Zeroth and second laws of black hole mechanics in Einstein-Maxwell-scalar effective field theory

Iain Davies*

Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

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There has been recent progress in extending the zeroth and second laws of black hole mechanics to gravitational effective field theories (EFTs). We generalize these results to a much larger class of EFTs describing gravity coupled to electromagnetism and a real scalar field. We also show that the zeroth law holds for the EFT of gravity coupled to electromagnetism and a charged scalar field.

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I. INTRODUCTION

The laws of black hole mechanics are a set of theorems determining the classical properties of black holes. Their striking resemblance to the laws of thermodynamics leads to an interpretation of black holes as thermodynamic objects, which is made concrete through the mechanism of Hawking radiation.

The original proofs of the laws [1,2] require that the theory of gravity is the two-derivative Einstein-Hilbert action, with a theory of matter, such as Maxwell theory or a minimally coupled scalar field, that satisfies suitable energy conditions. However, we know that two-derivative Einstein-Maxwell theory cannot be the complete description of gravity and electromagnetism on all scales as it is not a UV complete theory. Generically we expect any low energy limit of a UV theory of gravity to come with higher derivative corrections, which will invalidate the standard proofs of the laws of black hole mechanics. Since we do not expect these corrections to change the physical interpretation of black holes as thermodynamic objects, this is a problem.

There have been a variety of attempts to reconcile the laws in higher derivative theories of gravity, some of which are reviewed by Sarkar in 2019 in [3]. In particular, Wald proved in [4] that a modified, but still geometric, definition of black hole entropy could be used to prove the "equilibrium state" version of the first law in any diffeomorphism-invariant theory of gravity with arbitrary matter fields. However, this definition of the entropy fails to satisfy a second law, and he had to assume that the zeroth law holds via the assumption of a bifurcate Killing horizon.

Recently, however, there were several developments in proving the zeroth, first, and second laws in the setting of effective field theory (EFT). This setting requires assuming two things: (a) the Lagrangian is a series of terms with increasing derivatives coming with coefficients that scale in appropriate powers of some UV length scale l, and (b) any time or length scale L associated with the solution satisfies $L \gg l$. The first assumption is physically reasonable if we view our theory as a low energy limit of some UV complete theory of gravity. The second assumption means that higher derivative terms are less important, and so our solution remains in the regime of validity of the EFT.

Let us briefly review the spate of recent results.

Zeroth law. Bhattacharyya *et al.* [5] proved that the zeroth law holds for any diffeomorphism-invariant EFT of gravity without matter. The zeroth law states that the surface gravity κ of a stationary black hole is constant across the horizon.[1] The proof uses Gaussian null coordinates and the concept of "boost weight" to show that derivatives of κ tangent to the horizon are proportional to a component of the equations of motion, which is set to 0.

First law. Biswas *et al.* [8] generalized the Wald entropy [4] to prove the "physical process" version of the first law for an arbitrary higher derivative diffeomorphism-invariant, gauge-independent theory of gravity, electromagnetism, and a real scalar field. The physical process version of the first law concerns a stationary black hole that is

*id318@cam.ac.uk

[1] Note that, in order to define the surface gravity, the horizon of the black hole must be a Killing horizon. Hawking proved this is always the case for two-derivative general relativity (GR) in his rigidity theorem [6], under the assumption of analyticity. Recent work by Hollands *et al.* [7] has proved the rigidity theorem also holds in the EFT of gravity with no matter, also under the assumption of analyticity.

perturbed by some matter before settling down to a new stationary configuration. It relates the change in entropy δS to the mass δM, angular momentum δJ, and charge δQ of the matter perturbation,

$$\frac{\kappa}{2\pi}\delta S = \delta M - \Omega_H \delta J - \Phi_{\text{bh}}\delta Q, \tag{1}$$

where Ω_H is the angular velocity of the horizon and Φ_{bh} is the electrostatic potential.

Second law. Hollands *et al.* (HKR) [9], following up from work by Wall [10] and Bhattacharyya *et al.* [11], proved a version of the second law for any diffeomorphism-invariant EFT of gravity and a real scalar field. The second law states that the entropy of a dynamical black hole is nondecreasing in time, $\dot{S} \geq 0$. HKR consider a dynamical black hole settling down to an equilibrium stationary state and that remains in the regime of validity of the EFT as described above. They were able to define an entropy which is nondecreasing to quadratic order in perturbations around the stationary state, up to $O(l^N)$ terms, where l^N is the order up to which we know our EFT. Furthermore, this entropy reduces to the Wald entropy in equilibrium.

Finally, in the companion to this paper [12], Davies and Reall were able to show that a further extension of the HKR procedure can strengthen the second law result significantly by dropping its perturbative nature. They define an entropy which satisfies a *nonperturbative* second law in vacuum gravity EFT, up to $O(l^N)$ terms. This entropy reduces to the Wald entropy in equilibrium, satisfies the first law, and is purely geometrically defined for theories with up to six derivatives.

Taken together, these results mean we now have a much better understanding of the laws of black hole mechanics in EFT. However, the results we have for the zeroth law and second law are only applicable to gravity with minimal matter couplings or to gravity with the simplest matter field, a scalar field. Here we ask, are these results robust to the addition of nonminimal couplings of some more complicated matter field? The only field other than the metric and scalar field for which we know the classical approximation may be valid is the Maxwell field, and hence this seems like an important addition to make.

In this paper we extend the aforementioned works by completing the story for the EFT of gravity, electromagnetism, and a real (uncharged) scalar field. We prove a generalized zeroth law holds exactly and that the second law holds in the sense of Davies and Reall. Taken all together, this means there is now a zeroth, first, and second law for such theories. Along the way we will also show the zeroth law still holds even if the scalar field is charged and discuss how the second law could be generalized in this case. This gives further evidence that these proofs are robust to more complicated matter models and that, even in higher derivative theories of gravity, black holes will still obey the laws.

The paper is broken down as follows. In Sec. II, we define our Einstein-Maxwell-scalar EFT. In Sec. III, we define two distinct choices of Gaussian null coordinates (GNCs) and the notion of boost weight. We will work in GNCs throughout the paper. In Sec. IV, we state the generalized zeroth law and sketch its proof for Einstein-Maxwell-scalar EFT. Section V contains the details of this proof. We also show how the proof can be modified if the scalar field is charged. In Sec. VI, we make precise the scenario in which we will prove the second law and review the previous work on the matter. In Sec. VII, we prove the second law for Einstein-Maxwell-scalar EFT and discuss its generalization if the scalar is charged.

