# Horizon as a natural boundary

Qi-Yuan Mao,[1,*] Liang Ma,[1,†] and H. Lü[1,2,3,‡]

[1]*Center for Joint Quantum Studies and Department of Physics, School of Science, Tianjin University, Tianjin 300350, China*
[2]*Joint School of National University of Singapore and Tianjin University, International Campus of Tianjin University, Binhai New City, Fuzhou 350207, China*
[3]*Peng Huanwu Center for Fundamental Theory, Hefei, Anhui 230026, China*

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We consider Einstein gravity extended with Riemann-squared term and construct the leading-order perturbative solution to the rotating black hole with all equal angular momenta in $D = 7$. We find that in the extremal limit, the linear perturbation involves irrational powers in the near-horizon expansion. We argue that, despite that all curvature tensor invariants are regular on the horizon, the irrational power implies that the inside of the horizon is destroyed and the horizon becomes the natural boundary of the spacetime. We demonstrate that this vulnerability of the horizon regularity is an innate part of Einstein theory, and can arise in Einstein theory with minimally coupled matter. However, in fine-tuned theories such as supergravities, the black hole inside is preserved, which may be one of the criteria for a consistent theory of quantum gravity. We also show that the vulnerability occurs in general higher dimensions, which only a few sporadically distributed dimensions can evade.

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## I. INTRODUCTION

Black holes, predicted by Einstein's theory of general relativity, are characterized by having an event horizon, i.e., a boundary where the escape velocity equals the speed of light. In all the well-known black holes such as the Schwarzschild or the Kerr black holes, in four or higher dimensions, the curvature invariants on the horizon are regular and the near-horizon geometry is described by analytic metric functions that are infinitely differentiable. (see e.g., [1–8] for some Ricci-flat or Einstein metrics.) This implies that the horizon is not the boundary of spacetime itself. One can extend the horizon geometry, e.g., [9] to include the inside of a black hole and indeed this is confirmed by analyzing the geodesic motion.

Recently, the linear perturbation of higher-derivative gravity to $D = 5$ dimensional rotating black holes with equal angular momenta was obtained [10]. In the extremal limit, the perturbed solution involves a function like $(c + \log(1 - r_0/r))$, which is not analytic on the horizon $r = r_0$. In this particular case, the $i\pi$ factor generated by crossing over the horizon can be absorbed into the integration constant $c$. This, however, leads to some obvious questions; can worse nonanalyticities arise on the horizon and what are their implications?

Black hole extremal limit typically arises when the attractive force of gravity balances the repulsive force in the theory. The best known example of such balance is exhibited by the Reissner-Nordström (RN) black hole, where gravity and electric Coulomb repulsion balance precisely in the extremal limit. This can not only be verified by geodesic motion of charged test particles [11], but also allows the construction of multicenter black holes in harmonic superposition. This no-force condition was recently shown to be preserved under appropriate higher-order curvature corrections [12]. Rotations provide a repulsive centrifugal force and extremal rotating black holes are the results of its balance against the gravitational attraction. However, in higher dimensions, gravitational force becomes weaker, whilst the centrifugal force, associated with rotation on a plane, remains the same strength. This Newtonian picture suggests that the balance between gravity and centrifugal force becomes more strenuous in higher dimensions so that any perturbation might ruin the near-horizon geometry.

In this paper, we therefore consider Einstein gravity extended with a Riemann-squared term in general odd $D = 2n + 1$ dimensions, with a coupling constant $\alpha$. We construct the leading $\alpha$-order perturbation to the Ricci-flat

*qiyuan_mao@tju.edu.cn
†liangma@tju.edu.cn
‡mrhonglu@gmail.com

rotating black holes. For simplicity, we consider only the cohomogeneity-one metrics with all equal angular momenta. We find that in the extremal limit, the linearized solutions in general higher-odd dimensions involve a nonanalytic term of the type $(r-r_0)^{\Delta_+}$, where $\Delta_+$ is an irrational number. In fact, we find that it is generally irrational in higher $D=2n+1$ dimensions lying between $1/2$ and 1, but it can be rational sporadically.

We study the implication of these irrational structures and we argue that despite that curvature tensor invariants are all regular and geodesics are incomplete on the horizon, we cannot extend the spacetime beyond horizon, which therefore forms a natural boundary of spacetime. Similar singularities were studied in the context of cosmological models with homogeneous spatial section and it was referred to as the "wimper singularity" in that the universe terminates at a singularity where all physical quantities are well-behaved (a "whimper" rather than a "bang") and an associated Cauchy horizon [13,14].

It should be emphasized that having an irrational $\Delta_+$ is not a consequence of higher-derivative corrections. This is because the dynamics of perturbative equation is governed by the linearized equation of Einstein gravity in the background of the rotating black hole. Our conclusion therefore applies to Einstein gravity with minimally coupled matter, unless the matter Lagrangian is so fine-tuned that these irrational terms all drop out. The reason that we focus on higher-derivative corrections in this paper is to restrict our tension to only pure gravities, which however are not necessary to demonstrate the vulnerability of the horizon regularity.

The paper is organized as follows. In Sec. II, we review the Reall-Santos procedure to obtain the corrected black hole thermodynamic variables without solving for the perturbative solutions. This allows us to double check our later numerical calculations of the perturbative solutions. In Sec. III, we construct linear perturbation of the $D=7$ rotating black hole with all equal angular momenta in Einstein gravity extended by a Riemann-squared term. We analyze both the horizon and asymptotic structure and obtain the numerical solution that validates the result in Sec. II. We generalize the $D=7$ discussion to general $D=2n+1$ dimensions in Sec. IV. We then study the implication of having irrational powers in the near-horizon structure in Sec. V. We conclude our paper in Sec. VI. In Appendixes A–C, we present some detailed complicated formulas that would interrupt the discussion if presented in the main text.

## II. CORRECTIONS TO THERMODYNAMICS

In this paper, we focus on the study of rotating black holes with all equal angular momenta in general $D=2n+1$ dimensions. We shall restrict ourselves to pure gravity for simplicity. Many exact solutions of Ricci-flat or Einstein metrics have been constructed, e.g., [1–8]. The most general quadratic curvature extension to Einstein gravity,

$$S_{\rm Ein}=\frac{1}{16\pi}\int d^Dx\sqrt{-g}R, \tag{1}$$

involves three terms, i.e., $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R^{\mu\nu}R_{\mu\nu}$, and $R^2$. However, in the effective field theory approach, the couplings of these terms are all small, and hence the latter two terms can be removed by the field redefinition $g_{\mu\nu}\to g_{\mu\nu}+c_1R_{\mu\nu}+c_2Rg_{\mu\nu}$, leading to an equivalent description of quadratic curvature correction with simply just one term,

$$S_{\rm quad}=\frac{1}{16\pi}\int d^Dx\sqrt{-g}(\alpha R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}). \tag{2}$$

Rotating black holes that are asymptotically flat in general dimensions were constructed in [3]. In odd $D=2n+1$ dimensions, when all the angular momenta are equal, the metric reduces to cohomogeneity-one depending only on the radial variable $r$. Following the notation in [15], this solution can be expressed as

$$ds^2_{2n+1}=-\frac{h(r)}{W(r)}dt^2+\frac{dr^2}{f(r)}+r^2W(r)\big(\sigma+\omega(r)dt\big)^2+r^2ds^2_{CP^{n-1}}. \tag{3}$$

Here $ds^2_{CP^{n-1}}$ is the metric of an $(n-1)$-dimensional complex projective space. (We adopt the convention of [16] for the $\mathbb{CP}^n$ metrics.) The metric functions are given by $h=f=\bar f$, $W=\bar W$, and $\omega=\bar\omega$, with

$$\bar W=1+\frac{\nu^2}{r^{D-1}},\qquad \bar\omega(r)=\frac{\sqrt{\mu}\nu}{r^{D-1}\bar W},\qquad \bar f=1-\frac{\mu}{r^{D-3}}+\frac{\nu^2}{r^{D-1}}. \tag{4}$$

The solution has two integration constants $(\mu,\nu)$, parametrizing the mass $M_0$ and the angular momentum $J_0$. The event horizon $r_0$ is located at the largest root of $\bar f$ and thermodynamical quantities can all be easily obtained, given by

$$\begin{aligned}
&M_0=\frac{(D-2)\Omega_{D-2}}{16\pi}\mu,\qquad J_0=\frac{(D-1)\Omega_{D-2}}{16\pi}\sqrt{\mu}\nu,\\
&\Omega_0=\frac{\nu}{r_0\sqrt{r_0^{D-1}+\nu^2}},\qquad T_0=\frac{(D-3)r_0^{D-1}-2\nu^2}{4\pi r_0^{\frac12(D+1)}\sqrt{r_0^{D-1}+\nu^2}},\\
&S_0=\frac{\Omega_{D-2}}{4}r_0^{\frac12(D-3)}\sqrt{r_0^{D-1}+\nu^2},
\end{aligned} \tag{5}$$

where $\Omega_{D-2}\equiv\frac{(D-1)}{\Gamma((D+1)/2)}\pi^{(D-1)/2}$ is the volume of the unit round $(D-2)$-sphere. The parameters $(\mu,\nu)$ and the horizon location $r_0$ are related by $\bar{f}(r_0) = 0$. The Gibbs free energy associated with the Euclidean action is

$$G_0 = M_0 - T_0 S_0 - \Omega_0 J_0. \tag{6}$$

The leading $\alpha$-order correction from the quadratic extension (2) to these thermodynamic quantities can be computed using the Reall-Santos method [17] without having to construct the corresponding corrected solution. Specifically, we need first to fix the temperature and angular velocity,

$$T = T_0 + \mathcal{O}(\alpha^2), \qquad \Omega = \Omega_0 + \mathcal{O}(\alpha^2). \tag{7}$$

