virtual shape modes.

However, this contribution is suppressed by a factor of $\sqrt{\lambda}$, and so one must consider evolution $U_1(t)$ at order $O(\sqrt{\lambda})$. This evolution has been comprehensively studied in Refs. [31,33]. The conclusion is that, if the kink starts in its ground state, the only allowed process is the creation of two quanta. If both are continuum mesons, this process is called meson multiplication. If one is a shape mode, this is called Stokes scattering.

We thus conclude that Stokes scattering, included in $U_1(t)|t=0\rangle_0$, may, in principal, lead to a nonvanishing inner product with respect to a nonforward meson and so contribute to our process.

Of course, this cannot really happen, as the conservation of energy would imply $\omega_{k_2} = \omega_{k_1} - \omega_S$, which is the wrong energy for the recoil meson. However, in this appendix, we will try to show how this contribution vanishes.

2. Stokes scattering

Consider the interaction $H^{(1)\prime}$ from Eq. (3.17). It acts on meson 1 as in Eq. (3.20). At leading order, this leads to the final state

$$\begin{aligned} U_1(t)|t=0\rangle_0 &= -i\int_0^t d\tau_1 e^{-iH_2'(t-\tau_1)}H_3^{(1)\prime}e^{-iH_2'\tau_1}\int\frac{dk_1}{2\pi}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_0}|k_1\rangle_0 \\ &= -i\frac{\sqrt{\lambda}}{4}\int\frac{dk_1}{2\pi}\sum\!\!\!\!\!\!\int\frac{dk_2}{2\pi}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}e^{-i(\omega_{k_2}+\omega_{k'})t}\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2}-\omega_{k'})\tau_1} \\ &\quad\times\frac{V_{-k_1k_2k'}}{\omega_{k_1}}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2k'\rangle_0. \end{aligned} \tag{B1}$$

Stokes scattering corresponds to the case $k'=S$ and k_2 is a continuum mode. Of course, this expression is symmetric in k' and k_2 and so if $k_2=S$ then one can rename it k'. This freedom leads to a factor of 2.

Abusing our notation again, we will define $U_1(t)$ be the Stokes scattering part, which is equivalent to considering only Stokes terms in the definition of $H_3^{(1)\prime}$,

$$\begin{aligned} U_1(t)|t=0\rangle_0 &= -i\frac{\sqrt{\lambda}}{2}\int\frac{d^2k}{(2\pi)^2}e^{-i(\omega_{k_2}+\omega_S)t}\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2}-\omega_S)\tau_1} \\ &\quad\times\sum_S\frac{V_{-k_1k_2S}}{\omega_{k_1}}e^{-\sigma^2(k_1-k_0)^2-i(k_1-k_0)x_{\tau_1}}|k_2S\rangle_0. \end{aligned} \tag{B2}$$

Now, we make the usual approximation that ω_{k_1} in the denominator is ω_{k_0}, and again using the fact that $\mathfrak{g}_S(x)$ has compact support, we find

$$U_1(t)|t=0\rangle_0 = -i\frac{\sqrt{\lambda}}{2}\frac{\sqrt{\pi}}{2\pi\sigma}\int\frac{dk_2}{2\pi}\frac{V_{-k_0k_2S}}{\omega_{k_0}}e^{-i(\omega_{k_2}+\omega_S)t}\times\int_0^t d\tau_1 e^{-i(\omega_{k_0}-\omega_{k_2}-\omega_S)\tau_1}e^{-x_{\tau_1}^2/(4\sigma^2)}|k_2S\rangle_0$$
$$= -i\frac{\sqrt{\lambda}}{2}e^{-i\omega_{k_0}t_c}\int\frac{dk_2}{2\pi}\sum_S\frac{V_{-k_0k_2S}}{k_0}e^{-\sigma^2(\omega_{k_0}/k_0)^2(\omega_{k_0}-\omega_{k_2}-\omega_S)^2-i(\omega_{k_2}+\omega_S)(t-t_c)}|k_2S\rangle_0. \quad (B3)$$

3. Reduced inner product

To obtain the corresponding contribution to the probability, one needs to project this final state onto one meson final states, using the projection

$$\mathcal{P} = \frac{1}{\sqrt{Q_0}}\int\frac{dk_2}{2\pi}2\omega_{k_2}|k_2\rangle\langle k_2|. \quad (B4)$$

What is the reduced inner product of this final state with a single meson state $|k_2\rangle$?