II. EINSTEIN-MAXWELL-SCALAR EFT

We consider the EFT of gravity, electromagnetism, and a real[2] scalar field ϕ, which we shall refer to as Einstein-Maxwell-scalar EFT. In EFT, the Lagrangian is a sum of terms ordered by their number of derivatives. We assume diffeomorphism invariance and electromagnetic gauge invariance,[3] so that the Lagrangian consists only of contractions of $R_{\alpha\beta\gamma\delta}$, $F_{\alpha\beta}$, ϕ, and their covariant derivatives,

$$\mathcal{L} = \mathcal{L}(g_{\alpha\beta}, R_{\alpha\beta\gamma\delta}, \nabla_\alpha R_{\alpha\beta\gamma\delta}, \ldots, F_{\alpha\beta}, \nabla_\alpha F_{\alpha\beta}, \ldots, \phi, \nabla_\alpha \phi, \ldots). \tag{2}$$

The most general Lagrangian of this form with up to two derivatives can be written as[4]

$$\mathcal{L}_2 = R - V(\phi) - \frac{1}{2}\nabla_\alpha \phi \nabla^\alpha \phi - \frac{1}{4}c_1(\phi) F_{\alpha\beta}F^{\alpha\beta} + c_2(\phi) F_{\alpha\beta}F_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}. \tag{3}$$

An arbitrary function of ϕ multiplying R can be eliminated by redefining the metric, while an arbitrary function of ϕ multiplying $\nabla_\alpha \phi \nabla^\alpha \phi$ can be eliminated by redefining ϕ [14]. Here we have taken units with $16\pi G = 1$ and rescaled $F_{\alpha\beta}$ appropriately. The final term only appears in $d = 4$, where the volume form $\epsilon_{\alpha\beta\gamma\delta}$ has four indices; in higher dimensions, it is taken that this term is not present.

[2]The EFT of a charged, complex scalar field is discussed in Sec. V E.

[3]The assumption that the Lagrangian is invariant under an electromagnetic gauge transformation $A_\alpha \to A_\alpha + \nabla_\alpha \chi$ will rule out, for example, Chern-Simons terms, which are not themselves gauge invariant but do produce gauge-invariant equations of motion. See very recent work [13] for a linearized second law for Chern-Simons terms.

[4]We have included the zero-derivative term $V(\phi)$ in the two-derivative Lagrangian $\mathcal{L}_2$. Naively, in EFT, we should expect $V(\phi)$ to come with a factor $1/l^2$. However, we assume $V(\phi)$ is comparable to the cosmological constant Λ, which is extremely small for somewhat mysterious reasons. More precisely, we assume $|V| \leq 1/L^2$, where L is any typical length scale of the solution, and so $V(\phi)$ is of no larger scale than the two-derivative terms.

The only condition we put on the arbitrary functions $V(\phi)$, $c_1(\phi)$, and $c_2(\phi)$ is that $c_1(\phi) > 0$. This is a sufficient condition for the energy-momentum tensor of the leading order two-derivative theory to satisfy the null energy condition (NEC). For Einstein-Maxwell theory without a scalar field, $c_1 = 1$, so this positivity condition is also motivated on the grounds that we do not expect the scalar field to change the sign of c_1. If we were additionally to impose $V(\phi) \geq 0$ then the two-derivative energy-momentum tensor would also satisfy the dominant energy condition (DEC), which is the condition assumed in the original proof of the zeroth law by Bardeen *et al.* [2]. However, the proof still goes through if the two-derivative energy-momentum tensor minus any parts proportional to the metric satisfies the DEC. Since $V(\phi)$ only appears in $T_{\mu\nu}$ multiplying $g_{\mu\nu}$, this is satisfied by our two-derivative theory regardless of the sign of $V(\phi)$ (for example, we can include the case of a negative cosmological constant which is excluded by the DEC). Indeed, in the following proofs we will require no condition on $V(\phi)$.

In the full EFT action, higher derivative terms come with a factor of some UV scale l for each extra derivative,

$$S = \int \mathrm{d}^d x \sqrt{-g}\left(\mathcal{L}_2 + \sum_{n=1}^{\infty} l^n \mathcal{L}_{n+2}\right), \tag{4}$$

where $\mathcal{L}_{n+2}$ contains all terms with $n+2$ derivatives.

The equations of motion for this action are $E_{\alpha\beta} = 0$, $E_\alpha = 0$, $E = 0$, where

$$\begin{aligned} E_{\alpha\beta} &\equiv \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}} = E^{(0)}_{\alpha\beta} + \sum_{n=1}^{\infty} l^n E^{(n)}_{\alpha\beta}, \\ E_\alpha &\equiv -\frac{1}{\sqrt{-g}} g_{\alpha\beta}\frac{\delta S}{\delta A_\beta} = E^{(0)}_\alpha + \sum_{n=1}^{\infty} l^n E^{(n)}_\alpha, \\ E &\equiv \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta \phi} = E^{(0)} + \sum_{n=1}^{\infty} l^n E^{(n)}, \end{aligned} \tag{5}$$

where $E^{(0)}_{\alpha\beta}, E^{(0)}_\alpha, E^{(0)}$ are the result of varying the two-derivative terms from $\mathcal{L}_2$, i.e.,

$$\begin{aligned} E^{(0)}_{\alpha\beta} = {} & R_{\alpha\beta} - \frac{1}{2}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}c_1(\phi)F_{\alpha\delta}F_\beta{}^\delta \\ & - \frac{1}{2}g_{\alpha\beta}\left(R - V(\phi) - \frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi - \frac{1}{4}c_1(\phi)F_{\gamma\delta}F^{\gamma\delta}\right), \end{aligned} \tag{6}$$

$$E^{(0)}_\alpha = \nabla^\beta\left[c_1(\phi)F_{\alpha\beta} - 4c_2(\phi)F^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}\right], \tag{7}$$

$$\begin{aligned} E^{(0)} = {} & \nabla^\alpha\nabla_\alpha\phi - V'(\phi) - \frac{1}{4}c_1'(\phi)F_{\alpha\beta}F^{\alpha\beta} \\ & + c_2'(\phi)F_{\alpha\beta}F_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}. \end{aligned} \tag{8}$$

III. GAUSSIAN NULL COORDINATES

We will be concerned with quantities on the event horizon $\mathcal{N}$ of a black hole, which is a null hypersurface. We assume $\mathcal{N}$ is smooth and has generators that have affine parameters extending to the infinite future. The smoothness assumption will always be true in the stationary setting of the zeroth law, but is not generally true when considering dynamical black holes, as in the second law. However, it seems a reasonable assumption for the situation of a black hole settling down to equilibrium, as envisioned in [9–11].