The Gibbs free energy is then shifted by the correction from the quadratic extension,

$$G(T_0, \Omega_0, \alpha) = G_0 + \Delta G + \mathcal{O}(\alpha^2). \tag{8}$$

This shifted term can be calculated from the quadratic Euclidean action evaluated on the leading-order solution (4). We find

$$\Delta G = \frac{\Delta I}{\beta}, \quad \Delta I = -\frac{1}{16\pi}\int_0^\beta d\tau \int_0^{r_0} dr \times \int d\Omega_{D-2}\sqrt{g}(\alpha R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}). \tag{9}$$

We therefore have

$$\Delta G = -\frac{(D-3)\Omega_{D-2}}{16\pi r_0^{D+3}}\left((D-2)^2 r_0^{2(D-1)} - 2(2D-3)r_0^{D-1}\nu^2 + \nu^4\right)\alpha. \tag{10}$$

This allows one to obtain the complete set of thermodynamic variables at the $\alpha$-order correction, namely

$$J = -\left(\frac{\partial G}{\partial \Omega_0}\right)\Big|_{T_0,\alpha}, \qquad S = -\left(\frac{\partial G}{\partial T_0}\right)\Big|_{\Omega_0,\alpha},$$
$$M = G + T_0 S + \Omega_0 J. \tag{11}$$

In the extremal limit, $\bar{f}(r_0) = 0 = \bar{f}'(r_0)$, i.e.,

$$\mu = \frac{1}{2}(D-1)r_0^{D-3}, \qquad \nu^2 = \frac{1}{2}(D-3)r_0^{D-1}, \tag{12}$$

the mass and angular momentum are no longer independent, but satisfies the relation

$$M_{\rm ext} = \frac{(D-2)\left(\frac{\Omega_{D-2}}{32\pi}\right)^{\frac{1}{D-2}}}{(D-3)^{\frac{D-3}{2(D-2)}}(D-1)^{\frac{D-5}{2(D-2)}}} J_{\rm ext}^{\frac{D-3}{D-2}}\left(1 + \eta\alpha J^{-\frac{2}{D-2}}\right),$$
$$\eta = \frac{3D-11}{2(D-2)}(D-1)^{\frac{3}{D-2}}(D-3)^{\frac{D-1}{D-2}}\left(\frac{\Omega_{D-2}}{32\pi}\right)^{\frac{2}{D-2}}. \tag{13}$$

The $D = 5$ result was obtained in [10]. This is an elegant approach that makes finding the perturbative solution unnecessary. However, this procedure assumes that such a solution necessarily exists, which may not be guaranteed. Even if such a solution does exist, the procedure only uses the information of the spacetime regions outside the horizon. It does not tell how the inside of the horizon changes under the perturbation and how the geodesics can be extended on the horizon.

## III. LEADING-ORDER CORRECTION: THE $D = 7$ EXAMPLE

We now consider the leading $\alpha$-order correction to Ricci-flat rotating black holes (3) while keeping the full isometry. The perturbative *Ansätze* of the metric functions are chosen to be

$$W(r) = \bar{W} + \alpha\delta W, \qquad \omega(r) = \bar{\omega} - \alpha\frac{\bar{\omega}}{\bar{W}}\delta W + \alpha\delta\omega,$$
$$f(r) = \bar{f}(1 + \alpha\delta f), \qquad h(r) = \bar{f}(1 + \alpha\delta h). \tag{14}$$

The linear perturbative equations of $(\delta f, \delta h, \delta W, \delta\omega)$ are not solvable analytically in general higher dimensions. We therefore choose to solve the perturbative equations with the horizon radius $r_0$ fixed for an easier numerical approach. In this set of *Ansätze*, the perturbative functions $(\delta f, \delta h, \delta W, \delta\omega)$ should be all finite in all the regions from the outer horizon to asymptotic infinity. The $D = 5$ case can be solved exactly, and was obtained and analyzed in [10]. However, analytic solutions for general $D$ do not seem to exist, and we focus on the $D = 7$ case in this section and summarize the main results for higher dimensions in the next section.

### A. Perturbative equations

The four independent coupled linear equations of $(\delta f, \delta h, \delta W, \delta\omega)$ in general dimensions were given in Appendix A. By the standard procedure of eliminating variables, we obtain a fourth-order linear differential equation with a source for $\delta f$,

$$P_4\delta f'''' + P_3\delta f''' + P_2\delta f'' + P_1\delta f' + P_0\delta f = Q. \tag{15}$$

In $D = 7$ dimensions, we have

$$
\begin{aligned}
P_4 &= 5r^{10}(\nu^2 + r^6 - \mu r^2)^2(8\mu\nu^2 + 27r^{10} - 10\mu r^6 - 5\mu^2 r^2),\\
P_3 &= 10r^9(\nu^2 + r^6 - \mu r^2)(-32\mu\nu^4 + 324r^{16} - 86\mu r^{12} - 243\nu^2 r^{10} - 80\mu^2 r^8\\
&\quad + 206\mu\nu^2 r^6 + 10\mu^3 r^4 + \mu^2\nu^2 r^2),\\
P_2 &= 5r^8\big(168\mu\nu^6 + 3915r^{22} - 1848\mu r^{18} - 2376\nu^2 r^{16} - 974\mu^2 r^{14} + 1584\mu\nu^2 r^{12}\\
&\quad + 3r^{10}(200\mu^3 + 819\nu^4) - 712\mu^2\nu^2 r^8 + r^6(35\mu^4 - 774\mu\nu^4) - 96\mu^3\nu^2 r^4 + \mu^2\nu^4 r^2\big),\\
P_1 &= 5r^7\big(-168\mu\nu^6 + 2565r^{22} + 504\mu r^{18} - 2808\nu^2 r^{16} - 466\mu^2 r^{14} + 1776\mu\nu^2 r^{12}\\
&\quad - 7r^{10}(40\mu^3 + 351\nu^4) + 136\mu^2\nu^2 r^8 + 3r^6(15\mu^4 + 386\mu\nu^4) - 64\mu^3\nu^2 r^4 - \mu^2\nu^4 r^2\big),\\
P_0 &= -480r^{12}(8\mu\nu^4 + 135r^{16} + 14\mu r^{12} - 108\nu^2 r^{10} - 25\mu^2 r^8 + 40\mu\nu^2 r^6 - 4\mu^2\nu^2 r^2),
\end{aligned} \tag{16}
$$

together with the source

$$
\begin{aligned}
Q = 384\big(&8960\mu^2\nu^6 + 1782\mu\nu^2 r^{16} + 135r^{14}(101\mu^3 + 96\nu^4) - 62340\mu^2\nu^2 r^{12}\\
&+ r^{10}(54720\mu\nu^4 - 5250\mu^4) + 23610\mu^3\nu^2 r^8 - r^6(1125\mu^5 + 24352\mu^2\nu^4)\\
&+ 60r^4(121\mu^4\nu^2 + 24\mu\nu^6) - 14080\mu^3\nu^4 r^2\big).
\end{aligned} \tag{17}
$$

It is clear that the left-hand side of the equation, the linear sector, comes from the linearization of the Einstein tensor $\delta G_{\mu\nu}$ on the rotating black hole background (4). The right-hand side of the equation, the source $Q$, comes from the contributions of $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, but evaluated on the original metric functions (4).

For the remaining perturbations, $(\delta h, \delta\omega)$ reduce to quadratures and $\delta W$ becomes purely algebraic,

$$
\begin{aligned}
\delta h'(r) = &-\frac{1}{5r^9(8\mu\nu^2 + 27r^{10} - 10\mu r^6 - 5\mu^2 r^2)}\big(96(-416\mu\nu^4 + 135\mu^2 r^8 - 312\mu\nu^2 r^6\\
&- 15r^4(11\mu^3 + 8\nu^4) + 588\mu^2\nu^2 r^2\big) + 120r^{12}(-2\nu^2 + r^6 + \mu r^2)\delta f(r)\\
&+ 5r^7(7\nu^4 + 19r^{12} - 8\mu r^8 + 17\nu^2 r^6 - 7\mu^2 r^4 + \mu\nu^2 r^2)\delta f'(r)\\
&+ 35r^8(2r^6 - \nu^2)(\nu^2 + r^6 - \mu r^2)\delta f''(r) + 5r^9\big(\nu^2 + r^6 - \mu r^2)^2\delta f^{(3)}(r)\big),
\end{aligned} \tag{18}
$$

$$
\delta W = -\frac{36\mu(5\mu r^2 - 7\nu^2)}{5r^{12}} + \left(\frac{\nu^2}{r^6} - 2\right)\delta f(r) - \frac{(\nu^2 + r^6 - \mu r^2)}{4r^5}(\delta f'(r) + \delta h'(r)), \tag{19}
$$

$$
\begin{aligned}
\delta\omega'(r) = &\frac{1}{6\sqrt{\mu}\nu r^{11}(\nu^2 + r^6)^2}\big(24(\nu^2 + r^6)^2(16\nu^4 + 15\mu^2 r^4 - 36\mu\nu^2 r^2) + 4r^8(5r^6 - \nu^2)\\
&\times(\nu^2 + r^6)^2\delta f(r) - 18\mu\nu^2 r^{16}\delta h(r) + r^9(2\nu^4 + 5r^{12} + 7\nu^2 r^6)(\nu^2 + r^6 - \mu r^2)\delta h'(r)\\
&+ 2r^{14}(8\nu^4 + 2r^{12} + \nu^2 r^6 + 9\mu\nu^2 r^2)\delta W(r) - r^{15}(\nu^2 + r^6)(-2\nu^2 + r^6 - 3\mu r^2)\delta W'(r)\big).
\end{aligned} \tag{20}
$$

These equations are sufficiently complicated that we need to apply numerical methods to connect the near horizon geometry to the asymptotic infinity, both of which can be analyzed analytically.

