$\mathcal{N}=8$ invariant interaction of dynamical and semidynamical $\mathcal{N}=4$ multiplets

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We present a new model of $\mathcal{N}=8$ mechanics with semidynamic supermultiplets. The model is constructed as an interaction of $\mathcal{N}=4$ supermultiplets which carry an implicit $\mathcal{N}=4$ supersymmetry. The initial field content consists of three dynamical $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplets: one bosonic and two fermionic. To ensure implicit $\mathcal{N}=4$ supersymmetry, we introduce the superfields describing three semidynamical $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets: one fermionic and two bosonic. To avoid the second-order Lagrangian for fermions from the fermionic $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplets, we convert their velocities into new auxiliary fields. After conversion, these multiplets turn into semidynamical mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets without noncanonical terms in the $\mathcal{N}=8$ Lagrangian at the component level. The final $\mathcal{N}=8$ multiplet content is $(\mathbf{1},\mathbf{8},\mathbf{7})\oplus(\mathbf{8},\mathbf{8},\mathbf{0})$. As a first step to the ultimate $\mathcal{N}=4$ superfield formulation of the model, we recall a natural description of the standard and mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets in the framework of $\mathcal{N}=4, d=1$ biharmonic superspace.

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I. INTRODUCTION

Models of supersymmetric (quantum) mechanics play an important role as a training base for the study of systems with supersymmetry in higher space-time dimensions, involving the proper supersymmetrizations of gauge, gravitation, and cosmological theories. They are also closely related to diverse superextensions of $d=1$ integrable systems.

The most advanced and suggestive method to deal with supersymmetric theories is the superfield approach. While there exists a huge corpus of references on the $d=1$ superfield description of $\mathcal{N}\leq 4$ supersymmetric mechanics models and the related superextended $d=1$ integrable systems (see, e.g., [1–3]), much less is known about a similar approach to $\mathcal{N}>4$ models, in particular, to $\mathcal{N}=8$ ones. Until now, $\mathcal{N}=8$ models (see, e.g., [4–7]) have been constructed in both $\mathcal{N}=4$ and $\mathcal{N}=8$ superspace approaches.[1] The superfields used in these models encompass as a rule $\mathcal{N}=8$ supermultiplets of the same type. The latter are dynamical, that is, they possess Lagrangians that are of the second order in the time derivatives of the component bosonic fields. The $\mathcal{N}=8$ models involving interactions of different types of $\mathcal{N}=4$ multiplets, with an additional hidden $\mathcal{N}=4$ supersymmetry that mixes up these multiplets and extends the manifest $\mathcal{N}=4$ supersymmetry to $\mathcal{N}=8$, were only considered, to the best of our knowledge, in [15,16]. Yet, all of the involved $\mathcal{N}=4$ supermultiplets were of the dynamical type.

On the other hand, a number of models with $\mathcal{N}=4$ supersymmetry also include, in addition to dynamical supermultiplets, semidynamical ones. The bosonic fields of the latter are described by the $d=1$ Wess-Zumino (or Chern-Simons)-type Lagrangians of the first order in the time derivatives. The basic goal of the present work is to construct the first example of $\mathcal{N}=8$ supersymmetric models of this sort, with some fields being semidynamical.

In [17], an $\mathcal{N}=4$ generalization of the n-particle rational Calogero system was proposed (see Ref. [18] for the review). This $\mathcal{N}=4$ Calogero model employs the dynamical $n\times n$ matrix $(\mathbf{1},\mathbf{4},\mathbf{3})$ supermultiplet and n semidynamical $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplets. The one-particle $(n=1)$ limit of the model of Ref. [17] was considered in [19]. The more general form of the kinetic term of the matrix $(\mathbf{1},\mathbf{4},\mathbf{3})$ supermultiplet in the model of [17] gives rise to the $\mathcal{N}=4$ supersymmetric hyperbolic Calogero-Sutherland model [20]. Superconformal mechanics with $D(2,1,\alpha)$ supersymmetry was constructed in [21] as a generalization of the one-particle system of [19], such that the Lagrangian of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfields is a power function of the latter.

In this paper, we construct an $\mathcal{N}=8$ generalization of the $\mathcal{N}=4$ system suggested in [19]. We basically use

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[1] For a description of $\mathcal{N}=8$ supersymmetric systems at the component level, see, e.g., [8–14].

$\mathcal{N}=4, d=1$ harmonic superspace [22], which is a reduction of $\mathcal{N}=2, d=4$ harmonic superspace [23,24].

Modulo gauge transformations of the involved superfields, the model of Ref. [19] is governed by the cubic action (schematically)

$$\int \mu_{\mathcal{H}} v^2 + \int \mu_{\mathcal{A}}^{(-2)} \mathcal{V} Z^{+A} Z^{+B} c_{(AB)}, \qquad (1.1)$$

where $c_{(AB)}$ are some constants. The superfield $v(t,\theta,\bar\theta)$ encapsulates the $(\mathbf{1},\mathbf{4},\mathbf{3})$ supermultiplet, $\mathcal{V}(t_{\mathcal{A}},\theta^+,\bar\theta^+,u)$ is its analytic harmonic gauge prepotential, and $Z^{+A}(t_{\mathcal{A}},\theta^+,\bar\theta^+,u)$, $A=1,\ 2$ amounts to the $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet, which is semidynamical in the model with the action (1.1). In this paper, we just find the $\mathcal{N}=8$ generalization of the model (1.1). We build this model by making use of $\mathcal{N}=4$ superfields carrying an additional implicit $\mathcal{N}=4$ supersymmetry. Similar to [15,16], we extend the original $\mathcal{N}=4$ superfield content by adding an odd superfield $\Psi^{+A}(t_{\mathcal{A}},\theta^+,\bar\theta^+,u)$ as a superpartner of the superfield $v(t,\theta,\bar\theta)$ with respect to the implicit $\mathcal{N}=4$ supersymmetry. Analogously, the superfield $\mathcal{Z}^{+A}(t_{\mathcal{A}},\theta^+,\bar\theta^+,u)$ is extended to a wider $\mathcal{N}=4$ superfield set in which it occupies the same place as the superfield $\Psi^{+A}(t_{\mathcal{A}},\theta^+,\bar\theta^+,u)$ in the first set. To ensure $\mathcal{N}=8$ supersymmetry, we add one more superfield $w(t,\theta,\bar\theta)$ which also has the field content $(\mathbf{1},\mathbf{4},\mathbf{3})$ but it is Grassmann odd. In order to construct the $\mathcal{N}=8$ invariant action, it also turns out to be necessary to make use of at least two superfields $\mathcal{Z}_a^{+A}(t_{\mathcal{A}},\theta^+,\bar\theta^+,u)$, $a=1,2$ and two superfields $w_a(t,\theta,\bar\theta)$, $a=1,\ 2$. Requiring $\mathcal{N}=8$ symmetry for the total extended $\mathcal{N}=4$ superfield set, we finally derive the $\mathcal{N}=8$ invariant action as a generalization of (1.1).

The plan of the paper is as follows. In Sec. II we present the $\mathcal{N}=4$ harmonic superfield description of the multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{0},\mathbf{4},\mathbf{4})$. Each type of supermultiplets involves both even and odd superfields and we describe the implicit $\mathcal{N}=4$ supersymmetry transformations realized on these superfields. In Sec. III we present the $\mathcal{N}=4$ superfield formulation of the $\mathcal{N}=8$ invariant coupling of these supermultiplets. Section IV shows that the superfield model constructed leads to the component Lagrangian in which some fermionic fields enter only through their first-order time derivative and no such fields without derivatives are present. After applying the oxidation procedure of replacing this time derivative by a new auxiliary fermionic field [4,25], the Lagrangian yields the $\mathcal{N}=8$ invariant model. The resulting $\mathcal{N}=8$ invariant model involves on the mass shell one dynamic bosonic field and eight real fermionic fields, as well as three sets of semidynamical bosonic SU(2)-doublet fields. Some concluding remarks are collected in Sec. V. In the Appendix we demonstrate that the result of the oxidation procedure mentioned above can be reproduced by using off-shell superfields involving, as elementary components, the auxiliary fermionic fields which imitate the time derivative of the original fermionic fields. This is in agreement with the general proposition of Ref. [26].

II. $\mathcal{N}=4$ SUPERFIELDS USED IN CONSTRUCTING THE $\mathcal{N}=8$ MODEL

We build the $\mathcal{N}=8$ model in terms of the $\mathcal{N}=4$ superfields defined in both the usual $\mathcal{N}=4$ superspace and the $\mathcal{N}=4$ harmonic superspace [22–24]. In this section, we describe the main features of the objects used throughout the paper.

A. Brief information about $\mathcal{N}=4$, $d=1$ harmonic superspace

The powerful approach to constructing $\mathcal{N}=4$, $d=1$ supersymmetric models and finding interrelations between them is the $\mathcal{N}=4$, $d=1$ harmonic formalism, which was proposed in [22]. Compared with the description in the usual superspace with the coordinates $z=(t,\theta_i,\bar\theta^i)$, $(\theta_i)^*=\bar\theta^i$, and covariant derivatives

$$D^i=\frac{\partial}{\partial\theta_i}-i\bar\theta^i\partial_t, \qquad \bar D_i=\frac{\partial}{\partial\bar\theta^i}-i\theta_i\partial_t,$$
$$(D^i)^*=-\bar D_i, \qquad \{D^i,\bar D_k\}=-2i\delta^i_k\partial_t, \qquad (2.1)$$

the harmonic description involves additional commuting harmonic variables,

$$u^\pm_i, \qquad (u^+_i)^*=u^{-i}, \qquad u^{+i}u^-_i=1. \qquad (2.2)$$

In the harmonic analytic basis,

$$(z_{\mathcal{A}},u^\pm_i)=(t_{\mathcal{A}},\theta^\pm,\bar\theta^\pm,u^\pm_i), \qquad t_{\mathcal{A}}=t+i(\theta^+\bar\theta^-+\theta^-\bar\theta^+),$$
$$\theta^\pm=\theta^i u^\pm_i, \qquad \bar\theta^\pm=\bar\theta^i u^\pm_i, \qquad (2.3)$$

half of the $\mathcal{N}=4$ covariant spinor derivatives $D^\pm=u^\pm_i D^i$, $\bar D^\pm=u^\pm_i\bar D^i$ become short:

$$D^+=\frac{\partial}{\partial\theta^-}, \qquad \bar D^+=-\frac{\partial}{\partial\bar\theta^-}. \qquad (2.4)$$