To describe quantities near $\mathcal{N}$, we will use two appropriate choices of Gaussian null coordinates. The first applies to both the stationary and dynamical setting, while the second will only be used in the stationary case to prove the zeroth law.

A. Affinely parametrized GNCs

Here we use the same notation as [9,15]. Assume all generators intersect a spacelike cross section C exactly once, and take x^A to be a codimension-2 coordinate chart on C. Let the null geodesic generators have affine parameter v and future-directed tangent vector l^α such that $l = \partial_v$ and $v = 0$ on C. We can transport C along the null geodesic generators a parameter distance v to obtain a foliation $C(v)$ of $\mathcal{N}$. Finally, we uniquely define the null vector field n^α by $n \cdot (\partial/\partial x^A) = 0$ and $n \cdot l = 1$. The coordinates (r, v, x^A) are then assigned to the point affine parameter distance r along the null geodesic starting at the point on $\mathcal{N}$ with coordinates (v, x^A) and with tangent n^α there. The metric in these GNCs is given by

$$\begin{aligned} g = {} & 2\mathrm{d}v\mathrm{d}r - r^2\alpha(r, v, x^C)\mathrm{d}v^2 - 2r\beta_A(r, v, x^C)\mathrm{d}v\mathrm{d}x^A \\ & + \mu_{AB}(r, v, x^C)\mathrm{d}x^A\mathrm{d}x^B, \qquad l = \partial_v, \qquad n = \partial_r. \end{aligned} \tag{9}$$

This choice of coordinates will be referred to as "affinely parametrized GNCs." $\mathcal{N}$ is the surface $r = 0$, and C is the surface $r = v = 0$. The inverse of μ_{AB} is denoted by μ^{AB}, and we raise and lower $A, B, C, \ldots$ indices with μ^{AB} and μ_{AB}. We denote the induced volume form on $C(v)$ by $\epsilon_{A_1 \ldots A_{d-2}} = \epsilon_{rvA_1 \ldots A_{d-2}}$ where d is the dimension of the spacetime. The covariant derivative on $C(v)$ with respect to μ_{AB} is denoted by D_A. We also define

$$\begin{aligned} K_{AB} &\equiv \frac{1}{2}\partial_v \mu_{AB}, \qquad \bar{K}_{AB} \equiv \frac{1}{2}\partial_r \mu_{AB}, \\ K &\equiv K^A{}_A, \qquad \bar{K} \equiv \bar{K}^A{}_A. \end{aligned} \tag{10}$$

K_{AB} describes the expansion and shear of the horizon generators. $\bar{K}_{AB}$ describes the expansion and shear of the ingoing null geodesics orthogonal to a horizon cut $C(v)$.

Affinely parametrized GNCs are not unique: we are free to change the affine parameter on each generator of $\mathcal{N}$ by $v' = v/a(x^A)$ with arbitrary $a(x^A) > 0$. This will lead to a change $(v, r, x^A) \to (v', r', x'^A)$ with $v' = v/a(x^A) + O(r)$,

$r' = a(x^A)r + O(r^2)$, $x'^A = x^A + O(r)$ near the horizon. Details of how this transformation changes the quantities above are given in [9]. The remaining freedom in our affinely parametrized GNCs is to change our coordinate chart x^A on C; however, all calculations in this paper are manifestly covariant in $A, B, \ldots$ indices and so this freedom will not change any of the expressions.

1. Boost weight

An important concept in this set of GNCs is the boost weight of a quantity. Suppose we take a to be constant and consider the rescaling $v' = v/a$, $r' = ar$, which preserves the form of the GNCs above. If a quantity T transforms as $T' = a^b T$, then T is said to have boost weight b. See [9] for a full definition. Some important facts are stated here:

(i) A tensor component $T^{\mu_1\ldots\mu_n}_{\beta_1\ldots\beta_m}$ has boost weight given by the sum of $+1$ for each v subscript and each r superscript and -1 for each r subscript and v superscript. $A, B, \ldots$ indices contribute 0, e.g., T^A_{vvrB} has boost weight $+1$.

(ii) α, β_A, and μ_{AB} have boost weight 0. K_{AB} and $\bar{K}_{AB}$ have boost weight $+1$ and -1, respectively.

(iii) If T has boost weight b, then $D_{A_1}\ldots D_{A_n}\partial_v^p\partial_r^q T$ has boost weight $b + p - q$.

(iv) If X_i has boost weight b_i and $T = \prod_i X_i$, then T has boost weight $b = \sum_i b_i$.

In Lemma 2.1 of [9], it is proved that boost weight is independent of the choice of affinely parametrized GNCs on $\mathcal{N}$. More precisely, a quantity of certain boost weight in (r, v, x^A) GNCs on $\mathcal{N}$ can be written as the sum of terms of the same boost weight in (r', v', x'^A) GNCs on $\mathcal{N}$, where $v' = v/a(x^A)$ on $\mathcal{N}$.

B. Killing vector GNCs

In the stationary setting of the zeroth law we will also use another choice of GNCs, hereafter referred to as "Killing vector GNCs."

In standard two-derivative GR with a wide range of matter models, it can be proved that the future event horizon $\mathcal{N}$ of a stationary analytic black hole spacetime is a Killing horizon whose normal is some Killing vector ξ [1]. Recent work by Hollands *et al.* [7] has extended this result to arbitrary higher derivative effective field theories of gravity with no matter fields present. Here we assume this result still holds for our Einstein-Maxwell-scalar EFT and that we can drop the analyticity assumption.

Therefore, we can take a similar construction to the above, except with the null geodesic generators having nonaffinely parametrized, future-directed tangent vectors $\xi = \partial_\tau$. In the notation of [5], this leads to coordinates (ρ, τ, x^A) with metric

$$g = 2\mathrm{d}\tau\mathrm{d}\rho - \rho X(\rho, x^C)\mathrm{d}\tau^2 + 2\rho\omega_A(\rho, x^C)\mathrm{d}\tau\mathrm{d}x^A + h_{AB}(\rho, x^C)\mathrm{d}x^A\mathrm{d}x^B, \qquad \xi = \partial_\tau, \qquad \chi = \partial_\rho. \tag{11}$$

$\mathcal{N}$ is the surface $\rho = 0$, and C is the surface $\rho = \tau = 0$. In these coordinates, we raise $A, B, C, \ldots$ indices with h^{AB} and h_{AB} and denote the induced volume form on $C(\tau)$ by $\varepsilon_{A_1\ldots A_{d-2}} = \epsilon_{\rho\tau A_1\ldots A_{d-2}}$. The covariant derivative on $C(\tau)$ with respect to h_{AB} is denoted by $\mathcal{D}_A$.

