Euler fluid in 2 + 1 dimensions as a gauge theory and an action for the Euler fluid in any dimension

Horatiu Nastase[1,*] and Jacob Sonnenschein[2,†]

[1]*Instituto de Física Teórica, UNESP-Universidade Estadual Paulista, Rua Dr. Bento Teobaldo Ferraz 271, Bloco II, Sao Paulo 01140-070, SP, Brazil*

[2]*School of Physics and Astronomy, The Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 69978, Israel*

(Received 10 January 2024; accepted 7 March 2024; published 1 April 2024)

In this paper we parallel the construction of Tong of a gauge theory for shallow water, by writing a gauge theory for the Euler fluid in $2+1$ dimensions. We then extend it to a Euler fluid coupled to an electromagnetic background. We argue that the gauge theory formulation provides a topological argument for the quantization of $2+1$ dimensional Euler Hopfion solution. In the process, we find a (nongauge) action for the Euler fluid that can be extended to any dimension, including the physical $3+1$ dimensions. We also discuss several aspects of the Arnold-Beltrami-Childress flow.

DOI: 10.1103/PhysRevD.109.085006

I. INTRODUCTION

In a very interesting paper [1], Tong rewrote the shallow water equations, with variables: $2+1$ dimensional velocity $\vec{u}(x,y,t)$ and height $h(x,y,t)$, in terms of a gauge theory.[1] In the case of linearized theory, one obtains a Maxwell-Chern-Simons theory, which is known to have boundary chiral modes, that are now identified with the coastal Kelvin waves of the shallow water equations, giving them a topological reason. An action for the shallow water and the equivalent gauge theory is also obtained. He also suggests that a similar analysis could be made for Euler fluids with equation of state $P = C\rho^\gamma$ [so barotropic, $P = P(\rho)$], though that is not done.

In this paper, we show that indeed, similar to the construction in [1], one can rewrite the $2+1$ dimensional Euler fluids as a gauge theory. For that purpose we map the magnetic field B to the density of the fluid ρ, instead of the height function, and find that the magnetic term in the action is linear, and not quadratic in B. The resulting action describes fluids, not necessarily barotropic, such as to include the case of the $2+1$ dimensional fluid Hopfion with constant ρ, but nontrivial P, considered in [3,4]. We also consider the case of Euler fluids coupled to electromagnetism, as considered by Abanov and Wiegmann [5] and for which nontrivial a Hopfion solution was found in [6]. The quantization of the Hopfion solution will then be related to the quantization of the Chern-Simons level.

In the process, we write an action principle for the Euler fluid, which generalizes to any dimension, though only in $2+1$ dimensions is rewritten as a gauge theory.[2] Writing an effective action for an Euler fluid was considered before in quantum field theory formalisms, starting with [8] (based on AdS/CFT holographic arguments), and various constructions were attempted in [9–13] (see also other references therein), but here the action principle is based simply on the fluid variables. We also briefly discuss several aspects of the Arnold-Beltrami-Childress (ABC) flow. In particular, we review its Clebsh formulation, and for a special case of the ABC flow, we map it to electric and magnetic fields, which we also write in terms of the Bateman construction.

The paper is organized as follows. In Sec. II we put the Euler fluid in gauge theory form, and we also compare the energy-momentum tensors of the fluid and gauge forms. In Sec. III we couple the system to electromagnetism, and find that this is very natural in the gauge picture. As a further application, we also couple the shallow water equations to electromagnetism, and write it in the gauge picture. In Sec. IV we describe the main application of the gauge theory picture, describing the Euler fluid Hopfions as topological modes in the gauge theory. In Sec. V we first write an action for the Euler fluid in any dimension, and

*horatiu.nastase@unesp.br
†cobi@tauex.tau.ac.il

[1]The gauge formulation was found to be related to a 2-dimensional area preserving diffeomorphisms in [2].

[2]The action is in fact equivalent to an action written in [7], as we became aware after the paper was first posted on the arXiv.

then write a gauge theory form for it, and we consider the ABC flow, and present its Clebsh parametrization and a gauge field representation. We conclude in Sec. VI.

II. EULER FLUID IN GAUGE THEORY FORM

We consider the $2+1$ dimensional Euler (nondissipative) fluids, with the following variables: the flow velocity $\vec{u}(x,y,t)$, the fluid density $\rho(x,y,t)$, and the pressure $P(x,y,t)$. The well-known equations of motion are

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0,$$
$$\rho\partial_t u^i + \partial_i P + \rho(\vec{u}\cdot\vec{\nabla})u^i = 0 \Leftrightarrow \rho\frac{D\vec{u}}{Dt} = -\vec{\nabla}P, \tag{2.1}$$

where we have defined the "covariant derivative,"

$$\frac{D}{Dt} = \partial_t + \vec{u}\cdot\vec{\nabla}. \tag{2.2}$$

We then define the chemical potential μ, as usual, by

$$\frac{dP}{\rho} = \frac{d\mu}{m}. \tag{2.3}$$

It would seem like we need a barotropic fluid, $P = P(\rho)$, in order to integrate the relation and find μ, but actually, we see that the other case of interest for us, nontrivial P and ρ constant, is also included in it. In general then, all we need is to be able to integrate dP/ρ.

We note that in the appendix of [1] it was stated that we could consider the barotropic Euler fluid with $P = C\rho^\gamma$, in which case, in the action below (2.5), we would replace the term with $B\partial_0\tilde{\mu}$ by a term with $B^{\gamma-1}$. However, one of the reasons we consider our gauge theory action for the Euler fluid is that we want to obtain a topological gauge description for the fluid Hopfion (just like Tong obtained a topological gauge description for the coastal Kelvin waves), and for the Hopfion P as a function of space, but ρ is constant.

Further, we define $\tilde{\mu}$ by

$$\partial_0\tilde{\mu} = \frac{\mu}{m}, \tag{2.4}$$

which is introduced because we want to make $\tilde{\mu}$ a variable in the action, and by partial integration its equation of motion is ∂_0 on what it multiplies equals zero, rather than just what it multiplies equals zero.

The result of this is that the $\tilde{\mu}$ equation of motion will enforce $\partial_0 B = 0$, so $\partial_0\rho = 0$ (density constant in time), and by the continuity equation this also means $\vec{\nabla}\cdot(\rho\vec{u}) = 0$. We could find no way to avoid such a restriction, so we obtain a gauge description of the Euler fluid only in this case.

We then write the gauge theory action for the fluid

$$S = \int dt \int d^2x \left[\frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu} - \epsilon^{\mu\nu\rho}A_\mu\partial_\nu\tilde{A}_\rho\right], \tag{2.5}$$

where $\vec{E}, B$ is the gauge field strength that describes the fluid (note that, since $\vec{u},\rho$ are observables, they could only be related to field strengths, not to gauge fields themselves), via the definition

$$B = \rho, \qquad E_i = \epsilon_{ij}\rho u^j, \tag{2.6}$$

and $\tilde{A}_\mu$ is an *auxiliary* gauge field, defined through the usual Clebsch parametrization as[3]

$$\tilde{A}_\mu = \partial_\mu\chi + \beta\partial_\mu\alpha, \tag{2.7}$$

with α, β, χ real functions that are considered the actual variables (instead of $\tilde{A}_\mu$).