This implies the existence of the harmonic analytic superfields defined on the analytic subspace of the full harmonic superspace:

$$(\zeta,u)=(t_{\mathcal{A}},\theta^+,\bar\theta^+,u^\pm_i), \qquad u^{+i}u^-_i=1. \qquad (2.5)$$

It is closed under $\mathcal{N}=4$ supersymmetry and some generalized conjugation, $\widetilde{(t_{\mathcal{A}})}=t_{\mathcal{A}}, \widetilde{(\theta^+)}=\bar\theta^+, \widetilde{(\bar\theta^+)}=-\theta^+$, $\widetilde{u^{\pm i}}=u^\pm_i, \widetilde{u^\pm_i}=-u^{\pm i}$. The integration measure in the harmonic analytic subspace is defined as $dud\zeta^{(-2)} = dudt_{\mathcal{A}}d\theta^+ d\bar\theta^+$. An important tool of the formalism is the harmonic derivatives:

$$D^{\pm\pm} = \partial^{\pm\pm} + 2i\theta^\pm\bar\theta^\pm\partial_{t_A} + \theta^\pm\frac{\partial}{\partial\theta^\mp} + \bar\theta^\pm\frac{\partial}{\partial\bar\theta^\mp},$$
$$\partial^{\pm\pm} = u_i^\pm\frac{\partial}{\partial u_i^\mp}. \tag{2.6}$$

The harmonic derivative D^{++} is distinguished in that it commutes with the spinor derivatives (2.4) and so preserves the analyticity.

The integration measures are defined as

$$\mu_{\mathcal{H}} = dudtd^4\theta = \mu_{\mathcal{A}}^{(-2)}D^+\bar D^+, \qquad \mu_{\mathcal{A}}^{(-2)} = dud\zeta^{(-2)},$$
$$d\zeta^{(-2)} = dt_{\mathcal{A}}D^-\bar D^-. \tag{2.7}$$

Here we presented only the definitions of the basic notions to be used below. The full exposition of the harmonic superspace formalism of $d=1$ models can be found in Ref. [22].

B. $\mathcal{N}=4$ superfields

When building the model, we use the following $\mathcal{N}=4$ superfields:

Bosonic superfield $v(z)$	Fermionic superfield $\Psi^{iA}(\zeta,u)$
Fermionic superfields $w_a(z)$	Bosonic superfields $\mathcal{Z}_a^{iA}(\zeta,u)$

Below we describe these superfields in some detail.

1. Multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$

The multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is described by the $\mathcal{N}=4$ even superfield $v(z)$, $\tilde v = v$, obeying the constraints [27]

$$D^iD_iv = \bar D_i\bar D^iv = 0, \qquad [D^i,\bar D_i]v = 0. \tag{2.8}$$

The solution of the constraints (2.8) is

$$v(t,\theta_i,\bar\theta^i) = \mathrm{v} + \theta_i\varphi^i + \bar\theta_i\bar\varphi^i + i\theta_i\bar\theta_kA^{ik} - \frac{i}{2}(\theta)^2\bar\theta_i\dot\varphi^i$$
$$-\frac{i}{2}(\bar\theta)^2\theta_i\dot{\bar\varphi}^i + \frac14(\theta)^2(\bar\theta)^2\ddot{\mathrm{v}}, \tag{2.9}$$

where $(\theta)^2 = \theta_k\theta^k$, $(\bar\theta)^2 = \bar\theta^k\bar\theta_k$. The component fields in the expansion (2.9) satisfy the reality conditions $v^\dagger = v$, $(\varphi^i)^\dagger = \bar\varphi_i$, $(A^{ik})^\dagger = A_{ik} = A_{(ik)}$. In the harmonic superspace, the constraints (2.8) are rewritten as

$$D^{++}v = 0, \qquad D^+D^-v = \bar D^+\bar D^-v = 0,$$
$$(D^+\bar D^- + \bar D^+D^-)v = 0. \tag{2.10}$$

The $\mathcal{N}=4$ supersymmetry transformations of the component fields in (2.9) are given by

$$\delta_\varepsilon\mathrm{v} = -\varepsilon_i\varphi^i + \bar\varepsilon^i\bar\varphi_i,$$
$$\delta_\varepsilon\varphi^i = i\bar\varepsilon^i\dot{\mathrm{v}} - i\bar\varepsilon_kA^{ki}, \qquad \delta_\varepsilon\bar\varphi_i = -i\varepsilon_i\dot{\mathrm{v}} - i\varepsilon^kA_{ki},$$
$$\delta_\varepsilon A_{ik} = -2(\varepsilon_{(i}\dot\varphi_{k)} + \bar\varepsilon_{(i}\dot{\bar\varphi}_{k)}), \tag{2.11}$$

where ε_i, $\bar\varepsilon^i = (\varepsilon_i)^*$ are odd parameters of the explicit $\mathcal{N}=4$ supersymmetry.

As was shown in [28], the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet can also be described through the real analytic gauge superfield prepotential $\mathcal{V}(\zeta,u)$, which is defined up to the Abelian gauge transformations,

$$\mathcal{V} \Rightarrow \mathcal{V}' = \mathcal{V} + D^{++}\Lambda^{--}, \qquad \Lambda^{--} = \Lambda^{--}(\zeta,u). \tag{2.12}$$

They allow passing to the Wess-Zumino gauge:

$$\mathcal{V}(\zeta,u) = \mathrm{v}(t_{\mathcal{A}}) - 2\theta^+\varphi^i(t_{\mathcal{A}})u_i^- - 2\bar\theta^+\bar\varphi^i(t_{\mathcal{A}})u_i^-$$
$$+ 3i\theta^+\bar\theta^+A^{(ik)}(t_{\mathcal{A}})u_i^-u_k^-. \tag{2.13}$$

The superfield $v(z)$ is constructed from the superfield $\mathcal{V}(\zeta,u)$ through the transform

$$v(t,\theta^i,\bar\theta_k) = \int du\mathcal{V}(t + 2i\theta^i\bar\theta^ku^+_{(i}u^-_{k)}, \theta^iu_i^+, \bar\theta^ku_k^+, u_l^\pm). \tag{2.14}$$

The constraints (2.8) now prove to be a consequence of the harmonic analyticity constraints $D^+\mathcal{V} = \bar D^+\mathcal{V} = 0$. The inverse expression of $\mathcal{V}$ through the superfield v is given by the relation [15]

$$\mathcal{V} = v + D^{++}V^{--}, \tag{2.15}$$

where V^{--} is some general harmonic superfield with the transformation law $\delta V^{--} = \Lambda^{--}$ with respect to the gauge transformations (2.12). In what follows, we make use of the identity [15]

$$(D^+\bar D^- - \bar D^+D^-)v = -2D^+\bar D^+V^{--}. \tag{2.16}$$

In addition to the superfield $v(z)$, we also incorporate the $\mathcal{N}=4$ *odd* superfields $w_a(z)$, $a=1,2$, $\tilde w_a = -w_a$, obeying the constraints (2.8)

$$D^iD_iw_a = \bar D_i\bar D^iw_a = 0, \qquad [D^i,\bar D_i]w_a = 0. \tag{2.17}$$

Similarly to (2.9), the constraints (2.17) have the solution

$$w_a(t,\theta_i,\bar\theta^i) = \mathrm{w}_a + \theta_iB_a^i + \bar\theta_i\bar B_a^i + \theta_i\bar\theta_k\rho_a^{ik} - \frac{i}{2}(\theta)^2\bar\theta_i\dot B_a^i$$
$$-\frac{i}{2}(\bar\theta)^2\theta_i\dot{\bar B}_a^i + \frac14(\theta)^2(\bar\theta)^2\ddot{\mathrm{w}}_a, \tag{2.18}$$

where the reality conditions for the component fields are $(w_a)^\dagger = -w_a$, $(B^i_a)^\dagger = \bar{B}_{ai}$, $(\rho^{ik}_a)^\dagger = \rho_{aik} = \rho_{a(ik)}$. In the harmonic superspace, the constraints (2.8) read

$$D^{++}w_a = 0, \qquad D^+D^-w_a = \bar{D}^+\bar{D}^-w_a = 0, \qquad (D^+\bar{D}^- + \bar{D}^+D^-)w_a = 0. \tag{2.19}$$

The transformation properties of the component fields in the expansion (2.18) under $\mathcal{N}=4$ supersymmetry are given by

$$\delta_\varepsilon \mathrm{w}_a = -\varepsilon_i B^i_a + \bar{\varepsilon}^i \bar{B}_{ai},$$
$$\delta_\varepsilon B^i_a = i\bar{\varepsilon}^i \dot{\mathrm{w}}_a - \bar{\varepsilon}_k \rho^{ki}_a, \qquad \delta_\varepsilon \bar{B}_{ai} = -i\varepsilon_i \dot{\mathrm{w}}_a - \varepsilon^k \rho_{aki},$$
$$\delta_\varepsilon \rho^{ik}_a = -2i(\varepsilon^{(i}\dot{B}^{k)}_a + \bar{\varepsilon}^{(i}\dot{\bar{B}}^{k)}_a). \tag{2.20}$$

As in (2.12), we can introduce the analytic prepotential superfields $\mathcal{W}_a(\zeta,u)$ defined up to the proper gauge transformations,

$$\mathcal{W}_a \Rightarrow \mathcal{W}'_a = \mathcal{W}_a + D^{++}\tilde{\Lambda}^{--}_a, \qquad \tilde{\Lambda}^{--}_a = \tilde{\Lambda}^{--}_a(\zeta,u). \tag{2.21}$$

In the Wess-Zumino gauge, these superfields read

$$\mathcal{W}_a(\zeta,u) = \mathrm{w}_a(t_A) - 2\theta^+ B^i_a(t_A)u^-_i - 2\bar{\theta}^+\bar{B}^i_a(t_A)u^-_i + 3\theta^+\bar{\theta}^+\rho^{(ik)}_a(t_A)u^-_i u^-_k. \tag{2.22}$$