### B. Asymptotic behavior

Although the peturbative equations cannot be solved exactly, we can nevertheless obtain the asymptotic behavior. Assuming that the leading-order behavior is $\delta f = r^\gamma f_0$, we have

$$
(\gamma - 2)(\gamma + 4)(\gamma + 6)(\gamma + 10)r^{\gamma+12}f_0 = -\frac{25344}{5}\mu\nu^2. \tag{21}
$$

Therefore, we have $\gamma = 2, -4, -6, -10$ for the source-free contributions and $\gamma = -12$ for the source contribution, leading to the general solution with four integration constants,

$$\delta f = -\frac{132\mu\nu^2}{35r^{12}}\tilde{f}_0(r) + \frac{c_{10}}{r^{10}}\tilde{f}_{10}(r) + \frac{c_6}{r^6}\tilde{f}_6(r) + \frac{c_4}{r^4}\tilde{f}_4(r) + c_{-2}r^2\tilde{f}_{-2}(r), \tag{22}$$

where $\tilde{f}_i$'s all take the form of $\tilde{f}_i \sim 1 + \#_1/r + \#_2/r^2 + \cdots$ at the large $r$ expansion. (We use $\#_i$ to denote some generic constants.)

The first term in (22) comes from the quadratic curvature extension. It has faster falloff than any other terms, which come from the perturbation of the Einstein tensor. The $c_{-2}$ term is rather intriguing since it appears like a cosmological constant in the $g_{rr}$ metric component. This term however will not arise from the linear perturbation equation of the Schwarzschild black hole, while keeping the static and spherical symmetry. The connection between rotation and cosmological constant is worth further exploring.

In our case, however, in order for $\delta f$ to be regular at asymptotic infinity, we must set $c_{-2} = 0$. Therefore, general asymptotically-flat perturbations involve three independent free parameters $(c_4, c_6, c_{10})$. The low-lying falloff orders of $\tilde{f}_i$, so that $\delta f$ is up to order $1/r^{16}$, are given by

$$\begin{aligned}
\tilde{f}_0 &= 1 - \frac{21\mu}{11r^4} + \cdots, \\
\tilde{f}_4 &= 1 + \frac{\mu}{r^4} + \frac{\mu^2}{r^8} - \frac{59\mu\nu^2}{160r^{10}} + \frac{\mu^3 - \frac{2\nu^4}{5}}{r^{12}}\cdots, \\
\tilde{f}_6 &= 1 - \frac{\nu^2}{r^6} - \frac{101\mu^2}{160r^8} - \frac{3\mu\nu^2}{5r^{10}} + \cdots, \\
\tilde{f}_{10} &= 1 + \frac{261\mu}{160r^4} - \frac{7\nu^2}{5r^6} + \cdots.
\end{aligned} \tag{23}$$

Analogously, we can obtain the leading falloffs of the $(\delta h, \delta W, \delta\omega)$ functions,

$$\begin{aligned}
\delta h = {} & \left(\frac{48\mu^2}{5r^{10}} - \frac{564\mu\nu^2}{35r^{12}} + \frac{3(693\mu^3 + 608\nu^4)}{280r^{14}} - \frac{744\mu^2\nu^2}{35r^{16}} + \cdots\right) \\
& + \frac{c_{10}}{5r^{10}}\left(1 + \frac{61\mu}{32r^4} - \frac{2\nu^2}{r^6} + \cdots\right) + \frac{c_6}{r^6}\left(1 + \frac{4\mu}{5r^4} + \frac{99\mu^2}{160r^8} - \frac{\nu^2}{r^6} - \frac{8\mu\nu^2}{5r^{10}} + \cdots\right) \\
& + \frac{c_4}{r^4}\left(1 + \frac{\mu}{r^4} + \frac{\mu^2}{r^8} + \frac{\mu^3}{r^{12}} - \frac{4\nu^2}{5r^6} - \frac{259\mu\nu^2}{160r^{10}} + \frac{3\nu^4}{5r^{12}} + \cdots\right),
\end{aligned} \tag{24}$$

$$\begin{aligned}
\delta W = {} & \left(-\frac{12\mu^2}{r^{10}} - \frac{12\mu\nu^2}{7r^{12}} + \frac{456\mu^2\nu^2}{35r^{16}} - \frac{3(25\mu^3 - 32\nu^4)}{8r^{14}} + \cdots\right) \\
& + \frac{c_{10}}{r^{10}}\left(1 + \frac{25\mu}{32r^4} - \frac{2\nu^2}{5r^6} + \cdots\right) + \frac{c_6}{r^6}\left(1 - \frac{\mu}{r^4} - \frac{25\mu^2}{32r^8} + \frac{2\mu\nu^2}{5r^{10}} + \cdots\right) \\
& + \frac{c_4\nu^2}{r^{10}}\left(1 + \frac{25\mu}{32r^4} - \frac{2\nu^2}{5r^6} + \cdots\right),
\end{aligned} \tag{25}$$

$$\begin{aligned}
\delta\omega = {} & \frac{8\mu^{3/2}}{\nu r^6}\left(1 - \frac{5\nu^2}{2r^6} - \frac{3(7105\mu^3 + 3312\nu^4)}{15680\mu r^8} + \frac{\nu^2(18027\mu^3 + 4256\nu^4)}{4480\mu^2 r^{10}} + \cdots\right) \\
& - \frac{2c_{10}}{3\sqrt{\mu}\nu r^6}\left(1 - \frac{7\mu^2}{64r^8} - \frac{\nu^2}{r^6} - \frac{291\mu\nu^2}{640r^{10}} + \cdots\right) + \frac{7\sqrt{\mu}c_6}{6\nu r^6}\left(1 - \frac{\nu^2}{r^6} + \frac{2109\mu\nu^2}{1120r^{10}}\right. \\
& \left. - \frac{87\mu^3 + 80\nu^4}{112r^8\mu} + \cdots\right) - \frac{7\nu c_4}{6\sqrt{\mu}r^6}\left(1 - \frac{247\mu^2}{112r^8} + \frac{5\mu^4}{7r^{10}\nu^2} - \frac{\nu^2}{r^6} + \frac{2109\mu\nu^2}{1120r^{10}} + \cdots\right).
\end{aligned} \tag{26}$$

From these asymptotic behavior, we can read off the mass $M$ and angular momentum $J$, at their $\alpha$-order correction,

$$M = \frac{5\pi^2}{16}(\mu - \alpha c_4), \qquad J = \frac{3\pi^2}{8}\sqrt{\mu}\nu\left(1 + \frac{\alpha}{6\mu\nu^2}(48\mu^2 - 7\nu^2 c_4 + 7\mu c_6 - 4c_{10})\right). \tag{27}$$

### C. Near-horizon structure

Note that in our perturbative solution, we hold the horizon radius $r_0$ fixed; therefore, both the mass and angular momentum acquire higher-order corrections. At the first sight, the perturbed mass and angular momentum (27) depends on three parameters $(c_4, c_6, c_{10})$, which would violate the no-hair theorem, since we would expect that the solution should only have two independent parameters. Indeed not all the parameters $(c_4, c_6, c_{10})$ leads to black hole solutions.

In order to determine the parameter choices of $(c_4, c_6, c_{10})$ that yield black holes, we can study the near-horizon geometry. For general nonextremal black holes, the analysis will be given in Appendix B. Here, we shall focus only on the extremal case, corresponding to

$$\mu = 3r_0^4, \qquad \nu = \sqrt{2}r_0^3. \tag{28}$$

In other words, the solution depends only on one parameter, such that mass and angular momentum are no longer independent but they satisfy,

$$M_0 = \frac{5\pi^{2/5}}{4\sqrt[5]{3}} J_0^{\frac{4}{5}}. \tag{29}$$

To determine the leading-order behavior of $\delta f$ as $r \to r_0$, we can adopt the same trick by assuming $\delta f = (r - r_0)^\gamma \hat{f}$, where $\tilde{f}$ is analytic, satisfying the usual Taylor expansion at $r = r_0$. We find that the leading-order equation for small $(r - r_0)$ is

$$(\gamma+1)(\gamma+2)(\gamma-\Delta_+)(\gamma-\Delta_-)r^{-\gamma}\hat{f}(r_0) = -\frac{168}{5r_0^2}, \tag{30}$$

where

$$\Delta_\pm = -\frac{3}{2} \pm \frac{\sqrt{21}}{2}. \tag{31}$$

Thus, the general solution near $r_0$ takes the form,

$$\delta f = \frac{28}{5r_0^2}\hat{f}_0 + \frac{d_{-1}}{r - r_0}\hat{f}_{-1} + \frac{d_{-2}}{(r - r_0)^2}\hat{f}_{-2} + d_{\Delta_-}(r - r_0)^{\Delta_-}\hat{f}_{\Delta_-} + d_{\Delta_+}(r - r_0)^{\Delta_+}\hat{f}_{\Delta_+}. \tag{32}$$

Note that $\Delta_+ > 0$ and $\Delta_- < 0$. The regularity at $r_0$ requires that we set coefficients $d_{-1}$, $d_{-2}$ and $d_{\Delta_-}$ all to zero. The low-lying orders of the near-horizon expansions at $r = r_0$ are thus given by

$$\delta f = \frac{28}{5r_0^2}\left(1 + \frac{3658}{21r_0}(r - r_0) - \frac{63167}{441r_0^2}(r - r_0)^2 + \cdots\right) + d_{\Delta_+}(r - r_0)^{\Delta_+}\left(1 + \frac{117 - 7\sqrt{21}}{180r_0}(r - r_0) - \frac{1282\sqrt{21} - 6627}{18360r_0^2}(r - r_0)^2 + \cdots\right). \tag{33}$$

Analogous solutions can be obtained for $(\delta h, \delta W, \delta\omega)$, which we present in Appendix C. The irrational power of $(r - r_0)^{\Delta_+}$ should cause our concern since it indicates that the near-horizon geometry is not analytic. We shall come back to this point in Sec. V.

