Quantum phase transition and absence of quadratic divergence in generalized quantum field theories

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In ordinary thermodynamics, around first-order phase transitions, the intensive parameters such as temperature and pressure are automatically fixed to the phase transition point when one controls the extensive parameters such as total volume and total energy. From the microscopic point of view, the extensive parameters are more fundamental than the intensive parameters. Analogously, in conventional quantum field theory (QFT), coupling constants (including masses) in the path integral correspond to intensive parameters in the partition function of the canonical formulation. Therefore, it is natural to expect that, in a more fundamental formulation of QFT, coupling constants are dynamically fixed *a posteriori*, just as the intensive parameter in the microcanonical formulation. Here, we demonstrate that the automatic tuning of the coupling constants is realized at a quantum phase transition point at zero temperature, even when the transition is of higher order, due to the Lorentzian nature of the path integral. This naturally provides a basic foundation for the multicritical point principle. As a concrete toy model for solving the Higgs hierarchy problem, we study how the mass parameter is fixed in the ϕ^4 theory at the one-loop level in the microcanonical or further generalized formulation of QFT. We find that there are two critical points for the renormalized mass: zero and of the order of ultraviolet cutoff. In the former, the Higgs mass is automatically tuned to be zero and thus its fine-tuning problem is solved. We also show that the quadratic divergence is absent in a more realistic two-scalar model that realizes the dimensional transmutation. Additionally, we explore the possibility of fixing quartic coupling in ϕ^4 theory and find that it can be fixed to a finite value.

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I. INTRODUCTION

By the discovery of the Higgs boson at LHC, it has been confirmed that the electroweak symmetry breaking is triggered by the Higgs mechanism. However, the question of why there is a huge hierarchy between the electroweak scale 10^2 GeV and the Planck scale 10^{18} GeV is not unveiled yet. In the Standard Model (SM), the electroweak scale is merely obtained by fine-tuning the mass parameter of the Higgs potential, which makes the Higgs boson mass sensitive to ultraviolet (UV) scales such as the grand unification or Planck scale. Although supersymmetry has been discussed as one of the most promising new physics scenarios beyond the SM because of the stabilization of the Higgs boson mass to be the electroweak scale, it does not explain the smallness of the electroweak scale, nor is it found near the expected TeV scale. In order to confront this fine-tuning problem, more radical and fundamental approaches beyond ordinary quantum field theory (QFT) appear to be required [1–14].

Nature might already give us some important clues to tackle this problem. Let us recall a basic notion of statistical mechanics: there are several different formulations depending on which parameters are used as control parameters, and they are equivalent in the thermodynamic limit $V \to \infty$. In particular, the most fundamental formulation is obtained with the microcanonical ensemble where all the extensive parameters, e.g., energy E,

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volume V, and number of particles N are chosen as control parameters, while intensive parameters, e.g., temperature T, pressure p, and chemical potential μ, are determined as functions of these extensive parameters in the thermodynamic limit. Here, the important point is that this correspondence is not injective when a system undergoes phase transitions. For example, in a first-order phase transition, the temperature stays at the critical temperature $T_{\rm cri}(E)$ until the system releases or absorbs all the latent heat, which means that the critical point spans the finite region of E in the microcanonical formulation.[1] More generally, with a finite probability, intensive parameters are fixed at the point where the extensive quantities become discontinuous.

In this paper, we explore a correspondence in QFT similar to that observed in statistical mechanics. Our objective is to address the fine-tuning problem by determining parameters in the canonical partition function of QFT from the microcanonical partition function, thereby eliminating the need for tuning. In essence, the fine-tuning problem is resolved when a finite region in the parameter space of microcanonical (or further generalized) QFT corresponds to a specific point in the canonical-QFT parameter space that appears fine-tuned. We also acknowledge various attempts to construct microcanonical QFT, as discussed in previous studies [1–3,11–14] and the references therein.

The concept of generalized QFT is both fascinating and intriguing; however, the actual fine-tuning of couplings remains unclear, necessitating further research. As a first step, we investigate the free scalar theory to explore how the bare mass-squared parameter $m_{\rm B}^2$ is fixed in the generalized QFT. We calculate the generalized partition function and find two critical points in the large volume limit: $m_{\rm B}^2 = 0$ and $\mathcal{O}(\Lambda^2)$, where Λ denotes the cutoff scale. In particular, while the latter corresponds to a saddle point of the vacuum energy and depends on the regularization schemes in general, the former corresponds to its discontinuity and does not depend on the regularization schemes. In this regard, one might find $m_{\rm B}^2 = 0$ physically more preferable, meaning that massless theory is naturally realized in the microcanonical (or further generalized) picture. This can be also interpreted as a theoretical explanation of the origin of the so-called “classical scale invariance” or “classical conformality” in the literature [15–30], which is also the basic assumption behind the Coleman-Weinberg mechanism [31].

We proceed to examine the ϕ^4 theory at the one-loop level and generalize it to a large-N model. In both cases, the mass term receives UV divergent corrections $\delta m_{\rm UV}^2 \sim \Lambda^2$, prompting us to investigate at which value the renormalized mass $m^2 := m_{\rm B}^2 + \delta m_{\rm UV}^2$ is fixed in the generalized QFT. A key distinction from the above free theory lies in the vacuum transition at $m^2 = 0$; for $m^2 > 0$, we observe a trivial vacuum expectation value (VEV) $\langle\phi\rangle = 0$, while it becomes nonzero for $m^2 < 0$. This behavior corresponds to the discontinuity of the (second) derivative of the vacuum energy at $m^2 = 0$, which remains a critical point in both cases. Although our analysis is limited to the one-loop level or the large-N limit, we anticipate that the conclusions will remain unaffected by higher-order corrections as long as the theory is in the perturbative region.

In our next nontrivial example, we examine a two-real-scalar model at the one-loop level. This model has been extensively studied in a phenomenological context [32–37] because it can realize the Coleman-Weinberg mechanism [31] under the assumption of classical conformality, i.e., $m^2 = 0$. However, in the generalized QFT, this is not an assumption but a point of discussion for determining where and how renormalized masses are fixed. To streamline our analysis, we assume $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry and focus on the coupling space where one real scalar ϕ can develop a nonzero VEV $\langle\phi\rangle \neq 0$ due to the radiative potential. We discover that the classical conformal point $m_\phi^2 = m_S^2 = 0$ is not a critical point in the generalized partition function. Instead, we identify another critical point with $m_\phi^2 \neq 0$ and $m_S^2 = 0$, which corresponds to a quantum first-order phase transition point. From a phenomenological perspective, this critical point may serve as an alternative possibility for a dimensional transmutation mechanism [36,37] compared to the conventional classical conformal point; see also Ref. [35] for other possible critical points without $\mathbb{Z}_2$ symmetry.

The organization of this paper is as follows: In Sec. II, we introduce the generalized QFT and explain how the fine-tuning of coupling constants can be automatically achieved within this framework. We also review the standard discussion on the equivalence between different ensembles in statistical mechanics to clarify the underlying concept. In Sec. III, we investigate the free scalar theory in the context of generalized QFT, focusing on how the bare mass-squared parameter is fixed in the large volume limit. In Sec. IV, we delve into a more nontrivial example by considering the ϕ^4 theory, discussing how the renormalized mass squared is fixed at the one-loop level, as well as in the large-N model. In Sec. V, we explore the possibility of automatically tuning the quartic coupling constant. In Sec. VI, we analyze the two-real-scalar model within a simple parameter space where only one scalar can develop a VEV. Finally, in Sec. VII, we summarize our result.

Throughout the paper, we work in natural units $\hbar = c = 1$ and employ the metric convention $\eta_{\mu\nu} = \mathrm{diag}(-, +, \cdots, +)$.

II. GENERALIZED QUANTUM FIELD THEORY

In this section, we briefly review the basics of statistical mechanics, introduce a generalized partition function in the

[1]Although the terminology “critical point” is often used to express the end point of a phase equilibrium curve, such as a vapor-liquid critical point, we do not use the term in this sense here, but use it in the sense that a phase transition of any order occurs.

microcanonical formulation of QFT, and illustrate how the tuning of the coupling constants becomes possible. See, e.g., Ref. [3] and Appendix D in Ref. [38] for other reviews.

A. Statistical mechanics

To clarify the idea, let us briefly recall statistical mechanics. When control parameters are temperature T and volume V, the system is described by the canonical partition function

$$Z_V(T) = e^{-F_V(T)/T} = \sum_n e^{-E_n/T}, \tag{1}$$

where E_n denotes the energy eigenvalue and $F_V(T) = -T \log Z_V(T)$ is the Helmholtz free energy.

On the other hand, we can also consider the microcanonical formulation where E is a control parameter instead of T. The corresponding partition function, the number of states, is given by

$$\Omega_V(E) = \Delta E \sum_n \delta(E_n - E) = e^{S_V(E)}, \tag{2}$$

where $S_V(E)$ denotes the entropy and ΔE denotes a sufficiently small energy interval whose effects on observables vanish in the thermodynamic limit $V \to \infty$. In the following, we omit the subscript V for simplicity.

