Tunneling potentials to nothing

J. J. Blanco-Pillado
Department of Theoretical Physics, University of the Basque Country UPV/EHU EHU Quantum Center, University of the Basque Country, UPV/EHU and IKERBASQUE, Basque Foundation for Science, 48011, Bilbao, Spain

J. R. Espinosa and J. Huertas
Instituto de Física Teórica, IFT-UAM/CSIC, C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049, Madrid, Spain

K. Sousa
University of Alcalá, Department of Physics and Mathematics, Plaza de San Diego, s/n, 28801, Alcalá de Henares (Madrid), Spain

(Received 13 December 2023; accepted 22 March 2024; published 24 April 2024)

The catastrophic decay of a spacetime with compact dimensions, via bubbles of nothing (BoNs), is probably a generic phenomenon. BoNs admit a four-dimensional description as singular Coleman–de Luccia bounces of the size modulus field, stabilized by some potential $V(\phi)$. We apply the tunneling potential approach to this 4D description to provide a very simple picture of BoNs. Using it, we identify four different types of BoN, corresponding to different classes of higher-dimensional theories. We also identify 4D theories featuring a new type of quenching of BoN decay, which may be present even for dS vacua, and discuss the viability of embedding such models in a higher-dimensional theory. The present approach allows us to treat in a single framework BoN nucleation and other decay channels, and we study the interplay between the different nonperturbative instabilities comparing their decay rates.

DOI: 10.1103/PhysRevD.109.084057

I. INTRODUCTION

Multiple vacua are common in theories beyond the Standard Model, and their decay has been widely studied using Euclidean bounces [1,2]. In theories with compact extra dimensions, a qualitatively new decay process, mediated by the so-called *bubble of nothing* (BoN), was first discussed by Witten [3] for the $\mathbb{M}^4 \times S^1$ Kaluza-Klein (KK) model. For BoNs in cases with more general internal manifolds and dimensions, see, e.g., [4–8]. A BoN describes a hole in spacetime, where the size of a compact dimension vanishes at the surface of the bubble, leaving nothing in the interior. Once nucleated, the BoN expands, ultimately destroying the parent spacetime.

BoNs are also relevant for the swampland program, which aims to characterize which effective field theories can be consistently coupled to quantum gravity [9,10]. In particular, the swampland conjecture in [11,12] states that nonsupersymmetric vacua are metastable at best, and BoN decay has been postulated as a universal decay channel for all nonsupersymmetric compactifications [6,13]. The generality of BoN decay is supported by the swampland cobordism conjecture [14], which states that all consistent quantum gravity theories are cobordant between them, and thus, they must admit a cobordism to nothing. BoNs are such configurations, with spacetime ending smoothly on the BoN core. In other words, the cobordism conjecture ensures that BoN decay is always topologically allowed. Therefore, to be able to establish the universality of BoN decay, it is imperative to understand any possible obstructions to BoN nucleation which have a dynamical origin, such as gravitational quenching [2], a well-known effect in standard false vacuum decay.

BoNs admit an effective 4D description as singular Euclidean bounces of the modulus field ϕ that controls the compactification size [15]. This bottom-up approach, quite useful to study the impact of the potential $V(\phi)$ present in realistic models, has been used to get some of the necessary conditions on $V(\phi)$ for the existence of BoNs [16]. We follow this 4D approach but use the *tunneling potential* method [17,18] (see Sec. II). Vacuum decay is described by a tunneling potential, $V_t(\phi)$, that minimizes a simple action functional in field space. In this language, BoNs are described by V_t's which are unbounded in the region where the extra dimension disappears. We describe Witten's BoN in the V_t formalism in Sec. III.

This technique allows us to efficiently explore possible BoNs. We identify (Sec. IV) four types with characteristic asymptotics as $\phi \to \infty$ (the BoN core) corresponding to different $4 + d$ origins (depending on the compact geometry and possible presence of a UV defect). We study (Secs. V and VI) the action and structure of these BoNs, contrasting them with other decay channels, like Coleman–de Luccia (CdL) [2] or pseudobounces [19]. We also identify and study two kinds of BoN quenching. In the first, the action diverges (CdL suppression), and the BoN becomes an end-of-the-world brane, while in the second, the action remains finite. We summarize in Sec. VII. For further details on our work, see Ref. [20].