The original superfield $w_a(z)$ is related to $\mathcal{W}_a(\zeta,u)$ by the transform

$$w_a(t,\theta^i,\bar{\theta}_k) = \int du \mathcal{W}_a(t + 2i\theta^i\bar{\theta}^k u^+_{(i}u^-_{k)}, \theta^i u^+_i, \bar{\theta}^k u^+_k, u^\pm_l). \tag{2.23}$$

The constraints (2.17) emerge as a consequence of the harmonic analyticity of $\mathcal{W}_a$: $D^+\mathcal{W}_a = \bar{D}^+\mathcal{W}_a = 0$. The superfields $\mathcal{W}_a$ are expressed through the superfields w_a as

$$\mathcal{W}_a = w_a + D^{++}W^{--}_a, \tag{2.24}$$

where W^{--}_a are some general Grassmann-odd harmonic superfields, such that $\delta W^{--}_a = \tilde{\Lambda}^{--}_a$ with respect to the gauge transformation (2.21). In what follows, we use the relations

$$(D^+\bar{D}^- - \bar{D}^+D^-)w_a = -2D^+\bar{D}^+W^{--}_a. \tag{2.25}$$

2. Multiplets $(\mathbf{0},\mathbf{4},\mathbf{4})$ and $(\mathbf{4},\mathbf{4},\mathbf{0})$

The multiplet $(\mathbf{0},\mathbf{4},\mathbf{4})$ is described by the fermionic analytic superfield Ψ^{+A}, $(\widetilde{\Psi^{+A}}) = \Psi^+_A$, $A = 1, 2$, which satisfies the constraint [22]

$$D^{++}\Psi^{+A} = 0. \tag{2.26}$$

The constraint (2.26) has the general solution

$$\Psi^{+A} = \psi^{iA}u^+_i + \theta^+F^A + \bar{\theta}^+\bar{F}^A - 2i\theta^+\bar{\theta}^+\dot{\psi}^{iA}u^-_i, \tag{2.27}$$

where component fields satisfy the reality conditions $(\psi^{iA})^\dagger = -\psi_{iA}$, $(F^A)^\dagger = \bar{F}_A$. The doublet index $A = 1, 2$ is rotated by some Pauli-Gürsey group $\mathrm{SU}(2)_{\mathrm{PG}}$ commuting with supersymmetry. The $\mathcal{N}=4$ supersymmetry transformations of the component fields have the form (see, e.g., [29])

$$\delta_\varepsilon\psi^{iA} = -(\varepsilon^i F^A + \bar{\varepsilon}^i\bar{F}^A), \qquad \delta_\varepsilon F^A = 2i\bar{\varepsilon}^k\dot{\psi}^A_k,$$
$$\delta_\varepsilon\bar{F}_A = 2i\varepsilon_k\dot{\psi}^k_A. \tag{2.28}$$

In the central basis, the constraint (2.26) and the analyticity conditions $D^+\Psi^{+A} = \bar{D}^+\Psi^{+A} = 0$ imply

$$\Psi^{+A}(z,u) = \Psi^{iA}(z)u^+_i, \qquad D^{(i}\Psi^{k)A}(z) = \bar{D}^{(i}\Psi^{k)A}(z) = 0, \tag{2.29}$$

where $(\Psi^{iA})^\dagger = -\Psi_{iA}$.

The multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ are described by the bosonic analytic superfields $\mathcal{Z}^{+A}_a$, $(\widetilde{\mathcal{Z}^{+A}_a}) = -\mathcal{Z}^+_{aA}$, $A = 1, 2$, $a = 1$, 2, which satisfy the harmonic constraint [22]

$$D^{++}\mathcal{Z}^{+A}_a = 0. \tag{2.30}$$

As a solution to this constraint, the superfields $\mathcal{Z}^{+A}_a$ have the following component expansions:

$$\mathcal{Z}^{+A}_a = z^{iA}_a u^+_i + \theta^+\pi^A_a + \bar{\theta}^+\bar{\pi}^A_a - 2i\theta^+\bar{\theta}^+\dot{z}^{iA}_a u^-_i, \tag{2.31}$$

where $(z^{iA}_a)^\dagger = z_{aiA}$, $(\pi^A_a)^\dagger = \bar{\pi}_{aA}$. The $\mathcal{N}=4$ supersymmetry transformations are realized on the component fields as (see, e.g., [29])

$$\delta_\varepsilon z^{iA}_a = -(\varepsilon^i\pi^A_a + \bar{\varepsilon}^i\bar{\pi}^A_a), \qquad \delta_\varepsilon\pi^A_a = 2i\bar{\varepsilon}^k\dot{z}^A_{ak},$$
$$\delta_\varepsilon\bar{\pi}_{aA} = 2i\varepsilon_k\dot{z}^k_{aA}. \tag{2.32}$$

In the central basis, the constraint (2.30) and the analyticity conditions $D^+\mathcal{Z}^{+A}_a = \bar{D}^+\mathcal{Z}^{+A}_a = 0$ imply

$$\mathcal{Z}^{+A}_a(z,u) = \mathcal{Z}^{iA}_a(z)u^+_i, \qquad D^{(i}\mathcal{Z}^{k)A}_a(z) = \bar{D}^{(i}\mathcal{Z}^{k)A}_a(z) = 0, \tag{2.33}$$

where nonharmonic $\mathcal{N}=4$ superfields $\mathcal{Z}^{iA}_a(z)$ are subject to the reality conditions $(\mathcal{Z}^{iA}_a)^\dagger = \mathcal{Z}_{aiA}$.

3. Implicit $\mathcal{N}=4$, $d=1$ supersymmetry

The extra implicit $\mathcal{N}=4$ supersymmetry is realized on the superfields $v(z)$ and $\Psi^{iA}(z)$ by the transformations [6,15]

$$\delta_\xi v = -\xi_{iA}\Psi^{iA}, \quad \delta_\xi \Psi^{iA} = \frac{1}{2}\xi^A_k(D^i\bar{D}^k - \bar{D}^i D^k)v, \qquad (2.34)$$

where $\xi_{iA} = (\xi^{iA})^*$ are fermionic parameters of the second $\mathcal{N}=4$ supersymmetry. In terms of the harmonic superfields $\mathcal{V}(\zeta,u)$ and $\Psi^{+A}(\zeta,u)$, the transformations (2.34) take the form [15]

$$\delta_\xi v = \xi^{-A}\Psi^+_A - \xi^{+A}\Psi^-_A, \qquad \delta_\xi \mathcal{V} = 2\xi^{-A}\Psi^+_A,$$
$$\delta_\xi \Psi^{+A} = D^+\bar{D}^+(\xi^{-A}v + \xi^{+A}V^{--}), \qquad (2.35)$$

where $\xi^{\pm A} = \xi^{iA}u^\pm_i$. The superfield transformations (2.34) amount to the following ones for the component fields:

$$\delta_\xi \mathrm{v} = -\xi_{iA}\psi^{iA}, \qquad \delta_\xi \varphi^i = \xi^{iA}F_A, \qquad \delta_\xi \bar{\varphi}_i = -\xi_{iA}\bar{F}^A,$$
$$\delta_\xi A_{ik} = 2\xi_{(iA}\dot{\psi}^A_{k)}, \qquad \delta_\xi \psi^{iA} = i\xi^{iA}\dot{\mathrm{v}} + i\xi^A_k A^{ik},$$
$$\delta_\xi F^A = -2i\xi^A_k\dot{\varphi}^k, \qquad \delta_\xi \bar{F}_A = 2i\xi^k_A\dot{\bar{\varphi}}_k. \qquad (2.36)$$

Thus, the $\mathcal{N}=4$ multiplets $(\mathbf{1},\mathbf{4},\mathbf{3})$ and $(\mathbf{0},\mathbf{4},\mathbf{4})$ in the model under consideration together constitute the $\mathcal{N}=8$ multiplet $(\mathbf{1},\mathbf{8},\mathbf{7})$ [6,9,11,12].

Similar implicit $\mathcal{N}=4$ supersymmetry transformations can be defined for the $\mathcal{N}=4$ superfields $w_a(z)$ and $\mathcal{Z}^{iA}_a(z)$. In the conventional superspace, these read

$$\delta_\xi w_a = -\xi_{iA}\mathcal{Z}^{iA}_a, \quad \delta_\xi \mathcal{Z}^{iA}_a = \frac{1}{2}\xi^A_k(D^i\bar{D}^k - \bar{D}^i D^k)w_a, \qquad (2.37)$$

whereas in harmonic space, the superfields $\mathcal{W}_a(\zeta,u)$ and $\mathcal{Z}^{+A}_a(\zeta,u)$ transform as

$$\delta_\xi w_a = \xi^{-A}\mathcal{Z}^+_{aA} - \xi^{+A}\mathcal{Z}^-_{aA}, \qquad \delta_\xi \mathcal{W}_a = 2\xi^{-A}\mathcal{Z}^+_{aA},$$
$$\delta_\xi \mathcal{Z}^{+A}_a = D^+\bar{D}^+(\xi^{-A}w_a + \xi^{+A}W^{--}_a). \qquad (2.38)$$

For the component fields, these transformations amount to

$$\delta_\xi \mathrm{w}_a = -\xi_{iA}z^{iA}_a, \qquad \delta_\xi B^i_a = \xi^{iA}\pi_{aA}, \qquad \delta_\xi \bar{B}_{ai} = -\xi_{iA}\bar{\pi}^A_a,$$
$$\delta_\xi \rho^{ik}_a = 2i\xi^{(i}_A\dot{z}^{k)A}_a, \qquad \delta_\xi z^{iA}_a = i\xi^{iA}\dot{\mathrm{w}}_a + \xi^A_k\rho^{ik}_a,$$
$$\delta_\xi \pi^A_a = -2i\xi^A_k\dot{B}^k_a, \qquad \delta_\xi \bar{\pi}_{aA} = 2i\xi^k_A\dot{\bar{B}}_{ak}. \qquad (2.39)$$

In the next section, we construct the interaction of all of these superfields, which will be invariant under the implicit $\mathcal{N}=4$ supersymmetry.