We now continue to focus on the $\delta f$ equation. On the horizon, the general regular solution of $\delta f$ is specified by two parameters, namely $r_0$ and coefficient $d_{\Delta_+}$. However, for a given $r_0$, a generic value of $d_{\Delta_+}$ will excite the coefficient $c_{-2}$ of $r^2$ in the large-$r$ expansion (22). We need to fine-tune the value $d_{\Delta_+}$ for each given $r_0$ so that $c_2$ vanishes and the metric remains asymptotically flat. If such a $d_{\Delta_+}$ exists, then we obtain a leading-order perturbation the extremal rotating black hole. Thus, in the linearly perturbed solution, both the horizon and asymptotic parameters $(d_{\Delta_+}, c_4, c_6, c_{10})$ all depend on the horizon radius $r_0$, analogous to the unperturbed extremal black hole, which has only one independent parameter. This is consistent with the no-hair theorem. Consequently, it follows from (27) that both mass and angular momentum depend only on $r_0$,

$$M = \frac{15}{16}\pi^2 r_0^4 + \alpha\delta M(r_0), \qquad J = \frac{3}{4}\sqrt{\frac{3}{2}}\pi^2 r_0^5 + \alpha\delta J(r_0). \tag{34}$$

Therefore, at the linear $\alpha$ order, the mass-angular momentum relation becomes,

$$M = \frac{5\pi^{2/5}}{4\sqrt[5]{3}} J^{\frac{4}{5}}(1 + \eta\alpha J^{-\frac{2}{5}}), \tag{35}$$

where $\eta$ is some order-one dimensionless constant. Analytical calculation using the Reall-Santos procedure was discussed in Sec. II. It follows from (13) that for $D = 7$, we have

$$\eta = 2\,3^{3/5}\pi^{4/5} \sim 9.661. \tag{36}$$

From the leading-order perturbative solution, we deduce

$$\eta = \frac{\pi^{4/5}}{30\,3^{2/5}}\left(-\frac{c_4}{r_0^2} - \frac{21c_6}{r_0^4} + \frac{4c_{10}}{r_0^8} - 432\right). \tag{37}$$

We use numerical approach to show that the perturbative solution outside the horizon correctly reproduce the value of the $\eta$ coefficient (36).

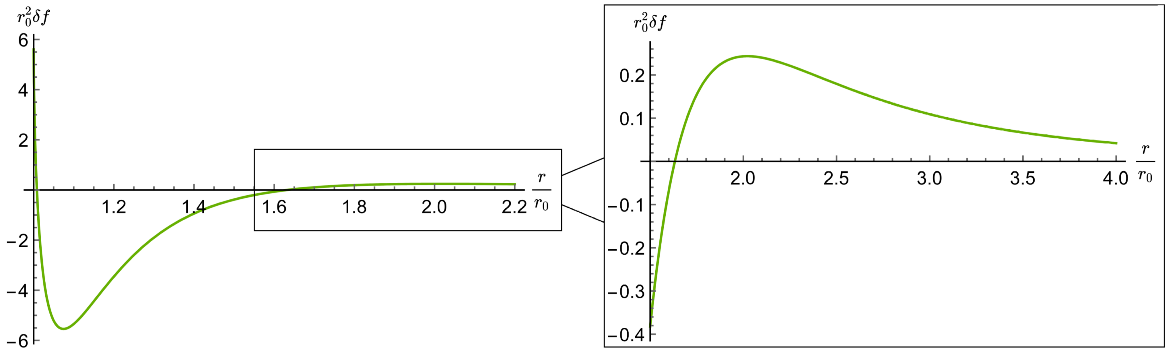


FIG. 1. The dimensionless function $r_0^2\delta f$ is not a monotonous function of the dimensionless radial variable $\tilde{r} = r_0/r$. The numerical result allows us to read off the asymptotic falloffs and determine both corrected mass and angular momentum.

### D. Numerical result

Since we do not have analytical solution of $\delta f$, we adopt the numerical approach to determine the coefficient $d_{\Delta_+}$ so that $c_2$ vanishes. Specifically, we use the shooting method by beginning with the analytical result of the power series expansion (33) up to $(r - r_0)^4$ order as the initial data, and then integrate over to large $r$, e.g., $200r_0$. We search for the appropriate $d_{\Delta_+}$ to shoot the target of $\delta f \to 0$. We then do curve fitting against the asymptotic structure (22) up to the order $1/r^{14}$. We first find a fine-tuned result of $d_{\Delta_+}$ so that $c_{-2}$ vanishes. We then drop off the $c_{-2}$ term and do curve fitting against the remaining coefficients and read off $c_4$, $c_6$, and $c_{10}$. This allows us to obtain the $\alpha$-order correction to the mass-angular momentum relation.

The result depends on $r_0$ trivially, since it is the only scale parameter in the linearized equation. To see this explicitly, we can define a dimensionless radial coordinate $\tilde{r} = r/r_0$, in which case, $r_0$ drops out from the linear equation completely for the dimensionless function $(r_0^2\delta f)$. We can therefore perform the numerical analysis for $r_0 = 1$ without loss of generality. We plot dependence of the dimensionless function $r_0^2\delta f$ on the dimensionless radial variable $\tilde{r}$ in Fig. 1. We see that $r_0^2\delta f$ is not a monotonous function of $\tilde{r}$. From this numerical solution, we can read off the asymptotic structure and determine both corrected mass and angular momentum. We obtain the expected $\eta$ within 1% of accuracy. We summarize the result in Table I.

## IV. GENERAL $D = 2n + 1$ DIMENSIONS

In the previous section, we found that in $D = 7$ dimensions, the near-horizon geometry of the extremal rotating black hole involves $(r - r_0)^{\Delta_+}$ terms with irrational $\Delta_+$. Such an irrational power does not arise in $D = 5$. In this section, we study whether such an irrational power occurs in general higher dimensions or it is a special case in $D = 7$. The linear perturbative equations in general $D = 2n + 1$ dimensions are given in Appendix A. By eliminating variables, we obtain the decoupled $\delta f$ equation, as in the $D = 7$ case. It is given by

$$P_4\delta f'''' + P_3\delta f''' + P_2\delta f'' + P_1\delta f' + P_0\delta f = Q, \tag{38}$$

where $P_i$ and $Q$ are polynomials of $r$. We find

$$\begin{aligned}
P_4 &= (D-2)r^{D+4}(r^D - \mu r^3 + \nu^2 r)^2(-8(D-2)\mu r^{D+3} + 3(D-1)^2 r^{2D} - 4(D-2)\mu^2 r^6 \\
&\quad + 4(D+1)\mu\nu^2 r^4), \\
P_3 &= (D-2)r^{D+3}(r^D - \mu r^3 + \nu^2 r)(-8(5D^2 - 29D + 38)\mu^2 r^{D+6} \\
&\quad + 8(5D^2 - 5D - 4)\mu\nu^2 r^{D+4} - 4(D^2 - 13D + 22)\mu^3 r^9 + 4(2D^2 - 11D - 19)\mu^2\nu^2 r^7 \\
&\quad + (9D^3 - 109D^2 + 247D - 163)\mu r^{2D+3} - 9(D-1)^3\nu^2 r^{2D+1} + 12(D-1)^3 r^{3D} \\
&\quad - 4(D+1)^2\mu\nu^4 r^5), \\
P_2 &= (D-2)r^{D+2}(-3(D-1)^2(2D^2 - D - 3)\nu^2 r^{3D+1} + 3(D-1)^2(5D^2 - 16D + 12)r^{4D} \\
&\quad + 4(5D^2 - 38D + 56)\mu^4 r^{12} + 4(-7D^2 + 27D + 58)\mu^3\nu^2 r^{10} - 4(D^2 - 8D + 6)\mu^2\nu^4 r^8 \\
&\quad - 4(D^3 - 43D^2 + 200D - 236)\mu^3 r^{D+9} + 4(2D^3 - 32D^2 + 23D + 9)\mu^2\nu^2 r^{D+7} \\
&\quad - 4(D^3 + 11D^2 - 13D - 17)\mu\nu^4 r^{D+5} - 2(6D^4 - 73D^3 + 157D^2 - 17D - 109)\mu\nu^2 r^{2D+4}
\end{aligned}$$