II. TUNNELING POTENTIAL METHOD

The tunneling potential method calculates the action for the decay of a false vacuum of $V(\phi)$ at ϕ_+ by finding the (tunneling potential) function $V_t(\phi)$, (going from ϕ_+ to some ϕ_0 on the basin of the true vacuum at ϕ_-) that minimizes the functional [18]

$$S[V_t] = \frac{6\pi^2}{\kappa^2}\int_{\phi_+}^{\phi_0} d\phi \frac{(D + V_t')^2}{V_t^2 D}, \tag{1}$$

where $\kappa = 1/m_P^2$, with m_P being the reduced Planck mass. We take $\phi_+ < \phi_0 < \phi_-$, $x' \equiv dx/d\phi$, and

$$D^2 \equiv V_t'^2 + 6\kappa(V - V_t)V_t. \tag{2}$$

When V_t solves its "equation of motion" (EoM),

$$(4V_t' - 3V')V_t' + 6(V - V_t)[V_t'' + \kappa(3V - 2V_t)] = 0, \tag{3}$$

$S[V_t]$ reproduces the Euclidean bounce result [2].

The shape of V_t depends on $V_+ \equiv V(\phi_+)$. For $V_+ \leq 0$ (Minkowski or AdS false vacua), V_t is monotonic with $V_t, V_t' \leq 0$; see Fig. 1, lower curve. Here, ϕ_0 is the core value of the Euclidean bounce, and it is found so as to satisfy the boundary conditions

$$V_t(\phi_+) = V(\phi_+), \qquad V_t'(\phi_+) = 0 \tag{4}$$

at the false vacuum ϕ_+, and

$$V_t(\phi_0) = V(\phi_0), \qquad V_t'(\phi_0) = 3V'(\phi_0)/4 \tag{5}$$

at ϕ_0.

For $V_+ > 0$ (dS vacua), V_t is not monotonic and has the shape of the upper curve in Fig. 1, with two parts: a Hawking-Moss (HM)-like [21] part (ϕ_+, ϕ_{0+}), with $V_t = V$, and a CdL-like part $(\phi_{0+}, \phi_0 = \phi_{0-})$, with $V_t < V$. The field values $\phi_{0\pm} \neq \phi_\pm$ are found so as to satisfy the boundary conditions $V_t(\phi_{0\pm}) = V(\phi_{0\pm})$ and $V_t'(\phi_{0\pm}) = 3V'(\phi_{0\pm})/4$ and they coincide with the extreme values of the Euclidean CdL bounce. If V_+ grows, the CdL interval shrinks to zero, there is no CdL decay, and the action tends toward the HM value [18].

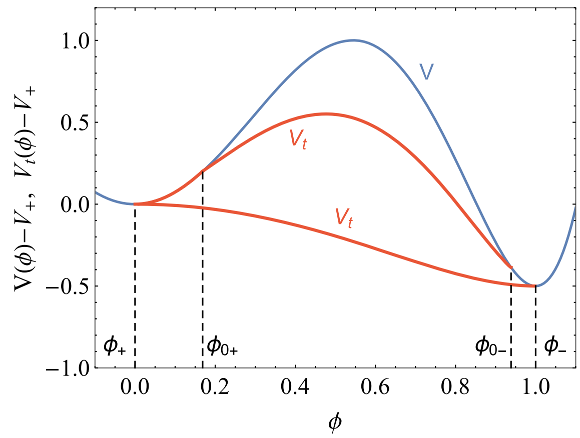

FIG. 1. Typical shape of tunneling potentials for the decay of AdS/Minkowski (lower curve) or dS (upper) vacua.

Gravitational quenching of decay, and thus vacuum stabilization, occurs if the condition $D^2 > 0$ (needed for a real $S[V_t]$) cannot be satisfied for any V_t. This can happen for Minkowski or AdS vacua if gravitational effects are strong. We define $\overline{V_t}(\phi)$ as the solution to $D \equiv 0$ (we set $\kappa = 1$ from now on),

$$\overline{V_t}' = -\sqrt{6(V - \overline{V_t})(-\overline{V_t})}, \tag{6}$$

with $\overline{V_t}(\phi_+) = V_+$. To have $D^2 > 0$, V_t should have a slope steeper than $\overline{V_t}$, so that $V_t(\phi) < \overline{V_t}(\phi)$. If, after leaving ϕ_+, $\overline{V_t}$ does not intersect V again, we have quenching. If $\overline{V_t}$ reaches V right at the minimum ϕ_- (critical case), then $V_t = \overline{V_t}$ describes a flat and static domain wall between false and true vacua, its action is infinite, and gravity also forbids the decay.

In the Euclidean approach, assuming $O(4)$ symmetry, vacuum decay is described by a bounce configuration, $\phi(\xi)$, which extremizes the Euclidean action, and a metric function, $\rho(\xi)$, entering the Euclidean metric

$$ds^2 = d\xi^2 + \rho(\xi)^2 d\Omega_3^2. \tag{7}$$

Here, ξ is a radial coordinate, and $d\Omega_3^2$ is the line element on a unit three-sphere. A dictionary between Euclidean and V_t methods follows from the key link between both formalisms,

$$V_t(\phi) = V(\phi) - \dot{\phi}^2/2, \tag{8}$$

where $\dot{x} \equiv dx/d\xi$, and $\dot{\phi}$ is expressed in terms of the field via the profile $\phi(\xi)$. The profiles $\phi(\xi)$ and $\rho(\xi)$ can be derived from V_t using the previous link and the Euclidean EoMs [17,18].