The action in terms of these variables reads

$$S = \int dt \int d^2x \left[\frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu} - \epsilon^{\mu\nu\rho}A_\mu\partial_\nu(\beta\partial_\rho\alpha)\right], \tag{2.8}$$

Thus, in fact, χ does not show up in action.

This action is manifestly not Lorentz invariant and instead it is invariant under rotation transformations, and space-time translations. Due to the CS term the action is also not invariant under parity and time-reversal transformations. It is obviously also invariant under gauge transformation of $A_\mu \to A_\mu + \partial_\mu\lambda$ assuming that $\epsilon^{\mu\nu\rho}\lambda\partial_\nu\tilde{A}_\rho$ vanishes on the boundary of space-time, and separately under gauge transformations of $\tilde{A}_\mu$.

A comparison between this action and the one used in [1][4] reveals the following differences: (i) here we have a term linear in B whereas in [1] it is quadratic; (ii) in the latter there is a term linear in A_0 that couples to the Coriolis parameter that our action lacks. We have not introduced this term, since it is not present in (exactly) $2+1$ dimensional Euler fluids, only in the shallow water fluids ($2+1$ dimensional velocity, but also height of the water, however small). Formally, we could consider such a term without worrying about where it comes from, just an $f\epsilon^{ij}u^j$ on the right-hand side of $\frac{Du^i}{Dt}$, and then, like for [1], this would give an fA_0 term in the action. But this is implicitly added in the next section, where we couple to external electromagnetic

[3]The Clebsch parametrization in the case of fluids was described in gauge theory language in [14], which also discusses anomalies. In [15], the group theory version was used to construct various topological terms.

[4]In [16], the fluid equations of [15] were also found to be equivalent to a Chern-Simons action, using a Clebsch parametrization for the gauge field.

fields $\vec{E}_e, B_e$: we just need to shift the external magnetic field as

$$\frac{e}{m}B_e \to \frac{e}{m}B_e + f, \tag{2.9}$$

and we obtain the desired term.

Next we consider the equations of motion associated with the variations of $A_\rho, \tilde{\mu}, \chi, \alpha$, and β. We see now that the $\tilde{\mu}$ equation of motion gives

$$\partial_0 B = \partial_0 \rho = 0 \Rightarrow B = \text{constant}(t), \tag{2.10}$$

instead of just $B = \rho = 0$ in the case we wrote simply μ/m and not $\partial_0\tilde{\mu}$ in the action, which would have made it a trivial fluid. As it is though, the action *only describes time-independent densities* ρ, *but that is enough for our purposes*.[5]

Then, first we see that χ is a pure gauge, so it has no equation of motion, whereas the equations of motion of β and α are

$$B\dot{\alpha} + \epsilon^{ij}E_i\partial_j\alpha = 0, \qquad B\dot{\beta} + \epsilon^{ij}E_i\partial_j\beta = 0, \tag{2.11}$$

and then defining $\tilde{E}_i$ and $\tilde{B}$ as the "electric" and "magnetic" fields of $\tilde{A}_\mu$, such that for instance

$$\tilde{E}_i = \dot{\beta}\partial_i\alpha - (\partial_i\beta)\dot{\alpha}; \tag{2.12}$$

from the above α and β equations of motion we get

$$\tilde{E}_i = \frac{E_i}{B}\tilde{B}. \tag{2.13}$$

Then the Gauss constraint, i.e., the A_0 equation of motion, gives

$$\partial_i\left(\frac{E_i}{B}\right) = \tilde{B}, \tag{2.14}$$

and, considering that $E_i/B = \epsilon_{ij}u^j$ and that $\partial_i(\epsilon^{ij}u_j) \equiv \omega$ is the $2+1$ dimensional vorticity, replacing in the above equation of motion we get

$$\tilde{E}_i = \epsilon_{ij}u^j\omega. \tag{2.15}$$

The Bianchi identity for the gauge field A_μ is

$$\epsilon^{\mu\nu\rho}\partial_\mu(\partial_\nu A_\rho) = 0, \tag{2.16}$$

and it becomes

$$\partial_0 B + \epsilon^{ij}(\partial_i E_j) = 0 \Rightarrow \partial_0\rho + \partial_i(\rho u^i) = 0, \tag{2.17}$$

i.e., the continuity equation. Since from the equation of motion of $\tilde{\mu}$, $\partial_0\rho = 0$, it implies that the action describes flows for which

$$\vec{\nabla}\cdot(\rho\vec{u}) = 0. \tag{2.18}$$

Finally, the Euler equation appears from the A_i equation of motion of the action, combined with the relation $\tilde{E}_i = \epsilon_{ij}u^j\omega$, obtained previously. Indeed, the A_i equation is

$$\partial_0\left(\frac{E_i}{B}\right) + \frac{1}{2}\epsilon_{ij}\partial_j\left(\frac{\vec{E}^2}{B^2}\right) + \epsilon_{ij}\partial_j\mu/m - \epsilon_{ij}\tilde{E}_j = 0, \tag{2.19}$$

which translates into

$$\partial_0(\epsilon_{ij}u^j) + \frac{1}{2}\epsilon_{ij}\partial_j(\vec{u}^2) + \epsilon_{ij}\frac{\partial_j P}{\rho} = \epsilon_{ij}\tilde{E}_j = -u_i\omega, \tag{2.20}$$

and it is easy to see that, by multiplying the equation by ϵ^{ki}, the left-hand side, minus the term on the right-hand side, becomes equal to

$$-\partial_t(u^k) - u_j\partial_j u_k - \partial_k P = 0, \tag{2.21}$$

i.e., the Euler equation.

An interesting observation is that, taking ∂_i on the A_i equation of motion (2.19), and using (2.14), we obtain

$$\partial_0\partial_i\left(\frac{E_i}{B}\right) = \partial_0\tilde{B} = \epsilon^{ij}\partial_i\tilde{E}_j, \tag{2.22}$$

which is a Maxwell equation in $2+1$ dimensions for $\tilde{A}_\mu$.