The differences between the two GNCs are twofold. First, since $\xi = \partial_\tau$ is a Killing vector, the unknown metric coefficients X, ω_A and h_{AB} are independent of τ. Second, the fact that τ is not necessarily an affine parameter means the coefficient of $\mathrm{d}\tau^2$ only comes with a factor of ρ, whereas $\mathrm{d}v^2$ comes with a factor of r^2 in the affinely parametrized GNCs.

The relationship between these two forms of GNCs is crucial to prove the zeroth law for this theory, following the method of [5].

IV. THE GENERALIZED ZEROTH LAW

We proceed to prove a generalized zeroth law of black hole mechanics for this theory. The zeroth law concerns stationary black hole solutions $(g_{\alpha\beta}, F_{\alpha\beta}, \phi)$ to the equations of motion above.

A. Assumptions

We make the following assumptions in order to prove the generalized zeroth law:

(1) The rigidity theorem of [7] can be extended to this theory, i.e., that the future event horizon $\mathcal{N}$ of the black hole is a Killing horizon with Killing vector ξ.

(2) The matter fields are invariant under this Killing vector, i.e.,

$$\mathcal{L}_\xi F = 0, \qquad \mathcal{L}_\xi \phi = 0. \tag{12}$$

In Killing vector GNCs, these imply that $\partial_\tau F_{\mu\nu} = 0$ and $\partial_\tau \phi = 0$.

(3) The black hole solution is analytic in l, i.e., we can write

$$\begin{aligned} g_{\alpha\beta} &= g^{(0)}_{\alpha\beta} + l g^{(1)}_{\alpha\beta} + l^2 g^{(2)}_{\alpha\beta} + \ldots, \\ F_{\alpha\beta} &= F^{(0)}_{\alpha\beta} + l F^{(1)}_{\alpha\beta} + l^2 F^{(2)}_{\alpha\beta} + \ldots, \\ \phi &= \phi^{(0)} + l\phi^{(1)} + l^2\phi^{(2)} + \ldots. \end{aligned} \tag{13}$$

In particular, we can write the Killing vector GNC metric components as series in l,

$$\begin{aligned} X &= X^{(0)} + lX^{(1)} + l^2 X^{(2)} + \ldots, \\ \omega_A &= \omega^{(0)}_A + l\omega^{(1)}_A + l^2\omega^{(2)}_A + \ldots, \\ h_{AB} &= h^{(0)}_{AB} + lh^{(1)}_{AB} + l^2 h^{(2)}_{AB} + \ldots. \end{aligned} \tag{14}$$

(4) Any spacelike cut C of the horizon is compact and simply connected. The second assumption implies every closed 1-form on C is exact. These assumptions

hold, e.g., if C has spherical S^{d-2} topology with $d \geq 4$ but not, e.g., if C has the topology of a black ring $S^1 \times S^{d-3}$.

B. Statement of generalized zeroth law

The surface gravity of the horizon $\mathcal{N}$ of a stationary black hole is defined by

$$\xi^\beta \nabla_\beta \xi_\alpha \Big|_{\mathcal{N}} = \kappa \xi_\alpha. \tag{15}$$

The zeroth law of black hole mechanics is the statement that κ is constant on $\mathcal{N}$. One can compute both sides of this equation in the Killing vector GNCs of Sec. III B and find

$$\kappa = \frac{1}{2} X(\rho, x^C)\Big|_{\rho=0}. \tag{16}$$

From this we see that κ is clearly independent of τ. Therefore, to prove the zeroth law we must show that

$$\partial_A X(\rho, x^C)\Big|_{\rho=0} = 0. \tag{17}$$

When electromagnetic fields are included in a black hole theory, the zeroth law is usually generalized to include a statement about their behavior on the horizon. Our "generalized zeroth law" formulation is

$$\partial_A X(\rho, x^C)\Big|_{\rho=0} = 0, \quad \text{and} \quad F_{\tau A}(\rho, x^C)\Big|_{\rho=0} = 0. \tag{18}$$

The interpretation of the second condition can be seen as follows. By the Cartan formula, $\mathcal{L}_\xi F = \mathrm{d}(\iota_\xi F) + \iota_\xi \mathrm{d}F$. $\mathrm{d}F = 0$ because F is a Maxwell field. We have also assumed $\mathcal{L}_\xi F = 0$ above. Hence $\mathrm{d}(\iota_\xi F) = 0$, and so, at least locally, $\iota_\xi F = \mathrm{d}\Phi$ for some scalar Φ. This scalar is the "electric potential" from the definition of the first law. The condition $F_{\tau A}|_{\rho=0} = 0$ is then equivalent to $\partial_A \Phi|_{\rho=0} = 0$, which says that the electric potential is constant on the horizon.

In the course of the proof, we will see that the two conditions in (18) are not independent. In fact, we will need to show $F_{\tau A}\Big|_{\rho=0} = 0$ in order to prove $\partial_A X\Big|_{\rho=0} = 0$.

C. Plan of the proof

In [5], Bhattacharyya *et al.* prove the zeroth law for gravitational EFTs without matter. Here we generalize their method to apply to our Einstein-Maxwell-scalar EFT. The gravitational parts go through largely unchanged, while additional steps are needed to deal with the Maxwell and scalar fields. Here, we sketch the main ideas of the proof.

Let Φ_I denote the collection of fields $(g_{\mu\nu}, F_{\mu\nu}, \phi)$. We have assumed we can write this as a series in l: $\Phi_I = \Phi_I^{(0)} + l\Phi_I^{(1)} + l^2\Phi_I^{(2)} + \cdots$. Let $E_I[\Phi_J]$ denote the collection of variations of the action S defined in (5). The equations of motion are

$$E_I[\Phi_J] \equiv E_I^{(0)}[\Phi_J] + \sum_{n=1}^{\infty} l^n E_I^{(n)}[\Phi_J] = 0, \tag{19}$$

where $l^n E_I^{(n)}[\Phi_J]$ comes from varying $l^n \mathcal{L}_{n+2}$. This equation must hold to each order in l individually. The order l^0 part is simply

$$E_I^{(0)}[\Phi_J^{(0)}] = 0. \tag{20}$$

We will show that in Killing vector GNCs on the horizon, $E_{\tau A}^{(0)}$ and $E_{\tau\tau}^{(0)}$ evaluate to

$$\begin{aligned} E_{\tau A}^{(0)}[\Phi_J]\Big|_{\rho=0} &= -\frac{1}{2}\partial_A X - \frac{1}{2} c_1(\phi)(F_{AB}h^{BC} - F_{\tau\rho}\delta_A^C)F_{\tau C}, \\ E_{\tau\tau}^{(0)}[\Phi_J]\Big|_{\rho=0} &= -\frac{1}{2} c_1(\phi) F_{\tau A} F_{\tau B} h^{AB}. \end{aligned} \tag{21}$$