Finally, the V_t approach also describes pseudobounces [19] as solutions of (3) with $V_t'(\phi_0) = 0$. These decay modes are not extremals of the action (they would be if ϕ_0 were held fixed), and they have actions larger than the CdL one. They are relevant when there is no CdL solution [19].

III. WITTEN'S BUBBLE OF NOTHING

The 5D KK spacetime (4D Minkowski $\times S^1$) is unstable against semiclassical decay via the nucleation of a BoN, described by the instanton metric

$$ds^2 = \frac{dr^2}{1-\frac{\mathcal{R}^2}{r^2}} + r^2 d\Omega_3^2 + R_{KK}^2\left(1-\frac{\mathcal{R}^2}{r^2}\right)d\theta_5^2, \quad (9)$$

where R_{KK} is the KK radius, $\mathcal{R}$ is the size of the nucleated bubble, $r \in [\mathcal{R}, \infty)$, and $\theta_5 \in [0, 2\pi)$ parametrizes the KK circle. For $r \to \infty$, this metric tends to $\mathbb{M}^4 \times S^1$. This instanton solution, analytically continued to Lorentzian signature, describes the tunneling from the homogeneous $\mathbb{M}^4 \times S^1$ to a spacetime in which the radius of the fifth dimension shrinks to zero as $r \to \mathcal{R}$ [3]. This BoN "hole" at $r = \mathcal{R}$ then expands and destroys the KK spacetime. The decay rate per unit volume is $\Gamma/V \sim e^{-\Delta S_E}$, with $\Delta S_E = (\pi m_P R_{KK})^2$ being the difference between the Euclidean action of the bounce and the KK vacuum.

The BoN (9) can be reduced to a 4D description [15] integrating the fifth dimension θ_5, and introducing the modulus scalar ϕ with

$$e^{-2\sqrt{2/3}\phi} \equiv 1 - R_{KK}^2/r^2. \quad (10)$$

A Weyl rescaling puts the BoN metric into CdL form (7), with

$$\frac{d\xi}{dr} \equiv \frac{1}{(1-R_{KK}^2/r^2)^{1/4}} \quad (11)$$

and

$$\rho(\xi)^2 \equiv r^2(1-R_{KK}^2/r^2)^{1/2}. \quad (12)$$

This maps (9) into a field profile, $\phi(\xi)$, with the BoN core at $\phi \to \infty$ ($\xi \to 0$), and the KK vacuum at $\phi \to 0$ ($\xi \to \infty$). This CdL solution is not of the standard form, as the field diverges at $\xi = 0$. Nevertheless, its Euclidean action is finite and equal to Witten's (after including a boundary term of 5D origin).

Finding the description in the V_t approach is straightforward, using $V_t = V - \dot{\phi}^2/2$. One gets

$$V_t(\phi) = -(6/R_{KK}^2)\sinh^3(\sqrt{2/3}\phi), \quad (13)$$

with $V_t(0) = 0$, $V_t(\phi \to \infty) \sim -e^{\sqrt{6}\phi}$, so that V_t diverges at $\phi \to \infty$. This is a generic property of the V_t's of BoNs. Furthermore, the action in the V_t formalism, Eq. (1), gives the correct result without the need of additional boundary terms. This is true for any other BoN solution in this formalism; see Ref. [20].

IV. BONS WITH NONZERO POTENTIAL

The modulus field potential, $V(\phi)$, needed to stabilize the extra dimensions, affects the existence and shape of BoNs. In the spirit of [16], we derive the conditions that $V(\phi)$ must satisfy to allow BoN decays. The single function $V_t(\phi)$, on the same footing as $V(\phi)$, captures the key BoN asymptotics in a simple way. Without assumptions about the origin of $V(\phi)$, we first identify four different types of asymptotics of V and V_t compatible with BoNs.

Any V_t describing a BoN solves Eq. (3) with standard boundary conditions at the false vacuum ϕ_+, and $V_t \to -\infty$ at $\phi \to \infty$ (the BoN core). The four different types of core asymptotics depending on the value of $\lim_{\phi\to\infty} V/|V_t|$ are listed in Table I:

Type 0: $\lim_{\phi\to\infty} V/|V_t| = 0$. Whether V is positive or negative at $\phi \to \infty$, Eq. (3) gives $V_t(\phi \to \infty) \sim V_{tA} e^{\sqrt{6}\phi}$, with $V_{tA} < 0$. V is irrelevant for $\phi \to \infty$, and these BoNs behave as Witten's BoN.