A. The energy-momentum tensor

For the calculation of $T_{\mu\nu}$, the CS term does not count (it is independent of the metric), so consider the action

$$\hat{S} = \int dt \int d^2x \left[\frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu}\right]. \tag{2.23}$$

We will consider the Belinfante energy-momentum tensor. In general, we first couple to a metric, and then write

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = -2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}. \tag{2.24}$$

First, we need to understand the fluid system better, to understand what we will be comparing against. For a fluid, we have

$$T_{\mu\nu} = \rho u^\mu u^\nu + P(\eta_{\mu\nu} + u_\mu u_\nu), \tag{2.25}$$

[5]Note that [2] consider a term BP in the Lagrangian for their *shallow water* action, but that is necessarily an *external* pressure field, not a dynamical one as we take here.

where, however, ρ refers to the *comoving* (in the local center of mass of the fluid) total energy density, so is really

$$\rho \to \rho_0 = m_0 n_0 = m_0 \frac{dN}{dV_0}, \tag{2.26}$$

where m_0 is the rest mass of the fluid particle, and dV_0 is the volume element in the center-of-mass system, related to the moving frame by $dV_0 = \gamma dV$. In a nonrelativistic approximation, $u^0 = \gamma$, so $u_0 \simeq -1 - v^2/2$, and $u^i = \gamma v^i$, so $u_i \simeq v^i$. Then, in the moving frame,

$$\begin{aligned} &T_{00} = \gamma^2 \rho \simeq \rho + \rho v^2, \qquad T_{0i} \simeq -\rho v^i, \\ &T_{ij} \simeq \rho v_i v_j + P\delta_{ij}, \end{aligned} \tag{2.27}$$

and ρ really refers to ρ_0. The energy of the fluid is then

$$E = \int dV \gamma^2 \rho = \gamma \int dV_0 \rho \simeq \int dV_0 \left[\rho + \rho \frac{v^2}{2}\right]. \tag{2.28}$$

However, to this rest + kinetic energy of the fluid particles we can add the subleading term of the potential energy $\int dV_0 P$.

In the nonrelativistic approximation, the total energy contains the term with the rest energy density ρ, and we integrate over the rest volume, as well as the kinetic energy density. With the map $E_i = \epsilon_{ji}\rho v_j$, $B = \rho$, the action $\hat{S}$ above contains only the kinetic energy density (thus omitting the rest mass energy), minus a potential term which, for constant ρ, is just $P dV_0$. The rest mass energy is neglected since we consider the case $\partial_t \rho = 0$.

The continuity equation and the Euler equation appear, as usual, from the 0 and i components of $\partial^\mu T_{\mu\nu} = 0$ for the fluid, respectively (after we use the 0 component in the i component also). The difference now is that $\partial_t \rho = 0$ and, instead, we keep just the subleading term with $\partial_t(\rho v^2/2)$, in the 0 component.

To construct $T_{\mu\nu}$ from the action $\hat{S}$, we couple to the metric as follows. First, since $-E_i = F_{0i}$ and $F_{ij} = \epsilon_{ij} B$, $\epsilon_{12} = +1$, and we keep these relations even in the case of a nontrivial metric, we write

$$\begin{aligned} \vec{E}^2 &= -F_{0i}F^{0i} = -F_{0i}F_{\mu\nu}g^{0\mu}g^{i\nu} \\ &= -\left(F_{0i}F_{0j}g^{00}g^{ij} + F_{0j}F_{ik}g^{0i}g^{jk}\right), \\ 2B &= g^{ik}g^{kl}\epsilon_{jl}F_{ik}, \end{aligned} \tag{2.29}$$

and we obtain

$$\begin{aligned} T_{00} &= -\frac{2}{\sqrt{-g}}\frac{\delta \hat{S}}{\delta g^{00}} = \frac{2\vec{E}^2}{2B} - \left(\frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu}\right) \\ &= \frac{\vec{E}^2}{2B} + B\partial_0\tilde{\mu} \leftrightarrow \rho\frac{\vec{v}^2}{2} + \rho\partial_0\tilde{\mu}. \end{aligned} \tag{2.30}$$

The T_{0i} components are obtained from the Lagrangian density term

$$\mathcal{L} = -\left[\frac{g^{0j}g^{ik}F_{0i}F_{jk}}{2B}\right] + \cdots, \tag{2.31}$$

so we obtain

$$T_{0i} = -2\frac{\delta \mathcal{L}}{\delta g^{0i}} = -\epsilon_{ij}E^j \leftrightarrow -\rho v_i. \tag{2.32}$$

The T_{ij} components are obtained by varying both the $\vec{E}^2$ terms and the B terms with respect to g^{ij}, so

$$\begin{aligned} T_{ij} &= -\frac{2}{\sqrt{g}}\frac{\delta \hat{S}}{\delta g^{ij}} = -\frac{E_iE_j}{B} + \frac{\vec{E}^2}{B}\delta_{ij} + 2B\partial_0\tilde{\mu}\delta_{ij} \\ &\quad + \left(\frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu}\right)\delta_{ij} \\ &= \frac{-E_iE_j + \vec{E}^2\delta_{ij}}{B} + \left(\frac{\vec{E}^2}{2B} + B\partial_0\tilde{\mu}\right)\delta_{ij}. \end{aligned} \tag{2.33}$$

Explicitly, this gives

$$\begin{aligned} T_{11} &= \frac{E_2^2}{B} + \left(\frac{\vec{E}^2}{2B} + B\partial_0\tilde{\mu}\right) \leftrightarrow \rho\frac{v_1^2}{2} + \left(\frac{\rho\vec{v}^2}{2} + \rho\partial_0\tilde{\mu}\right) \\ T_{22} &= \frac{E_1^2}{B} + \left(\frac{\vec{E}^2}{2B} + B\partial_0\tilde{\mu}\right) \leftrightarrow \rho\frac{v_2^2}{2} + \left(\frac{\rho\vec{v}^2}{2} + \rho\partial_0\tilde{\mu}\right) \\ T_{12} &= -\frac{E_1E_2}{B} \leftrightarrow \rho\frac{v_1v_2}{2}. \end{aligned} \tag{2.34}$$

We see that we obtain the correct fluid energy $T_{\mu\nu}$, except for an extra term $\rho v^2/2\delta_{ij}$, an extra momentum flux that should be related to the nonrelativistic corrections to ρ that we kept: in $\partial^0 T_{0i} + \partial^j T_{ji} = 0$, we get an extra $-v_i\partial_t(\rho v^2/2)$ from the first, and an extra $\partial_i(\rho v^2/2)$ from the second, cancelling under the assumption of a kinetic energy depending explicitly only on time.

A much more interesting case would have been the viscous Navier-Stokes case. However, it is not easy to generalize to the presence of the viscous term in the energy-momentum tensor, because of the complicated nature of the equations in the gauge theory form. We have not been able to find the gauge theory action that corresponds to the Navier-Stokes fluid.

III. EULER FLUID COUPLED TO ELECTROMAGNETISM IN GAUGE THEORY FORM

In this section we consider the Euler fluid equations coupled to external electromagnetism, as considered by Abanov and Wiegmann [5][6] in $3+1$ dimensions, and then by us [6] in $2+1$ dimensions, with equations of motion

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0,$$
$$\rho\partial_t u^i + \partial_i P + \rho(\vec{u}\cdot\vec{\nabla})u^i = \frac{e}{m}\left(\vec{E}_e + \vec{u}\times\vec{B}_e\right), \quad (3.1)$$

which in $2+1$ dimensions gives the Euler equation

$$\frac{Du^i}{Dt} + \partial_i P = \frac{e}{m}\left(E_e^i + B_e\epsilon^{ij}u_j\right), \quad (3.2)$$

where $\vec{E}_e, B_e$ are the true, external (i.e., nondynamical), electromagnetic fields, and we have put $c=1$ for simplicity.