$$
\begin{aligned}
&+3(2D^4-37D^3+139D^2-215D+119)\mu r^{3D+3}+3(D-1)^2D(2D-1)\nu^4 r^{2D+2}\\
&+\mu r^6((6D^4-131D^3+702D^2-1199D+626)\mu r^{2D}+12D(D+1)\nu^6)),\\
P_1 =& (D-2)r^{D+1}(12(D^2-6D+8)\mu^4 r^{12}-4(D^2+11D-62)\mu^3\nu^2 r^{10}\\
&+4(D^2-8D+6)\mu^2\nu^4 r^8-3(D-1)^2(D^3-5D^2+6)\nu^2 r^{3D+1}+3(D-1)^2(2D^3-17D^2\\
&+38D-24)r^{4D}+4(D^4-18D^3+102D^2-245D+210)\mu^3 r^{D+9}-4(2D^4-25D^3\\
&+93D^2-142D+74)\mu^2\nu^2 r^{D+7}-(6D^4-67D^3+278D^2-495D+282)\mu^2 r^{2D+6}\\
&+4(D^4-7D^3+25D^2-5D-32)\mu\nu^4 r^{D+5}+2(6D^4-25D^3-99D^2+415D\\
&-333)\mu\nu^2 r^{2D+4}+(3D^5-59D^4+421D^3-1267D^2+1676D-798)\mu r^{3D+3}\\
&-3(D-1)^2D(2D-1)\nu^4 r^{2D+2}-12D(D+1)\mu\nu^6 r^6),\\
P_0 =& -2(D-3)(D-2)(D-1)r^{2D}(-4(D^2-8D+15)\mu^2\nu^2 r^7+4(D^2-4D-5)\mu\nu^4 r^5\\
&+2(D^3-16D^2+57D-58)\mu^2 r^{D+6}-2(D^3-10D^2-9D+50)\mu\nu^2 r^{D+4}+(5D^3\\
&-51D^2+147D-133)\mu r^{2D+3}-3(D-1)^2(D+1)\nu^2 r^{2D+1}+6(D-2)(D-1)^2 r^{3D}),
\end{aligned}
\tag{39}
$$

together with the source contribution,

$$
\begin{aligned}
Q =& 2(D-1)(-12(D-1)^3(D+1)^2(D^2-5D-24)\nu^4 r^{3D}-4(D-3)^2(D-2)^3(D^2\\
&-5D+4)\mu^5 r^{13}-4(D+1)^3(D^4-42D^2-64D+105)\mu\nu^8 r^5-2(D^2-5D+6)^2(11D^3\\
&-86D^2+203D-140)\mu^4 r^{D+10}-3(D-1)^3(D+1)^2(3D^3-11D^2-67D-21)\nu^6 r^{2D+1}\\
&+6(D-1)^3(2D^4-15D^3+8D^2+19D-6)\mu\nu^2 r^{3D+2}+2(D+1)^2(11D^5-64D^4\\
&-192D^3+658D^2+709D-1122)\mu\nu^6 r^{D+4}+4(D+1)^2(4D^5-16D^4-91D^3\\
&+107D^2+279D-315)\mu^2\nu^6 r^7-8(3D^7-21D^6+112D^4-127D^3-23D^2\\
&+156D-36)\mu^3\nu^4 r^9+3(D-3)^2(3D^6-38D^5+194D^4-516D^3+759D^2-586D\\
&+184)\mu^3 r^{2D+7}+2(33D^7-434D^6+2090D^5-4734D^4+4987D^3-1396D^2-1350D\\
&+804)\mu^3\nu^2 r^{D+8}-2(33D^7-280D^6+311D^5+1046D^4-945D^3-1988D^2+729D\\
&+1350)\mu^2\nu^4 r^{D+6}+4(4D^7-48D^6+201D^5-341D^4-53D^3+1109D^2-1580D\\
&+732)\mu^4\nu^2 r^{11}-(27D^8-378D^7+2101D^6-6336D^5+10905D^4-10042D^3+4139D^2\\
&-524D+108)\mu^2\nu^2 r^{2D+5}+(27D^8-252D^7+568D^6-284D^5-1562D^4+1132D^3\\
&+4192D^2-596D-3225)\mu\nu^4 r^{2D+3}).
\end{aligned}
\tag{40}
$$

In the extremal limit (12), we can take an *Ansatz* $\delta f=(r-r_0)^\gamma \hat f(r)$, where $\hat f(r)$ is analytic at $r=r_0$. In the limit of $r\to r_0$, we find that the leading-order equation is

$$
(\gamma+1)(\gamma+2)(\gamma-\Delta_-)(\gamma-\Delta_+)\hat f(r_0) = \frac{(D-9)(D-1)(D+7)}{(D-2)r_+^2}, \tag{41}
$$

where

$$
\Delta_\pm = \frac{1}{2}\left(-3\pm\sqrt{\frac{17D-35}{D-3}}\right), \tag{42}
$$

which reproduces the earlier $D=7$ result. Since only $\Delta_+>0$, the near-horizon structure of $\delta f$ takes the form,

$$
\delta f = \frac{(9-D)(D-3)(D+7)}{4(D-2)r_0^2}\hat f_0(r)+d_{\Delta_+}(r-r_0)^{\Delta_+}\hat f_{\Delta_+}(r), \tag{43}
$$

where $d_{\Delta_+}$ a certain appropriate coefficient that is to be determined. Both functions $\hat f(r)$ and $\hat f_{\Delta_+}(r)$ are analytic, satisfying the Taylor expansion of the form $1+\#_1(r-r_0)+\#_2(r-r_0)^2+\cdots$. Thus we see that in general higher dimensions, $\Delta_+$ is irrational as in the $D=7$ case. However, rational numbers can arise for sporadically

TABLE I. Numerical data of the $\alpha$-correction of the extremal rotating black hole.

| $d_{\Delta_+} r_0^{2+\Delta_+}$ | $c_4/r_0^2$ | $c_6/r_0^4$ | $c_{10}/r_0^8$ | $\delta M/r_0^2$ | $\delta J/r_0^3$ | $\eta$ |
|---|---|---|---|---|---|---|
| −604.856 | 13.440 | −45.165 | −80.72 | −41.452 | −96.133 | 9.669 |

TABLE II. Low-lying examples of sporadic dimensions that give rise to rational $\Delta_+$.

| $D$ | 5 | 101 | 1027 | 10661 | 428741 |
|---|---|---|---|---|---|
| $\Delta_+$ | 1 | $\frac{4}{7}$ | $\frac{9}{16}$ | $\frac{41}{73}$ | $\frac{260}{463}$ |

distributed old $D$, and we give a few low-lying examples in Table 2.

It should be pointed out that there exists a decoupling limit $\epsilon \to 0$ associated with the coordinate transformation $r = r_0 + \epsilon\rho$. In this limit, the near-horizon geometry $AdS_3 \times \mathbb{CP}^n$ becomes the solution on its own where $AdS_3$ is written as a $U(1)$ bundle over $AdS_2$. The $\Delta_+$ term drops out completely from the metric of the homogeneous vacuum.

## V. NATURAL BOUNDARY ON THE HORIZON

We have found that the Riemann-squared correction to Einstein gravity has a consequence that the extremal rotating black holes in $D \geq 7$ odd dimensions with all equal angular momenta have in general irrational powers in the near-horizon expansion, as given in (43), where $\Delta_+$ is in general irrational. In this section we discuss its implication.

### A. Horizon as a natural boundary

In a typical black hole, e.g., the Schwarzschild black hole, the horizon ($g_{tt} = 0$) represents a coordinate singularity that can be removed by coordinate extension such as the Kruskal extension. Such a procedure is problematic when the near-horizon geometry is not analytic with irrational powers. To illustrate this, we consider a simpler toy model with the metric,

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2,$$
$$f = \left(1 - \frac{r_0}{r}\right)^2 \left(1 + \frac{\alpha}{r^3}\left(1 - \frac{r_0}{r}\right)^{\Delta_+}\right). \tag{44}$$

We require that $\alpha > 0$ so that the metric satisfies both the strong and weak energy conditions on and outside the event horizon, i.e., $r \geq r_0$. When $\alpha = 0$, the metric is simply the extremal RN black hole. The $\alpha$ term arises from the exotic matter energy-momentum tensor $T^\mu_\nu = \mathrm{diag}\{-\rho, p_r, p_T, p_T\}$, with

$$\rho^m = -p_r^m = \frac{\alpha(r - r_0)(2r - (\Delta_+ + 4)r_0)}{r^7}\left(1 - \frac{r_0}{r}\right)^{\Delta_+},$$
$$\rho_T^m = \frac{\alpha(6r^2 - 6(\Delta_+ + 4)r_0 r + (\Delta_+^2 + 9\Delta_+ + 20)r_0^2)}{2r^7} \times \left(1 - \frac{r_0}{r}\right)^{\Delta_+}. \tag{45}$$

This of course is a poor imitation of the rotating solution we constructed, since in this toy model, the irrational power existed in the matter energy-momentum tensor, whilst in the rotating black holes, no such matter is needed. Nevertheless, this simpler toy model can help us understand the global structure of the similar metric.

If $\Delta_+$ is a positive integer, then the function $f$ is analytic at $r = r_0$, and it is infinitely differentiable. However, in our case, $\Delta_+$ is an irrational positive number lying between 0 and 1. The function $f$ is therefore only second-order differentiable. However, the metric is still infinitely differentiable under covariant derivatives, owing to the fact that,

$$g^{rr}\nabla_r\nabla_r = f\partial_r^2 + \cdots. \tag{46}$$

Therefore, there is no curvature or covariant-derivative curvature singularity at any order on the $r = r_0$ horizon. In other words, any invariant polynomial of the curvature tensor and its covariant derivatives near $r = r_0$ takes the form $X(r) + Y(r)(r - r_0)^{\Delta_+}$ for regular some analytic functions $(X(r), Y(r))$ at $r = r_0$. Nevertheless we cannot extend the coordinate beyond the region $r \geq r_0$. The $r = r_0$ horizon with irrational $\Delta_+$ thus forms a natural boundary of the spacetime. Although there is no infinite tidal force on the horizon, this boundary is necessarily singular since the geodesics are incomplete on the horizon and they cannot extended beyond horizon.