Types $\pm$: $\lim_{\phi\to\infty} V/|V_t|$ is a constant of sign $\pm$, which labels the type. For $V \sim V_A e^{a\sqrt{6}\phi}$ and $V_t \sim V_{tA} e^{a\sqrt{6}\phi}$ at $\phi \to \infty$, with $a > 0$, $V_{tA} < 0$, Eq. (3) gives

$$[V_A + (a^2-1)V_{tA}](3V_A - 2V_{tA}) = 0. \quad (14)$$

The first option is $V_{tA} = V_A/(1-a^2)$. For type $-$, $V_A < 0, a < 1$. For type $+$, $V_A > 0, a > 1$.

Type $-^*$: The second option to satisfy Eq. (14) is $V_{tA} = 3V_A/2$. One needs $V_A < 0$, as $V_t < V$.

TABLE I. For $V(\phi \to \infty) = V_A e^{a\sqrt{6}\phi}$, we show, for the four different types of BoN, the asymptotics of $V_t(\phi \to \infty)$, parameter constraints (Param. Constr.), the exponent β in $\rho(\xi \to 0) \sim \xi^\beta$, the exponent δ in $D(\phi \to \infty) \sim D_\infty e^{\delta\phi}$, and their possible UV origin. The label "Sing." indicates the need for a defect to avoid a singularity.

Type	0	$-$	$+$	$-^*$
$V_t(\infty)$	$V_{tA} e^{\sqrt{6}\phi}$	$\frac{V_A e^{a\sqrt{6}\phi}}{(1-a^2)}$	$\frac{V_A e^{a\sqrt{6}\phi}}{(1-a^2)}$	$\frac{3V_A e^{a\sqrt{6}\phi}}{2}$
Param. Constr.	$V_{tA} < 0$ $a < 1$	$V_A < 0$ $\frac{1}{\sqrt{3}} < a < 1$	$V_A > 0$ $a > 1$	$V_A < 0$ $a > \frac{1}{\sqrt{3}}$
β	$\frac{1}{3}$	$\frac{1}{3a^2}$	$\frac{1}{3a^2}$	1
δ	$\frac{(1+\beta)}{\sqrt{2\beta}}$	$\frac{(1+\beta)}{\sqrt{2\beta}}$	$\frac{(1+\beta)}{\sqrt{2\beta}}$	$a\sqrt{6}$
UV	S^1	S^d	Sing.	Sing.

Table I also shows additional parameter constraints on a obtained by requiring the finiteness of the BoN action.

V and V_t determine the asymptotics of the Euclidean BoN functions $\phi(\xi)$ and $\rho(\xi)$ at $\xi \to 0$. We get

$$\rho \simeq c_\rho \xi^\beta, \qquad \phi \simeq -\frac{1}{a}\sqrt{\frac{2}{3}}\log\left[\xi a\sqrt{3(V_A - V_{tA})}\right], \tag{15}$$

with β as given in Table I. For type-0 BoNs, this holds with $a = 1$, $V_A = 0$ and agrees with [16]. Thus, the 4D instanton is singular, with the leading behavior near the singularity determined by V_{tA}.

From a $4 + d$ BoN geometry, we can integrate over the compact space to get a reduced 4D metric and a modulus field [with potential $V(\phi)$] that tracks the size of the extra dimensions. This gives a 4D picture of the BoN as a singular CdL bounce $\phi(\xi)$ [15,16], or as a divergent tunneling potential, $V_t(\phi)$. Via such a top-down approach, we explore the $4 + d$ origin of the parameters in the BoNs found above.

Consider first a BoN with a d-dimensional sphere, S^d, of radius R_{KK}, as compact space. Imposing the smoothness of the $4 + d$ BoN solution at $r \to 0$, and reducing to 4D, we obtain the $\xi \to 0$ scaling

$$\phi \simeq -\sqrt{\frac{2d}{(d+2)}}\log \xi_d, \qquad \rho \simeq \mathcal{R}\xi_d^{d/(d+2)}, \tag{16}$$

where $\xi_d \equiv (d+2)\xi/(2R_{KK})$, which agrees with [16]. Comparing with the scalings found above using V_t, we get

$$a = \sqrt{\frac{d+2}{3d}}, \qquad \beta = \frac{d}{d+2}, \tag{17}$$

and

$$D_\infty = \frac{3}{R_{KK}\mathcal{R}}\sqrt{\frac{d(d+2)}{2}}, \tag{18}$$

where $D(\phi \to \infty) \sim D_\infty e^{\delta\phi}$, with δ given in Table I. V_A and V_{tA} are also determined by (16), giving

$$V \simeq \frac{-d(d-1)}{2R_{KK}^2} e^{\sqrt{\frac{2(d+2)}{d}}\phi}, \qquad V_t \simeq \frac{-3d^2}{4R_{KK}^2} e^{\sqrt{\frac{2(d+2)}{d}}\phi}, \tag{19}$$

at $\phi \to \infty$. Thus, the smoothness condition imposes V to be of the form one gets in the 4D reduced action from the curvature $\mathcal{R}_d$ of the compact space,

$$\delta V(\phi) = -\frac{\mathcal{R}_d}{2} e^{\sqrt{\frac{2(d+2)}{d}}\phi}. \tag{20}$$

Type-0 BoNs are realized for $d = 1$, and type $-$ for $d > 1$ [as $1/\sqrt{3} < a = \sqrt{(d+2)/(3d)} < 1$].