The equivalence between the canonical and microcanonical formulations can be shown as

$$\begin{aligned} e^{-F(T)/T} &= \sum_n e^{-E_n/T} = \int dE \sum_n \delta(E - E_n) e^{-E/T} \\ &= \frac{1}{\Delta E} \int dE e^{S(E) - E/T} \\ &= \frac{V}{\Delta E} \int d\varepsilon e^{V(s(\varepsilon) - \varepsilon/T)}, \end{aligned} \tag{3}$$

where $s = S/V$ ($\varepsilon = E/V$) represents the entropy (energy) density. The integration is dominated by the saddle point in the thermodynamic limit $V \to \infty$ as

$$Z_V(T) = e^{-F(T)/T} \approx \frac{T(2\pi C)^{1/2}}{\Delta E} e^{V(s(\varepsilon_*) - \varepsilon_*/T)}, \tag{4}$$

$$\therefore \frac{F(T)}{T} = \frac{E_*}{T} - S(E_*) + \mathcal{O}(\log V), \tag{5}$$

where $C = \partial E/\partial T$ is the specific heat and $E_* = \varepsilon_* V$ is the solution of

$$\frac{\partial S}{\partial E} = \frac{1}{T}. \tag{6}$$

Equations (5) and (6) are nothing but the Legendre transformation between the free energy and the entropy in thermodynamics. It is also straightforward to check the equivalence of the ensemble averages of a general observable $\hat{x}$. The canonical ensemble average is given by

$$\langle \hat{x} \rangle_{\mathrm{can}} = \frac{1}{Z_V(T)} \sum_n x_n e^{-E_n/T}, \tag{7}$$

where $x_n = \langle E_n | \hat{x} | E_n \rangle$. This can be written as

$$\langle \hat{x} \rangle_T^{\mathrm{can}} = \frac{1}{Z_V(T)} \sum_n x_n \int dE e^{-E/T} \delta(E - E_n) \tag{8}$$

$$= \frac{1}{Z_V(T) \Delta E} \int dE e^{S(E) - E/T} \langle \hat{x} \rangle_E^{\mathrm{mic}}, \tag{9}$$

where

$$\langle \hat{x} \rangle_E^{\mathrm{mic}} = \frac{\sum_n x_n \delta(E - E_n)}{\Omega_V(E)} \tag{10}$$

is the microcanonical ensemble average. Because the integrand in Eq. (9) has a strong peak at $E = E_*$ as in Eq. (4), we obtain

$$\langle \hat{x} \rangle_T^{\mathrm{can}} = \langle \hat{x} \rangle_E^{\mathrm{mic}} \tag{11}$$

in the thermodynamic limit $V \to \infty$. In particular, during the first-order phase transition, T is equal to the critical temperature $T_c(E)$, but E varies between the two phases. This means that the coexisting phases are described in the finite parameter region of the microcanonical scheme. In this sense, a fine-tuning to a first-order phase transition point $T = T_c(E)$ is automatically realized as long as E is the finite region. The correspondence is summarized in Table I.

B. Microcanonical formulation of QFT

Now let us return to QFT. Conventionally, we start from the "canonical" partition function with the bare coupling constants $\{\lambda\} \coloneqq \{\lambda_j\}_{j=0,1,2,\ldots}$,

TABLE I. Relation between canonical and microcanonical formulations in statistical mechanics.

	Parameter	Partition function	Thermodynamic function
Canonical	T	$Z(T) = \sum_n e^{-E_n/T}$	$F(T) = -T \log Z(T)$
Microcanonical	E	$\Omega(E) = \Delta E \sum_n \delta(E_n - E)$	$S(E) = \log \Omega(E)$

$$Z(\{\lambda\}) = \int \mathcal{D}\phi \exp\left(i\sum_j \lambda_j O_j[\phi]\right), \tag{12}$$

where $O_j[\phi]$ $(j = 0, 1, 2, \ldots)$ denotes a spacetime integral of a local operator such as $\int d^d x \frac{1}{2}(\partial_\mu \phi(x))^2$, $\int d^d x \frac{1}{2}(\phi(x))^2$, $\int d^d x \frac{1}{4!}(\phi(x))^4$, etc. After the renormalization procedure, we can calculate physical observables finitely as functions of (renormalized) couplings. However, the problem is that there is no principle to pick up specific values of $\{\lambda\}$ theoretically, and this is the very origin of the fine-tuning problems. Here the analogy between QFT and statistical mechanics comes into play: What if we start from the "microcanonical" picture in QFT?

The number of states $\Omega(E)$ in statistical mechanics can naturally be promoted to the partition function in QFT as

$$\Omega(\{A\}) = \int \mathcal{D}\phi \prod_j \delta(O_j[\phi] - A_j), \tag{13}$$

where we write $\{A\} \coloneqq \{A_j\}_{j=0,1,2,\ldots}$ and each A_j is an "extensive parameter" corresponding to $O_j[\phi]$, which is proportional to the spacetime volume V_d. By using the Fourier transform of the δ function, Eq. (14) can be written as

$$\begin{aligned}\Omega(\{A\}) &= \int \left(\prod_j d\lambda_j\right) \int \mathcal{D}\phi \exp\left(i\sum_l \lambda_l (O_l[\phi] - A_l)\right) \\ &= \int \left(\prod_j d\lambda_j\right) Z(\{\lambda\}) e^{-i\sum_l \lambda_l A_l}. \end{aligned} \tag{14}$$

Now, one can see that we have an ensemble average of various coupling constants and their weight is proportional to the canonical partition function (12).

C. Further generalized QFT

The essence of the above discussion is the Fourier transform of the δ function[2]

$$\delta(\{x\}) = \int_{-\infty}^{\infty} \left(\prod_i \frac{d\lambda_j}{2\pi} e^{i\sum_j \lambda_j x_j}\right). \tag{15}$$

We can generalize it to an arbitrary function,

$$W(\{x\}) = \int \left(\prod_j d\lambda_j e^{i\lambda_j x_j}\right) \omega(\{\lambda\}). \tag{16}$$

Here we assume that the function $W(\{x\})$ does not contain any extensive parameters and consider the following generalized field theory [3]:

[2]Here, it is understood that $\{x\} \coloneqq \{x_j\}_{j=0,1,2,\ldots}$.

$$\begin{aligned}\Omega(\{A\}) &= \int \mathcal{D}\phi W(\{O[\phi] - A\}) \\ &= \int \left(\prod_j d\lambda_j\right) Z(\{\lambda\}) e^{-i\sum_l \lambda_l A_l} \omega(\{\lambda\}). \end{aligned} \tag{17}$$

If there exists a strong peak $\{\lambda^*\}$ of $Z(\{\lambda\})$ in the infinite volume limit $V \to \infty$, the generalized QFT is equivalent to the ordinary canonical QFT whose (bare) coupling constants are fixed at $\{\lambda^*\}$. In other words, the fine-tuning of coupling constants is automatically realized in the generalized partition function. We will see that only the behavior of the canonical partition function $Z(\{\lambda\})$ matters for the realization of the Higgs fine-tuning, regardless of whether it is in the microcanonical QFT (14) or in the generalized QFT (17). See Table II for the summary of naive correspondence. In the following sections, we will verify this fine-tuning mechanism for the (bare) mass term m_{B}^2 in scalar field theory.

III. FIXING MASS IN FREE SCALAR THEORY

We study the free scalar theory in the generalized QFT and show how the bare mass parameter is fixed.

A. Partition function of free scalar theory

We consider the free scalar theory in the d-dimensional spacetime,

$$S_0 = -\int d^d x \frac{1}{2}(\partial_\mu \phi)^2. \tag{18}$$

Here, we introduce the bare mass term according to the generalized QFT, while leaving the kinetic term (18) in the ordinary canonical way.[3] Then the generalized partition function is defined by

[3]One may treat the kinetic term in the framework of generalized QFT as

$$\int \mathcal{D}\phi W(S_0) \cdots = \int_{-\infty}^{\infty} \frac{dz}{2\pi} \int \mathcal{D}\phi \omega(z) e^{izS_0} \cdots.$$

There are several possibilities for $\omega(z)$. If $\omega(z)$ has support only for positive values of z, then we can always take the canonical form as in Eq. (19) by the field redefinition. If $\omega(z)$ has support only for negative values of z, then we can take the coefficient of the kinetic term to be -1 by the field redefinition, and such a theory would not have a ground state and lead to instability. If $\omega(z)$ has support at $z = 0$, such a scalar field should be regarded as an auxiliary field, which can be eliminated by using the equation of motion. If $\omega(z)$ has support for both positive and negative values of z, we need to compare the partition functions of both the contributions. Throughout this paper, we consider the first case, i.e., $\omega(z)$ having support only for positive values of z. Note also that, after the field redefinition, z appears in the coupling constants so that the case $z = 0$ can be regarded as a limit where the coupling constants are large.

TABLE II. Naive correspondence in QFT.

	Parameter	Partition function	Generating function
Canonical QFT	λ	$Z(\lambda) = \int \mathcal{D}\phi e^{i\lambda O[\phi]}$	$F(\lambda) = -i^{-1} \log Z(\lambda)$
Microcanonical QFT	A	$\Omega(A) = \int \mathcal{D}\phi\delta(O[\phi] - A)$	$S(A) = \log \Omega(A)$
Generalized QFT	A	$\Omega(A) = \int \mathcal{D}\phi W(O[\phi] - A)$	$S(A) = \log \Omega(A)$

$$\Omega(A) \coloneqq \int \mathcal{D}\phi e^{iS_0} W\left(A - \frac{1}{2}\int d^d x\phi^2(x)\right) \tag{19}$$

$$= \int_{-\infty}^{\infty} \frac{dm^2}{2\pi} \int \mathcal{D}\phi\omega(m^2) \times \exp\left\{i\left[S_0 + m^2\left(A - \frac{1}{2}\int d^d x\phi^2(x)\right)\right]\right\} \tag{20}$$

$$= \int_{-\infty}^{\infty} \frac{dm^2}{2\pi} \omega(m^2) e^{im^2 A} Z(m^2), \tag{21}$$

where $\omega(m^2)$ is the Fourier transform of $W(x)$ and $Z(m^2)$ denotes the ordinary canonical partition function

$$Z(m^2) = \int \mathcal{D}\phi \exp\left[i\left(S_0 - \frac{m^2}{2}\int d^d x\phi^2(x)\right)\right]. \tag{22}$$

Note that, since we are studying the free theory here, there is no distinction between bare mass and renormalized mass. We can now perform the path integral as

$$\Omega(A) \propto \int_{-\infty}^{\infty} \frac{dm^2}{2\pi} \omega(m^2) \times \exp\left[im^2 A - \frac{1}{2}\mathrm{tr}\log\left[i(-\Box + m^2 - i\varepsilon)\right]\right] \tag{23}$$

$$= \int_{-\infty}^{\infty} \frac{dm^2}{2\pi} \omega(m^2) \exp\left[iV_d(am^2 - F(m^2))\right], \tag{24}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, $a = A/V_d$, $i\varepsilon$ is Feynman's prescription, and

$$F(m^2) = \frac{1}{2}\int \frac{d^d p_{\mathrm{E}}}{(2\pi)^d} \log(p_{\mathrm{E}}^2 + m^2 - i\varepsilon) + \mathrm{const}, \tag{25}$$

with p_{E}^μ being the Euclidean momentum. This is apparently UV divergent and we need a regularization.