Physically, one may argue that an irrational number $\Delta_+$ can be approximated by a rational number $\Delta_+ = p/q$, with $(p, q)$ being coprime positive integers, in arbitrary accuracy. In this case, we can redefine the radial coordinate,

$$r = r_0 + \rho^q, \to (r - r_0)^{\frac{p}{q}} = \rho^p. \tag{47}$$

Under the new radial coordinate, the $r > r_0$ spacetime is described by $\rho > 0$, and the geodesics can be extended into the “inside” of the horizon where $\rho$ is negative.

However, three distinct situations with equally good approximation can arise with the above scenario: (1) $p$ is even and $q$ is odd; (2) $p$ is odd and $q$ is even; (3) $p$ and $q$ are both odd. In the first case, $(r - r_0)^{p/q} \sim \rho^p$ is an even function of $\rho$, i.e., it is the same inside or outside the horizon, but the function $r$ is not since $r \sim r_0 + \rho^q$ is an odd function of $\rho$. In the second case, the situation reverses, and $r$ remains the same inside or outside of the horizon, but not for $(r - r_0)^{p/q} \sim \rho^p$. In the third case, both are odd functions of $\rho$. These three approximations of the irrational

$\Delta_+$ therefore lead to three distinct insides of the horizon, since different components of the metric involve both integer powers of $r$ as well as $(r-r_0)^{p/q}$. Furthermore, the more accurate the fraction is to the irrational number, the larger the integers $q$ and $p$ must be, corresponding to larger number of branch cuts. An irrational power implies an infinite number of branch cuts; therefore, the global structures imply that the irrational power cannot be approximated by rational numbers in the black hole discussion.

### B. Origin of irrational $\Delta_+$

In this paper, we considered the higher-derivative correction to rotating black holes, in order to restrict ourselves to a pure gravity discussion. We may therefore falsely accuse that the emerging of irrational $\Delta_+$ is a consequence of higher-derivative gravities. We now show that the vulnerability of the horizon regularity is actually an innate part of Einstein gravity. The general equation of motion can be expressed as

$$G_{\mu\nu}(\Phi) + \alpha E_{\mu\nu}(\Phi) = 0, \tag{48}$$

where $G_{\mu\nu}$ is the Einstein tensor, and $E_{\mu\nu}$ represents the differential operator associated with the higher-derivative correction. $\Phi$ denotes the metric functions to be solved. Perturbatively, we take $\Phi = \Phi_0 + \alpha\Phi_1$, with $G_{\mu\nu}(\Phi_0) = 0$. At the linear $\alpha$ order, the perturbative equation is thus given by

$$\delta G_{\mu\nu}(\Phi_1) = -E_{\mu\nu}(\Phi_0). \tag{49}$$

Here $\delta G_{\mu\nu}$ represents the linearized Einstein tensor on the $\Phi_0$ background. Thus, we see that in the perturbative analysis, the dynamics of the higher-derivative correction to the solution, i.e., $\Phi_1$, is still governed by the Einstein's theory. The higher-derivative terms contribute only as a source to the linearized equation.

Therefore, the horizon regularity is not only vulnerable to higher-derivative corrections, but also to minimally coupled matter fields in Einstein gravity. This may partially explain why exact solutions of rotating black holes in higher dimensions do not exist in Einstein-Maxwell theory in higher dimensions, since it is hard to imagine an analytic special function can give rises to the horizon structure (43). It is also worth commenting that since curvature tensors and their covariant derivatives are all nonsingular at $r = r_0$, this vulnerability persists as long as we take perturbative approach, no matter how higher orders we consider in higher-derivative corrections.

### C. Fine-tuning away the horizon boundary

Exact solutions of charged rotating black holes in $D = 7$ do exist in gauged or ungauged supergravities [18–22] and there is no irrational horizon structure (43) in these examples, even in the extremal limit. Analogously, the nonanalytic logarithmic terms in extremal five-dimensional rotating black hole found in [10] do not arise in supergravity solutions constructed in [23–25] either. These examples contradict our earlier statements.

From our perturbative point of view, this can be explained that the source term $Q$ of (15) in these examples are fine-tuned in supergravities such that the divergent $c_{-2}$ vanishes at asymptotic infinity even when we set the coefficient $d_{\Delta_+} = 0$. We expect that for the simpler Einstein-Maxwell theory in general odd dimensions, $d_{\Delta_+}$ coefficient will be necessarily turned on.

To validate the above arguments, we consider Einstein gravity with a cubic Riemann tensor extension, namely

$$L_{\text{cubic}} = \beta(e_1 R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} R^{\alpha\beta}{}_{\mu\nu} + e_2 R^{\mu}{}_{\nu}{}^{\alpha}{}_{\beta} R^{\nu}{}_{\rho}{}^{\beta}{}_{\gamma} R^{\rho}{}_{\mu}{}^{\gamma}{}_{\alpha}). \tag{50}$$

We focus on $D = 7$ dimensions, and we find that the source term in (15) is now given by

$$\begin{aligned} Q = \frac{96}{r^{10}} \big( & 30720(12556e_1 + 1301e_2)\mu\nu^{10} + 6531840e_1\mu^2 r^{26} - 311040(263e_1 + 13e_2)\mu\nu^2 r^{24} \\ & - 675r^{22}\big(4e_1(34453\mu^3 - 57024\nu^4) - 27e_2(19\mu^3 + 768\nu^4)\big) + 1890(442324e_1 + 17957e_2) \\ & \times \mu^2\nu^2 r^{20} + 30\mu r^{18}\big(28e_1(200305\mu^3 - 2409444\nu^4) - e_2(36955\mu^3 + 5017716\nu^4)\big) \\ & - 36\nu^2 r^{16}\big(e_1(41738114\mu^3 - 39073536\nu^4) + 3e_2(464483\mu^3 - 1297152\nu^4)\big) \\ & + \mu^2 r^{14}\big(e_1(4349474736\nu^4 - 39410300\mu^3) + e_2(315875\mu^3 + 275913768\nu^4)\big) \\ & + 6\mu\nu^2 r^{12}\big(e_1(55820180\mu^3 - 832631552\nu^4) + e_2(1704055\mu^3 - 71862016\nu^4)\big) \\ & - 240r^{10}\big(e_1(65500\mu^6 + 4001976\mu^3\nu^4 - 8136288\nu^8) + e_2(-475\mu^6 + 243219\mu^3\nu^4 \\ & - 843048\nu^8)\big) + 40\mu^2\nu^2 r^8\big(e_1(4254997\mu^3 + 28078336\nu^4) + 32e_2(3601\mu^3 + 74962\nu^4)\big) \\ & - 72\mu\nu^4 r^6\big(e_1(9349998\mu^3 + 6333248\nu^4) + e_2(490623\mu^3 + 669808\nu^4)\big) \\ & + 20736(60494e_1 + 4387e_2)\mu^3\nu^6 r^4 - 128(8742916e_1 + 779963e_2)\mu^2\nu^8 r^2\big). \end{aligned} \tag{51}$$

In the extremal limit (28), we have

$$\delta f = \frac{1900e_1 - 571e_2}{5r_0^4}\hat{f}_0(r) + d_{\Delta_+}(r - r_0)^{\Delta_+}\hat{f}_{\Delta_+}(r), \quad (52)$$

with the irrational $\Delta_+$ given in (31). Now we have an extra nontrivial free parameter $\xi = e_2/e_1$. For generic $\xi$, the parameter $d_{\Delta_+}$ will be turned on appropriately for asymptotically flat spacetime. In these cases, the horizon is the natural boundary of the spacetime. However, we can fine-tune the $\xi$ parameter so that we can turn off $d_{\Delta_+}$, so that the horizon is analytic. By numerical analysis, we find, up to six significant figures, that

$$\xi \sim -2.51062. \quad (53)$$

In the linear perturbative solution, this quantity is independent of the size of the extremal black hole, namely the $r_0$ value.

## VI. CONCLUSION

In this paper, we obtained the leading-order perturbation of the extremal $D = 7$ Ricci-flat rotating black holes with all equal angular momenta, in Einstein gravity extended with Riemann-squared term. The resulting mass-angular momentum relation is the same as one derived from the Reall-Santos procedure. We found that in the extremal limit, the perturbative solution of $1/g_{rr}$ took the form (33) where $\Delta_+$ was an irrational number between half and one. We argued that the irrational $\Delta_+$ implied that the black hole inside of the original extremal black hole was destroyed by the perturbation and the horizon forms a natural boundary of spacetime. All curvature invariants of the perturbation are regular on the horizon and hence there is no divergent tidal force; however, the horizon boundary is nevertheless singular since the geodesic is incomplete there. The origin of this singularity comes from the infinity number of branch cuts owing to the irrational power. We showed that this feature generally continued in higher odd dimensions, except in sporadically distributed special dimensions.

We demonstrated that this vulnerability of horizon regularity and black hole inside was not a consequence of higher-derivative corrections, but it was an innate part of Einstein gravity. In other words, horizon natural boundaries will generally arise in Einstein gravity with minimally coupled matter in higher dimensions, unless the matter fields are fine-tuned as in supergravities. We also illustrated with a concrete example that we could also remove the nonanalytic structure on the extremal horizon by fine-tuning the coupling constants in higher-derivative gravities.

The phenomenon of singular horizon appears to confirm further that extremal black holes with zero temperature are not physical, under the third law of black hole thermodynamics. However, the fact that the nonanalytical terms that would generally exist in the near-horizon geometry of the rotating black holes in $D = 5$ or $D = 7$ actually do not arise in supergravities is tantalizing. Is this one of the criteria of a good quantum theory of gravity which should be fine-tuned so as not to generate the horizon natural boundary? It is clearly problematic to understand the microscopic origin of the black hole entropy in the extremal limit when the inside of the horizon does not exist. It is thus worth investigating whether the supersymmetric higher-order correction would also protect the inside of the black holes.