Other well-known sources of moduli potentials (see, e.g., [16]) give

$$\delta V(\phi) = \Lambda_{4+d} e^{\sqrt{\frac{2d}{d+2}}\phi} + \frac{Q^2}{2g^2\mathcal{V}_{(d)}} e^{3\sqrt{\frac{2d}{d+2}}\phi}. \tag{21}$$

The first term comes from a $4 + d$ cosmological constant, Λ_{4+d}. If this is the dominant term in V, then the a parameter of our V_t description (see Table I) would be $1/3 \leq a = \sqrt{d/[3(d+2)]} < 1/\sqrt{3}$, which is of type 0. The second term comes from a d-form flux on the compact space, $\int_{S^d} F_d = Q$ (with g being the gauge coupling and $\mathcal{V}_{(d)}$ the volume of the d sphere). This contribution gives $1 \leq a = \sqrt{3d/(d+2)} < \sqrt{3}$: the scaling of type-+ cases (provided $d > 1$).

However, for S^d compactifications, the flux contribution to V cannot dominate at the $\phi \to \infty$ limit, as the regularity conditions require V as in (19). Nevertheless, scalar fields present besides the modulus ϕ can modify the potential probed asymptotically by the BoN. Such an example for a BoN in a flux compactification model is given in [20]. There, the naive type-+ behavior of the flux contribution is tamed by the presence of a smooth source that effectively transforms the solution in a type-0 BoN at its core. See Ref. [4] for a higher-dimensional realization of this effect.

More exotic types of solutions leading to singular BoNs (like types $-^*$ and $+$) can also be realized. The BoN singularity signals the need of a brane, or another UV object, whose properties (tension and charge) could be inferred from the behavior of the solution in the limit $\phi \to \infty$; see Refs. [14,22–24].

V. LOW-FIELD SHOOTING AND BON QUENCH

For the numerical exploration of vacuum decay solutions, instead of starting at large field values using the overshoot/undershoot method as in [16], we solve the EoM for V_t starting at low-field values. Our solutions never under/overshoot but are always on target: all starting boundary conditions correspond to a solution, be it a BoN, a CdL, or a pseudobounce.

As V_t is a solution of the second-order differential equation (3), it depends on two integration constants—e.g., V_t and V_t' at some field value. For dS vacua, we can solve for V_t starting from the initial point of the CdL range of V_t, $\phi_i \neq \phi_+$, with $V_t(\phi_i) = V(\phi_i)$ and $V_t'(\phi_i) = 3V'(\phi_i)/4$. For Minkowski or AdS vacua, we start at ϕ_+ with $V_t(\phi_+) = V(\phi_+)$, but $V_t'(\phi_+) = 0$ does not fix completely the solution, as ϕ_+ is an accumulation point of an infinite family of solutions, and one needs to impose an additional condition to select a particular one; see below.

There is an interesting interplay between the boundary conditions satisfied by $V_t(\phi)$ at both ends of the field interval in which it is defined. In order to illustrate this, we use the simple type-0 potential

$$V(\phi) = m^2\phi^2/2. \tag{22}$$

The low-field expansion of V_t is

$$V_t(A;\phi) \simeq -m^2\phi^2/[1/A - (1/3)\log(A\phi^2)], \quad (23)$$

with $A > 0$ being a free parameter. (For AdS vacua, the behavior is similar, with a low-field expansion for V_t with a different parameter [20].) We find an infinite family of solutions, $V_t(A;\phi)$, describing BoN decays; see the top plot of Fig. 2. For $A \to 0$, we reach the critical $\overline{V_t}(\phi)$ (black dashed line), which has $D = 0$ and is an upper limit on allowed V_t's (with $D^2 > 0$).

The asymptotics of the V_t solutions is of type 0:

$$V_t \sim V_{tA}(A)e^{\sqrt{6}\phi}, \qquad D \sim D_\infty(A)e^{\sqrt{8/3}\phi}. \quad (24)$$

(For these BoNs, the two integration constants in the large-field regime can be chosen to be V_{tA} and D_∞.) The functions $V_{tA}(A)$ and $D_\infty(A)$ depend on $V(\phi)$ and are given for our case in the bottom plot of Fig. 2. Interestingly, $-V_{tA}$ is bounded below by the V_{tA} prefactor of $\overline{V_t} \sim \overline{V}_{tA}e^{\sqrt{6}\phi}$ (dashed line).