B. Mass squared in free scalar theory

To discuss how the mass is tuned, we employ cutoff and dimensional regularizations.

1. Cutoff regularization

First, let us study the cutoff regularization. As a function of m^2, the integrand in Eq. (24) has a branch cut at

$$-\Lambda^2 + i\varepsilon \leq m^2 \leq i\varepsilon, \tag{26}$$

and the integration line $-\infty < m^2 < +\infty$ is located below this branch cut, as shown in the left panel in Fig. 1. Then, Eq. (25) can be evaluated as

$$F(m^2) = \frac{S_{d-1}}{2(2\pi)^d} \int_0^\Lambda dp_{\mathrm{E}} p_{\mathrm{E}}^{d-1} \log(p_{\mathrm{E}}^2 + m^2 - i\varepsilon) \tag{27}$$

$$= \frac{S_{d-1}\Lambda^d}{4(2\pi)^d} \int_0^1 dx x^{\frac{d}{2}-1} \log(x + \xi - i\varepsilon), \qquad \xi = m^2/\Lambda^2 \tag{28}$$

$$= \frac{S_{d-1}\Lambda^d}{4(2\pi)^d} \times \begin{cases} \int_0^1 dx x^{\frac{d}{2}-1}\log(x+\xi) & \text{for } \xi > 0 \\ \int_0^1 dx x^{\frac{d}{2}-1}\log|x+\xi| - i\frac{2\pi}{d}(-\xi)^{\frac{d}{2}} & \text{for } 0 > \xi > -1 \\ \int_0^1 dx x^{\frac{d}{2}-1}\log|x+\xi| - i\frac{2\pi}{d} & \text{for } -1 > \xi \end{cases}, \tag{29}$$

where S_{d-1} denotes the area of a $d-1$-dimensional sphere. The negative imaginary part can be interpreted as the instability of vacuum.

We see that the function $F(m^2)$ contains the imaginary part for $m^2 < 0$, which gives a large suppression in the partition function when V_d is large. Qualitatively, the contribution from $m^2 < 0$ is

$$\int_{-\infty}^0 d\xi e^{-V_d\Lambda^d(-\xi)^{d/2}} = \mathcal{O}\left(\frac{1}{(V_d\Lambda^d)^{\frac{2}{d}}}\right). \tag{30}$$

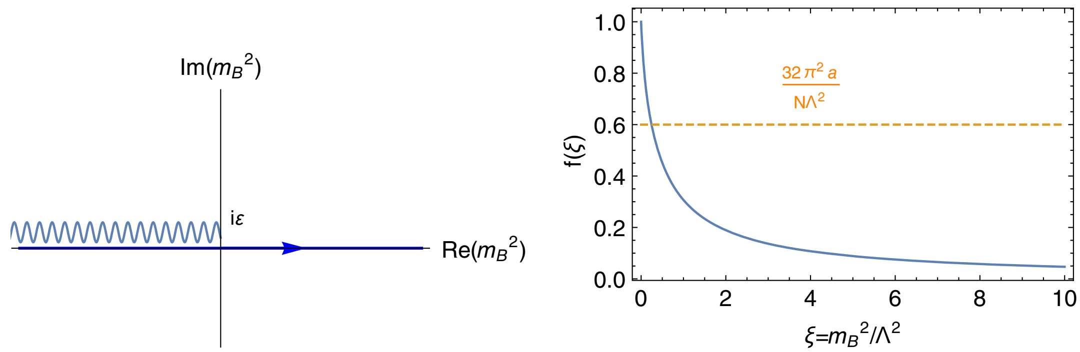

FIG. 1. Left: integration line. Right: $f(\xi)$ (blue) and $32\pi^2 a/(N\Lambda^2)$ (orange).

The exponential suppression in the large volume limit is due to the above-mentioned negative imaginary part.

On the other hand, there is no such suppression for $m^2 > 0$, and the integrand is a rapidly oscillating function of m^2. In this case, the saddle point can exist at the point determined by

$$a = \frac{dF}{dm^2} = \frac{S_{d-1}\Lambda^{d-2}}{4(2\pi)^d}\int_0^1 dx \frac{x^{\frac{d}{2}-1}}{x+\xi}. \tag{31}$$

When $d = 4$, this equation becomes

$$a = \frac{S_3\Lambda^2}{4(2\pi)^4}\left(1 - \xi\log\frac{1+\xi}{\xi}\right) =: \frac{\Lambda^2}{32\pi^2}f(\xi), \tag{32}$$

where $f(\xi)$ is a monotonic function satisfying $f(0) = 1$ and $f(\infty) = 0$; see the right panel in Fig. 1. Thus, there can be a saddle point when $a/\Lambda^2 \leq (32\pi^2)^{-1}$, while the exponent $am^2 - F(m^2)$ becomes a monotonic function when $a/\Lambda^2 > (32\pi^2)^{-1}$.

2. Dimensional regularization

Let us also calculate the partition function in the dimensional regularization. We define $\epsilon = 4 - d$. In this case, the free energy is calculated as

$$\begin{aligned}F(m^2) &= -\frac{(m^2)^2}{2(4\pi)^2}\left(\frac{1}{\epsilon} - \frac{\gamma}{2} + \frac{3}{4} + \frac{1}{2}\log 4\pi - \frac{1}{2}\log\left(\frac{m^2 - i\varepsilon}{\mu^2}\right)\right)\\ &=: -\frac{(m^2)^2}{2(4\pi)^2}\left(c_{\overline{\mathrm{MS}}} - \frac{1}{2}\log\left(\frac{m^2 - i\varepsilon}{\mu^2 e^{3/2}}\right)\right),\end{aligned} \tag{33}$$

where μ is the renormalization scale and $c_{\overline{\mathrm{MS}}}$ contains both the finite and divergent terms.

For $m^2 \leq 0$, we again have the imaginary part from the logarithmic term, and the partition function is highly suppressed. On the other hand, for $m^2 > 0$, the saddle point is determined by

$$a = \frac{dF}{dm^2} = \frac{m^2}{32\pi^2}\log\left(\frac{m^2}{\mu^2 e^{1+2c_{\overline{\mathrm{MS}}}}}\right), \tag{34}$$

which always has a solution, unlike the cutoff case. If we identify $\mu e^{c_{\overline{\mathrm{MS}}}}$ as the cutoff scale, the location of the saddle points is

$$m^2 \sim \Lambda^2 \sim \mu^2 e^{2c_{\overline{\mathrm{MS}}}} \tag{35}$$

in both of the regularizations.

3. Mass tuning in free scalar theory

As we have seen, the existence of a saddle point depends on the regularization scheme, but in any case, $m^2 = 0$ holds over a wide range of the parameter space. First, if there is no saddle point, the m^2 integration is dominated by the boundary $m^2 = 0$ by the mathematical formula (A2) for any smooth weight function $\omega(m^2)$. On the other hand, if there is a saddle point, $am^2 - F(m^2)$ is monotonic below the saddle point $m^2 \lesssim \Lambda^2$. Thus, when the weight function $\omega(m^2)$ has a finite support for $m_{\mathrm{B}}^2 \lesssim \Lambda^2$, the free energy $am^2 - F(m^2)$ is monotonic, and the boundary $m^2 = 0$ is always dominant due to the mathematical formula (A2). In conclusion, $m^2 = 0$ appears to be a unique critical point that is physically reasonable in the generalized partition function (24).

C. Equivalence in large volume limit

Finally, let us confirm the equivalence between generalized QFT and canonical QFT in the large volume limit. When $A \neq 0$, the first derivative of $m^2 A/V_d - F(m^2)$ is continuous and nonzero at $m^2 = 0$, while the second derivative is discontinuous. Then, we can use the mathematical formula (A6) and $\Omega(A)$ is evaluated as

$$\begin{aligned}\lim_{V_d\to\infty}\Omega(A) &= \lim_{V_d\to\infty}\int_{-\infty}^{\infty}\frac{dm^2}{2\pi}\omega(m^2)e^{im^2 A}Z(m^2)\\ &= \mathcal{N}\lim_{V_d\to\infty}\frac{Z(m^2=0)}{V_d^2},\end{aligned} \tag{36}$$

where $\mathcal{N}$ is an unimportant numerical factor. That is,

$$\lim_{V_d\to\infty}\log\Omega(A) = \lim_{V_d\to\infty}\log Z(m^2=0) + \mathcal{O}(\log V_d). \tag{37}$$

We apparently see the equivalence in the large volume limit.

We can also check the equivalence of correlation functions. We introduce a source term as

$$\begin{aligned}\Omega[A;J] &= \int \mathcal{D}\phi e^{iS_0 - i\int d^dx J(x)\phi(x)} W\left(A - \frac{1}{2}\int d^dx\phi^2(x)\right)\\ &= \int_{-\infty}^{\infty}\frac{dm^2}{2\pi}\omega(m^2)e^{iV_d(m^2a - F(m^2))}\exp\left(\frac{i}{2}\int d^dx J(x)(-\Box + m^2 - i\epsilon)^{-1}J(x)\right)\\ &= \int_{-\infty}^{\infty}\frac{dm^2}{2\pi}\omega(m^2)e^{iAm^2}Z(m^2)G(J;m^2).\end{aligned} \tag{38}$$

As long as $J(x)$ is a finite supported function, i.e., when it has no volume dependence, the factor $G(J;m^2)$ does not have exponentially large volume dependence, and the m^2 integral in Eq. (38) is dominated by the critical point of $Z(m^2)$. Namely, we have

$$\lim_{V_d\to\infty}\Omega[A;J] = \text{const} \times G(J;m^2=0). \tag{39}$$

Then, by taking the functional derivatives with respect to $J(x)$, we obtain

$$\langle T\{\phi(x_1)\cdots\phi(x_n)\}\rangle_{\text{mic}} = \langle T\{\phi(x_1)\cdots\phi(x_n)\}\rangle_{\text{can}}|_{m^2=0}, \tag{40}$$

which corresponds to Eq. (11) in statistical mechanics.