There are two contributions. The first comes from the leading part $|k_2\rangle_0$ of $|k_2\rangle$. Using the master formula (4.14) of Ref. [38] with $\gamma^{02}(k_2S)=1/2$, one finds that the term contracting the shape mode and the zero mode is

$${}_0\langle k_2|U_1(t)|t=0\rangle_{0\,\mathrm{red}} = -i\frac{\sqrt{\lambda}}{8}e^{-i\omega_{k_0}t_c}\sum_S\frac{V_{-k_0k_2S}\Delta_{SB}}{k_0\omega_S\omega_{k_2}}e^{-\sigma^2(\omega_{k_0}/k_0)^2(\omega_{k_0}-\omega_{k_2}-\omega_S)^2-i(\omega_{k_2}+\omega_S)(t-t_c)}. \quad (B5)$$

Here we have remembered the factor of $1/(2\omega_{k_2})$ from $B_{k_2}^{\ddagger\dagger}$, which is built into the normalization of our states $|k_2\rangle_0$.

However, there are also contributions, at the same order, from the two-meson quantum corrections $|k_2\rangle_1$,

$$\gamma_{1k_2}^{02}(k_1',k_2') = -\frac{2\pi\delta(k_2'-k_2)}{4}\left(\Delta_{k_1'B}+\sqrt{\lambda Q_0}\frac{V_{\mathcal{I}k_1'}}{\omega_{k_1'}}\right)+\frac{\sqrt{\lambda Q_0}V_{-k_2k_1'k_2'}}{4\omega_{k_2}(\omega_{k_2}-\omega_{k_1'}-\omega_{k_2'})}$$
$$-\frac{2\pi\delta(k_1'-k_2)}{4}\left(\Delta_{k_2'B}+\sqrt{\lambda Q_0}\frac{V_{\mathcal{I}k_2'}}{\omega_{k_2'}}\right). \quad (B6)$$

Here $\gamma_{i\psi}^{mn}(k_1\cdots k_n)$ is the coefficient that arises when decomposing the state $|\psi\rangle$ into the basis $\phi_0^m|k_1\cdots k_n\rangle_0$ at order $O(\lambda^{i/2})$. It is defined to include a factor of $Q_0^{i/2}$ so that it contains no powers of the coupling λ.

We are interested in the case where k_1' or k_2' is a shape mode. At late times, the meson k_2 is far from the kink and so cannot interact with a shape mode. As a result, after all of the usual integrations, the $V_{-k_2k_1'k_2'}$ term will vanish.

Now k_2 is the momentum of the asymptotic meson, so it is by assumption not a shape mode, as we are calculating the amplitude to produce an asymptotic meson. Therefore, in the $\delta(k_2'-k_2)$ term, it must be that k_1' is the shape mode and similarly for the other δ term.

We need to sum over whether k_1' or k_2' is the shape mode. Now, remembering the factors of $1/(2\omega_S)$ and $1/(2\omega_{k_2})$ from the contractions of the two mesons, one can see that the Δ terms cancel half of (B5). The other half is canceled by the contribution from Eq. (6.5),

$${}_0\langle k_2|U_1(t)|t=0\rangle_{0\,\mathrm{red}} = i\frac{\sqrt{\lambda}}{16}e^{-i\omega_{k_0}t_c}\sum\!\!\!\!\!\!\int\frac{dk'}{2\pi}\frac{V_{-k_0k_2k'}\Delta_{-k'B}}{k_0\omega_{k'}\omega_{k_2}}\times e^{-\sigma^2(\omega_{k_0}/k_0)^2(\omega_{k_0}-\omega_{k_2}-\omega_{k'})^2-i(\omega_{k_2}+\omega_{k'})(t-t_c)}. \quad (B7)$$

This leaves the two tadpole terms, which are proportional $V_{\mathcal{I}S}$. They yield an inner product of

$${}_0\langle k_2|U_1(t)|t=0\rangle_{0\,\mathrm{red}} = i\frac{\lambda\sqrt{Q_0}}{8}e^{-i\omega_{k_0}t_c}\frac{V_{-k_0k_2S}V_{\mathcal{I}S}}{k_0\omega_S^2\omega_{k_2}}e^{-\sigma^2(\omega_{k_0}/k_0)^2(\omega_{k_0}-\omega_{k_2}-\omega_S)^2-i(\omega_{k_2}+\omega_S)(t-t_c)}. \quad (B8)$$

This term appears to be a disaster, as it contributes to the final probability with a final energy ω_{k_2} that is not close to the initial energy ω_{k_0}.

However, the term looks reminiscent of the tadpole interactions studied in Sec. III C. Indeed, the inner product of A_1, the first term in the parentheses in Eq. (3.29), with ${}_0\langle k_2|$, exactly cancels this term.

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