From the first component, we see that $E_{\tau A}^{(0)}[\Phi_J^{(0)}]\Big|_{\rho=0} = 0$ implies $\partial_A X^{(0)}\Big|_{\rho=0} = 0$ if $F_{\tau A}^{(0)}\Big|_{\rho=0} = 0$. But from the second component, $E_{\tau\tau}^{(0)}[\Phi_J^{(0)}]\Big|_{\rho=0} = 0$ implies $F_{\tau A}^{(0)}\Big|_{\rho=0} = 0$ because $h^{(0)AB}$ is positive definite and we assumed $c_1 > 0$. Thus, the generalized zeroth law holds to zeroth order in l.

The proof will then proceed by induction. We will assume that $\partial_A X^{(n)}|_{\rho=0} = 0$ and $F_{\tau A}^{(n)}\Big|_{\rho=0} = 0$ for $n < k$, and then prove that $\partial_A X^{(k)}|_{\rho=0} = 0$ and $F_{\tau A}^{(k)}\Big|_{\rho=0} = 0$.

To do this, we will consider the order l^k part of two components of (19). In a similar fashion to Bhattacharyya *et al.*, we will show that $E_{\tau A}[\Phi_J]$ greatly simplifies on the horizon, regardless of the higher order terms in the EFT. In particular, its order l^k part is of the following form:

$$\text{at order } l^k, \quad E_{\tau A}[\Phi_J]\big|_{\rho=0} = -\frac{1}{2} l^k \partial_A X^{(k)} + l^k M_A{}^C F_{\tau C}^{(k)} = 0, \tag{22}$$

where $M_A{}^C$ is a function only of the lowest order fields $\Phi_I^{(0)}$. From this we can see that the two statements in the generalized zeroth law are not independent: if we can show that $F_{\tau A}^{(k)}\Big|_{\rho=0} = 0$, then we immediately have $\partial_A X^{(k)}\Big|_{\rho=0} = 0$.

To do this, final step we must look at another component of the equation of motion. We will show that the order l^k part of $E_\tau[\Phi_J]|_{\rho=0}$ can be brought into the following form:

$$\text{at order } l^k, \quad E_\tau[\Phi_J]\big|_{\rho=0} = l^k \mathcal{D}_A^{(0)}\left[N^{AB} F_{\tau B}^{(k)}\right] = 0, \tag{23}$$

where N^{AB} is a function only of the lowest order fields $\Phi_I^{(0)}$, and $\mathcal{D}_A^{(0)}$ is the covariant derivative with respect to $h_{AB}^{(0)}$. We will show that this equation has no nontrivial solutions for $F_{\tau A}^{(k)}\big|_{\rho=0}$ if every closed 1-form on $C(\tau)$ is exact. This condition follows from our assumptions on the topology of $C(\tau)$, and thus the generalized zeroth law is proved in this case.

In order to simplify $E_{\tau A}[\Phi_J]|_{\rho=0}$ and $E_{\tau}[\Phi_J]|_{\rho=0}$ to the forms in (22) and (23), we will need to prove a crucial fact: the generalized zeroth law implies[5] that all positive boost weight, gauge-independent quantities vanish on the horizon $\mathcal{N}$. In [5], it is shown that a relation between Killing vector GNCs and affinely parametrized GNCs can be used to show the zeroth law implies that positive boost weight quantities built only out of metric components vanish on $\mathcal{N}$. In Sec. V C we will show this relation can also be applied to positive boost weight quantities built out of the Maxwell field $F_{\mu\nu}$ and the scalar field ϕ.

V. PROOF OF THE GENERALIZED ZEROTH LAW

A. The base case: The generalized zeroth law for two-derivative Einstein-Maxwell-scalar theory

The first step in our proof will be to show that the generalized zeroth law holds at lowest order in l, i.e., that

$$\partial_A X^{(0)}\Big|_{\rho=0} = 0, \quad \text{and} \quad F_{\tau A}^{(0)}\Big|_{\rho=0} = 0. \tag{24}$$

This is equivalent to proving the generalized zeroth law for the two-derivative Einstein-Maxwell-scalar Lagrangian $\mathcal{L}_2$ given in (3). The original proof of the zeroth law by Bardeen *et al.* [2] would achieve this, because the two-derivative theory satisfies the dominant energy condition up to parts proportional to the metric, as discussed above. Here we give a reformulation of the proof that motivates many of the steps used in the later proof for full Einstein-Maxwell-scalar EFT.

We proceed by studying $E_I^{(0)}[\Phi_J]$, which is the part of the equation of motion arising from $\mathcal{L}_2$. Rewriting here for convenience, the $(\alpha\beta)$ component is

$$\begin{aligned} E_{\alpha\beta}^{(0)}[\Phi_J] = {} & R_{\alpha\beta} - \frac{1}{2}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}c_1(\phi)F_{\alpha\delta}F_\beta{}^\delta \\ & - \frac{1}{2}g_{\alpha\beta}\left(R - V(\phi) - \frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi - \frac{1}{4}c_1(\phi)F_{\gamma\delta}F^{\gamma\delta}\right). \end{aligned} \tag{25}$$

We will evaluate two components in Killing vector GNCs on the horizon. First we will evaluate $E_{\tau A}^{(0)}[\Phi_J]|_{\rho=0}$. The Ricci component $R_{\tau A}|_{\rho=0}$ is evaluated in [5],

$$R_{\tau A}|_{\rho=0} = -\frac{1}{2}\partial_A X\Big|_{\rho=0}. \tag{26}$$

The second term $\nabla_\tau\phi\nabla_A\phi$ vanishes because we assumed the scalar field is invariant under the Killing vector ξ, which implied $\partial_\tau\phi = 0$. The third term $c_1(\phi)F_{\tau\gamma}F_{A\delta}g^{\gamma\delta}$ simplifies on the horizon where $g^{\gamma\delta}$ is particularly straightforward,

$$\begin{aligned} c_1(\phi)F_{\tau\gamma}F_{A\delta}g^{\gamma\delta}|_{\rho=0} &= c_1(\phi)(F_{\tau\rho}F_{A\tau} + F_{\tau C}F_{AB}h^{BC}) \\ &= c_1(\phi)(F_{AB}h^{BC} - F_{\tau\rho}\delta_A^C)F_{\tau C}. \end{aligned} \tag{27}$$

The final bracketed term also vanishes on the horizon because the prefactor $g_{\tau A}|_{\rho=0} = 0$. This leaves us with the first equation from (21), and as discussed, we can substitute this into the order l^0 part of the equation of motion $E_I^{(0)}[\Phi_J^{(0)}] = 0$ to get that $\partial_A X^{(0)}|_{\rho=0} = 0$ if $F_{\tau C}^{(0)}|_{\rho=0} = 0$.