III. $\mathcal{N}=8$ INVARIANT COUPLING

As shown in [6,15], the action

$$-\frac{1}{2}\int \mu_{\mathcal{H}}v^2 + \frac{1}{2}\int \mu^{(-2)}_{\mathcal{A}}\Psi^{+A}\Psi^+_A \qquad (3.1)$$

is invariant with respect to the implicit $\mathcal{N}=4$ supersymmetry (2.34) and describes the free $\mathcal{N}=8$ multiplet $(\mathbf{1},\mathbf{8},\mathbf{7})$ in terms of $\mathcal{N}=4$ superfields.

Let us build the coupling of the multiplets v and Ψ^{iA} to the multiplets w_a, $a=1$, 2 and $\mathcal{Z}^{iA}_a$, $a=1$, 2. As the guiding principle, we take the requirement of implicit $\mathcal{N}=4$ supersymmetry [(2.34) and (2.37)]. The natural generalization of the second term in the action (1.1) is the action with the analytic Lagrangian $in^{ab}_{AB}\mathcal{V}\mathcal{Z}^{+A}_a\mathcal{Z}^{+B}_b$, where n^{ab}_{AB} are some constants. Then, the additional terms needed to ensure the implicit $\mathcal{N}=4$ supersymmetry (2.34) must have the form $im^{ab}_{AB}\mathcal{W}_a\mathcal{Z}^{+A}_b\Psi^{+B}$, where m^{ab}_{AB} are some constants. Thus, we start with the trial interaction Lagrangian in the form

$$i\int \mu^{(-2)}_{A}[n^{ab}_{AB}\mathcal{V}\mathcal{Z}^{+A}_a\mathcal{Z}^{+B}_b + m^{ab}_{AB}\mathcal{W}_a\mathcal{Z}^{+A}_b\Psi^{+B}]. \qquad (3.2)$$

Considering only variations $\delta_\xi\mathcal{V}$, $\delta_\xi\mathcal{W}_a$ and using (2.35) and (2.37), we obtain that the corresponding variation of the action (3.2) is equal to

$$-2i\int \mu^{(-2)}_{A}\xi^-_C[n^{ab}_{AB}\mathcal{Z}^{+A}_a\mathcal{Z}^{+B}_b\Psi^{+C} + m^{ab}_{AB}\mathcal{Z}^{+C}_a\mathcal{Z}^{+A}_b\Psi^{+B}]. \qquad (3.3)$$

The quantities ξ^-_1 and ξ^-_2 are independent. Therefore, the requirement that (3.3) vanishes amounts to the equations

$$(m^{ab}_{A1}\mathcal{Z}^{+2}_a\mathcal{Z}^{+B}_b)\Psi^{+1} + (n^{ab}_{AB}\mathcal{Z}^{+A}_a\mathcal{Z}^{+B}_b + m^{ab}_{A2}\mathcal{Z}^{+2}_a\mathcal{Z}^{+B}_b)\Psi^{+2} = 0,$$
$$(m^{ab}_{A2}\mathcal{Z}^{+1}_a\mathcal{Z}^{+B}_b)\Psi^{+2} + (n^{ab}_{AB}\mathcal{Z}^{+A}_a\mathcal{Z}^{+B}_b + m^{ab}_{A1}\mathcal{Z}^{+1}_a\mathcal{Z}^{+B}_b)\Psi^{+1} = 0. \qquad (3.4)$$

Since Ψ^{+1} and Ψ^{+2} are independent, these equations yield the following restrictions on the constants:

$$m^{ab}_{AB} = 2n^{ab}_{AB} = m\epsilon_{ab}\epsilon_{AB}, \qquad (3.5)$$

where m is a constant. Choosing $m=1/2$, we have[2]

$$i\int \mu^{(-2)}_{A}[\mathcal{V}\mathcal{Z}^{+A}_1\mathcal{Z}^+_{2A} + (\mathcal{W}_1\mathcal{Z}^{+A}_2 - \mathcal{W}_2\mathcal{Z}^{+A}_1)\Psi^+_A]. \qquad (3.6)$$

Let us check the invariance of (3.6) under the implicit $\mathcal{N}=4$ supersymmetry (2.34). Considering only variations

[2]The superfield action with other choices of the constant m is obtained from the action (3.6) by the following scale transformation: $\mathcal{Z}^{+A}_a \to (2m)^{1/2}\mathcal{Z}^{+A}_a$, $\mathcal{W}_a \to (2m)^{1/2}\mathcal{W}_a$.

$\delta_\xi \mathcal{V}$, $\delta_\xi \mathcal{W}_a$ and using Eqs. (2.35) and (2.37), we obtain that the corresponding variation of the action (3.6) is

$$2i \int \mu_{\mathcal{A}}^{(-2)} \xi^{-C} \Psi^{+D} \mathcal{Z}_1^{+A} \mathcal{Z}_2^{+B} (\epsilon_{AB}\epsilon_{CD} + \epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{DB}), \tag{3.7}$$

and it is identically zero. In addition, the nullifying of the set of such terms requires the use of two supermultiplets w_a and two supermultiplets $\mathcal{Z}_a^{+A}$ in our construction.

Let us next consider the variation of superfields $\mathcal{Z}_a^{+A}$, Ψ^{+A} in the action (3.6).

Consider first the variation of the first term in (3.6) under the transformation of $\mathcal{Z}_a^{+A}$ and the variation of the second term under that of Ψ^{+A}. Taking $\delta_\xi \mathcal{Z}_a^{+A}$ from (2.38) and $\delta_\xi \Psi^{+A}$ from (2.35) and using the relation $\mu_{\mathcal{H}} = \mu_{\mathcal{A}}^{(-2)} D^+ \bar{D}^+$ [see Eq. (2.7)] for the integration measures, we obtain

$$i \int \mu_{\mathcal{H}} [\mathcal{V}(\xi^{-A} w_1 + \xi^{+A} W_1^{--}) \mathcal{Z}_{2A}^+ - \mathcal{V}(\xi^{-A} w_2 + \xi^{+A} W_2^{--}) \mathcal{Z}_{1A}^+ + (\mathcal{W}_1 \mathcal{Z}_2^{+A} - \mathcal{W}_2 \mathcal{Z}_1^{+A})(\xi_A^- v + \xi_A^+ V^{--})]. \tag{3.8}$$

Now we make the following substitutions in (3.8): $\mathcal{V} = v + D^{++} V^{--}$ [Eq. (2.15)] and $\mathcal{W}_a = w_a + D^{++} W_a^{--}$ [Eq. (2.24)]. Half of the terms in the resulting expression contain the superfield $\mathcal{Z}_1^{+A}$, while the other half contains $\mathcal{Z}_2^{+A}$. Those terms in (3.8) that involve the superfield $\mathcal{Z}_2^{+A}$ are collected as

$$i \int \mu_{\mathcal{H}} [(\xi^{-A} w_1 + \xi^{+A} W_1^{--})(v + D^{++} V^{--}) \mathcal{Z}_{2A}^+ + (\xi^{-A} v + \xi^{+A} V^{--})(w_1 + D^{++} W_1^{--}) \mathcal{Z}_{2A}^+]. \tag{3.9}$$

Making in (3.9) the substitutions $\xi^{+A} = D^{++} \xi^{-A}$ and integrating by parts with respect to D^{++}, we find that the only surviving term is

$$-2i \int \mu_{\mathcal{H}} v w_1 \xi^{-A} \mathcal{Z}_{2A}^+.$$

It can be rewritten as

$$-i \int \mu_{\mathcal{H}} v w_1 \delta_\xi \mathcal{W}_2 = -i \int \mu_{\mathcal{H}} v w_1 (\delta_\xi w_2 + D^{++} \delta_\xi W_2^{--}) = -i \int \mu_{\mathcal{H}} v w_1 \delta_\xi w_2. \tag{3.10}$$

In a similar way, we can show that the terms in (3.8) that contain the superfield $\mathcal{Z}_1^{+A}$ are reduced to $-i \int \mu_{\mathcal{H}} v (\delta_\xi w_1) w_2$. Thus, the total variation (3.8) finally proves to be equal to $-i \int \mu_{\mathcal{H}} v \delta_\xi (w_1 w_2)$.

It remains to take into account the variation of the second term in (3.6) under the transformations $\delta_\xi \mathcal{Z}_a^{+A}$. Using Eq. (2.38) for $\delta_\xi \mathcal{Z}_a^{+A}$, we find that this variation takes the form

$$i \int \mu_{\mathcal{H}} [\mathcal{W}_1 (\xi^{-A} w_2 + \xi^{+A} W_2^{--}) \Psi_A^+ - \mathcal{W}_2 (\xi^{-A} w_1 + \xi^{+A} W_1^{--}) \Psi_A^+], \tag{3.11}$$

where we made use of the relation $\mu_{\mathcal{H}} = \mu_{\mathcal{A}}^{(-2)} D^+ \bar{D}^+$ [see Eq. (2.7)] for the integration measures. Substituting the expressions $\mathcal{W}_a = w_a + D^{++} W_a^{--}$ [Eq. (2.24)] here, using the conditions $D^{++} \Psi^{+A} = 0$ [Eq. (2.27)], and representing $\xi^{+A} = D^{++} \xi^{-A}$, we find that, modulo a total harmonic derivative in the integrand, the expression (3.11) is reduced to

$$-2i \int \mu_{\mathcal{H}} (\xi^{-A} \Psi_A^+) w_1 w_2 = -i \int \mu_{\mathcal{H}} (\delta_\xi \mathcal{V}) w_1 w_2 = -i \int \mu_{\mathcal{H}} (\delta_\xi v) w_1 w_2. \tag{3.12}$$

In deriving (3.12), we used that $\mathcal{V} = v + D^{++} V^{--}$ [Eq. (2.15)], $\delta_\xi \mathcal{V} = \delta_\xi v + D^{++} \delta_\xi V^{--}$, and omitted a total harmonic derivative thanks to the condition $D^{++} w_a = 0$ [Eq. (2.19)].

Thus, the total variation of the action (3.6) under the implicit $\mathcal{N} = 4$ supersymmetry is reduced to

$$-i \int \mu_{\mathcal{H}} \delta_\xi (v w_1 w_2). \tag{3.13}$$

As a result, the sum of the action (3.6) and the action

$$i \int \mu_{\mathcal{H}} v w_1 w_2 \tag{3.14}$$

is invariant with respect to the implicit $\mathcal{N} = 4$ supersymmetry [(2.34) and (2.37)].