As shown in Sec. III, a given $4 + d$ theory with fixed R_{KK} determines V_{tA} via Eq. (19) (with $d = 1$), thus selecting one member of the family of V_t's. When $-V_{tA}(A)$ is bounded below, $-V_{tA}(A) \geq -V_{tA*}$ (as in Fig. 2), BoN decay is allowed provided

$$R_{KK}^2 = \frac{3}{4(-V_{tA})} \leq \frac{3}{4(-V_{tA*})}, \quad (25)$$

and forbidden otherwise. The critical case V_{tA*} corresponds to the limit $A \to 0$, for which $D_\infty \to 0$, $S \to \infty$, and $\mathcal{R} \to \infty$ [see (18)], as expected for an infinite and static BoN: an end-of-the-world brane [25]. The situation is similar for the AdS case.

This dynamical obstruction to BoN decay [6,13] is similar to the CdL quenching of the standard decay of Minkowski or AdS false vacua, with the critical case corresponding to a domain wall of infinite action; see Sec. II. Although already [3] discussed possible topological obstructions to BoN decays, the cobordism conjecture [14] removes such obstructions. In that case, the only protection of a compactification against BoN formation must be dynamical [6,13].

For the dS case, an expansion of V_t near ϕ_i is used, and one can take ϕ_i as the free parameter for a family of V_t solutions. For regular CdL decay, we get a family of pseudobounces ending at the proper CdL [19]. For BoNs, we get a type-0 family with asymptotics $V_t \sim V_{tA}(\phi_i)e^{\sqrt{6}\phi}$. However, now there is no bound on $V_{tA}(\phi_i)$ (as there is no critical $\overline{V_t}$), and thus no dynamical constraint on BoN decay (see Ref. [20]).

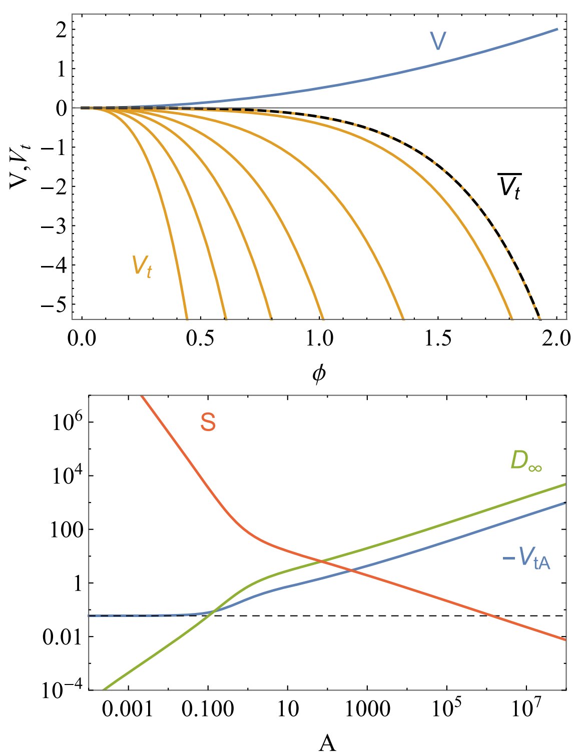

FIG. 2. Top: Potential $V = \phi^2/2$ and tunneling potentials $V_t(A;\phi)$ (bounded by $\overline{V_t}$, dashed line). Bottom: Tunneling action S and prefactors V_{tA} and D_∞.

VI. BONS VS. OTHER DECAY CHANNELS

To illustrate the interplay of BoNs with standard decay channels (CdL decay and pseudobounces), let us consider the potential

$$V(\phi) = V_+ + \frac{1}{2}m^2\phi^2 - \lambda\phi^4 + \lambda_6\phi^6, \quad (26)$$

which admits examples of type-0 BoNs. For numerics, we take $m = 1$, $\lambda = 17/4$, and $\lambda_6 = 8/3$. The potential has a false vacuum at $\phi_+ = 0$, separated from the true vacuum at $\phi_- = 1$ by a shallow barrier that peaks at $\phi_B = 0.25$. (Further examples, including dS cases with HM but no CdL decay, type-− BoNs, etc., are discussed in the companion paper [20].)