IV. FIXING MASS IN ϕ^4 THEORY

In this section, we analyze the ϕ^4 theory in the scope of generalized QFT. In particular, we study how the mass term is fixed by studying its critical point at the one-loop level and also in the large-N extended model. Our investigation reveals that the renormalized mass settles at zero in the one-loop analysis as well as in the large-N model, due to the discontinuity of the derivative of the vacuum energy.

In this section, we assume that the quartic coupling is prefixed at a certain value, as in the ordinary QFT. The possibility of its tuning will be discussed in the next section.

A. Mass squared at one-loop level

We first introduce a bare action without the bare mass term,

$$S_{\mathrm{B}}[\phi] = \int d^dx\left(-\frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda_{\mathrm{B}}}{4!}\phi^4 - \Lambda_{\mathrm{B}}\right), \tag{41}$$

where we have also included the bare cosmological constant term for later convenience. For simplicity, we have imposed the $\mathbb{Z}_2$ symmetry $\phi\to-\phi$, which leaves only the quadratic and quartic terms in the renormalizable potential.

Our starting point is the generalized partition function $\Omega(A)$, which can be expressed in terms of the canonical partition function $Z(m_{\mathrm{B}}^2)$ by the same procedure as in Sec. III A,

$$\Omega(A) = \int_{-\infty}^{\infty}\frac{dm_{\mathrm{B}}^2}{2\pi}\omega(m_{\mathrm{B}}^2)e^{im_{\mathrm{B}}^2A}Z(m_{\mathrm{B}}^2), \tag{42}$$

where

$$Z(m_{\mathrm{B}}^2) = \int\mathcal{D}\phi e^{iS[\phi]}, \qquad S[\phi] = S_{\mathrm{B}}[\phi] - \frac{m_{\mathrm{B}}^2}{2}\int d^dx\phi^2. \tag{43}$$

The canonical partition function Z can be obtained from the ordinary effective action $\Gamma[\phi]$ via $Z = e^{i\min_\phi\Gamma[\phi]}$. At the one-loop level, we obtain

$$\begin{aligned}\Gamma[\phi] &= S[\phi] - \frac{1}{2i}\mathrm{tr}\log\left(-\Box + m_{\mathrm{B}}^2 + \frac{\lambda_{\mathrm{B}}}{2}\phi^2 - i\varepsilon\right)\\ &= S[\phi] + \frac{V_d}{2}\int\frac{d^dp_{\mathrm{E}}}{(2\pi)^d}\log\left(p_{\mathrm{E}}^2 + m_{\mathrm{B}}^2 + \frac{\lambda_{\mathrm{B}}}{2}\phi^2 - i\varepsilon\right)\\ &= S[\phi] + \frac{V_d}{2}\frac{(M_\phi^2(\phi))^2}{(4\pi)^2}\left[c_{\overline{\mathrm{MS}}} - \frac{1}{2}\log\left(\frac{M_\phi^2(\phi) - i\varepsilon}{\mu^2e^{3/2}}\right)\right],\end{aligned} \tag{44}$$

where $M_\phi^2(\phi) = m_{\mathrm{B}}^2 + \frac{\lambda_{\mathrm{B}}}{2}\phi^2$.

Now the one-loop effective potential is given by

$$
\begin{aligned}
V_{\mathrm{eff}}(\phi) &:= -\frac{\Gamma[\phi]}{V_d}\bigg|_{\phi=\mathrm{const}} = \Lambda_{\mathrm{B}} + \frac{m_{\mathrm{B}}^2}{2}\phi^2 + \frac{\lambda_{\mathrm{B}}}{4!}\phi^4 - \frac{(M_\phi^2(\phi))^2}{32\pi^2}\left[c_{\overline{\mathrm{MS}}} - \frac{1}{2}\log\left(\frac{M_\phi^2(\phi) - i\varepsilon}{\mu^2 e^{3/2}}\right)\right] \\
&= \Lambda_{\mathrm{R}} + \frac{m_{\mathrm{R}}^2}{2}\phi^2 + \frac{\lambda_{\mathrm{R}}}{4!}\phi^4 + \frac{(M_\phi^2(\phi))^2}{64\pi^2}\log\left(\frac{M_\phi^2(\phi) - i\varepsilon}{\mu^2 e^{3/2}}\right),
\end{aligned} \tag{45}
$$

where

$$
\Lambda_{\mathrm{R}} = \Lambda_{\mathrm{B}} - c_{\overline{\mathrm{MS}}}\frac{m_{\mathrm{B}}^4}{32\pi^2}, \qquad m_{\mathrm{R}}^2 = m_{\mathrm{B}}^2 - \frac{c_{\overline{\mathrm{MS}}}}{16\pi^2}\lambda_{\mathrm{B}} m_{\mathrm{B}}^2, \qquad \lambda_{\mathrm{R}} = \lambda_{\mathrm{B}} - \frac{3c_{\overline{\mathrm{MS}}}}{16\pi^2}\lambda_{\mathrm{B}}^2 \tag{46}
$$

denote the renormalized parameters. We can regard these renormalized couplings as our parameters instead of the bare couplings $(m_{\mathrm{B}}^2, \lambda_{\mathrm{B}})$.

As a consistency check, let us consider the free theory limit: $\lambda_{\mathrm{B}} = \lambda_{\mathrm{R}} = 0$. In this case, there is no difference between the bare mass and the renormalized mass, and the effective potential is exactly given by

$$
V_{\mathrm{eff}}(\phi)|_{\lambda_{\mathrm{B}}=0} = \Lambda_{\mathrm{R}} + \frac{m_{\mathrm{B}}^2}{2}\phi^2 + \frac{m_{\mathrm{B}}^4}{64\pi^2}\log\left(\frac{m_{\mathrm{B}}^2 - i\varepsilon}{\mu^2 e^{3/2}}\right), \tag{47}
$$

which has a trivial minimum at $\phi = 0$ and the corresponding vacuum energy is

$$
V_{\min} = \Lambda_{\mathrm{B}} - \frac{(m_{\mathrm{B}}^2)^2}{32\pi^2}\left(c_{\overline{\mathrm{MS}}} - \frac{1}{2}\log\left(\frac{m_{\mathrm{B}}^2 - i\varepsilon}{\mu^2 e^{3/2}}\right)\right), \tag{48}
$$

which is nothing but Eq. (33).

Let us come back to the interacting scalar theory. We assume $\lambda_{\mathrm{R}} > 0$ to ensure the stability of the effective potential. As usual, we can obtain the renormalization group (RG) improved effective potential by choosing the renormalization scale μ appropriately. For $m_{\mathrm{R}}^2 \geq 0$, the VEV is trivial $v = 0$, and we take $\mu = m_R e^{-3/4}$, which will give a simple expression of the vacuum energy [as in the first line in Eq. (51) below]. On the other hand, for $m_{\mathrm{R}}^2 < 0$, the field ϕ develops a nonzero VEV, which is determined by

$$
\begin{aligned}
&\frac{\partial V_{\mathrm{eff}}(\phi)}{\partial\phi} = 0 \\
\Leftrightarrow\quad & m_{\mathrm{R}}^2 v + \frac{\lambda_{\mathrm{R}}}{6}v^3 + \frac{\lambda_{\mathrm{R}} v}{32\pi^2}\left(M_\phi^2(v)\log\left(\frac{M_\phi^2(v)}{\mu^2 e}\right)\right) = 0.
\end{aligned} \tag{49}
$$

By choosing the renormalization scale at $\mu^2 = M_\phi^2(v)e^{-1}$, the VEV is given by

$$
v^2 = -\frac{6m_{\mathrm{R}}^2}{\lambda_{\mathrm{R}}}. \tag{50}
$$

Now the vacuum energy as a function of m_{R}^2 is given by

$$
V_{\min}(m_{\mathrm{R}}^2) = \Lambda_{\mathrm{B}} - \frac{c_{\overline{\mathrm{MS}}}}{32\pi^2}(m_{\mathrm{R}}^2)^2 + \begin{cases} 0 & \text{for } m_{\mathrm{R}}^2 \geq 0 \\ -\frac{3(m_{\mathrm{R}}^2)^2}{2\lambda_{\mathrm{R}}} - \frac{(2m_{\mathrm{R}}^2)^2}{128\pi^2} & \text{for } m_{\mathrm{R}}^2 < 0 \end{cases}, \tag{51}
$$

which shows that the second derivative of $V_{\min}(m_{\mathrm{R}}^2)$ is discontinuous at $m_{\mathrm{R}}^2 = 0$ as

$$
\frac{\partial^2 V_{\min}}{\partial(m_{\mathrm{R}}^2)^2}\bigg|_{m_{\mathrm{R}}^2=0+} = -\frac{c_{\overline{\mathrm{MS}}}}{16\pi^2}, \qquad \frac{\partial^2 V_{\min}}{\partial(m_{\mathrm{R}}^2)^2}\bigg|_{m_{\mathrm{R}}^2=0-} \simeq -\frac{c_{\overline{\mathrm{MS}}}}{16\pi^2} - \frac{3}{\lambda_{\mathrm{R}}} + \frac{1}{16\pi^2}. \tag{52}
$$

Note that the first derivative of $m_{\mathrm{B}}^2 A/V_d - V_{\min}(m_{\mathrm{R}}^2)$ is continuous and nonzero at $m_{\mathrm{R}}^2 = 0$ as long as $A \neq 0$, which means that we can use the mathematical formula (A6) in Appendix A. Then, the partition function is evaluated as

$$
\begin{aligned}
\lim_{V_d\to\infty} e^{im_B^2 A + \log Z(m_{\mathrm{R}}^2)} &= \lim_{V_d\to\infty} e^{im_B^2 A - iV_d V_{\min}(m_{\mathrm{R}}^2)} \\
&\propto \lim_{V_d\to\infty}\frac{e^{-iV_d V_{\min}(m_{\mathrm{R}}^2=0)}}{V_d^2}\delta(m_{\mathrm{R}}^2),
\end{aligned} \tag{53}
$$

which leads to

$$
\begin{aligned}
\lim_{V_d\to\infty}\Omega(A) &= \int_{-\infty}^{\infty}\frac{dm_{\mathrm{R}}^2}{2\pi}\left(\frac{\partial m_{\mathrm{B}}^2}{\partial m_{\mathrm{R}}^2}\right)\omega(m_{\mathrm{B}}^2)e^{im_{\mathrm{B}}^2 A + \log Z(m_{\mathrm{R}}^2)} \\
&= \frac{\mathcal{N}}{V_d^2}e^{\log Z(m_{\mathrm{R}}^2=0)},
\end{aligned} \tag{54}
$$

where $\mathcal{N}$ is an unimportant factor. That is,

$$
\lim_{V_d\to\infty}\log\Omega(A) = \lim_{V_d\to\infty}\log Z(m_{\mathrm{R}}^2 = 0) + \mathcal{O}(\log V_d). \tag{55}
$$