Thus, we have obtained the $\mathcal{N} = 8$ supersymmetry-invariant action, which is the sum of the actions (3.1), (3.6), and (3.14),

$$S = -\frac{1}{2} \int \mu_{\mathcal{H}} v^2 + \frac{1}{2} \int \mu_{\mathcal{A}}^{(-2)} \Psi^{+A} \Psi_A^+ + \frac{i}{2} \epsilon_{ab} \int \mu_{\mathcal{H}} v w_a w_b + \frac{i}{2} \epsilon_{ab} \int \mu_{\mathcal{A}}^{(-2)} [\mathcal{V} \mathcal{Z}_a^{+A} \mathcal{Z}_{bA}^+ + 2 \mathcal{W}_a \mathcal{Z}_b^{+A} \Psi_A^+]. \tag{3.15}$$

Let us demonstrate that the action (3.15) is a generalization of the action (1.1) to the case of two semidynamic multiplets $\mathcal{Z}_a^{+A}$. Introducing the superfields Z^{+A} and Y^{+A} by the relations

$$\mathcal{Z}_1^{+A} = Z^{+A} + i(\sigma_3)^A{}_B Y^{+B}, \quad \mathcal{Z}_2^{+A} = Y^{+A} - i(\sigma_3)^A{}_B Z^{+B}, \tag{3.16}$$

we obtain

$$\mathcal{Z}_1^{+A}\mathcal{Z}_{2A}^{+} = -iZ^{+A}Z^{+B}(\sigma_3)_{AB} - iY^{+A}Y^{+B}(\sigma_3)_{AB}, \qquad (3.17)$$

where $(\sigma_3)_{AB} = (\sigma_3)_{(AB)} = \epsilon_{AC}(\sigma_3)^C{}_B$. Thus, in the limit $Y^{+A}=0$, $\Psi^{+A}=0$, $w_a=0$, the action (3.15) is reduced to the action (1.1) with $c_{AB}=(\sigma_3)_{AB}$. Of course, when performing the transition $\mathcal{Z}_a^{+A}\to(Z^{+A},Y^{+A})$ [(3.16)], the Pauli-Gürsey SU(2) symmetry acting on the capital indices A, B gets broken.

The superfield action (3.15) can be cast in a more suggestive form,

$$S = -\frac{1}{2}\int\mu_{\mathcal{H}}\left(v-\frac{i}{2}\epsilon_{ab}w_aw_b\right)^2 + \frac{1}{2}\int\mu_{\mathcal{A}}^{(-2)}(\Psi^{+A}+i\epsilon_{ab}\mathcal{W}_a\mathcal{Z}_b^{+A})(\Psi_A^{+}+i\epsilon_{cd}\mathcal{W}_c\mathcal{Z}_{dA}^{+})$$
$$+\frac{i}{2}\int\mu_{\mathcal{A}}^{(-2)}\left(\mathcal{V}-\frac{i}{2}\epsilon_{ab}\mathcal{W}_a\mathcal{W}_b\right)\epsilon_{cd}\mathcal{Z}_c^{+A}\mathcal{Z}_{dA}^{+}. \qquad (3.18)$$

The final action (3.18) contains the scalar composite superfield $v-\frac{i}{2}\epsilon_{ab}w_aw_b$, the scalar composite analytic superfield $\mathcal{V}-\frac{i}{2}\epsilon_{ab}\mathcal{W}_a\mathcal{W}_b$, the analytic composite superfields $\Psi^{+A}+i\epsilon_{ab}\mathcal{W}_a\mathcal{Z}_b^{+A}$, and the analytic superfields $\mathcal{Z}_a^{+A}$. It is worth pointing out that, although the superfield $\mathcal{V}$ and superfields $\mathcal{W}_a$ are prepotentials for the superfields v and w_a, respectively, the composite superfield $\mathcal{V}-\frac{i}{2}\epsilon_{ab}\mathcal{W}_a\mathcal{W}_b$ is by no means a prepotential for the composite superfield $v-\frac{i}{2}\epsilon_{ab}w_aw_b$.

IV. COMPONENT FORM OF THE $\mathcal{N}=8$ ACTION

The superfields entering the action (3.18) have the following component expansions:

$$v-\frac{i}{2}\epsilon_{ab}w_aw_b = \left(\mathrm{v}-\frac{i}{2}\epsilon_{ab}\mathrm{w}_a\mathrm{w}_b\right) + \theta_i(\varphi^i+i\epsilon_{ab}\mathrm{w}_aB_b^i) + \bar\theta_i(\bar\varphi^i+i\epsilon_{ab}\mathrm{w}_a\bar B_b^i)$$
$$+\frac{i}{4}(\theta)^2\epsilon_{ab}B_a^iB_{bi} - \frac{i}{4}(\bar\theta)^2\epsilon_{ab}\bar B_a^i\bar B_{bi} + i\theta_i\bar\theta_k[A^{ik}-\epsilon_{ab}(\mathrm{w}_a\rho_b^{ik}+B_a^i\bar B_b^k)]$$
$$-\frac{i}{2}(\theta)^2\bar\theta_i[\dot\varphi^i+\epsilon_{ab}(i\mathrm{w}_a\dot B_b^i+B_{ak}\rho_b^{ik})] - \frac{i}{2}(\bar\theta)^2\theta_i[\dot{\bar\varphi}^i+\epsilon_{ab}(i\mathrm{w}_a\dot{\bar B}_b^i+\bar B_{ak}\rho_b^{ik})]$$
$$+\frac{1}{4}(\theta)^2(\bar\theta)^2\left[\ddot{\mathrm{v}}-i\epsilon_{ab}\mathrm{w}_a\ddot{\mathrm{w}}_b-\frac{i}{2}\epsilon_{ab}\rho_a^{ik}\rho_{bik}+\epsilon_{ab}(B_a^i\dot{\bar B}_{bi}-\dot B_a^i\bar B_{bi})\right], \qquad (4.1)$$

$$\mathcal{V}-\frac{i}{2}\epsilon_{ab}\mathcal{W}_a\mathcal{W}_b = \left(\mathrm{v}-\frac{i}{2}\epsilon_{ab}\mathrm{w}_a\mathrm{w}_b\right) - 2\theta^+(\varphi^-+i\epsilon_{ab}\mathrm{w}_aB_b^-) - 2\bar\theta^+(\bar\varphi^-+i\epsilon_{ab}\mathrm{w}_a\bar B_b^-)$$
$$+i\theta^+\bar\theta^+(3A^{--}-3\epsilon_{ab}\mathrm{w}_a\rho_b^{--}-4\epsilon_{ab}B_a^-\bar B_b^-), \qquad (4.2)$$

$$\Psi^{+A}+i\epsilon_{ab}\mathcal{W}_a\mathcal{Z}_b^{+A} = (\psi^{+A}+i\epsilon_{ab}w_az_b^{+A})$$
$$+\theta^+[F^A+i\epsilon_{ab}(\mathrm{w}_a\pi_b^A+2z_a^+B_b^-)] + \bar\theta^+[\bar F^A+i\epsilon_{ab}(\mathrm{w}_a\bar\pi_b^A+2z_a^+\bar B_b^-)]$$
$$-2i\theta^+\bar\theta^+\left[\dot\psi^{-A}+\epsilon_{ab}\left(i\mathrm{w}_a\dot z_b^{-A}-\frac{3}{2}\rho_a^{--}z_b^{+A}-B_a^-\bar\pi_b^A-\bar B_a^-\pi_b^A\right)\right]. \qquad (4.3)$$

Inserting (4.1) into the first term of (3.18), we see that this term gives rise to the following component action:

$$\int dt(-\mathrm{v}\ddot{\mathrm{v}}+i\ddot{\mathrm{v}}\mathrm{w}_1\mathrm{w}_2+i\mathrm{v}\ddot{\mathrm{w}}_1\mathrm{w}_2+i\mathrm{v}\mathrm{w}_1\ddot{\mathrm{w}}_2).$$

Up to a total derivative, this action equals

$$\int dt(\dot{\mathrm{x}}\,\dot{\mathrm{x}}-i\epsilon_{ab}\mathrm{x}\dot{\mathrm{w}}_a\dot{\mathrm{w}}_b), \qquad (4.4)$$

where

$$\mathrm{x} := \mathrm{v}-\frac{i}{2}\epsilon_{ab}\mathrm{w}_a\mathrm{w}_b. \qquad (4.5)$$

Thus, the model under consideration contains two fermionic fields $\mathrm{w}_a(t)$, $a = 1, 2$, with the second-order Lagrangians for them.

We shall try to bring the action (3.18) to a form in which it only depends on $\dot{\mathrm{w}}_a(t)$. By performing the "oxidation procedure" [4,25,30], in which the quantities $\dot{\mathrm{w}}_a(t)$ are replaced by new auxiliary variables, we get rid of second-order terms in the derivatives of fermionic fields.