It is perhaps clear (at least *a posteriori*, after figuring it out), that the correct way to incorporate the electromagnetic fields into the gauge theory action is to add another Chern-Simons term (or rather, BF term) coupling the gauge fields A_μ and A_μ^e (electromagnetic), so the final action is

$$S = \int dt\int d^2x\left[\frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu} - \epsilon^{\mu\nu\rho}A_\mu\partial_\nu\tilde{A}_\rho + \frac{e}{m}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho^e\right]. \quad (3.3)$$

Then the α,β equations are unchanged, so is the Bianchi identity for A_μ, giving the continuity equation, but the Gauss law constraint (equation of motion for A_0) gets a new contribution from the added term, so is now

$$\partial_i\left(\frac{E_i}{B}\right) - \tilde{B} + \frac{e}{m}B^e = 0, \quad (3.4)$$

and gives

$$\tilde{B} = \omega + \frac{e}{m}B^e. \quad (3.5)$$

From the α and β equations, we get

$$\tilde{E}_i = \frac{E_i}{B}\tilde{B} = \epsilon_{ij}u^j\left(\omega + \frac{e}{m}B^e\right). \quad (3.6)$$

Then the equation of motion of A_i contains two extra terms, one directly from the coupling of A_i to E_e^j, and one indirectly, from the fact that now $\tilde{E}_i$ contains the extra B^e term above, giving

$$D_t(\epsilon^{ij}u^j) + \epsilon^{ij}\partial_j\frac{\mu}{m} + \frac{e}{m}u^iB^e = \frac{e}{m}\epsilon^{ij}E_j^e, \quad (3.7)$$

which is nothing but the Euler equation coupled to external electromagnetism, as we wanted.

The Maxwell equation for $\tilde{A}_\mu$ in (2.22) is then modified as

$$\partial_0\partial_i\left(\frac{E_i}{B}\right) = \partial_0\left(\tilde{B} - \frac{e}{m}B^e\right) = \epsilon^{ij}\partial_i\left(\tilde{E}_j - \frac{e}{m}E_j^e\right). \quad (3.8)$$

A. Shallow water equations coupled to electromagnetism, and gauge theory form

Here we make a small aside, and note that we can also couple the shallow water equations to electromagnetism, and write it in gauge theory form, just as above for the Euler case.

Indeed, the shallow water equations coupled to electromagnetism (for velocity u^i and height h of the fluid) are

$$\partial_t h + h\vec{\nabla}\cdot\vec{u} = 0,$$
$$\frac{Du^i}{Dt} = f\epsilon^{ij}u^j - g\partial_i h + \frac{e}{m}\left(E_e^i + \epsilon_{ij}u^jB_e\right), \quad (3.9)$$

where f is the Coriolis parameter, g is the gravitational acceleration, and E_e^i, B_e are the true external electric and magnetic fields.

Again, the correct way to introduce the coupling to E_e^i, B_e in the gauge theory formulation is to introduce an extra BF term, of the type AdA_e, namely to modify the gauge theory action in [1] to (here, as before, we have the Clebsch parametrization $\tilde{A}_\mu = \partial_\mu\chi + \beta\partial_\mu\alpha$)

$$S = \int dtd^2x\left[\frac{\vec{E}^2}{2B} - \frac{g}{2}B^2 + fA_0 - \epsilon^{\mu\nu\rho}A_\mu\partial_\nu\tilde{A}_\rho + \frac{e}{m}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho^e\right]. \quad (3.10)$$

Indeed, again one finds that only the Gauss constraint (equation of motion for A_0) is modified, adding a term $\frac{e}{m}B_e$ to $\tilde{B}$, which becomes

$$\tilde{B} = \omega + f + \frac{e}{m}B_e, \quad (3.11)$$

and also implies (from the α and β equations of motion)

$$\tilde{E}_i = \frac{E_i}{B}\tilde{B} = \epsilon^{ij}u_j\left(\omega + f + \frac{e}{m}B_e\right), \quad (3.12)$$

[6]See also the earlier paper [17] for Euler fluids coupled to electromagnetism and anomalies.

as well as the A_i equation of motion, in which we get an extra term from the coupling of A_i to E_e^j, and an indirect term, via the contribution of $\tilde{B}$ to $\tilde{E}_i$ above,

$$\partial_t\left(\frac{E_i}{B}\right)+\frac{1}{2}\epsilon_{ij}\partial_j\left(\frac{\vec{E}^2}{B^2}\right)+g\epsilon_{ij}\partial_j B-\epsilon_{ij}\tilde{E}_j+\frac{e}{m}\epsilon_{ij}E_e^j=0. \tag{3.13}$$

When substituting the map to the fluid, we get indeed the shallow water equation coupled to electromagnetism,

$$D_t(\epsilon^{ij}u^j)+fu^j+g\epsilon^{ij}\partial_j h+\frac{e}{m}u^i B_e-\frac{e}{m}\epsilon^{ij}E_e^j=0. \tag{3.14}$$

IV. 2+1 DIMENSIONAL HOPFION AS A GAUGE THEORY TOPOLOGICAL MODE

Similar to the way Tong found the coastal Kelvin waves of the shallow water equations as chiral boundary modes (topological modes) in the *effective* Chern-Simons theory obtained for small fluctuations [1], in this section we want to see if the $2+1$ dimensional Hopfion solution with constant ρ can be similarly understood from topological considerations in the corresponding gauge theory.

The $2+1$ dimensional Hopfion solution was obtained in [3,4] by dimensional reduction of the $3+1$ dimensional Hopfion, which is a null ($\vec{u}^2=1$) fluid solution obtained from the analogy to $3+1$ dimensional electromagnetism with a Hopfion solution (that is also null in electromagnetism).

The $2+1$ dimensional Hopfion solution has $\rho=1$, is time independent, and has

$$\begin{aligned}
P&=P_\infty-\frac{2}{1+x^2+y^2},\\
u_x&=\frac{2y}{1+x^2+y^2},\\
u_y&=-\frac{2x}{1+x^2+y^2},
\end{aligned} \tag{4.1}$$

which gives the vorticity

$$\omega=\epsilon^{ij}\partial_i u_j=-\frac{4}{(1+x^2+y^2)^2}, \tag{4.2}$$

which integrates to -4π.

A. The winding number in the gauge picture

We translate the winding number which is the integral over the vorticity to the gauge theory language,

$$\begin{aligned}
\mathcal{H}&=\int d^2x w(t,x,y)=\int d^2x\left(\partial_x u_y-\partial_y u_x\right)\\
&=\int d^2x\partial_i\left(\frac{E^i}{B}\right).
\end{aligned} \tag{4.3}$$

We would like to check if this quantity is conserved in time. For that purpose we use the equations of motion for A_i, or more precisely the Maxwell equations (2.22) derived from them, and find that

$$\begin{aligned}
\partial_t\mathcal{H}&=\int d^2x\partial_t\tilde{B}=\int d^2x\epsilon^{ij}\partial_i\tilde{E}_j\\
&=\int d^2x\epsilon^{ij}\partial_t\left(\partial_i\beta\partial_j\alpha-\partial_i\alpha\partial_j\beta\right).
\end{aligned} \tag{4.4}$$

Thus, the winding $\mathcal{H}$ is conserved in time, since it is a total derivative, so a boundary term, that can be put to zero. We also note that, in the more general case of coupling to an electromagnetic field $B_e, \vec{E}_e$, if we have a constant B_e and vanishing $\vec{E}_e$, the same result applies.