In pursuit of proving $F_{\tau C}^{(0)}|_{\rho=0} = 0$, let us now evaluate another component on the horizon, $E_{\tau\tau}^{(0)}[\Phi_J]|_{\rho=0}$. The first term is $R_{\tau\tau}|_{\rho=0} = R_{\mu\nu}\xi^\mu\xi^\nu|_{\mathcal{N}}$ which vanishes by the Raychaudhuri equation. The second term $\nabla_\tau\phi\nabla_\tau\phi$ once again vanishes by $\partial_\tau\phi = 0$. The third term is

$$c_1(\phi)F_{\tau\gamma}F_{\tau\delta}g^{\gamma\delta}\Big|_{\rho=0} = c_1(\phi)F_{\tau A}F_{\tau B}h^{AB}. \tag{28}$$

The bracketed term vanishes again since $g_{\tau\tau}|_{\rho=0} = 0$. Therefore, we retrieve the second equation from (21), which we can substitute into the l^0 part of the equation of motion to get

$$c_1(\phi^{(0)})F_{\tau A}^{(0)}F_{\tau B}^{(0)}h^{(0)AB} = 0. \tag{29}$$

Now, h_{AB} is the induced metric on the spacelike cut $C(\tau)$, therefore it is positive definite. This implies $h_{AB}^{(0)}$ is also positive definite since $h_{AB}^{(0)} = h_{AB}|_{l=0}$. Furthermore, we assumed $c_1 > 0$. Therefore, (29) implies $F_{\tau A}^{(0)}|_{\rho=0} = 0$, and so we have proved the generalized zeroth law to order l^0.

B. The inductive step

We prove the generalized zeroth law to all orders in l by induction. Let us assume $\partial_A X^{(n)}|_{\rho=0} = 0$ and $F_{\tau A}^{(n)}|_{\rho=0} = 0$ for $n < k$. We now aim to prove that $\partial_A X^{(k)}|_{\rho=0} = 0$ and $F_{\tau A}^{(k)}\Big|_{\rho=0} = 0$.

To do this, let us consider the order l^k part of the full equation of motion (19). This is not simply $l^k E_I^{(k)}[\Phi_J]$ because Φ_J is itself a series in l. However, it will certainly not depend on $E_I^{(n)}$ or $\Phi_J^{(n)}$ for $n > k$ because they come with too high powers of l. Let us introduce the notation

[5] In case it is confusing why we will assume the generalized zeroth law while in the middle of proving it, we will be using this result up to and including order l^{k-1} to complete an inductive loop at order l^k.

$$f^{[n]} = \sum_{m=0}^{n} l^m f^{(n)}. \tag{30}$$

Then the order l^k part of (19) will be a subset of the terms in $E_I^{[k]}[\Phi_J^{[k]}]$. Furthermore, since $\Phi_J^{(k)}$ already comes with a factor l^k, the only place $\Phi_J^{(k)}$ can appear is in $E_I^{(0)}[\Phi_J^{[k]}]$. In particular, it will appear as $E_I^{(0)}[\Phi_J^{(0)} + l^k\Phi_J^{(k)}]$ linearized around the background $\Phi_J^{(0)}$. Therefore, we can write the order l^k part of (19) as follows:

at order l^k,

$$E_I[\Phi_J] = E_I^{[k]}[\Phi_J^{[k-1]}] + l^k\left(\Phi_J^{(k)}\frac{\delta E_I^{(0)}}{\delta\Phi_J}[\Phi_J^{(0)}] + \partial_\mu\Phi_J^{(k)}\frac{\delta E_I^{(0)}}{\delta(\partial_\mu\Phi_J)}[\Phi_J^{(0)}] + \cdots\right), \tag{31}$$

where it is given that we only take order l^k terms in the first term, and the bracketed term is $E_I^{(0)}[\Phi_J^{(0)} + l^k\Phi_J^{(k)}]$ linearized around $\Phi_J^{(0)}$. Setting this to zero allows us to solve for the fields Φ_J order by order in l: once we have solved for $\Phi_J^{(0)}$ we can study (31) at order l to solve for $\Phi_J^{(1)}$, then at order l^2 to solve for $\Phi_J^{(2)}$ and so on.

It is difficult to study (31), however, because we do not know the form of $E_I^{[k]}$. It comes from the variation of the higher derivative parts of our EFT Lagrangian, which in theory could take a variety of forms. The only part we do know the form of is $E_I^{(0)}$. Therefore, we would like to find a scenario where the dependence on the unknown $E_I^{[k]}$ vanishes.

It turns out that, on the horizon, certain components of $E_I^{[k]}[\Phi_J^{[k-1]}]$ do necessarily vanish by our inductive hypothesis. In particular, $E_{\tau A}^{[k]}[\Phi_J^{[k-1]}]|_{\mathcal{N}} = 0$ and $E_\tau^{[k]}[\Phi_J^{[k-1]}]|_{\mathcal{N}} = 0$. The proof of these are left to the next two sections. In short, it follows from the fact that they are proportional to positive boost weight components in affinely parametrized GNCs. It will be shown in Sec. V C that positive boost weight quantities vanish on the horizon if the generalized zeroth law holds. But by our inductive hypothesis, the generalized zeroth law holds for the fields $\Phi_J^{[k-1]}$, which are all that $E_{\tau A}^{[k]}[\Phi_J^{[k-1]}]$ and $E_\tau^{[k]}[\Phi_J^{[k-1]}]$ depend on.

Let us assume for now that these components do indeed vanish on the horizon. Then at order l^k, $E_{\tau A}[\Phi_J]|_{\rho=0}$ and $E_\tau[\Phi_J]|_{\rho=0}$ are simply given by $E_{\tau A}^{(0)}[\Phi_J^{(0)} + l^k\Phi_J^{(k)}]|_{\rho=0}$ and $E_\tau^{(0)}[\Phi_J^{(0)} + l^k\Phi_J^{(k)}]|_{\rho=0}$ linearized around $\Phi_I^{(0)}$.