In terms of the new variables (4.5) and

$$\phi^i \coloneqq \varphi^i + i\epsilon_{ab}\mathrm{w}_a B^i_b, \qquad \bar{\phi}^i \coloneqq \bar{\varphi}^i + i\epsilon_{ab}\mathrm{w}_a \bar{B}^i_b, \qquad C^{ik} \coloneqq A^{ik} - \epsilon_{ab}\mathrm{w}_a\rho^{ik}_b, \tag{4.6}$$

$$\chi^{iA} \coloneqq \psi^{iA} + i\epsilon_{ab}\mathrm{w}_a z^{iA}_b, \qquad G^A \coloneqq F^A - i\epsilon_{ab}\mathrm{w}_a\pi^A_b, \qquad \bar{G}^A \coloneqq \bar{F}^A - i\epsilon_{ab}\mathrm{w}_a\bar{\pi}^A_b, \tag{4.7}$$

the component off-shell expansions of the superfields (4.1), (4.2), and (4.3) are written as

$$\begin{aligned} v - \frac{i}{2}\epsilon_{ab}w_a w_b = {} & \mathrm{x} + \theta_i\phi^i + \bar{\theta}_i\bar{\phi}^i + \frac{i}{4}(\theta)^2\epsilon_{ab}B^i_a B_{bi} - \frac{i}{4}(\bar{\theta})^2\epsilon_{ab}\bar{B}^i_a\bar{B}_{bi} + i\theta_i\bar{\theta}_k[C^{ik} - \epsilon_{ab}B^i_a\bar{B}^k_b] \\ & - \frac{i}{2}(\theta)^2\bar{\theta}_i[\dot{\phi}^i + \epsilon_{ab}(iB^i_a\dot{\mathrm{w}}_b + B_{ak}\rho^{ik}_b)] - \frac{i}{2}(\bar{\theta})^2\theta_i[\dot{\bar{\phi}}^i + \epsilon_{ab}(i\bar{B}^i_a\dot{\mathrm{w}}_b + \bar{B}_{ak}\rho^{ik}_b)] \\ & + \frac{1}{4}(\theta)^2(\bar{\theta})^2\left[\ddot{\mathrm{x}} + i\epsilon_{ab}\dot{\mathrm{w}}_a\dot{\mathrm{w}}_b - \frac{i}{2}\epsilon_{ab}\rho^{ik}_a\rho_{bik} + \epsilon_{ab}(B^i_a\dot{\bar{B}}_{bi} - \dot{B}^i_a\bar{B}_{bi})\right], \end{aligned} \tag{4.8}$$

$$\mathcal{V} - \frac{i}{2}\epsilon_{ab}\mathcal{W}_a\mathcal{W}_b = \mathrm{x} - 2\theta^+\phi^- - 2\bar{\theta}^+\bar{\phi}^- + i\theta^+\bar{\theta}^+(3C^{--} - 4\epsilon_{ab}B^-_a\bar{B}^-_b), \tag{4.9}$$

$$\begin{aligned} \Psi^{+A} + i\epsilon_{ab}\mathcal{W}_a\mathcal{Z}^{+A}_b = {} & \chi^{+A} + \theta^+(G^A + 2i\epsilon_{ab}z^{+A}_a B^-_b) + \bar{\theta}^+(\bar{G}^A + 2i\epsilon_{ab}z^{+A}_a\bar{B}^-_b) \\ & - 2i\theta^+\bar{\theta}^+\left[\dot{\chi}^{-A} - \epsilon_{ab}\left(i\dot{\mathrm{w}}_a z^{-A}_b + \frac{3}{2}\rho^{--}_a z^{+A}_b + B^-_a\bar{\pi}^A_b + \bar{B}^-_a\pi^A_b\right)\right]. \end{aligned} \tag{4.10}$$

In the expansions of the superfields (4.8), (4.9), (4.10), the derivatives $\dot{\mathrm{w}}_a(t)$ are present, but no fermionic fields $\mathrm{w}_a(t)$ appear on their own. Therefore, the component action contains only $\dot{\mathrm{w}}_a(t)$, which can be replaced [4,25,30] by new fields,

$$\zeta_a(t) \coloneqq \dot{\mathrm{w}}_a(t). \tag{4.11}$$

In terms of these variables, the $\mathcal{N} = 4$ supersymmetry transformation (2.20) takes the form

$$\begin{aligned} &\delta_\epsilon\zeta_a = -\epsilon_i\dot{B}^i_a + \bar{\epsilon}^i\dot{\bar{B}}_{ai}, \\ &\delta_\epsilon B^i_a = i\bar{\epsilon}^i\zeta_a - \bar{\epsilon}_k\rho^{ki}_a, \qquad \delta_\epsilon\bar{B}_{ai} = -i\epsilon_i\zeta_a - \epsilon^k\rho_{aki}, \\ &\delta_\epsilon\rho^{ik}_a = -2i(\epsilon^{(i}\dot{B}^{k)}_a + \bar{\epsilon}^{(i}\dot{\bar{B}}^{k)}_a). \end{aligned} \tag{4.12}$$

After combining B^i_a, $\bar{B}^i_a$, ζ_a, ρ^{ik}_a into the new fields

$$\begin{aligned} &f_a{}^i{}_{i'} = (f_a{}^i{}_{i'=1}, f_a{}^i{}_{i'=2}) \coloneqq (B^i_a, \bar{B}^i_a), \\ &\omega_{ai}{}^k \coloneqq i\zeta_a\delta^k_i + \rho_{ai}{}^k, \end{aligned} \tag{4.13}$$

the transformations (4.12) are rewritten as

$$\delta f^{ii'}_a = \varepsilon^{ki'}\omega_{ak}{}^i, \qquad \delta\omega^{ik}_a = -2i\varepsilon^{ij'}\dot{f}_a{}^k{}_{j'}, \tag{4.14}$$

where the infinitesimal parameters ε^i, $\bar{\varepsilon}^i$ are joined into the $\mathrm{SU}_L(2)\times\mathrm{SU}_R(2)$ bispinor $\varepsilon^{ii'}$: $\varepsilon^{ii'} = (\varepsilon^{ii'=1}, \varepsilon^{ii'=2}) = (\varepsilon^i, \bar{\varepsilon}^i)$. The indices $i = 1, 2$ and $i' = 1, 2$ are acted upon by the $\mathrm{SU}_L(2)$ and $\mathrm{SU}_R(2)$ groups, respectively, which form the automorphism group SO(4) of the $\mathcal{N} = 4$ superalgebra. For each value of the index $a = 1, 2$, the bosonic $d = 1$ fields $f_a{}^i{}_{i'}$ and fermionic $d = 1$ fields ω^{ik}_a are exactly component fields of the semidynamical $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ mirror (or twisted) multiplet, which is described by the superfield $q^{+A'}$ in the biharmonic space [see Eq. (4.7) in [31] and Appendix]. The transformations (4.14) are similar to the transformations (2.32). But, the Pauli-Gürsey group acting on the index A' of the mirror multiplet $q^{+A'}$ in the present case was chosen to coincide with the $\mathrm{SU}_L(2)$ group acting on the index i of the original $d = 1$ fields. Thus, after the oxidation procedure, two fermionic $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplets with noncanonical kinetic terms for fermions transform into two semidynamical $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ mirror multiplets with auxiliary fermions. The superfield meaning of the mutual conversion of different usual and mirror $\mathcal{N} = 4$ multiplets was clarified in [26].

It is important that the $\mathcal{N}=4$ supersymmetry transformations of the variables (4.5), (4.6), and (4.7) are expressed only through new variables:

$$\delta_\varepsilon \mathrm{x} = -\varepsilon_k \phi^k + \bar\varepsilon^k \bar\phi_k, \tag{4.15}$$

$$\delta_\varepsilon \phi^i = i\bar\varepsilon^i \dot{\mathrm{x}} - i\bar\varepsilon_k (C^{ki} + \epsilon_{ab}\bar B^k_a B^i_b) + \frac{i}{2}\varepsilon^i \epsilon_{ab} B_{ak} B^k_b, \tag{4.16}$$

$$\delta_\varepsilon \bar\phi_i = i\varepsilon_i \dot{\mathrm{x}} - i\varepsilon^k (C_{ki} - \epsilon_{ab} B_{ak}\bar B_{bi}) + \frac{i}{2}\varepsilon^i \epsilon_{ab} B_{ak} B^k_b, \tag{4.17}$$

$$\begin{aligned}\delta_\varepsilon C^{ik} &= -2(\varepsilon^{(i}\dot\phi^{k)} + \bar\varepsilon^{(i}\dot{\bar\phi}^{k)}) - 2i\epsilon_{ab}\zeta_a(\varepsilon^{(i}B^{k)}_b + \bar\varepsilon^{(i}\bar B^{k)}_b) \\ &\quad + \epsilon_{ab}(\varepsilon_j B^j_b + \bar\varepsilon_j \bar B^j_b)\rho^{ik}_b,\end{aligned} \tag{4.18}$$

$$\delta_\varepsilon \chi^{iA} = -\varepsilon^i G^A - \bar\varepsilon^i \bar G^A - i\varepsilon_k \epsilon_{ab} B^k_a z^{iA}_b - i\bar\varepsilon_k \epsilon_{ab}\bar B^k_a z^{iA}_b, \tag{4.19}$$

$$\begin{aligned}\delta_\varepsilon G^A &= 2i\bar\varepsilon^k \dot\chi^A_k + 2\bar\varepsilon^k \epsilon_{ab}\zeta_a z^A_{bk} + i\varepsilon_k \epsilon_{ab} B^k_a \pi^A_b \\ &\quad + i\bar\varepsilon_k \epsilon_{ab}\bar B^k_a \pi^A_b,\end{aligned} \tag{4.20}$$

$$\begin{aligned}\delta_\varepsilon \bar G^A &= 2i\varepsilon_k \dot\chi^{kA} + 2\varepsilon_k \epsilon_{ab}\zeta_a z^{kA}_b + i\varepsilon_k \epsilon_{ab} B^k_a \bar\pi^A_b \\ &\quad + i\bar\varepsilon_k \epsilon_{ab}\bar B^k_a \bar\pi^A_b.\end{aligned} \tag{4.21}$$

The full set of transformations of the new variables also includes the explicit $\mathcal{N}=4$ supersymmetry transformations $\delta_\varepsilon z^{iA}_a$, $\delta_\varepsilon \pi^A_a$, $\delta_\varepsilon \bar\pi_{aA}$ given in (2.32).