B. Linearized fluctuations

Again following the logic in [1], we write a linearized theory by expanding around a background with $\hat{\alpha},\hat{\beta},\rho_0,\hat{A}_\mu$, but with a small (perturbation) velocity $u^i=\delta u^i$ and $\omega=\delta\omega$, so $E_i/B=\epsilon^{ij}u_j$, as

$$\begin{aligned}
&\alpha=\hat{\alpha}+q, \qquad \beta=\hat{\beta}+p,\\
&\partial_1\hat{\beta}\partial_2\hat{\alpha}-\partial_2\hat{\beta}\partial_1\hat{\alpha}\equiv k\Rightarrow\partial_j\hat{\beta}\partial_l\hat{\alpha}-\partial_l\hat{\beta}\partial_j\hat{\alpha}=k\epsilon_{jl},\\
&B=\rho_0+\delta b, \qquad E_i=\delta e_i,\\
&A_\mu=\hat{A}_\mu+\delta A_\mu, \qquad \hat{A}_0=0\\
\Rightarrow\ &\partial_1\hat{A}_2-\partial_2\hat{A}_1=\rho_0,
\end{aligned} \tag{4.5}$$

and writing simply A_μ for δA_μ, so B for $\delta b=\delta\rho$ and E_i for $\delta e_i=\rho_0\epsilon^{ij}u_j$. Note that we have defined

$$\tilde{B}=\partial_1\hat{\beta}\partial_2\hat{\alpha}-\partial_2\hat{\beta}\partial_1\hat{\alpha}\equiv k, \tag{4.6}$$

but the Gauss constraint of the action (3.3) is, in the above notation for linearization,

$$k\equiv\tilde{B}=\omega+\frac{e}{m}B_e(+f), \tag{4.7}$$

where we have put an f which is just a shift of B_e, but we have assumed that ω is a perturbation, $\omega=\delta\omega$, so the only way to assume k is not (as we will see that we need) is to say that B_e or f are large. That, in turn, means that the Hopfion solution (4.1) is not valid, as it was derived at $B_e=f=0$. Nevertheless, we would still like to see if there is a quantization condition possible in this (as of yet not considered) case for the Hopfion.

Then the action becomes

$$S=\int dt\int d^2x\left[\frac{\vec{E}^2}{2\rho_0}-\rho_0\partial_0\tilde{\mu}-\delta b\partial_0\tilde{\mu}-\rho_0 p\dot{q}+\epsilon^{ij}E_i(q\partial_j\hat{\beta}-p\partial_j\hat{\alpha})\right]. \quad (4.8)$$

The equations of motion for p, q (coming originally from $\tilde{A}_\mu$) are

$$\rho_0\dot{q}=-\epsilon^{ij}E_i\partial_j\hat{\alpha},\qquad \rho_0\dot{p}=-\epsilon^{ij}E_i\partial_j\hat{\beta}, \quad (4.9)$$

the equation of motion for A_i is

$$\dot{E}_i=-\rho_0\epsilon_{ij}\partial_j\partial_0\tilde{\mu}-\rho_0\epsilon_{ij}\left(\partial_j\hat{\beta}\,\dot{q}-\partial_j\hat{\alpha}\,\dot{p}\right), \quad (4.10)$$

and the equation for A_0 (Gauss constraint, for the $A_0=0$ gauge) is

$$\partial_iE_i=\rho_0\epsilon_{ij}\left(\partial_i\hat{\beta}\partial_jq-\partial_i\hat{\alpha}\partial_jp\right). \quad (4.11)$$

Replacing $\rho_0\dot{q}$ and $\rho_0\dot{p}$ from the equations for q and p into the equation for A_i, and using $\partial_j\hat{\beta}\partial_l\hat{\alpha}-\partial_l\hat{\beta}\partial_j\hat{\alpha}=k\epsilon_{jl}$, we obtain

$$\dot{E}_i=\epsilon_{ij}[kE_j-\rho_0\partial_j\partial_0\tilde{\mu}]. \quad (4.12)$$

But also using the solutions

$$\rho q=-\epsilon^{ij}A_i\partial_j\hat{\alpha},\qquad \rho p=-\epsilon^{ij}A_i\partial_j\hat{\beta} \quad (4.13)$$

of the equations for p and q, and using $\partial_j\hat{\beta}\partial_l\hat{\alpha}-\partial_l\hat{\beta}\partial_j\hat{\alpha}=k\epsilon_{jl}$, we obtain that the Gauss constraint in the $A_0=0$ gauge becomes

$$\partial_iE_i=kB, \quad (4.14)$$

and the effective action for the linearized fluctuations becomes

$$S=\int dt\int d^2x\left[\frac{\dot{A}_i^2}{2\rho_0}-B\partial_0\tilde{\mu}+\frac{k}{2\rho_0}\epsilon^{ij}A_i\dot{A}_j\right], \quad (4.15)$$

whose A_i equation is the same as (4.12), so that the full action (including A_0) is

$$S=\frac{1}{2\rho_0}\int dt\int d^2x[E^2-2B\rho_0\partial_0\tilde{\mu}-k\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho], \quad (4.16)$$

which contains now a true CS term (not a BF term like the full nonlinear theory).

Note that in fluid variables, the Gauss constraint $\partial_iE_i=kB$ becomes, at this linearized level,

$$\partial_iE_i\simeq\rho_0\epsilon^{ij}\partial_iu_j\equiv\rho_0\omega=k\delta\rho\Rightarrow\omega=\frac{k}{\rho_0}\delta\rho. \quad (4.17)$$

But then, from the quantization of the coefficient of the CS term, we obtain $(k/\rho_0)\in 2\pi\mathbb{Z}$, so $\omega/\delta\rho\in 2\pi\mathbb{Z}$, which is the topological reason we wanted for the quantization of vorticity ω when we perturb the constant density ρ_0 by $\delta\rho$, in the background of a constant density $\rho_0=1$ and a constant external magnetic field B_e (or Coriolis term f).

Finally, that means that the $2+1$ dimensional fluid Hopfion with quantized vorticity appears because of the existence of quantized linearized modes in the gauge theory action.

V. AN ACTION FOR A EULER FLUID IN ANY DIMENSION AND A 2+1 DIMENSIONAL GAUGE ACTION

Like in the appendix of [1] (for the shallow water case $h,\vec{u}$), one can start with an action for the $2+1$ dimensional Euler fluid (with variables $\rho,\vec{u},P$), and derive the same gauge theory action directly from it. However, now we will focus on the Euler fluid action, and generalize it to higher dimensions.