We motivated ourself to study the horizon structure in the extremal limit of higher-dimensional rotating black hole by the fact that centrifugal force remains the same while gravity becomes weaker in higher dimensions. The extremal limit becomes a more strenuous balance between gravity and centrifugal force. We thus expected that the horizon became problematic under perturbation. However, it is curious to note that rational powers can nevertheless exist sporadically as indicated in Table 1, which provides a small number of counterexamples to our expectations. Our analysis of the event horizon expansion of general nonextremal black holes in the Appendix indicates that the metric functions are all analytic and infinitely differentiable. However, the situation with near-extremal cases where the nonextremality is of the order of $\alpha$ remains to be investigated further.

In this paper, for simplicity, we studied only the rotating black holes in odd dimensions with all equal angular momenta. These solutions are all cohomogeneity-one metrics depending only on one radial variable. Although we expect that the arising of horizon natural boundaries will also occur in general rotating cases, the subjects require further investigation.

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## APPENDIX A: PERTURBED EQUATION IN GENERAL $D = 2n + 1$ DIMENSIONS

The four perturbed equations of $(\delta f, \delta h, \delta W, \delta\omega)$ are coupled second-order differential equations,

$$
\begin{aligned}
&r^{2D}\big((D^2-2D-3)\nu^4r^2+2(D-5)\nu^2r^{D+1}+2(D-3)r^{2D}+(D-1)^2\mu\nu^2r^4\big)\delta W\\
&\quad+2(D^2-4D+3)(r^D+\nu^2r)^2\big((D^2-4D-5)\nu^4+(D^2-6D+8)\mu^2r^4-2(D^2\\
&\quad-5D+4)\mu\nu^2r^2\big)-(D-1)^2\mu\nu^2r^{2D+4}\delta h+2(D-3)r^D(r^D+\nu^2r)^2\big((D-2)r^D-\nu^2r\big)\delta f\\
&\quad-2(D-1)\sqrt{\mu}\nu r^{D+4}(r^D+\nu^2r)^2\delta\omega'+r^{D+1}(r^D+\nu^2r)(2(D-2)r^D+(D-3)\nu^2r)(r^D\\
&\quad-\mu r^3+\nu^2r)\delta h'-r^{2D+1}(r^D+\nu^2r)(2r^D+r^3(\mu-D\mu)-(D-3)\nu^2r\big)\delta W'=0,
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
&-2(D-3)(D-1)(r^D+\nu^2r)\big((D^2-6D+8)\mu^2r^{D+4}-(D^2+4D+3)\nu^4r^D\\
&\quad+2(D^2-1)\mu\nu^4r^3-(D^2+4D+3)\nu^6r-(D-4)D\mu^2\nu^2r^5\big)+(D-1)^2\mu\nu^2r^{2D+4}\delta h\\
&\quad-r^{2D}\big((D^2-2D-3)\nu^4r^2+2(D-5)\nu^2r^{D+1}+2(D-3)r^{2D}+(D-1)^2\mu\nu^2r^4\big)\delta W\\
&\quad-2(D-3)r^D(r^D+\nu^2r)^2\big((D-2)r^D-\nu^2r\big)\delta f+2(D-1)\sqrt{\mu}\nu r^{D+4}(r^D+\nu^2r)^2\delta\omega'\\
&\quad-r^{D+1}(r^D+\nu^2r)(2(D-2)r^D+(D-3)\nu^2r)(r^D-\mu r^3+\nu^2r)\delta f'\\
&\quad-r^{2D+1}(r^D+\nu^2r)\big(2(D-1)r^D-(D+1)\mu r^3+3(D-1)\nu^2r\big)\delta W'\\
&\quad-2r^{2D+2}(r^D+\nu^2r)(r^D-\mu r^3+\nu^2r)\delta W''=0,
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
&-4(D-3)(D-1)^2(r^D+\nu^2r)\big((D-4)\mu r^2-(D+1)\nu^2\big)\big(-\mu r^{D+2}-\nu^2r^D+\mu\nu^2r^3-\nu^4r\big)\\
&\quad-4r^{2D}\big(-(D^2-5D+6)\nu^4r^2+4(D-2)\nu^2r^{D+1}+(D-3)r^{2D}-(D-1)^2\mu\nu^2r^4\big)\delta W\\
&\quad-4(D-1)^2\mu\nu^2r^{2D+4}\delta h+4(D-3)r^D(r^D+\nu^2r)^3\delta f-8(D-1)\sqrt{\mu}\nu r^{D+4}(r^D+\nu^2r)^2\delta\omega'\\
&\quad-2r^{D+2}(r^D+\nu^2r)^2(r^D-\mu r^3+\nu^2r)\delta h''+4r^{2D+2}(r^D+\nu^2r)(r^D-\mu r^3+\nu^2r)\delta W''\\
&\quad+4r^{2D+1}(r^D+\nu^2r)\big((D-2)r^D+(2D-3)\nu^2r-\mu r^3\big)\delta W'+r^{D+1}(r^D+\nu^2r)\\
&\quad\times\big(-(D-1)\mu r^{D+3}-(D-5)\nu^2r^{D+1}+2r^{2D}+(D-1)\mu\nu^2r^4-(D-3)\nu^4r^2\big)\delta f'\\
&\quad-r^{D+1}(r^D+\nu^2r)\big((D-3)\mu r^{D+3}+(3D-11)\nu^2r^{D+1}+2(D-3)r^{2D}\\
&\quad-(D+1)\mu\nu^2r^4+(D-5)\nu^4r^2\big)\delta h'=0,
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
&-4(D-3)(D-1)^2\sqrt{\mu}\nu r^{-D-2}\big((D-4)\mu r^2-(D+1)\nu^2\big)-2r^{-D}(r^D+\nu^2r)^2\delta\omega''\\
&\quad-2r^{-D-1}(r^D+\nu^2r)\big(D(r^D-\nu^2r)+2\nu^2r\big)\delta\omega'+(D-1)\sqrt{\mu}\nu\delta f'-(D-1)\sqrt{\mu}\nu\delta h'\\
&\quad+2D\sqrt{\mu}\nu\delta W'+2\sqrt{\mu}\nu r\delta W''=0.
\end{aligned}
\tag{A4}
$$

By the standard procedure of eliminating variables, we can obtain a decoupled fourth-order differential equation for $\delta f$, given in Sec. IV. The functions $\delta h$ and $\delta\omega$ can be given as quadratures and $\delta W$ can be solved algebraically. They are

$$
\begin{aligned}
&(D-2)(-8(D-2)\mu r^{D+3}+3(D-1)^2r^{2D}-4(D-2)\mu^2r^6+4(D+1)\mu\nu^2r^4)\delta h'\\
&=-2(D-1)r^{-D-1}(3(D-3)^2(D^3-7D^2+14D-8)\mu^2r^{D+4}-2(D-3)^2(2D^3-15D^2\\
&\quad+34D-24)\mu^3r^7+4(D+1)^2(D^3-5D^2-17D+21)\nu^6r+2(-6D^5+39D^4+6D^2+38D\\
&\quad-13)\mu\nu^4r^3-2(3D^5-23D^4+26D^3+22D^2-29D+1)\mu\nu^2r^{D+2}+(3D^5-7D^4-94D^3\\
&\quad-122D^2+91D+129)\nu^4r^D+4(3D^5-30D^4+97D^3-161D^2+140D-49)\mu^2\nu^2r^5)\\
&\quad-2(D^3-6D^2+11D-6)r^{D-1}((D-5)r^D+2\mu r^3-4\nu^2r)\delta f+(2-D)(-2(2D^2\\
&\quad-15D+23)\mu r^{D+3}+2(2D^2-11D+13)\nu^2r^{D+1}+(5D^2-28D+27)r^{2D}\\
&\quad+4(7-2D)\mu^2r^6+4(D-6)\mu\nu^2r^4+4D\nu^4r^2)\delta f'+2(D-2)r(-r^D+\mu r^3-\nu^2r)\\
&\quad\times((5D-7)r^D+2D\mu r^3-2D\nu^2r-14\mu r^3)\delta f''-4(D-2)r^2(r^D-\mu r^3+\nu^2r)^2\delta f''',
\end{aligned}
\tag{A5}
$$

$$4(D-2)r^{2D}\delta W = 2(D-1)(-(D-7)(D-1)(D+1)\nu^4 - ((D-4)(D-3)(D-2)\mu^2 r^4) + 2(D-1)((D-7)D+7)\mu\nu^2 r^2) + (D-2)r^D\big((-2(D-3)r^D + 4\nu^2 r)\delta f - r(r^D - \mu r^3 + \nu^2 r)(\delta f + \delta h)\big), \tag{A6}$$

$$2(D-1)\sqrt{\mu}\nu(r^D+\nu^2 r)^2\delta\omega' = 2(D-3)(D-1)r^{-D-4}(r^D+\nu^2 r)^2\big((D-5)(D+1)\nu^4 + (D-4)(D-2)\mu^2 r^4 - 2(D-4)(D-1)\mu\nu^2 r^2\big) - (D-1)^2\mu\nu^2 r^D\delta h + \frac{2(D-3)}{r^4} \times \big((D-2)r^D - 2 - \nu^2 r\big)(r^D+\nu^2 r)^2\delta f + r^{D-4}(2(D-5)\nu^2 r^{D+1} + 2(D-3)r^{2D} + (D-1)^2\mu\nu^2 r^4 + (D-3)(D+1)\nu^4 r^2)\delta W + \frac{(r^D+\nu^2 r)}{r^3}(2Dr^D - 4r^D + D\nu^2 r - 3\nu^2 r) \times (r^D - \mu r^3 + \nu^2 r)\delta h' - r^{D-3}(r^D+\nu^2 r)(2r^D - D\mu r^3 - D\nu^2 r + \mu r^3 + 3\nu^2 r)\delta W'. \tag{A7}$$