We see that the equivalence between two formulations still holds in the ϕ^4 theory at the one-loop level in the large volume limit. In particular, the renormalized mass parameter m_{R}^2 is fixed at zero because of the discontinuity of the second derivatives of the vacuum energy.[4] This result

[4]Precisely speaking, there is also a saddle point solution $m_{\mathrm{R}}^2 \sim -\frac{A}{V_d c_{\overline{\mathrm{MS}}}}$, the value of which depends on the regularization schemes and the way the large volume limit $A/(c_{\overline{\mathrm{MS}}}V_d) = $ fixed is taken. On the other hand, the critical point $m_{\mathrm{R}}^2 = 0$ has no such uncertainty and is uniquely determined as in the free case.

implies that the quadratic divergence problem is absent in the microcanonical formulation. Moreover, in the simple ϕ^4 theory, the conclusion is not affected by higher-loop corrections as long as $\lambda_{\rm R} \lesssim 1$. However, the situation can be different in more general theories with more than one field because another mass scale can be radiatively generated by other fields. See Sec. VI for a concrete two-scalar model.

B. Mass squared in large-N model

In this section, we turn to the $O(N+1)$ symmetric scalar theory with ϕ_i ($i = 0, 1, \ldots, N$), and discuss how the mass parameter is fixed in the generalized partition function in the large-N limit. The bare action is given by

$$S_{\rm B} = \int d^d x \left(-\frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{\lambda_{\rm B}}{4!}(\phi_i^2)^2 \right), \tag{56}$$

where we have omitted the bare mass term as before and apply the Einstein summation convention for the field index i.

The generalized partition function now becomes

$$\begin{aligned}\Omega(A) &= \int_{-\infty}^{\infty} \frac{dm_{\rm B}^2}{2\pi} \omega(m_{\rm B}^2) e^{im_{\rm B}^2 NA} \int \mathcal{D}\phi e^{iS} \\ &= \int_{-\infty}^{\infty} \frac{dm_{\rm B}^2}{2\pi} \omega(m_{\rm B}^2) e^{im_{\rm B}^2 NA} Z(m_{\rm B}^2),\end{aligned} \tag{57}$$

where

$$S = \int d^d x \left(-\frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2} m_{\rm B}^2 \phi_i^2 - \frac{\lambda_{\rm B}}{4!}(\phi_i^2)^2 \right). \tag{58}$$

We can generally separate the original field ϕ_i as

$$\phi_i = \begin{cases} \sqrt{N} s & i = 0 \\ \pi_i & i = 1, 2, \ldots, N \end{cases}, \tag{59}$$

where s is the field that may acquire a VEV. The Lagrangian now becomes

$$\begin{aligned}\mathcal{L} = {}& N\left(-\frac{1}{2}(\partial_\mu)^2 - \frac{m_{\rm B}^2}{2} s^2 \right) - \frac{1}{2}(\partial_\mu \pi_i)^2 \\ & - \frac{m_{\rm B}^2}{2}\pi_i^2 - \frac{\lambda_{\rm B}}{4!}(Ns^2 + \pi_i^2)^2.\end{aligned} \tag{60}$$

We can introduce an auxiliary scalar field c in the Lagrangian such that the partition function does not change after performing the path integral over c,

$$\begin{aligned}\mathcal{L} = {}& N\left(-\frac{1}{2}(\partial_\mu)^2 - \frac{1}{2}(m_{\rm B}^2 + c)s^2 \right) \\ & - \frac{1}{2}(\partial_\mu \pi_i)^2 - \frac{1}{2}(m_{\rm B}^2 + c)\pi_i^2 + \frac{3}{2\lambda_{\rm B}} c^2.\end{aligned} \tag{61}$$

By performing the path integral of π_i, the partition function is given as

$$\begin{aligned}Z(m_{\rm B}^2) &= \int \mathcal{D}\phi e^{iS} \\ &= \int \mathcal{D}c \int \mathcal{D}s \exp\left[iN \int d^d x \left(-\frac{1}{2}(\partial_\mu)^2 - \frac{1}{2}(m_{\rm B}^2 + c)s^2 + \frac{1}{4\tilde{\lambda}_{\rm B}} c^2 - \frac{1}{2}\int \frac{d^d p_{\rm E}}{(2\pi)^d} \log(p_{\rm E}^2 + m_{\rm B}^2 + c - i\epsilon) \right) \right],\end{aligned} \tag{62}$$

where $\tilde{\lambda}_{\rm B} \coloneqq \lambda_{\rm B} N/6$. The variation of c gives

$$-s^2 + \frac{1}{\tilde{\lambda}_{\rm B}} c - \int \frac{d^d p_{\rm E}}{(2\pi)^d} \frac{1}{p_{\rm E}^2 + m_{\rm B}^2 + c - i\epsilon} = 0, \tag{63}$$

while the variation of s gives

$$(-\Box + m_{\rm B}^2 + c)s = 0. \tag{64}$$

As long as we focus on the ground state, we can omit $\Box s$. For a given value of $\tilde{\lambda}_{\rm B}$, the parameter space of $m_{\rm B}^2$ is divided into two regions as shown in Fig. 2, which correspond to the broken phase (blue) and the unbroken phase (uncolored), respectively. Now let us discuss the behavior of the partition function in each region.

1. Broken phase

When $s \neq 0$, we have $m_{\rm B}^2 + c = 0$. Equation (63) then becomes

$$s^2 = -\frac{1}{\tilde{\lambda}_{\rm B}} m_{\rm B}^2 - \int^\Lambda \frac{d^d p_{\rm E}}{(2\pi)^d} \frac{1}{p_{\rm E}^2} > 0, \tag{65}$$

which indicates the parameter space of broken phase. This is shown by the blue region in Fig. 2. We represent the critical value of $m_{\rm B}^2$ as $-m_\Lambda^2$. For $d = 4$, it is

$$m_\Lambda^2 = \frac{\tilde{\lambda}_{\rm B} \Lambda^2}{16\pi^2}. \tag{66}$$

For the parameters obeying Eq. (65), the canonical partition function (62) becomes very simple in the large-N analysis. In fact,

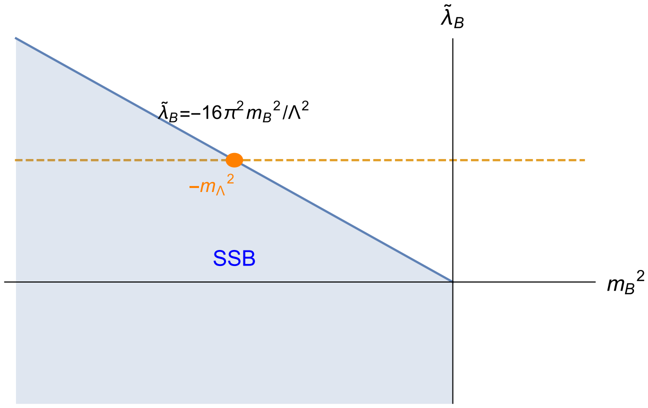

FIG. 2. Parameter space in the large-N limit.

$$e^{im_B^2 NA} Z(m_\mathrm{B}^2) \sim \exp\left(N\left\{\mathrm{const.} + im_\mathrm{B}^2 A + \frac{i}{4\tilde{\lambda}_\mathrm{B}}\int d^d x c^2\right\}\right) \tag{67}$$

$$= \exp\left[N\left\{\mathrm{const.} + i\frac{V_d}{4\tilde{\lambda}_\mathrm{B}}\left(m_\mathrm{B}^2 + \frac{2\tilde{\lambda}_\mathrm{B} A}{V_d}\right)^2\right\}\right]. \tag{68}$$

One can see that there is a saddle point at $m_\mathrm{B}^2 = -2\tilde{\lambda}_\mathrm{B} A/V_d$ if it is smaller than the critical value $-m_\Lambda^2$. Otherwise, the exponent in Eq. (68) is a monotonic function for $m_\mathrm{B}^2 \leq -m_\Lambda^2$. In particular, the second derivative is given by

$$\frac{\partial^2 \log Z(m_\mathrm{B}^2)}{\partial(m_\mathrm{B}^2)^2} = i\frac{NV_d}{2\tilde{\lambda}_\mathrm{B}}. \tag{69}$$

2. Unbroken phase

When $s = 0$, the system is in the unbroken phase. In this case, c is determined by the gap equation

$$c = \tilde{\lambda}_\mathrm{B}\int \frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{1}{p_\mathrm{E}^2 + m_\mathrm{B}^2 + c - i\epsilon}. \tag{70}$$