We start with the Euler fluid action

$$\mathcal{L}=\frac{\rho\vec{u}^2}{2}-\rho\partial_0\tilde{\mu}+\phi\left(\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{u}\right)-\rho\beta_a\frac{D\alpha^a}{Dt}, \quad (5.1)$$

where ϕ is a Lagrange multiplier and α^a, β_a are variables from a Clebsch parametrization of the velocity $\vec{u}$, as we will see.

In this action, the first two terms are obvious. Indeed, the first term is just the kinetic term of the fluid, the second is $=-\rho\mu/m$, where $-\partial\mu/m=-dP/\rho$, so the term is just (at least if ρ is constant) P, so for the Lagrangian L (not the Lagrangian density $\mathcal{L}$) this would be $-\int PdV$, giving a potential energy. The third term just enforces the conservation of the energy density ρ,

$$\frac{D\rho}{Dt}+\rho\vec{\nabla}\cdot\vec{u}=\frac{\partial\rho}{\partial t}+\vec{\nabla}(\rho\vec{u})=0, \quad (5.2)$$

with the Lagrange multiplier ϕ. The last term is more unusual but is the exact analog of the corresponding term for the shallow water action in [1].

We then have equations of motion for the variables $\rho,\vec{u},\phi,\alpha^a,\beta_a$.

We want to obtain the Euler equation from the Lagrangian, which in our definitions is

$$\frac{D\vec{u}}{Dt}=-\frac{\vec{\nabla}P}{\rho}=-\vec{\nabla}\mu=-\vec{\nabla}\partial_0\tilde{\mu}. \quad (5.3)$$

(i) The equation of motion for ρ gives

$$\frac{\vec{u}^2}{2} - \partial_0\tilde{\mu} - \frac{D\phi}{Dt} = 0 \Rightarrow \frac{D\phi}{Dt} = \frac{\vec{u}^2}{2} - \frac{\partial_t P}{\rho}. \tag{5.4}$$

(ii) The ϕ equation of motion gives, as we said, the conservation of ρ (the continuity equation),

$$\partial_t\rho + \vec{\nabla}\cdot(\rho\vec{u}) = 0. \tag{5.5}$$

(iii) The α^a equation of motion gives

$$\partial_t(\rho\beta_a) + \vec{\nabla}\cdot(\rho\beta_a\vec{u}) = 0, \tag{5.6}$$

but using the conservation of ρ above, this gives the conservation of β_a,

$$\frac{D\beta_a}{Dt} = \partial_t\beta_a + \vec{\nabla}\cdot(\beta_a\vec{u}) = 0. \tag{5.7}$$

(iv) The β_a equation of motion gives the conservation of α^a,

$$\frac{D\alpha^a}{Dt} = 0. \tag{5.8}$$

(v) The equation of motion of $\vec{u}$ gives the form of $\vec{u}$ in a Clebsch-like parametrization,

$$\vec{u} = \vec{\nabla}\phi + \beta_a\vec{\nabla}\alpha^a. \tag{5.9}$$

Now the Euler equation is obtained as a combination of the $\vec{u}, \alpha^a, \beta_a$, and ρ equations.

We first use the $\vec{u}$ equation of motion (the resulting Clebsch-like parametrization) to calculate first

$$\begin{aligned}\frac{Du^i}{Dt} &= \partial_t u^i + \vec{\nabla}\cdot\vec{\nabla}u^i\\ &= \partial_t\partial_i\phi + \partial_t(\beta_a\partial_i\alpha^a) + u^j\partial_j(\partial_i\phi + \beta_a\partial_i\alpha^a)\\ &= \partial_i\partial_t\phi + (u^j\partial_j)\partial_i\phi + (\partial_t\beta_a)(\partial_i\alpha^a) + (u^j\partial_j\beta_a)(\partial_i\alpha^a)\\ &\quad + \beta_a\partial_i\partial_t\alpha^a + \beta_a(u^j\partial_j\partial_i\alpha^a).\end{aligned} \tag{5.10}$$

In the last expression, we use the ρ equation of motion $D\phi/Dt = 0$ in the first line, the second line vanishes due to the α^a equation of motion (that used also the ϕ equation, the ρ conservation, as we saw) $D\beta_a/Dt = 0$, and in the third line we use the β_a equation of motion $D\alpha^a/Dt = 0$, to obtain

$$\begin{aligned}\frac{Du^i}{Dt} &= -\beta_a(\partial^i u^j)\partial_j\alpha^a - (\partial^i u^j)\partial_j\phi + \partial^i\frac{\vec{u}^2}{2} - \partial^i\partial_t\tilde{\mu}\\ &= -\partial^i\partial_t\tilde{\mu},\end{aligned} \tag{5.11}$$

where in the last equality we have used again the equation of motion of u_i, that $\partial_j\phi + \beta_a\partial_j\alpha^a = u_j$.

We see that we finally obtained the Euler equation, as we wanted.

The above action is trivially generalized to $3+1$ dimensions, as nothing in it, not the action, and not the above derivation, depends on dimension.

The equation of motion for $\vec{u}$ in the above action gives the Clebsch parametrization for the velocity (5.9), which is true in any dimension. In particular, it has been used in $3+1$ dimensions, as we note in the next subsection.

We can also integrate by parts $D\rho/Dt = \partial_t\rho + \vec{u}\cdot\vec{\nabla}\rho$, in order to obtain ρ as a common factor of the action, obtaining the Euler action in general dimension[7]

$$\mathcal{L} = \rho\left[\frac{\vec{u}^2}{2} - \partial_0\tilde{\mu} - \frac{D\phi}{Dt} - \beta_a\frac{D\alpha^a}{Dt}\right]. \tag{5.15}$$

The transition to a gauge theory action works, however, only in $2+1$ dimensions.

We introduce the Lorentz covariant parametrization $C_\mu \equiv \partial_\mu\phi$, and consider now C_μ as the independent field, and impose the parametrization with the new Lagrange multiplier = gauge field A_μ, via a new term in $\mathcal{L}$,

$$\mathcal{L} \to \rho\left[\frac{\vec{u}^2}{2} - \partial_0\tilde{\mu} - (C_0 + u^iC_i) - \beta_a\frac{D\alpha^a}{Dt}\right] + \epsilon^{\mu\nu\rho}A_\mu\partial_\nu C_\rho. \tag{5.16}$$

Then the equation of motion for C_0 gives

$$B \equiv \partial_1A_2 - \partial_2A_1 = \rho, \tag{5.17}$$

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[7]This action has essentially been used by [7], as we became aware after the first version of this paper was posted on the arXiv. Indeed, in [7], one uses the Lagrangian

$$\mathcal{L} = -j^\mu a_\mu + \frac{1}{2}\rho\vec{v}^2 - V, \tag{5.12}$$

where V is some potential giving a pressure (or force), and

$$j^\mu = (c\rho, \rho\vec{v}), \qquad a_\mu = \partial_\mu\theta + \alpha^i\partial_\mu\beta^i, \tag{5.13}$$

so the auxiliary gauge field a_μ is given in a Clebsch parametrization.