On the other hand, the implicit $\mathcal{N}=4$ supersymmetry transformations (2.39) involving the variables (4.5), (4.6), and (4.7) are of the form

$$\begin{aligned}\delta_\xi \mathrm{x} &= -\xi_{iA}\chi^{iA}, \\ \delta_\xi \chi^{iA} &= i\xi^{iA}\dot{\mathrm{x}} + i\xi^A_k C^{ik} + \xi_{kB}\epsilon_{ab} z^{kB}_a z^{iA}_b,\end{aligned} \tag{4.22}$$

$$\delta_\xi \zeta_a = -\xi_{iA}\dot z^{iA}_a, \qquad \delta_\xi z^{iA}_a = i\xi^{iA}\zeta_a + i\xi^A_k \rho^{ik}_a, \tag{4.23}$$

$$\delta_\xi C^{ik} = 2\xi^{(i}_A \dot\chi^{k)A} - 2i\xi^{(i}_A \epsilon_{ab}\zeta_a z^{k)A}_b + \xi_{jA}\epsilon_{ab} z^{jA}_a \rho^{ik}_b, \tag{4.24}$$

$$\begin{aligned}\delta_\xi \phi^i &= \xi^{iA} G_A - i\xi_{kA}\epsilon_{ab} z^{kA}_a B^i_b, \\ \delta_\xi \bar\phi^i &= \xi^{iA}\bar G_A - i\xi_{kA}\epsilon_{ab} z^{kA}_a \bar B^i_b.\end{aligned} \tag{4.25}$$

$$\begin{aligned}\delta_\xi G^A &= -2i\xi^A_k \dot\phi^k - 2\xi^A_k \epsilon_{ab}\zeta_a B^i_b + i\xi_{iB}\epsilon_{ab} z^{iB}_a \pi^A_b, \\ \delta_\xi \bar G^A &= -2i\xi^A_k \dot{\bar\phi}^k - 2\xi^A_k \epsilon_{ab}\zeta_a \bar B^i_b + i\xi_{iB}\epsilon_{ab} z^{iB}_a \bar\pi^A_b,\end{aligned} \tag{4.26}$$

and they also contain only new variables. The remaining variations $\delta_\xi B^i_a$, $\delta_\xi \bar B_{ai}$, $\delta_\xi \rho^{ik}_a$, $\delta_\xi \pi^A_a$, $\delta_\xi \bar\pi_{aA}$ from the complete set of transformations of new variables with respect to the implicit $\mathcal{N}=4$ supersymmetry are presented in (2.39).

Thus, the transformations of all new variables (4.12)–(4.26) with respect to both explicit and implicit $\mathcal{N}=4$ supersymmetries are expressed only in terms of the new variables. Therefore, the Lagrangian written in terms of new variables, involving (4.11), is invariant under the total $\mathcal{N}=8$ supersymmetry.

Using the component expansions (2.31), (4.9), and (4.10), we obtain

$$\begin{aligned}&\frac{\partial}{\partial\bar\theta^+}\frac{\partial}{\partial\theta^+}[(\Psi^{+A} + i\epsilon_{ab}\mathcal{W}_a \mathcal{Z}^{+A}_b)(\Psi^+_A + i\epsilon_{cd}\mathcal{W}_c \mathcal{Z}^+_{dA})] \\ &= -4i\chi^{+A}\left[\dot\chi^-_A - \epsilon_{ab}\left(i\zeta_a z^-_{bA} + \frac{3}{2}\rho^{--}_a z^+_{bA} + B^-_a \bar\pi_{bA} + \bar B^-_a \pi_{bA}\right)\right] \\ &\quad + 2(G^A + 2i\epsilon_{ab} z^{+A}_a B^-_b)(\bar G_A + 2i\epsilon_{cd} z^+_{cA}\bar B^-_d),\end{aligned} \tag{4.27}$$

$$\begin{aligned}&\frac{\partial}{\partial\bar\theta^+}\frac{\partial}{\partial\theta^+}\left[\left(\mathcal{V} - \frac{i}{2}\epsilon_{ab}\mathcal{W}_a\mathcal{W}_b\right)\epsilon_{cd}\mathcal{Z}^{+A}_c \mathcal{Z}^+_{dA}\right] \\ &= -2\mathrm{x}\epsilon_{ab}(2iz^{+A}_a \dot z^-_{bA} + \pi^A_a \bar\pi_{bA}) + 4\epsilon_{ab} z^{+A}_a(\phi^-\bar\pi_{bA} - \bar\phi^-\pi_{bA}) \\ &\quad + i\epsilon_{ab} z^{+A}_a z^+_{bA}(3C^{--} - 4\epsilon_{cd} B^-_c \bar B^-_d).\end{aligned} \tag{4.28}$$

Taking into account (4.8), (4.27), and (4.28) and performing integration over the Grassmann coordinates [we use $\int \mu_{\mathcal{H}}(\theta)^2(\bar\theta)^2 K(t) = -4\int dt K(t)$, $\int \mu^{(-2)}_{\mathcal{A}}\theta^+\bar\theta^+ N(t_{\mathcal{A}}) = \int dt_{\mathcal{A}} N(t_{\mathcal{A}})$], as well as over harmonics, we derive the off-shell component Lagrangian $L(t)$ corresponding to the action $S = \int dt L(t)$, defined in (3.18).

This Lagrangian has a somewhat cumbersome form because of the large number of terms present in it:

$$\begin{aligned}L &= \dot{\mathrm{x}}\dot{\mathrm{x}} + \mathrm{x}\epsilon_{ab}(z^{iA}_a \dot z_{biA} + \dot B^i_a \bar B_{bi} - B^i_a \dot{\bar B}_{bi}) - i\chi^{iA}\dot\chi_{iA} - i\bar\phi^i\dot\phi_i + i\dot{\bar\phi}^i\phi_i - i\mathrm{x}\epsilon_{ab}\zeta_a\zeta_b - \epsilon_{ab}(\bar\phi_i B^i_a - \phi_i \bar B^i_a - \chi^{iA} z_{aiA})\zeta_b \\ &\quad + \frac{i}{2}\mathrm{x}\epsilon_{ab}\rho^{ik}_a \rho_{bik} + i\epsilon_{ab}(\bar\phi^i B^k_a - \phi^i \bar B^k_a + \chi^{iA} z^k_{aA})\rho_{bik} - i\mathrm{x}\epsilon_{ab}\pi^A_a \bar\pi_{bA} + i\epsilon_{ab}\chi^{iA} B_{ai}\bar\pi_{bA} - i\epsilon_{ab}\pi^A_a \chi^i_A \bar B_{bi} \\ &\quad + \frac{1}{2}(C^{ik} - \epsilon_{ab} B^{(i}_a \bar B^{k)}_b)(C_{ik} - \epsilon_{cd} B_{ci}\bar B_{dk}) - \frac{1}{2}\epsilon_{ab} z^{iA}_a z^k_{bA} C_{ik} + G^A \bar G_A + iG^A \epsilon_{ab} z^i_{aA}\bar B_{bi} + i\epsilon_{ab} z^i_{aA} B_{bi}\bar G_A \\ &\quad + \frac{1}{4}(\epsilon_{ab} B^i_a \bar B_{bi})^2 - \frac{1}{4}\epsilon_{ab} B^i_a B_{bi}\epsilon_{cd}\bar B^k_c \bar B_{dk} - \frac{2}{3}\epsilon_{ab} z^{(i}_{aA} z^{k)A}_b \epsilon_{cd} B_{ci}\bar B_{dk} + \frac{4}{3}\epsilon_{ab}\epsilon_{cd} z^{(i}_{aA} z^{k)A}_c B_{bi}\bar B_{dk}.\end{aligned} \tag{4.29}$$

In (4.29) the bosonic fields C^{ik}, G^A, $\bar{G}^A$ as well as the fermionic fields ρ_a^{ik}, π_a^A, $\bar{\pi}_{aA}$, ζ_a are auxiliary. After eliminating these auxiliary fields by their equations of motion, we obtain the following on-shell Lagrangian:

$$
\begin{aligned}
L = {}& \dot{\mathrm{x}}\dot{\mathrm{x}} + \mathrm{x}\epsilon_{ab}(z_a^{iA}\dot{z}_{biA} + \dot{B}_a^i\bar{B}_{bi} - B_a^i\dot{\bar{B}}_{bi}) - i\chi^{iA}\dot{\chi}_{iA} - i\bar{\phi}^i\dot{\phi}_i + i\dot{\bar{\phi}}^i\phi_i \\
& - \frac{i}{2\mathrm{x}}\epsilon_{ab}(\bar{\phi}^i B_a^k - \phi^i\bar{B}_a^k + \chi^{iA}z_{aA}^k)(\bar{\phi}_i B_{bk} - \phi_i\bar{B}_{bk} + \chi_i^B z_{bkB}) - \frac{i}{\mathrm{x}}\epsilon_{ab}\chi^{iA}B_{ai}\chi_A^k\bar{B}_{bk} \\
& + \frac{1}{4}(\epsilon_{ab}B_a^i\bar{B}_{bi})^2 - \frac{1}{4}\epsilon_{ab}B_a^iB_{bi}\epsilon_{cd}\bar{B}_c^k\bar{B}_{dk} - \frac{1}{8}\epsilon_{ab}z_{aA}^{(i}z_b^{k)A}\epsilon_{cd}z_{ciB}z_{dk}^B \\
& - \frac{1}{6}\epsilon_{ab}z_{aA}^{(i}z_b^{k)A}\epsilon_{cd}B_{ci}\bar{B}_{dk} + \frac{4}{3}\epsilon_{ab}\epsilon_{cd}z_{aA}^{(i}z_c^{k)A}B_{bi}\bar{B}_{dk} - \epsilon_{ab}z_{aA}^iB_{bi}\epsilon_{cd}z_c^{kA}\bar{B}_{dk}.
\end{aligned}
\tag{4.30}
$$

As follows from the Lagrangian (4.30), the bosonic variable x and fermionic variables χ^{ia}, ϕ_i are dynamical, while the bosonic variables z_a^{iA}, B_a^i, $\bar{B}_a^i$ have the kinetic terms of the first order in ∂_t and so are semidynamical.

Thus, under $\mathcal{N}=8$ supersymmetrization, we deal with the $\mathcal{N}=4$ system which involves on the mass shell one dynamical bosonic field x, two semidynamical bosonic fields z_a^{iA}, and additional dynamical fermionic fields χ^{iA} and semidynamical bosonic fields B_a^i, $\bar{B}_a^i$. It follows from the transformations of the implicit $\mathcal{N}=4$ supersymmetry given above that the bosonic $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets, the standard one (z_a^{iA}, π_a^A) and the mirror one $(B_a^i, \bar{B}_a^i, \zeta_a, \rho_a^{(ik)})$, are transformed through each other and so together constitute the multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$ of $\mathcal{N}=8$ supersymmetry [6], while the remaining fields (as already mentioned) fit well in a kind of $(\mathbf{1},\mathbf{8},\mathbf{7})$ multiplet [6,9,11,12].

V. CONCLUDING REMARKS

In this paper, we have presented the $\mathcal{N}=8$ supersymmetric model with dynamical and semidynamical $d=1$ fields. The initial "trial" model (3.18) was composed from the dynamical $\mathcal{N}=4$ multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ (the superfield v), two semidynamical bosonic multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ (the superfields $\mathcal{Z}_a^{+A}$), and their partners with respect to the implicit $\mathcal{N}=4$ supersymmetry (the superfields Ψ^{+A} and w_a, respectively). The latter multiplets have the opposite Grassmann parity compared with the former ones.