This equation has a solution only for

$$-\frac{1}{\tilde{\lambda}_\mathrm{B}} m_\mathrm{B}^2 - \int_0^\Lambda \frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{1}{p_\mathrm{E}^2} < 0, \tag{71}$$

which is consistent with Eq. (65). In this case, the canonical partition function becomes

$$Z(m_\mathrm{B}^2) \sim \exp\left[iNV_d\left\{-\frac{1}{2}\int\frac{d^d p_\mathrm{E}}{(2\pi)^d}\log(p_\mathrm{E}^2 + m_\mathrm{B}^2 + c - i\varepsilon) + \frac{1}{4\tilde{\lambda}_\mathrm{B}}c^2\right\}\right], \tag{72}$$

where c depends on m_B^2 via the gap equation (70). The exponent in Eq. (72) is a monotonically decreasing function of m_B^2 and the second derivative with respect to m_B^2 at the critical point $-m_\Lambda^2$ is

$$\begin{aligned}
&\left.\frac{\partial^2 \log Z(m_\mathrm{B}^2)}{\partial(m_\mathrm{B}^2)^2}\right|_{m_\mathrm{B}^2 = -m_\Lambda^2} \\
&= \frac{iNV_d}{2}\left[\int\frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{1}{(p_\mathrm{E}^2 + m_\mathrm{B}^2 + c)^2}\left(1 + \frac{dc}{dm_\mathrm{B}^2}\right)^2 + \frac{1}{\tilde{\lambda}_\mathrm{B}}\left(\frac{dc}{dm_\mathrm{B}^2}\right)^2\right]\Big|_{m_\mathrm{B}^2 = -m_\Lambda^2} \\
&= \frac{iNV_d}{2\tilde{\lambda}_\mathrm{B}}\frac{\tilde{\lambda}_\mathrm{B}\int\frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{1}{(p_\mathrm{E}^2)^2}}{1 + \tilde{\lambda}_\mathrm{B}\int\frac{d^d p_\mathrm{E}}{(2\pi)^d}\frac{1}{(p_\mathrm{E}^2)^2}}.
\end{aligned} \tag{73}$$

By comparing Eqs. (69) and (73), one can see that the second derivative is discontinuous at the critical point $m_\mathrm{B}^2 = -m_\Lambda^2$. Thus, we conclude that m_B^2 is fixed at $-m_\Lambda^2$ by the mathematical formula (A6) (as long as $A \neq 0$) at which the renormalized mass $m_\mathrm{R}^2 = m_\mathrm{B}^2 + c$ is zero.

V. FIXING QUARTIC COUPLING IN ϕ^4 THEORY

We briefly comment on the possibility of fixing the quartic coupling. As well as the mass term, we can also consider the generalized partition function for the quartic coupling

$$\int dm_\mathrm{R}^2 \int d\lambda_\mathrm{R}\,\omega(m_\mathrm{R}^2, \lambda_\mathrm{R}) e^{im_\mathrm{R}^2 A_{m^2} + i\lambda_\mathrm{R} A_\lambda} Z(m_\mathrm{R}^2, \lambda_\mathrm{R}), \tag{74}$$

where we take the renormalized couplings as the integration parameters instead of the bare couplings for simplicity.[5] Note that A_λ is another extensive parameter proportional to the spacetime volume V. We will discuss the case where $\omega(m_\mathrm{R}^2, \lambda_\mathrm{R})$ has support only for non-negative values of λ_R.[6]

At the one-loop level of the ϕ^4 theory, the vacuum energy (51) vanishes at the critical point of the mass squared $m_\mathrm{R}^2 = 0$ for any $\lambda_\mathrm{R} \geq 0$, which means that the above partition function becomes

[5]The Jacobian from the change of variables has been absorbed by the redefinition of ω.

[6]We can also consider the case where ω has support for both positive and negative values of λ_R. In this case, we have to consider contributions to the partition function from both. However, it is possible that the negative values of λ_R lead to the exponential damping of the partition function for the large volume limit; this is because the vacuum energy density for negative λ_R has a negative imaginary part due to the instability of vacuum, as we have seen in Eq. (30) for the mass parameter.

$$\sim \int_0^\infty d\lambda_R \omega(\lambda_R) e^{i\lambda_R A_\lambda}. \tag{75}$$

As long as $A_\lambda \neq 0$, the exponent is a linear function of λ_R, and this integration is strongly dominated by $\lambda_R = 0$ by the formula (A2) in the large volume limit. Thus, the free scalar theory seems to be realized in the generalized QFT in the ϕ^4 theory at least at one-loop level. As long as the renormalized mass m_R^2 is fixed at zero and we focus on the cutoff independent part of the vacuum energy, this conclusion will not be significantly altered by the higher-loop effects because the cutoff independent part should be proportional to $(m_R^2)^2$ by dimensional analysis and they vanish at the critical point $m_R^2 = 0$.

On the other hand, a nontrivial saddle point can appear if we also include the cutoff-dependent parts. For example, at the three-loop level, we have the following contributions to the vacuum energy:

$$\lambda_R \frac{A_\lambda}{V_4} + (c_1\lambda_R - c_2\lambda_R^2)\Lambda^4, \tag{76}$$

where $c_i > 0$ are just constants containing the loop suppression factors. Thus, one can see that there exists a saddle point at

$$\lambda_R \sim \left(\frac{A_\lambda}{V_4} + c_1\right)/c_2. \tag{77}$$

This is a very interesting possibility but its physical meaning is subtle, as in the mass parameter case. More detailed studies are left for future investigations. See also Appendix B for the possibility of fixing λ_R in the large-N limit.

VI. DIMENSIONAL TRANSMUTATION IN TWO-REAL-SCALAR MODEL

As discussed above, the low energy effective theory of the generalized QFT is nothing but the ordinary QFT, whose parameters are tuned to be the critical values. In a wide range of parameter spaces of various models, one of the quartic couplings vanishes and a VEV is generated at a nonperturbatively small scale, which is known as the Coleman-Weinberg mechanism [31].

As an example, we study a two-real-scalar model [32–37] at the one-loop level. We show that an automatic tuning of the mass-squared parameter is realized so that the dimensional transmutation is successfully achieved. Note also that we treat the scalar quartic couplings in the usual canonical way and fix them to be positive values to guarantee the stability of the system for simplicity.

For simplicity, we again impose the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, which leads to the following Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial S)^2 - \frac{m_\phi^2}{2}\phi^2 - \frac{m_S^2}{2}S^2 - \frac{\lambda_{\phi S}}{4}\phi^2 S^2 - \frac{\lambda_\phi}{4!}\phi^4 - \frac{\lambda_S}{4!}S^4. \tag{78}$$

To simplify the discussion further, we focus on the parameter space $\lambda_\phi \ll \lambda_S \sim 1$ and $\lambda_{\phi S} > 0$, which means that the ϕ direction is almost flat for $m_\phi^2 \sim 0$. ϕ and S play the roles of the scalar and gauge fields in the original Coleman-Weinberg mechanism. On the other hand, the S direction is always dominated by the tree-level potential; that is, its VEV is well determined by $m_S^2 S^2 + \frac{\lambda_S}{4!}S^4$.

Under these conditions, we will show that there is a critical point at $m_S^2 = 0$ and $m_\phi^2 \neq 0$ which corresponds to a quantum first-order phase transition point. Although the conventional classical conformal point $m_S^2 = m_\phi^2 = 0$ is not realized in the current formulation, our result serves as a concrete way of dimensional transmutation without assuming an artificial symmetry such as classical conformality.

Now let us discuss the details. The one-loop effective potential in the $\overline{\rm MS}$ scheme is given by

$$V_{\rm eff}(\phi, S) = \frac{m_\phi^2}{2}\phi^2 + \frac{m_S^2}{2}S^2 + \frac{\lambda_{\phi S}}{4}\phi^2 S^2 + \frac{\lambda_\phi}{4!}\phi^4 + \frac{\lambda_S}{4!}S^4 + \frac{(M_+^2(\phi,S))^2}{64\pi^2}\log\left(\frac{M_+^2(\phi,S)}{\mu^2 e^{3/2}}\right) + \frac{(M_-^2(\phi,S))^2}{64\pi^2}\log\left(\frac{M_-^2(\phi,S)}{\mu^2 e^{3/2}}\right), \tag{79}$$

where

$$M_\pm^2(\phi, S) = \frac{1}{2}\left(M_\phi^2 + M_S^2 \pm \sqrt{(M_\phi^2 - M_S^2)^2 + 4\lambda_{\phi S}^2\phi^2 S^2}\right) \tag{80}$$

with

$$M_\phi^2 = m_\phi^2 + \frac{\lambda_{\phi S}}{2}S^2 + \frac{\lambda_\phi}{2}\phi^2, \quad M_S^2 = m_S^2 + \frac{\lambda_{\phi S}}{2}\phi^2 + \frac{\lambda_S}{2}S^2. \tag{81}$$

Here, all the coupling constants are the renormalized ones. As usual, we can take the renormalization scale at the point $\mu = M$ where λ_ϕ vanishes. In the following, we first examine how m_ϕ^2 is fixed for a given value of m_S^2 and then discuss the fixing of m_S^2.

First, let us consider the parameter space $m_S^2 \geq 0$. In this case, $\langle S\rangle = 0$ as long as the tree-level potential dominates in the S direction. The effective potential for ϕ is then

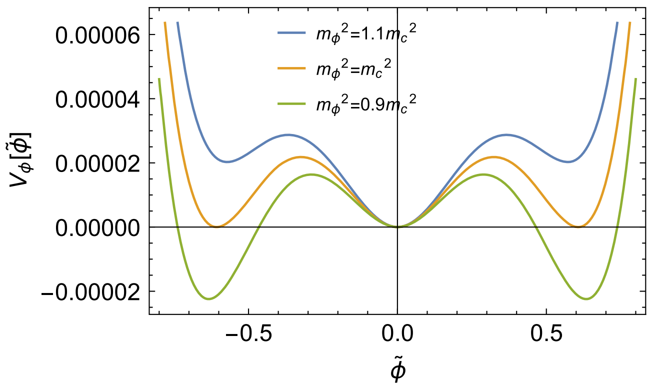

FIG. 3. One-loop effective potential of $\tilde{\phi}$. Different colors correspond to the different values of m_ϕ^2.

$$V_{\rm eff}(0,\phi) = \frac{(m_\phi^2)^2}{64\pi^2}\log\left(\frac{m_\phi^2}{M^2e^{3/2}}\right) + \frac{m_\phi^2}{2}\phi^2 + \frac{(m_S^2+\frac{\lambda_{\phi S}}{2}\phi^2)^2}{64\pi^2}\log\left(\frac{m_S^2+\frac{\lambda_{\phi S}}{2}\phi^2}{M^2e^{3/2}}\right). \quad (82)$$