Writing explicitly, we have

$$\mathcal{L} = \rho\left[\frac{\vec{v}^2}{2} - \frac{V}{\rho} - \frac{D\theta}{Dt} - \alpha^i\frac{D\beta^i}{Dt}\right]. \tag{5.14}$$

We see that it is basically our action, since $-\int V/\rho = \int\partial_0\tilde{\mu}$. It is not clear to us whether [7] meant for it to be used in any dimension (since in fact, they did not have indices i on α^i, β^i, but rather a single set).

Note that the extension of [7] to cases with spin was done in [18].

the equation of motion for C_i gives

$$E_i \equiv -(\partial_0 A_i - \partial_i A_0) = \epsilon_{ij}\rho u^j, \tag{5.18}$$

and the rest are as before.

Imposing these equations of motion amounts to dropping the terms $-\rho(C_0 + u^i C_i)$ and $\epsilon^{\mu\nu\rho}A_\mu \partial_\nu C_\rho$ from the action, and replacing (ρ, u^i) with (B, E_i) according to the above map, which gives

$$\begin{aligned}
\frac{\rho\vec{u}^2}{2} &= \frac{\vec{E}^2}{2B}, \\
\beta_a \frac{D\alpha^a}{Dt} &= (\rho)\beta_a\partial_t\alpha^a + (\rho u^j)\beta_a\partial_j\alpha^a \\
&= (\partial_1 A_2 - \partial_2 A_1)\beta_a\partial_t\alpha^a - (\partial_t A_i - \partial_i A_0)\epsilon^{ik}\beta_a\partial_j\alpha^a \\
&= \epsilon^{\mu\nu\rho}(\partial_\mu A_\nu)\beta_a\partial_\rho\alpha^a.
\end{aligned} \tag{5.19}$$

Thus the action becomes

$$\mathcal{L} = \frac{\vec{E}^2}{2B} - B\partial_0\tilde{\mu} - \epsilon^{\mu\nu\rho}(\partial_\mu A_\nu)\beta_a\partial_\rho\alpha^a, \tag{5.20}$$

which is just the action (2.5) with the Clebsch parametrization (2.7).

A. The ABC flow, in Clebsh and gauge field representations

As an example of the Clebsch parametrization for Euler flow in the case of $3+1$ dimensions, the standard ABC flow[8] solution of the Euler equations in $3+1$ dimensions,

$$\begin{aligned}
u_x &= \dot{x} = b\sin y - c\cos z, \\
u_y &= \dot{y} = c\sin z - a\cos x, \\
u_z &= \dot{z} = a\sin x - b\cos y,
\end{aligned} \tag{5.21}$$

can be written in the Clebsch parametrization,

$$\vec{u} = \vec{\nabla}\phi + \beta_1\vec{\nabla}\alpha^1 + \beta_2\vec{\nabla}\alpha^2, \tag{5.22}$$

where [19]

$$\begin{aligned}
\phi &= z(a\sin x - b\cos y), \\
\beta_1 &= b\sin y - c\cos z - az\cos x, \\
\beta_2 &= c\sin z - a\cos x - bz\sin y, \\
\alpha_1 &= x, \\
\alpha_2 &= y.
\end{aligned} \tag{5.23}$$

[8] We thank P. Wiegmann for bringing the ABC flow to our attention.

Can one map the ABC flow to a gauge field configuration? In [3,4] we proposed a map between fluid and gauge dynamics by mapping the corresponding components of the energy-momentum tensor. This map to ordinary Maxwell theory in $3+1$ dimensions is valid for $\vec{u}^2 = 1$ (null motion, $u^\mu u_\mu = 0$) and takes the form (here $a = 1, 2, 3$)

$$T_{00} = \rho \leftrightarrow \frac{1}{2}(\vec{E}^2 + \vec{B}^2), \qquad T_{a0} = \rho u_a \leftrightarrow [\vec{E}\times\vec{B}]_i. \tag{5.24}$$

We start with the degenerate ABC flow where $b = c = 0, a = 1$. In this case the flow vector is given by

$$\vec{v} = (0, -\cos x, \sin x), \qquad \rho = 1. \tag{5.25}$$

It is easy to check that in this case the corresponding electric and magnetic fields are given by

$$\begin{aligned}
\vec{E} &= (1, 0, 0), \\
\vec{B} &= -(0, \sin x, \cos x) = \vec{\nabla}\times\vec{A} \Rightarrow \vec{A} \\
&\quad = -(0, \sin x, \cos x) = \vec{B}.
\end{aligned} \tag{5.26}$$

It is interesting to note that the dual (electromagnetic) of this special ABC flow is a "null configuration" [20] obeying

$$\vec{E}\cdot\vec{B} = 0, \qquad \vec{E}^2 = \vec{B}^2. \tag{5.27}$$

As such the electromagnetic fields admit conserved nontrivial helicities, in particular the nonzero magnetic-magnetic helicity,

$$\mathcal{H}_{\mathrm{mm}} = \int d^3x\vec{A}\cdot\vec{B} = \int d^3x(\cos^2 x + \sin^2 x) = \int d^3x. \tag{5.28}$$

Note however that this magnetic field $\vec{B}$ does not obey the (free, vacuum) Maxwell equations, since

$$\vec{\nabla}\times\vec{B} = -(0, \sin x, \cos x) = \vec{B}, \tag{5.29}$$

so at most it can be interpreted as being sourced by a current $\vec{J} = -(0, \sin x, \cos x) = \vec{B}$ (but the electromagnetic helicities are usually defined for Maxwell fields in vacuum).

Next we can express the electric and magnetic fields in terms of two complex scalar fields α and β using the Bateman formulation [20], namely

$$\vec{F} = \vec{E} + i\vec{B} = \vec{\nabla}\alpha\times\vec{\nabla}\beta. \tag{5.30}$$

The complex scalar fields that yield the electric and magnetic fields given above are

$$\alpha = y - i\cos x, \qquad \beta = z + i\sin x. \tag{5.31}$$

In the Bateman formulation, half of the (free, vacuum) Maxwell's equations are automatic, $\vec{\nabla}\cdot\vec{F} = 0$, while the other half are

$$\vec{\nabla}\times\vec{F} = i\partial_t\vec{F} \Rightarrow i(\partial_t\alpha\vec{\nabla}\beta - \partial_t\beta\vec{\nabla}\alpha) = \vec{F} = \vec{\nabla}\alpha\times\vec{\nabla}\beta, \tag{5.32}$$

and these are not satisfied, since $\partial_t\alpha = \partial_t\beta = 0$, a statement equivalent to

$$\vec{\nabla}\times\vec{B} - \partial_t\vec{E} \neq 0, \qquad \vec{\nabla}\times\vec{E} + \partial_t\vec{B} \neq 0, \tag{5.33}$$

which we already noted happens in this case ($\vec{\nabla}\times\vec{B} - \partial_t\vec{E} = \vec{B}$ now).