The $\mathcal{N}=8$ model constructed describes a system with the kinetic term of the second order in the "velocities" of fermionic fields belonging to w_a. To get rid of this drawback, we carried out the oxidation procedure, which amounts to replacing the derivatives of fermionic fields with new auxiliary fields [4,25]. We have shown that, after passing to some suitable new variables, such a procedure works perfectly well for our system. As a result of this procedure, we obtained the new $\mathcal{N}=8$ supersymmetric system (4.30).

On the mass shell, the obtained $\mathcal{N}=8$ invariant model (4.30) describes one dynamical bosonic field x and eight real fermionic dynamical fields ϕ^i, $\bar{\phi}^i$, χ^{iA}, as well as three sets of semidynamical bosonic SU(2)-doublet fields z_a^{iA}, B_a^i, $\bar{B}_a^i$.

Surely, the $\mathcal{N}=8$ superfield system (3.18) and the $\mathcal{N}=8$ supersymmetric component system (4.30) are not equivalent to each other, because the directly applied oxidation does not preserve the canonical structure of the model. We obtained the $\mathcal{N}=8$ supersymmetric system (4.30) only at the component level. Rederiving this system at the complete superfield level is the next interesting task. A clue to this construction might be the fact that the fields in the action (4.30) naturally fall into a set of one dynamical $\mathcal{N}=8$ multiplet $(\mathbf{1},\mathbf{8},\mathbf{7})$ and one semidynamical $\mathcal{N}=8$ multiplet $(\mathbf{8},\mathbf{8},\mathbf{0})$. When constructing the superfield action, it may also be necessary to involve some extra auxiliary supermultiplets. A hint for constructing the self-consistent superfield formulation is the observation that the transformations (4.12) can be identified with the transformations (4.14) of the component fields of two semidynamical mirror (or twisted) $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets. In the Appendix, we demonstrate that such a multiplet has the natural description in the framework of the $\mathcal{N}=4, d=1$ biharmonic superfield formalism developed in [26,31]. Capitalizing on this property, we conjecture that the self-consistent superfield formulation of our system can be achieved within such a biharmonic approach.[3]

Another prospective task in the further development of the constructed model is to work out the $\mathcal{N}=8$ covariant procedure of gauging isometries in systems of this type. The $\mathcal{N}=4$ supersymmetric gauging procedure [28] proved to be an important tool for the construction of $\mathcal{N}=4$ supersymmetric generalizations of integrable many-particle systems of the Calogero type [20]. Being generalized to the $\mathcal{N}=8$ case, it would hopefully provide an opportunity to find new $\mathcal{N}=8$ supersymmetric extensions of these notorious systems.

[3]It is worth noting that the set of fields of all eventual $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets is closed under both manifest and implicit $\mathcal{N}=4$ supersymmetries, while the remaining fields are transformed through both themselves and fields of $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets. This indicates that in the present case we are dealing with some not fully reducible representation of $\mathcal{N}=8$ supersymmetry and the constraints on $\mathcal{N}=4$ superfields belonging to the $(\mathbf{1},\mathbf{8},\mathbf{7})$ subset should be nonlinear and properly include the $(\mathbf{4},\mathbf{4},\mathbf{0})$ superfields.

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APPENDIX: (4,4,0) MULTIPLETS IN BIHARMONIC SUPERSPACE

The automorphism group of the $\mathcal{N}=4, d=1$ supersymmetry algebra is the $SO(4) \cong SU_L(2) \times SU_L(2)$ group. Throughout our article, the group $SU_L(2)$ is implemented explicitly on the doublet $SU_L(2)$ indices $i=1, 2$. The harmonics $u_i^{\pm}$ are only associated with this group. At the same time, the group $SU_R(2)$ is implicit, but there exists a formulation in which both $SU_L(2)$ and $SU_R(2)$ symmetries are explicit. Such a description is achieved in $\mathcal{N}=4$ biharmonic superspace [31], which well suits describing models where both $\mathcal{N}=4$ ordinary and mirror multiplets participate.

In such a description, the odd superspace coordinates θ^i, $\bar{\theta}^i$, which are $SU_L(2)$ doublets, are added to the $SU_L(2)\times SU_R(2)$ quartet $\theta^{ii'}$: $(\theta^i,\bar{\theta}^i)=(\theta^{ii'=1},\theta^{ii'=2})$, where $i=1, 2$ and $i'=1, 2$ are doublet indices of $SU_L(2)$ and $SU_R(2)$, respectively.

In the biharmonic formulation, in addition to the harmonics $u_i^{\pm}\in SU_L(2)/U(1)$ (2.2), additional commuting harmonic variables $v_{i'}^{\pm}\in SU_R(2)/U(1)$ are introduced, with the defining relations

$$v_{i'}^{\pm}, \quad (v_{i'}^{+})^* = v^{-i'}, \qquad v^{+i'}v_{i'}^{-}=1. \tag{A1}$$

In the central basis, $\mathcal{N}=4$, $d=1$ biharmonic superspace is parametrized by the coordinates $(t,\theta^{ii'},u_i^{\pm},v_{i'}^{\pm})$. In this superspace, we define the harmonic projections of $\theta^{ii'}$ as

$$\theta^{\pm,\pm}=\theta^{ii'}u_i^{\pm}v_{i'}^{\pm}, \qquad \theta^{\pm,\mp}=\theta^{ii'}u_i^{\pm}v_{i'}^{\mp}. \tag{A2}$$

Given this, one of two analytical bases in the biharmonic superspace can be defined: either with the coordinates

$$(z_+,u_i^{\pm},v_{i'}^{\pm}), \qquad z_+=(t_+,\theta^{\pm,\pm},\theta^{\pm,\mp}),$$
$$t_+=t-i(\theta^{+,+}\theta^{-,-}+\theta^{-,+}\theta^{+,-}) \tag{A3}$$

or with

$$(z_-,u_i^{\pm},v_{i'}^{\pm}), \qquad z_-=(t_-,\theta^{\pm,\pm},\theta^{\pm,\mp}),$$
$$t_-=t-i(\theta^{+,+}\theta^{-,-}-\theta^{-,+}\theta^{+,-}). \tag{A4}$$

Note that t_+ coincides with the coordinate $t_{\mathcal{A}}$ introduced in (2.3): $t_+=t_{\mathcal{A}}$.

In the analytic bases (A3) and (A4), half of the $\mathcal{N}=4$ covariant spinor derivatives become short. This is a reflection of the fact that the spaces (A3) and (A4) contain the $\mathcal{N}=4$ invariant subspaces with half of the initial Grassmann coordinates. Namely, the analytic superspace parametrized by supercoordinates

$$(\zeta_+,u_i^{\pm},v_{i'}^{\pm}), \qquad \zeta_+=(t_+,\theta^{+,+},\theta^{+,-}) \tag{A5}$$

is closed under the full $\mathcal{N}=4$ supersymmetry. Another analytic superspace,

$$(\zeta_-,u_i^{\pm},v_{i'}^{\pm}), \qquad \zeta_-=(t_-,\theta^{+,+},\theta^{-,+}), \tag{A6}$$

is also closed.

The ordinary $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet is described by the superfield $q^{(0,+)A}(\zeta_+,u,v)$ living in the analytic superspace (A5), while a mirror multiplet is represented by a superfield $q^{(0,+)A'}(\zeta_-,u,v)$ defined on the analytic superspace (A6). Here, the indices A and A' are transformed by two Pauli-Gürsey groups, which are generically different. These superfields are subject only to the harmonic conditions:

$$D^{++,0}q^{(+,0)A}=D^{0,++}q^{(+,0)A}=0,$$
$$D^{++,0}q^{(0,+)A'}=D^{0,++}q^{(0,+)A'}=0, \tag{A7}$$

where $D^{++,0}$ and $D^{0,++}$ are the harmonic derivatives $\partial^{++,0}=u_i^{+}\partial/\partial u_i^{-}$ and $\partial^{0,++}=v_{i'}^{+}\partial/\partial v_{i'}^{-}$ rewritten in the analytic bases (A3) and (A4), respectively [see Eq. (2.6)].

Solving the conditions (A7) yields the component expansions of the superfields $q^{(+,0)A}$ and $q^{(+,0)A'}$. The ordinary $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplet is described by the superfield

$$q^{(+,0)A}(\zeta_+,u,v)=z^{iA}(t_+)u_i^{+}+\theta^{+,-}\pi^{i'A}(t_+)v_{i'}^{+}$$
$$-\theta^{+,+}\pi^{i'A}(t_+)v_{i'}^{-}$$
$$-2i\theta^{+,+}\theta^{+,-}\partial_{t_+}z^{iA}u_i^{-}, \tag{A8}$$

while the mirror multiplet is described by the superfield

$$q^{(0,+)A'}(\zeta_-,u,v)=f^{i'A'}(t_-)v_{i'}^{+}+\theta^{-,+}\omega^{iA'}(t_-)u_i^{+}$$
$$-\theta^{+,+}\pi^{iA'}(t_-)u_i^{-}$$
$$-2i\theta^{+,+}\theta^{-,+}\partial_{t\_}\omega^{i'A'}v_{i'}^{+}. \tag{A9}$$

The expansion (2.31) for the superfield $\mathcal{Z}_a^{+A}$ at an arbitrary value of $a=1, 2$ coincides with the expansion (A8) for the superfield $q^{(+,0)A}$ after the following identification of the component fields: $\pi^{i'A}=(\pi^{i'=1A},\pi^{i'=2A})=(\pi^A,\bar{\pi}^A)$. The mirror $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets correspond to identifying the index A' in (A9) with the $SU(2)_L$ index j. The linear off-shell transformations of the explicit $\mathcal{N}=4$ supersymmetry on the component fields can be easily obtained from the standard superfield transformations.[4]

More details on $\mathcal{N}=4$ supermultiplets in biharmonic superspace can be found in [31].

[4]The realizations of implicit $\mathcal{N}=4$ supersymmetry linearly mixing both $(\mathbf{4},\mathbf{4},\mathbf{0})$ superfields can also be easily defined [31].

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