In Fig. 3, we show the plots of $V_{\rm eff}(0,\phi)$ where the different colors correspond to the different values of m_ϕ^2. Here, $\tilde{\phi}$ and $V_\phi(\tilde{\phi})$ are defined by

$$\tilde{\phi}^2 = (m_S^2 + \lambda_{\phi S}\phi^2/2)/M^2e^{3/2}, \quad (83)$$

$$V_\phi(\tilde{\phi}) \coloneqq (M^2e^{3/2})^2\left(\frac{\tilde{m}_\phi^2}{2}\tilde{\phi}^2 + \frac{\tilde{\phi}^4}{64\pi^2}\log\tilde{\phi}^2\right). \quad (84)$$

In particular, one can see that there exists a quantum first-order phase transition at

$$m_\phi^2 = \frac{\lambda_{\phi S}e^{1/2}}{64\pi^2}M^2 \eqqcolon m_c^2, \quad (85)$$

meaning that ϕ achieves a nonzero VEV, $\langle\phi\rangle = v_\phi$, for $m_\phi^2 < m_c^2$. See Appendix C for the derivation of Eq. (85). Correspondingly, the VEV and the vacuum energy are given by

$$\langle\phi\rangle = \begin{cases} 0 & \text{for } m_\phi^2 \geq m_c^2 \\ v_\phi & \text{for } m_\phi^2 < m_c^2 \end{cases}, \quad (86)$$

$$V_{\rm eff}(0,\langle\phi\rangle) - \frac{(m_\phi^2)^2}{64\pi^2}\log\left(\frac{m_\phi^2}{M^2e^{3/2}}\right) = \begin{cases} \frac{(m_S^2)^2}{64\pi^2}\log\left(\frac{m_S^2}{M^2e^{3/2}}\right) & \text{for } m_\phi^2 \geq m_c^2 \\ \frac{m_\phi^2}{2}v_\phi^2 + \frac{(m_S^2+\frac{\lambda_{\phi S}}{2}v_\phi^2)^2}{64\pi^2}\log\left(\frac{m_S^2+\frac{\lambda_{\phi S}}{2}v_\phi^2}{M^2e^{3/2}}\right) & \text{for } m_\phi^2 < m_c^2 \end{cases}. \quad (87)$$

As in the simple ϕ^4 theory, the additional contribution in the second line gives the discontinuity of the second derivative of the vacuum energy at $m_\phi^2 = m_c^2$, and this point is dominant in the generalized partition function.

Second, we turn to the case $m_S^2 < 0$. In this case, S already has a VEV at $\langle S\rangle^2 = -6m_S^2/\lambda_S$ at tree level, and there exists the tree-level vacuum energy $-3(m_S^2)^2/(2\lambda_S)$ as in the simple ϕ^4 theory discussed in the previous sections. As for the ϕ potential, nonzero VEV of S just changes the effective mass in Eq. (81) as

$$M_S^2 = m_S^2 + \frac{\lambda_S}{2}\langle S\rangle^2 + \frac{\lambda_{\phi S}}{2}\phi^2 = 2|m_S^2| + \frac{\lambda_{\phi S}}{2}\phi^2. \quad (88)$$

Note that M_ϕ^2 is just a constant because of $\lambda_\phi = 0$. Thus, we can repeat the same discussion as above and conclude that m_ϕ^2 is still fixed at the critical point $m_\phi^2 = m_c^2$.

Finally, at such a critical point of m_ϕ^2, the vacuum energy as a function of m_S^2 is given by

$$V_{\rm min}|_{m_\phi^2=m_c^2} \simeq \begin{cases} \frac{(m_S^2)^2}{64\pi^2}\log\left(\frac{m_S^2}{M^2e^{3/2}}\right) & \text{for } m_S \geq 0 \\ -\frac{3(m_S^2)^2}{2\lambda_S} + \frac{(m_S^2+\frac{\lambda_{\phi S}}{2}v_\phi^2)^2}{64\pi^2}\log\left(\frac{m_S^2+\frac{\lambda_{\phi S}}{2}v_\phi^2}{M^2e^{3/2}}\right) & \text{for } m_S < 0 \end{cases}. \quad (89)$$

As in the simple ϕ^4 theory, m_S^2 is fixed at zero because the second derivative of this equation is discontinuous at $m_S^2 = 0$. The conventional Coleman-Weinberg point $m_S^2 = m_\phi^2 = 0$ is not realized in the current formation because such a point $m_S^2 = m_\phi^2 = 0$ corresponds to the degeneracy of false vacua as discussed in Ref. [35]. We leave the discussion on the criticality of general extrema for future investigation.

VII. SUMMARY

In this paper, we have investigated the Higgs hierarchy problem in the generalized QFT. We first studied the free scalar theory and found that there are two critical points in the large volume limit: one is $m_{\mathrm{B}}^2 = 0$ and the other is $m_{\mathrm{B}}^2 = \mathcal{O}(\Lambda^2)$. While the former does not depend on the regularization methods, the latter does, implying that $m_{\mathrm{B}}^2 = 0$ is a physically reasonable critical point. This can be a theoretical origin/explanation of classical conformality, which is implicitly assumed in many literatures.

We then studied the ϕ^4 theory at the one-loop level, as well as the large-N model. In this case, we found that a critical point exists at the point where the renormalized mass $m^2 = m_{\mathrm{B}}^2 + \delta m_{\mathrm{UV}}^2$ vanishes due to the discontinuity of the vacuum energy's derivative. This further backs up the fine-tuning of the (Higgs) mass squared being automatically accomplished in the generalized QFT.

We have also discussed the possibility that the quartic coupling is automatically fixed, with a positive result.

As a next nontrivial example, we examined the $\mathbb{Z}_2$ invariant two-real-scalar model at the one-loop level by focusing on a simple parameter space, where only one real scalar ϕ can develop a nonzero VEV $\langle\phi\rangle \neq 0$. Under these conditions, we found that there exists a critical point where $m_\phi^2 = m_c^2 > 0$ and $m_{\mathrm{S}}^2 = 0$, which corresponds to a quantum first-order phase transition point of the theory. This result gives a complete realization of dimensional transmutation without assuming classical conformality that has been implicitly assumed in many literatures.

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APPENDIX A: MATHEMATICAL FORMULAS

In the following, we often use mathematical formulas of generalized functions. We summarize them here. The first one is well known as the saddle point approximation,

$$\lim_{V\to\infty} e^{-Vg(x)} \simeq e^{-Vg(x_0)} \sqrt{\frac{2\pi}{Vg''(x_0)}} \times \delta(x), \tag{A1}$$

where $g(x)$ is a smooth function and x_0 is a saddle point, $g'(x_0) = 0$. The proof of this equation is textbook level and we will not repeat it here. The second formula is

$$\lim_{V\to\infty} e^{iVg(x)}\theta(x) \simeq \frac{i}{V}\left|\frac{dg}{dx}\right|^{-1} e^{iVg(0)} \times \delta(x), \tag{A2}$$

where $g(x)$ is now a real, smooth, and monotonic function for $x \geq 0$ and satisfies $g'(0) \neq 0$. The proof is as follows. By multiplying a test function $f(x)$ with a finite support and integrating from 0 to ∞, we have

$$\int_0^\infty dx e^{iVg(x)} f(x) = \int_{g(0)}^{g(\infty)} dg \left|\frac{dg}{dx}\right|^{-1} e^{iVg} f(x = x(g)) \tag{A3}$$

$$\begin{aligned} &= \left[\frac{e^{iVg}}{iV}\left|\frac{dg}{dx}\right|^{-1} f(x(g))\right]_{g(0)}^{g(\infty)} - \frac{1}{iV}\int_{g(0)}^{g(\infty)} dg \frac{d}{dg}\left(\left|\frac{dg}{dx}\right|^{-1} f(x(g))\right) e^{iVg} \\ &= i\frac{e^{iVg(0)}}{V}\left|\frac{dg}{dx}\right|^{-1} f(x)|_{x=0} - \frac{1}{(iV)^2}\left[\frac{d}{dg}\left(\left|\frac{dg}{dx}\right|^{-1} f(x(g))\right) e^{iVg}\right]_{g(0)}^{g(\infty)} + \cdots \\ &= i\frac{e^{iVg(0)}}{V}\left|\frac{dg}{dx}\right|^{-1} f(x)|_{x=0} + \mathcal{O}(V^{-2}), \end{aligned} \tag{A4}$$

which confirms Eq. (A2). Note that the contribution from $x = \infty$ is zero because we have assumed that $f(x)$ has a finite support. More generally, when $g(x)$ is a monotonic and smooth function for both sides $x > 0$ and $x < 0$, but its derivative $g'(x)$ is discontinuous at $x = 0$, we have

$$\lim_{V\to\infty} e^{iVg(x)} = \lim_{V\to\infty} \frac{i}{V}\left[\left|\frac{dg}{dx}\right|_{0+}^{-1} - \left|\frac{dg}{dx}\right|_{0-}^{-1}\right] e^{iVg(0)} \times \delta(x). \tag{A5}$$

Note that when $g'(x)$ is also continuous at $x = 0$, the right-hand side vanishes and higher-order terms dominate. In general, when the derivatives of $g(x)$ are continuous and nonzero up to $(n-1)$th order but the nth derivative of $g(x)$ is discontinuous, we have

$$\lim_{V\to\infty} e^{iVg(x)} = c \lim_{V\to\infty} \frac{i^n}{V^n} e^{iVg(0)} \times \delta(x), \tag{A6}$$

where c is a coefficient determined by $g'(0), g''(0), \ldots, d^{(n-1)}g(0)/dx^{n-1}, dg^{(n)}(0+)/dx^n, dg^{(n)}(0-)/dx^n$.

APPENDIX B: FIXING QUARTIC COUPLING IN LARGE-N MODEL

In this appendix, we study a possibility of fixing the quartic coupling by taking the N^{-1} corrections. In the following, the mass term m^2 is fixed at the critical point $m^2 = m_{\rm B}^2 + c_{\rm cl} = 0$ and we focus on $d = 4$.