During the proof of the base case, we calculated

$$E_{\tau A}^{(0)}[\Phi_J]\Big|_{\rho=0} = -\frac{1}{2}\partial_A X - \frac{1}{2}c_1(\phi)(F_{AB}h^{BC} - F_{\tau\rho}\delta_A^C)F_{\tau C}. \tag{32}$$

We can now replace the fields $X = X^{(0)} + l^k X^{(k)}$, $F_{\mu\nu} = F_{\mu\nu}^{(0)} + l^k F_{\mu\nu}^{(k)}$, etc., linearize around the order l^0 fields $X^{(0)}$, $F_{\mu\nu}^{(0)}$, etc., and use that $F_{\tau C}^{(0)}|_{\rho=0} = 0$ to get that,

at order l^k,

$$E_{\tau A}[\Phi_J]\Big|_{\rho=0} = -\frac{1}{2}l^k\partial_A X^{(k)} - \frac{1}{2}c_1(\phi^{(0)})(F_{AB}^{(0)}h^{(0)BC} - F_{\tau\rho}^{(0)}\delta_A^C)l^k F_{\tau C}^{(k)} = 0. \tag{33}$$

Therefore,

$$\partial_A X^{(k)}\Big|_{\rho=0} = M_A{}^C F_{\tau C}^{(k)}\Big|_{\rho=0}, \tag{34}$$

where $M_A{}^C = -c_1(\phi^{(0)})\left(F_{AB}^{(0)}h^{(0)BC} - F_{\tau\rho}^{(0)}\delta_A^C\right)$ is a function only of the lowest order fields $\Phi_I^{(0)}$. Once again we see that $\partial_A X^{(k)}|_{\rho=0} = 0$ if $F_{\tau C}^{(k)}|_{\rho=0} = 0$.

We now turn to $E_\tau^{(0)}[\Phi_J]$ in order to prove $F_{\tau C}^{(k)}|_{\rho=0} = 0$. Rewriting from above, it is given by

$$E_\tau^{(0)}[\Phi_J] = \nabla^\beta\left[c_1(\phi)F_{\tau\beta} - 4c_2(\phi)F^{\gamma\delta}\epsilon_{\tau\beta\gamma\delta}\right]. \tag{35}$$

We can again evaluate this on the horizon in Killing vector GNCs. The calculation involves evaluating Christoffel symbols and is given in Appendix A 1 as an example of using Killing vector GNCs. The result is

$$E_\tau^{(0)}[\Phi_J]\Big|_{\rho=0} = h^{AB}\mathcal{D}_A\left[c_1(\phi)F_{\tau B} - 8c_2(\phi)\epsilon_B{}^C F_{\tau C}\right], \tag{36}$$

where $\mathcal{D}_A$ is the covariant derivative with respect to h_{AB}. We again linearize all the fields around the background $\Phi_J^{(0)}$ and use $F_{\tau C}^{(0)}|_{\rho=0} = 0$ to get,

at order l^k,

$$E_\tau[\Phi_J]\Big|_{\rho=0} = l^k h^{(0)AB}\mathcal{D}_A^{(0)} \times\left[c_1(\phi^{(0)})F_{\tau B}^{(k)} - 8c_2(\phi^{(0)})\epsilon_B^{(0)C}F_{\tau C}^{(k)}\right] = 0, \tag{37}$$

where $\mathcal{D}^{(0)}$ is the covariant derivative with respect to $h_{AB}^{(0)}$. For each τ, this is a differential equation for $F_{\tau C}^{(k)}|_{\rho=0}$ on the spacelike 2-slice $C(\tau)$. We will show that it has no nontrivial solutions.

Note that since $F_{\alpha\beta}$ is a Maxwell field, it satisfies $\partial_{[\alpha}F_{\beta\gamma]} = 0$. We can combine this with $\partial_\tau F_{\alpha\beta} = 0$ to get

$$\partial_{[\tau}F_{AB]} = 0 \Longrightarrow \partial_A F_{\tau B} - \partial_B F_{\tau A} = 0. \tag{38}$$

This must hold to all orders in l, so

$$\partial_A F^{(k)}_{\tau B} - \partial_B F^{(k)}_{\tau A} = 0. \tag{39}$$

Now, we can view $F^{(k)}_{\tau A}|_{\rho=0}$ as a 1-form on the submanifold $C(\tau)$. Call this 1-form $V_A = F^{(k)}_{\tau A}|_{\rho=0}$. Then Eq. (39) becomes

$$\mathrm{d}V = 0, \tag{40}$$

where d is the exterior derivative on $C(\tau)$. We assumed that every closed 1-form on $C(\tau)$ is exact, and hence there exists some function f on the whole of $C(\tau)$ such that

$$V = \mathrm{d}f, \tag{41}$$

where f is essentially just the order l^k part of the electric potential. The crucial thing is that we know we can define f on the whole of $C(\tau)$, whereas we could not necessarily define the electric potential globally.

We can substitute this into (37) to get

$$h^{(0)AB}\mathcal{D}^{(0)}_A\left[c_1(\phi^{(0)})\mathcal{D}^{(0)}_B f - 8c_2(\phi^{(0)})\epsilon^{(0)C}_B\mathcal{D}^{(0)}_C f\right] = 0. \tag{42}$$

We now integrate this against $\sqrt{h^{(0)}}f$ over $C(\tau)$,

$$\int_{C(\tau)} \mathrm{d}^{d-2}x\sqrt{h^{(0)}}f h^{(0)AB}\mathcal{D}^{(0)}_A \times\left[c_1(\phi^{(0)})\mathcal{D}^{(0)}_B f - 8c_2(\phi^{(0)})\epsilon^{(0)C}_B\mathcal{D}^{(0)}_C f\right] = 0. \tag{43}$$

Apply the divergence theorem to get

$$-\int_{C(\tau)} \mathrm{d}^{d-2}x\sqrt{h^{(0)}}\Big[c_1(\phi^{(0)})h^{(0)AB}\mathcal{D}^{(0)}_A f\mathcal{D}^{(0)}_B f - 8c_2(\phi^{(0)})\epsilon^{(0)AB}\mathcal{D}^{(0)}_A f\mathcal{D}^{(0)}_B f\Big] = 0, \tag{44}$$

where the boundary term vanished because we assumed $C(\tau)$ was compact. The second term in the brackets is 0 by the antisymmetry of $\epsilon^{(0)AB}$, which just leaves the first term. Since $h^{(0)}_{AB}$ is positive definite and $c_1 > 0$, for the integral to be 0 we must have $\mathcal{D}^{(0)}_A f = 0$, or equivalently,

$$F^{(k)}_{\tau A}\Big|_{\rho=0} = 0. \tag{45}$$

Therefore, the inductive step is proven.