The Minkowski case ($V_+ = 0$) is shown in Fig. 3. In the upper plot, the ($D = 0$) $\overline{V_t}$ of (6) (black dashed line) touches the potential beyond the barrier, signaling a CdL instability of the false vacuum. Below $\overline{V_t}$, we find $V_t(A;\phi)$ solutions, labeled by the parameter A of the low-field V_t expansion: the CdL instanton solution (red line) for $A = A_{\rm CdL} \simeq 2.8$, pseudobounce solutions (green lines) for $A < A_{\rm CdL}$, and unbounded BoN solutions (orange lines) for $A > A_{\rm CdL}$.

The bottom plot of Fig. 3 gives the tunneling action of the V_t solutions just described. The action of pseudobounces diverges at $A \to 0$ (when $V_t \to \overline{V_t}$) and interestingly, the BoN action beyond the CdL point can be larger or

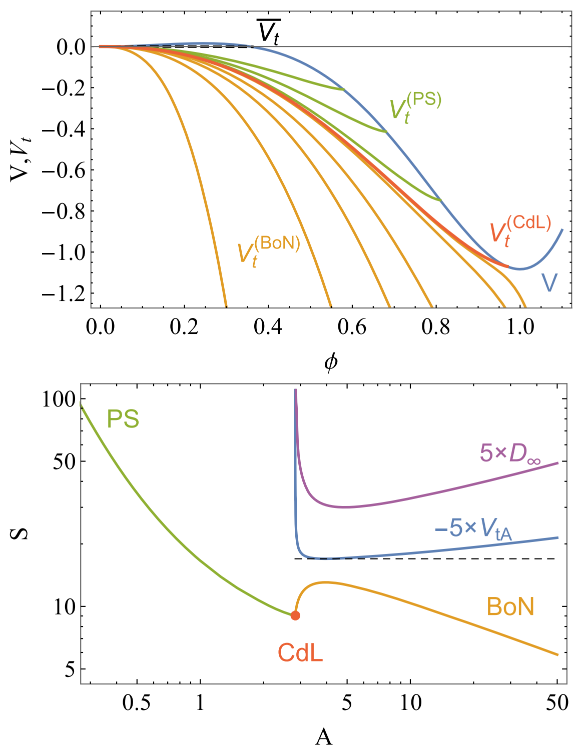

FIG. 3. Top: Potential [Eq. (26)] with $V_+ = 0$ and tunneling potentials $V_t(A;\phi)$: $\overline{V_t}$ (black dashed), pseudobounces (green), CdL bounce (red), and BoNs (orange). Bottom: Tunneling action S for the $V_t(A;\phi)$. For the BoN range of A, (rescaled) prefactors V_{tA} and D_∞.

smaller than $S_{\rm CdL}$. Thus, the BoN decay channel does not always dominate.

The BoNs obtained are of type 0, with $\phi \to \infty$ asymptotic behavior as in (24). The lower plot of Fig. 3 shows $D_\infty(A)$ and $-V_{tA}(A)$, which is bounded below by $-V_{tA*} = -V_{tA}(A_*)$ (black dashed line). That minimum is reached when S is maximal.

In a given $4+1$ theory, V_{tA} is determined by (19) (with $d=1$). When the bound (25) is satisfied, there are two possible BoNs, corresponding to the two solutions of $R_{KK}^2 = \frac{3}{4(-V_{tA}(A))}$. The solution with the lowest tunneling action (thus the relevant one) lies in the branch of solutions extending from the action maximum to values below $S_{\rm CdL}$ $(A > A^*)$.

BoN decays are forbidden if $R_{KK}^2 > \frac{3}{4(-V_{tA*})}$ (although CdL decay is still open). This dynamical quench with finite S can happen even for a dS vacuum [20], in contrast with the standard quenching of decay, which only occurs for $V_+ \le 0$. If we require the KK and 4d EFT scales to be well separated (R_{KK} small compared to the typical EFT length scale), this needs large $-V_{tA}$ due to (25). In this limit, where the EFT is well under control, BoN decay is always allowed, and it becomes the fastest decay channel.

VII. SUMMARY

The V_t method greatly facilitates the study of which modulus potentials $V(\phi)$ admit BoN decays and which types of BoN exist. We identify four types of BoN, with different asymptotics in the compactification limit ($\phi \to \infty$, the BoN core); see Table I. Type-0/− BoNs can appear if the compact space is a S^d sphere, while Type-+ or $-^*$ BoNs need more complicated compact geometries, and/or the presence of some UV defect at the BoN core.

For BoNs of types 0 or −, there are simple relations between the asymptotics of V, V_t, and the BoN geometry in the $4+d$ theory (like the KK radius, R_{KK}). Such relations tell which BoNs are relevant for a given theory. For potentials not growing as fast as $e^{\sqrt{6\kappa}\phi}$, we find a continuous family of type-0 BoN solutions labeled by some parameter p, with $V_t(p;\phi) \simeq V_{tA}(p)e^{\sqrt{6}\phi}$ and $D(p;\phi) \simeq D_\infty(p)e^{\sqrt{8/3}\phi}$ for $\phi \to \infty$. Fixing the compactification scale, R_{KK}, selects a finite number of BoNs from the family (each with different action). The number of such selected BoNs is model dependent in the following way.