Around the large-N solution $(c, s) = (c_{\rm cl}, s_{\rm cl})$ studied in Sec. IV B, the effective action in Eq. (62) becomes

$$S = \frac{N\Lambda^4 V_4}{4\tilde{\lambda}_{\rm B}} \left(\frac{\tilde{\lambda}_{\rm B}}{16\pi^2}\Lambda^2\right)^2 + N\int d^4x \left(-\frac{1}{2}(\partial_\mu \delta s)^2 - \frac{1}{2}\delta c \delta s^2 + \frac{1}{4}\left(\frac{1}{\tilde{\lambda}_{\rm B}} + \int \frac{d^4 p_{\rm E}}{(2\pi)^4}\frac{1}{(p_{\rm E}^2)^2}\right)\delta c^2 + \cdots\right), \tag{B1}$$

where $\delta s = s - s_{\rm cl}, \delta c = c - c_{\rm cl}$, and $\cdots$ represents the higher-order terms. Note that the first term is the leading vacuum energy contribution and it is proportional to $\tilde{\lambda}_{\rm B}$. Thus, this can be absorbed into the definition of A_λ. In the following, we represent

$$\frac{1}{\tilde{\lambda}_{\rm B}} + \int \frac{d^4 p_{\rm E}}{(2\pi)^4}\frac{1}{(p_{\rm E}^2)^2} = \frac{1}{\tilde{\lambda}_{\rm B}} + \frac{1}{8\pi^2}\log\left(\frac{\Lambda}{\mu}\right) \coloneqq \frac{1}{N\lambda(\mu)}, \tag{B2}$$

where μ is some (IR) scale. We can check that $\lambda(\mu)$ corresponds to the usual renormalized quartic coupling because Eq. (B2) is nothing but the solution of the RG equation in the large-N limit,

$$\frac{d\lambda}{d\log\mu} \approx \frac{N}{8\pi^2}\lambda^2. \tag{B3}$$

In Eq. (B1), the δc part can be rewritten as

$$\frac{1}{4N\lambda(\mu)}(\delta c - N\lambda(\mu)\delta s^2)^2 - \frac{N\lambda(\mu)}{4}\delta s^4. \tag{B4}$$

By the field redefinitions $N^{1/2}\delta s \to \delta s$ and $N^{1/2}(\delta c - N\lambda(\mu)\delta s^2) \to \delta c$, the action becomes

$$S = \int d^4x \left(-\frac{1}{2}(\partial_\mu \delta s)^2 - \frac{\lambda(\mu)}{4}\delta s^4 + \frac{1}{4N\lambda(\mu)}\delta c^2 + \mathcal{O}(N^{-1})\right). \tag{B5}$$

Now we can perform the one-loop integration of δc as

$$\begin{aligned} -\frac{V_4}{2}\int \frac{d^4 q_{\rm E}}{(2\pi)^4}\log(N\lambda(\mu)/2)^{-1} &= -\frac{F}{2}\log\left(\frac{2}{N\lambda(\mu)}\right) \\ &= -\frac{F}{2}\log\left(\frac{1}{\lambda_{\rm B}} + \frac{N}{8\pi^2}\log\left(\frac{\Lambda}{\mu}\right)\right) + \text{const}, \end{aligned} \tag{B6}$$

where $F = V_4\Lambda^4/(32\pi^2)$. Note that we have singular points at

$$\lambda(\mu) = 0, \qquad \pm\infty. \tag{B7}$$

Now the $\lambda_{\rm B}$ integration in the microcanonical partition function is given by

$$\begin{aligned} \Omega(A) &= \int_{-\infty}^{\infty} \frac{d\lambda_{\rm B}}{2\pi} \exp\Bigg(i\lambda_{\rm B} N A_\lambda \\ &\qquad - i\frac{F}{2}\log\left(\frac{1}{\lambda_{\rm B}} + \frac{N}{8\pi^2}\log\left(\frac{\Lambda}{\mu}\right)\right)\Bigg) \end{aligned} \tag{B8}$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda_{\rm B}}{2\pi} e^{iNA_\lambda(\lambda(\mu))}, \tag{B9}$$

where

$$G(\lambda(\mu)) \coloneqq \frac{\lambda(\mu)}{1 - \frac{N\lambda(\mu)}{8\pi^2}\log\left(\frac{\Lambda}{\mu}\right)} + \frac{F}{2NA_\lambda}\log\lambda(\mu). \tag{B10}$$

One can check that there exist two saddle points,

$$N\lambda(\mu) = N\lambda_\pm = \frac{1}{B}\left(1 - \frac{A_\lambda}{FB} \pm \sqrt{\left(1 - \frac{A_\lambda}{FB}\right)^2 - 1}\right), \tag{B11}$$

where

$$B = \frac{1}{8\pi^2}\log\left(\frac{\Lambda}{\mu}\right). \tag{B12}$$

We can see that $N\lambda_\pm$ is real and positive if we take the following large-N limit:

$$N \to \infty \quad N\lambda_{\mathrm{R}}(\mu) = \text{fixed}, \qquad \frac{A_\lambda}{F} = \text{fixed}, \qquad \left|1 - \frac{A_\lambda}{FB}\right| \geq 1. \tag{B13}$$

More detailed analysis is necessary to verify the validity of this limit because we now have an additional parameter A_λ that is absent in the usual large-N analysis.

APPENDIX C: DETAILS OF TWO-REAL-SCALAR MODEL

By putting $\tilde{\phi}^2 = (m_S^2 + \lambda_{\phi S}\phi^2/2)/M^2 e^{3/2}$, we can rewrite Eq. (82) as

$$V_{\mathrm{eff}}(0,\phi) = \Lambda(m_\phi^2, m_S^2) + \tilde{V}_\phi(\tilde{\phi}), \tag{C1}$$

where

$$\Lambda(m_\phi^2, m_S^2) \coloneqq \frac{(m_\phi^2)^2}{64\pi^2}\log\left(\frac{m_\phi^2}{\mu^2 e^{3/2}}\right) - \frac{m_S^2 m_\phi^2}{\lambda_{\phi S}}, \tag{C2}$$

$$\tilde{V}_\phi(\tilde{\phi}) \coloneqq (M^2 e^{3/2})^2\left(\frac{\tilde{m}_\phi^2}{2}\tilde{\phi}^2 + \frac{\tilde{\phi}^4}{64\pi^2}\log\tilde{\phi}^2\right), \tag{C3}$$

in which

$$\tilde{m}_\phi^2 \coloneqq \frac{2m_\phi^2}{\lambda_{\phi S} M^2 e^{3/2}}. \tag{C4}$$

The effective potential has minima at $\phi = 0$ and

$$\tilde{\phi} = \tilde{v}_\phi, \tag{C5}$$

where $\tilde{v}_\phi^2$ is a solution of

$$\tilde{m}_\phi^2 + \frac{\tilde{\phi}^2}{16\pi^2}\log(\tilde{\phi}^2 e^{1/2}) = 0. \tag{C6}$$

Correspondingly, v_ϕ will denote the VEV of the original field ϕ below.

Note that $\langle\phi\rangle = 0$ is always the true vacuum for $m_S^2 > \frac{\lambda_{\phi S}}{2}v_\phi^2$ because $\tilde{\phi}^2 > \tilde{v}_\phi^2$ for any values of $\phi^2 > 0$. On the other hand, the minimum of $\tilde{V}_\phi(\tilde{\phi})$ depends on m_ϕ^2 for $\frac{\lambda_{\phi S}}{2}v_\phi^2 \geq m_S^2 \geq 0$. In particular, it has a critical point at

$$0 = \tilde{V}_\phi(0) = \tilde{V}_\phi(\tilde{v}_\phi) \quad \to \quad \tilde{m}_\phi^2 = \frac{1}{32\pi^2 e} \eqqcolon \tilde{m}_c^2. \tag{C7}$$

See Fig. 3 for the explicit plots of $\tilde{V}_\phi(\tilde{\phi})$.

As a consistency check of $\langle S\rangle = 0$ for $m_S^2 \geq 0$, let us also check the positivity of the effective mass of S at $S = 0$,

$$m_{\mathrm{eff},S}^2 = \left.\frac{\partial V_{\mathrm{eff}}}{\partial S^2}\right|_{S=0,\phi=\langle\phi\rangle} \tag{C8}$$

$$= m_S^2 + \frac{\lambda_{\phi S}}{2}\langle\phi\rangle^2 + \frac{\lambda_S}{32\pi^2}\left(m_S^2 + \frac{\lambda_{\phi S}}{2}\langle\phi\rangle^2\right) \times\log\left(\frac{m_S^2 + \frac{\lambda_{\phi S}}{2}\langle\phi\rangle^2}{M^2 e}\right) + \frac{\lambda_{\phi S} m_\phi^2}{32\pi^2}\log\left(\frac{m_\phi^2}{M^2 e}\right). \tag{C9}$$

The last term is negligible at around the critical point $m_\phi^2 = m_c^2$ because

$$\frac{\lambda_{\phi S} m_\phi^2}{32\pi^2}\log\left(\frac{m_\phi^2}{M^2 e}\right) \sim -\frac{\lambda_{\phi S}^2}{(32\pi)^2}\langle\phi\rangle^2 \ll \frac{\lambda_{\phi S}^2}{2}\langle\phi\rangle^2. \tag{C10}$$

On the other hand, the second term in Eq. (C9) seems to become negative at around $m_S^2 \sim 0$ for $\langle\phi\rangle = 0$, but it is just an illusion of looking at the first term of leading log corrections. By summing up all the leading log terms (which correspond to the RG improvement), we have

$$1 + \frac{\lambda_S}{32\pi^2}\log\left(\frac{m_S^2}{M^2 e}\right) + \left(\frac{\lambda_S}{32\pi^2}\log\left(\frac{m_S^2}{M^2 e}\right)\right)^2 + \cdots = \frac{1}{1 - \frac{\lambda_S}{32\pi^2}\log\left(\frac{m_S^2}{M^2 e}\right)}, \tag{C11}$$

which is always positive for $m_S^2 \geq 0$.[7] Thus, $\langle S\rangle = 0$ is justified at this one-loop level calculation and m_ϕ^2 is fixed at the critical point m_c^2 by the discontinuity of the vacuum energy.

[7]This is nothing but the well-known fact [31] that the Coleman-Weinberg mechanism does not work for a single (real) scalar.

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