## APPENDIX B: NEAR-HORIZON GEOMETRY OF NONEXTREMAL BLACK HOLES

For nonextremal black holes, we can solve for $\mu$ in terms of $\nu$ and $r_0$, i.e.,

$$\mu = r_0^{D-3} + \frac{\nu^2}{r_0^2}. \tag{B1}$$

To obtain the behavior of $\delta f$ in the near-horizon region, we can assume an *Ansatz* $\delta f = (r-r_0)^\gamma \hat{f}$. We find that at the leading order of $(r-r_0)$, we have

$$(\gamma-1)\gamma(\gamma+1)^2\hat{f} = Q_0(r-r_0)^{2-\gamma}, \tag{B2}$$

where

$$Q_0 = \frac{2(D-1)r_0^{-D-4}}{3(D-2)((D-3)r_0^D - 2\nu^2 r_0)^2}\big(2(119D^3 - 1406D^2 + 1739D - 336)\nu^6 r_0^3 - (289D^4 - 3528D^3 + 11220D^2 - 11684D + 3519)\nu^4 r_0^{D+2} - (D-3)^2(9D^4 - 86D^3 + 289D^2 - 406D + 200)r_0^{3D} + 2(52D^5 - 685D^4 + 3162D^3 - 6617D^2 + 6488D - 2436)\nu^2 r_0^{2D+1}\big). \tag{B3}$$

The modes associated with the double roots $\gamma = -1$ is clearly unacceptable for the regular horizon, we thus have

$$\delta f = d_0\hat{f}_0(r) + d_1\hat{f}_1(r) + \frac{Q_0}{18}\hat{f}_s(r). \tag{B4}$$

Here $d_0$ and $d_1$ are constants to be determined so that the $c_2$ term at asymptotic infinity will not be generated. The last term above represents the contribution from the source. All $f_i(r)$ functions are analytic with Taylor expansions of the form $\hat{f} \sim 1 + \#_1(r-r_0) + \#_2(r-r_0)^2 + \cdots$.

At the first sight, there appears to have two many parameters on the horizon, namely $(r_0, \nu, d_0, d_1)$. However, in a typical discussion of perturbative solutions, we need to hold the mass and angular momentum fixed. In our case, we have already hold the $r_0$ fixed, we can now hold either mass or angular momentum fixed. In this case, we need fix one combination of the coefficients $(\nu, d_0, d_1)$. Furthermore, we need fine-tune these parameters so that the asymptotic divergent $c_{-2}$ term vanishes, and this fix another combination, leaving only one combination, together with $r_0$.

We present the near-horizon structure at low-lying orders of the Taylor expansion for all the functions only in $D = 7$,

$$\delta f = d_0 + (r - r_0)d_1 + (r - r_0)^2\left(-\frac{16(1815r_0^{18} - 2037r_0^{12}\nu^2 - 1397r_0^6\nu^4 + 1015\nu^6)}{45r_0^{10}(2r_0^6 - \nu^2)^2} + \frac{4r_0^4(31r_0^6 + 4\nu^2)d_0}{9(2r_0^6 - \nu^2)^2} - \frac{(74r_0^6 + 47\nu^2)d_1}{54r_0(2r_0^6 - \nu^2)}\right) + (r - r_0)^3\left(+\frac{2(95475r_0^{24} - 150483r_0^{18}\nu^2 - 64897r_0^{12}\nu^4 + 91711r_0^6\nu^6 - 15910\nu^8)}{45r_0^{11}(2r_0^6 - \nu^2)^3} - \frac{r_0^3(605r_0^{12} + 637r_0^6\nu^2 + 32\nu^4)d_0}{18(2r_0^6 - \nu^2)^3} + \frac{(547r_0^{12} + 587r_0^6\nu^2 + 40\nu^4)d_1}{108r_0^2(2r_0^6 - \nu^2)^2}\right),$$

$$\delta h = h_0 + (r - r_0)\left(-\frac{d_1}{3} - \frac{8d_0r_0^5}{2r_0^6 - \nu^2} + \frac{32(7\nu^4 - 18\nu^2r_0^6 + 15r_0^{12})}{5r_0^9(2r_0^6 - \nu^2)}\right) + (r - r_0)^2\left(-\frac{16(-788\nu^6 + 3037\nu^4r_0^6 - 3540\nu^2r_0^{12} + 1635r_0^{18})}{45r_0^{10}(2r_0^6 - \nu^2)^2} + \frac{76d_0r_0^4(\nu^2 + r_0^6)}{9(2r_0^6 - \nu^2)^2} + \frac{d_1(13\nu^2 - 122r_0^6)}{54r_0(2r_0^6 - \nu^2)}\right) + \cdots,$$

$$\delta W = -\frac{36(-2\nu^4 + 3\nu^2r_0^6 + 5r_0^{12})}{5r_0^{14}} + d_0\left(-2 + \frac{\nu^2}{r_0^6}\right) + (r - r_0)\left(\frac{4}{3}d_1\left(-2 + \frac{\nu^2}{r_0^6}\right) + \frac{d_0(-6\nu^2 + 4r_0^6)}{r_0^7} + \frac{8(-167\nu^4 + 108\nu^2r_0^6 + 195r_0^{12})}{5r_0^{15}}\right) + (r - r_0)^2\left(\frac{6d_1(-\nu^2 + r_0^6)}{r_0^7} - \frac{3d_0(7\nu^4 - 16\nu^2r_0^6 + 16r_0^{12})}{r_0^8(2r_0^6 - \nu^2)} - \frac{12(671\nu^6 - 1810\nu^4r_0^6 + 489\nu^2r_0^{12} + 810r_0^{18})}{5r_0^{16}(2r_0^6 - \nu^2)}\right) + \cdots,$$

$$\delta\omega = \omega_0 + (r - r_0)\left(-\frac{3\nu h_{t_0}r_0^4}{(\nu^2 + r_0^6)^{3/2}} - \frac{2d_1(-5\nu^4 + 8\nu^2r_0^6 + 4r_0^{12})}{9\nu r_0(\nu^2 + r_0^6)^{3/2}} + \frac{d_0r_0^4(-29\nu^4 + 8\nu^2r_0^6 + 10r_0^{12})}{3\nu(\nu^2 + r_0^6)^{5/2}} + \frac{4(-604\nu^6 - 544\nu^4r_0^6 + 912\nu^2r_0^{12} + 525r_0^{18})}{15\nu r_0^{10}(\nu^2 + r_0^6)^{3/2}}\right) + \cdots. \tag{B5}$$

Note that all the expansion powers are integers and the functions therefore are all analytic and infinitely differentiable. Taking an extremal limit, corresponding to setting $\nu = \sqrt{2}r_0^3$, is singular in this expansion. In the above expansion, we have assumed that $(\nu^2 - 2r_0^6)/r_0^4 \gg \alpha$. The near-extremal case with $(\nu^2 - 2r_0^6)/r_0^4 \sim \alpha$ remains further study.

## APPENDIX C: NEAR-HORIZON EXPANSION OF THE $D = 7$ EXTREMAL BLACK HOLE

In Sec. III C, we give the near-horizon expansion of the perturbative function $\delta f$ for the $D = 7$ extremal rotating black hole. Here, we give the these expansions for the remaining functions,

$$\delta h = \left(\hat{h}_0 - \frac{15704(r - r_0)}{15r_0^3} + \frac{68164(r - r_0)^2}{315r_0^4} + \cdots\right) - d_{\Delta_+}(r - r_0)^{\frac{1}{2}(\sqrt{21}-3)}\left(\frac{1}{6}(\sqrt{21} + 3) + \frac{(11\sqrt{21} + 24)(r - r_0)}{90r_0} + \frac{(27551\sqrt{21} - 121011)(r - r_0)^2}{36720r_0^2} + \cdots\right),$$

$$\delta W = \left(-\frac{108}{5r_0^2} - \frac{456(r - r_0)}{r_0^3} - \frac{27444(r - r_0)^2}{5r_0^4} + \cdots\right) + d_{\Delta_+}(r - r_0)^{\frac{1}{2}(\sqrt{21}-1)}\left(-\frac{3(\sqrt{21} + 3)}{2r_0} + \frac{3(31\sqrt{21} + 99)(r - r_0)}{20r_0^2} - \frac{(57947\sqrt{21} + 114813)(r - r_0)^2}{4080r_0^3} + \cdots\right),$$

$$\delta\omega = \left(\hat{\omega}_0 - \frac{\left(\sqrt{\frac{2}{3}}(5\hat{h}_0r_0^2 + 1082)\right)(r - r_0)}{5r_0^4} - \frac{(5\hat{h}_0r_0^2 + 11226)(r - r_0)^2}{5(\sqrt{6}r_0^5)} + \cdots\right) + d_{\Delta_+}(r - r_0)^{\frac{1}{2}(\sqrt{21}-1)}\left(-\frac{2\sqrt{5 + \sqrt{21}}}{3r_0^2} + \frac{(33\sqrt{14} - 41\sqrt{6})(r - r_0)}{90r_0^3} + \frac{(47953\sqrt{6} + 261\sqrt{14})(r - r_0)^2}{18360r_0^4} + \cdots\right). \tag{C1}$$

We have two new integration constants $(\hat{h}_0, \hat{\omega}_0)$. The former should be chosen such that the speed of light at the asymptotic infinity remains unit, whilst the latter should be chosen so that the asymptotic spacetime is nonrotating.


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