When the modulus has a single vacuum (or if gravity forbids its decay), the BoN is unique (for fixed R_{KK}). If the vacuum is a Minkowski or AdS one, there is an upper critical limit R_{KK}^* for which the BoN has infinite action and radius and turns into an end-of-the-world brane. For $R_{KK} > R_{KK}^*$, BoN decay is forbidden (CdL-like dynamical quenching).

When the scalar potential has additional vacua and admits standard decay channels (CdL/HM) to them, there are (at least) two BoNs (the one with the lowest action being the relevant one). In this case, there is also a critical R_{KK}^* which corresponds to the merging of the two BoN solutions into one with finite action. For $R_{KK} > R_{KK}^*$, BoN decay is again dynamically forbidden. It would be interesting to understand this new quenching of the BoN decay channel from a higher-dimensional theory—in particular, in models of flux compactifications within a string theory context.

ACKNOWLEDGMENTS

J. J. B.-P., J. R. E., and J. H. are supported in part by Grants No. PID2021-123703NB-C21, No. PID2022-142545NB-C22, and No. PID2021-123017NB-I00, funded by "ERDF, A way of making Europe" and by MCIN/AEI/10.13039/501100011033. J. J. B.-P. is supported by Basque Government Grant No. IT-1628-22 and the Basque Foundation for Science (IKERBASQUE). J. R. E. and J. H. are supported by IFT Centro de Excelencia Severo Ochoa No. CEX2020-001007-S. J. H. is supported by FPU Grant No. FPU20/01495 from the Spanish Ministry of Education and Universities.

[1] S. R. Coleman, Phys. Rev. D **15**, 2929 (1977); **16**, 1248(E) (1977).
[2] S. R. Coleman and F. De Luccia, Phys. Rev. D **21**, 3305 (1980).
[3] E. Witten, Nucl. Phys. **B195**, 481 (1982).
[4] J. J. Blanco-Pillado and B. Shlaer, Phys. Rev. D **82**, 086015 (2010).
[5] J. J. Blanco-Pillado, H. S. Ramadhan, and B. Shlaer, J. Cosmol. Astropart. Phys. 10 (**2010**) 029.
[6] J. J. Blanco-Pillado, B. Shlaer, K. Sousa, and J. Urrestilla, J. Cosmol. Astropart. Phys. 10 (**2016**) 002.
[7] H. Ooguri and L. Spodyneiko, Phys. Rev. D **96**, 026016 (2017).
[8] G. Dibitetto, N. Petri, and M. Schillo, J. High Energy Phys. 08 (2020) 040.
[9] C. Vafa, arXiv:hep-th/0509212.
[10] M. van Beest, J. Calderón-Infante, D. Mirfendereski, and I. Valenzuela, arXiv:2102.01111.
[11] H. Ooguri and C. Vafa, Adv. Theor. Math. Phys. **21**, 1787 (2017).
[12] B. Freivogel and M. Kleban, arXiv:1610.04564.
[13] I. García Etxebarria, M. Montero, K. Sousa, and I. Valenzuela, arXiv:2005.06494.
[14] J. McNamara and C. Vafa, arXiv:1909.10355.
[15] M. Dine, P. J. Fox, and E. Gorbatov, J. High Energy Phys. 09 (2004) 037.
[16] P. Draper, I. G. Garcia, and B. Lillard, Phys. Rev. D **104**, L121701 (2021); J. High Energy Phys. 12 (2021) 154.
[17] J. R. Espinosa, J. Cosmol. Astropart. Phys. 07 (2018) 036.
[18] J. R. Espinosa, Phys. Rev. D **100**, 104007 (2019).
[19] J. R. Espinosa, Phys. Rev. D **100**, 105002 (2019); J. R. Espinosa and J. Huertas, J. Cosmol. Astropart. Phys. 12 (2021) 029.
[20] J. J. Blanco-Pillado, J. R. Espinosa, J. Huertas, and K. Sousa, J. Cosmol. Astropart. Phys. 03 (2024) 029.
[21] S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).
[22] G. T. Horowitz, J. Orgera, and J. Polchinski, Phys. Rev. D **77**, 024004 (2008).
[23] P. Bomans, D. Cassani, G. Dibitetto, and N. Petri, SciPost Phys. **12**, 099 (2022).
[24] B. Friedrich, A. Hebecker, and J. Walcher, arXiv:2310.06021.
[25] R. Angius, J. Calderón-Infante, M. Delgado, J. Huertas, and A. M. Uranga, J. High Energy Phys. 06 (2022) 142.