Can we map a more general ABC flow to electromagnetism? The answer turns out to be no, as we see now. Basically, the problem is that the map is only valid for $\vec{u}^2 = 1$, which for the ABC flow becomes

$$a^2 + b^2 + c^2 - 2bc\sin y\cos z - 2ac\sin y\cos x - 2ab\sin x\cos y = 1, \tag{5.34}$$

which we can easily see that it cannot be solved for arbitrary x, y, z in the case of nonzero a, b, c. In fact, in order to have a solution at arbitrary x, y, z, we need to have at least two of a, b, c vanish, so that $ab = bc = ac = 0$.

We also note that the special case considered, with $b = c = 0$, obeys the two-dimensional relations (here $i = 1, 2$)

$$u_i = \epsilon_{ij}\frac{E_i}{|\vec{E}|}, \qquad \rho = |\vec{E}|^2, \qquad B = |\vec{E}|, \tag{5.35}$$

where $B = B_z$ and $E_i = (E_x, E_y)$ correspond to the formulation of the $2+1$ dimensional electromagnetism (though, of course, it is not quite $2+1$ dimensional, since all are functions of the third coordinate, z).

Finally, in this case, also the space-space components of the energy-momentum tensor match between the fluid and electromagnetism, since (replacing $B_a = B\delta_{a3} = |\vec{E}|\delta_{a3}$ in the electromagnetic T_{ab})

$$T_{ab} = \rho u_a u_b = |\vec{E}^2|\delta_{ab} - E_a E_b - |\vec{E}^2|\delta_{a3}\delta_{b3}, \tag{5.36}$$

which we can check separately for the cases $a = 3 = z$ and $a, b \neq 3 = z$.

VI. CONCLUSIONS AND DISCUSSION

In this paper, we have written a gauge theory for the Euler fluids in $2+1$ dimensions, with or without a coupling to electromagnetism. Using it, we were able to obtain the quantization of vorticity of $2+1$ dimensional Hopfion solutions from the quantization of the level of Chern-Simons. As a small aside, we have also coupled the shallow water equations to electromagnetism, and wrote the resulting equations in gauge theory form. We have also written an action for the Euler fluid in any dimension, using the Clebsch parametrization for the velocity.[9]

There are several open questions that deserve further research. In particular, we note the following:

(i) An important challenge is the quest for gauge theory formulation that corresponds to the Navier-Stokes fluid in $2+1$ dimensions.

(ii) We derived the action for an Euler fluid coupled to an electromagnetic background. A not less interesting phenomenon is the incorporation of the electromagnetic interactions of the fluid itself.

(iii) The action we derived for the Euler fluid in any dimensions includes a Lagrange multiplier term for the continuity equation. It would be more elegant to have an action that yields that equation not due to a Lagrange multiplier.

(iv) The study of the interplay between fluid flows and topologically nontrivial electromagnetic solutions has been touched upon in this paper. There are several additional questions about it, in particular, the use of special conformal transformation to derive novel solutions following [20].

ACKNOWLEDGMENTS

The work of H. N. is supported in part by CNPq Grant No. 301491/2019-4 and FAPESP Grants No. 2019/21281-4 and No. 2019/13231-7. H. N. would also like to thank the ICTP-SAIFR for their support through FAPESP Grant No. 2016/01343-7. The work of J. S. was supported by a center of excellence of the Israel Science Foundation (Grant No. 2289/18).

[9]Though an equivalent form was written before, in [7].

[1] D. Tong, A gauge theory for shallow water, SciPost Phys. **14**, 102 (2023).

[2] M. M. Sheikh-Jabbari, V. Taghiloo, and M. H. Vahidinia, Shallow water memory: Stokes and Darwin drifts, SciPost Phys. **15**, 115 (2023).

[3] D. W. F. Alves, C. Hoyos, H. Nastase, and J. Sonnenschein, Knotted solutions for linear and nonlinear theories: Electromagnetism and fluid dynamics, Phys. Lett. B **773**, 412 (2017).

[4] D. W. F. Alves, C. Hoyos, H. Nastase, and J. Sonnenschein, Knotted solutions, from electromagnetism to fluid dynamics, Int. J. Mod. Phys. A **32**, 1750200 (2017).

[5] A. G. Abanov and P. B. Wiegmann, Axial-current anomaly in Euler fluids, Phys. Rev. Lett. **128**, 054501 (2022).

[6] H. Nastase and J. Sonnenschein, Fluid-electromagnetic helicities and knotted solutions of the fluid-electromagnetic equations, J. High Energy Phys. 12 (2022) 144.

[7] B. Bistrovic, R. Jackiw, H. Li, V. P. Nair, and S. Y. Pi, Non-Abelian fluid dynamics in Lagrangian formulation, Phys. Rev. D **67**, 025013 (2003).

[8] D. Nickel and D. T. Son, Deconstructing holographic liquids, New J. Phys. **13**, 075010 (2011).

[9] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, Effective field theory for hydrodynamics: Thermodynamics and the derivative expansion, Phys. Rev. D **85**, 085029 (2012).

[10] F. M. Haehl, R. Loganayagam, and M. Rangamani, Effective actions for anomalous hydrodynamics, J. High Energy Phys. 03 (2014) 034.

[11] M. Crossley, P. Glorioso, and H. Liu, Effective field theory of dissipative fluids, J. High Energy Phys. 09 (2017) 095.

[12] P. Glorioso, M. Crossley, and H. Liu, Effective field theory of dissipative fluids (II): Classical limit, dynamical KMS symmetry and entropy current, J. High Energy Phys. 09 (2017) 096.

[13] F. M. Haehl, R. Loganayagam, and M. Rangamani, Effective action for relativistic hydrodynamics: Fluctuations, dissipation, and entropy inflow, J. High Energy Phys. 10 (2018) 194.

[14] V. P. Nair, R. Ray, and S. Roy, Fluids, anomalies and the chiral magnetic effect: A group-theoretic formulation, Phys. Rev. D **86**, 025012 (2012).

[15] V. P. Nair, Topological terms and diffeomorphism anomalies in fluid dynamics and sigma models, Phys. Rev. D **103**, 085017 (2021).

[16] G. M. Monteiro, V. P. Nair, and S. Ganeshan, Topological fluids with boundaries and FQH edge dynamics: A fluid dynamics derivation of the chiral boson action, arXiv: 2203.06516.

[17] G. M. Monteiro, A. G. Abanov, and V. P. Nair, Hydrodynamics with gauge anomaly: Variational principle and Hamiltonian formulation, Phys. Rev. D **91**, 125033 (2015).

[18] D. Karabali and V. P. Nair, Relativistic particle and relativistic fluids: Magnetic moment and spin-orbit interactions, Phys. Rev. D **90**, 105018 (2014).

[19] Z. Yoshida and P. J. Morrison, Epi-two-dimensional fluid flow: A new topological paradigm for dimensionality, Phys. Rev. Lett. **119**, 244501 (2017).

[20] C. Hoyos, N. Sircar, and J. Sonnenschein, New knotted solutions of Maxwell's equations, J. Phys. A **48**, 255204 (2015).