Now all that remains is to fill the gap in our proof and show that $E^{[k]}_{\tau A}[\Phi^{[k-1]}_J]|_{\mathcal{N}} = 0$ and $E^{[k]}_{\tau}[\Phi^{[k-1]}_J]|_{\mathcal{N}} = 0$. To do this, we will need to prove a statement about positive boost weight quantities on the horizon.

C. Positive boost weight quantities on the horizon

Following the method of [5], we will prove that the generalized zeroth law implies that all positive boost weight, electromagnetic gauge-independent quantities vanish on the horizon. Somewhat counterintuitively, this will allow us to complete the inductive step and prove the generalized zeroth law itself.

The boost weight of a quantity (defined in Sec. III A 1) is determined in the affinely parametrized GNCs of Sec. III A. The most basic electromagnetic gauge-independent quantities we can make from the metric and matter fields in these coordinates are of the form $\partial_{A_1}\ldots\partial_{A_n}\partial_r^p\partial_v^q\varphi$ with $\varphi\in\{\alpha,\beta_A,\mu_{AB},\mu^{AB},F_{vr},F_{AB},F_{vA},F_{rA},\phi\}$. We call such terms "building blocks." On the horizon $r=0$, all quantities in our theory can be expanded out as expressions in building blocks,[6] e.g., $\nabla_v F_{rA}|_{\rho=0} = \partial_v F_{rA} - \frac{1}{2}F_{AB}\beta_C\mu^{BC} - \frac{1}{2}F_{rB}\partial_v\mu_{AC}\mu^{BC} - \frac{1}{2}F_{rv}\beta_A$.

The boost weights of these building blocks are as follows:

(i) F_{vA} has boost weight $+1$.
(ii) α, β_A, μ_{AB}, μ^{AB}, F_{vr}, F_{AB}, and ϕ have boost weight 0.
(iii) F_{rA} has boost weight -1.
(iv) ∂_v derivatives each add $+1$ to the boost weight, ∂_r derivatives each add -1, and ∂_A derivatives add 0.

Therefore, positive boost weight building blocks are of the form

$$\partial_{A_1}\ldots\partial_{A_n}\partial_r^p\partial_v^q\varphi \text{ with } \varphi\in\{F_{vA},\partial_v\alpha,\partial_v\beta_A,\partial_v\mu_{AB},\partial_v\mu^{AB},\partial_v F_{vr},\partial_v F_{AB},\partial_v\phi,\partial_v^2F_{rA}\} \text{ and } q\geq p. \tag{46}$$

The terms in the expansion of a positive boost weight quantity on the horizon must all have at least one factor of the positive boost weight building blocks listed above. Therefore, if we can show that all positive boost weight building blocks vanish on the horizon, then we have shown that all positive boost weight quantities vanish on the horizon.

To do this, we shall employ a relation between affinely parametrized GNCs and Killing vector GNCs.

[6] There is no explicit appearance of the coordinates (v, x^A) because they do not appear explicitly in the metric.

Let us assume the generalized zeroth law holds. Then $F_{\tau A}|_{\rho=0} = 0$ and $X|_{\rho=0} = 2\kappa$ with κ constant. By smoothness,

$$X(\rho, x^C) = 2\kappa + \rho f(\rho, x^C), \qquad F_{\tau A}(\rho, x^C) = \rho f_A(\rho, x^C), \tag{47}$$

where $f(\rho, x^C)$ and $f_A(\rho, x^C)$ are regular on the horizon. Then, in Killing vector GNCs, $(g_{\alpha\beta}, F_{\alpha\beta}, \phi)$ are given by (with x^C dependence suppressed)

$$\begin{aligned} g &= 2\mathrm{d}\tau\mathrm{d}\rho - [2\kappa\rho + \rho^2 f(\rho)]\mathrm{d}\tau^2 + 2\rho\omega_A(\rho)\mathrm{d}x^A\mathrm{d}\tau + h_{AB}(\rho)\mathrm{d}x^A\mathrm{d}x^B, \\ F &= F_{\tau\rho}(\rho)\mathrm{d}\tau \wedge \mathrm{d}\rho + F_{\rho A}(\rho)\mathrm{d}\rho \wedge \mathrm{d}x^A + \rho f_A(\rho)\mathrm{d}\tau \wedge \mathrm{d}x^A + F_{AB}(\rho)\mathrm{d}x^A \wedge \mathrm{d}x^B, \\ \phi &= \phi(\rho). \end{aligned} \tag{48}$$

We now make the coordinate transformation[7]

$$\rho = r(\kappa v + 1), \qquad \tau = \frac{1}{\kappa}\log(\kappa v + 1). \tag{49}$$

with the x^C coordinates unchanged. In these new coordinates, the horizon is $r = 0$ and C is $v = r = 0$. The transformation also has the effect of putting the metric in affinely parametrized form,

$$\begin{aligned} g &= 2\mathrm{d}v\mathrm{d}r - r^2 f(r(\kappa v + 1))\mathrm{d}v^2 + 2r\omega_A(r(\kappa v + 1))\mathrm{d}v\mathrm{d}x^A + h_{AB}(r(\kappa v + 1))\mathrm{d}x^A\mathrm{d}x^B, \\ F &= F_{\tau\rho}(r(\kappa v + 1))\mathrm{d}v \wedge \mathrm{d}r + (\kappa v + 1)F_{\rho A}(r(\kappa v + 1))\mathrm{d}r \wedge \mathrm{d}x^A \\ &\quad + r\Big[\kappa F_{\rho A}(r(\kappa v + 1)) + f_A(r(\kappa v + 1))\Big]\mathrm{d}v \wedge \mathrm{d}x^A + F_{AB}(r(\kappa v + 1))\mathrm{d}x^A \wedge \mathrm{d}x^B, \\ \phi &= \phi(r(\kappa v + 1)). \end{aligned} \tag{50}$$

Thus, the (r, v, x^C) are a choice of affinely parametrized GNCs with (again suppressing x^C dependence)

$$\begin{aligned} \alpha(r, v) &= f(r(\kappa v + 1)), \qquad \beta_A(r, v) = -\omega_A(r(\kappa v + 1)), \\ \mu_{AB}(r, v) &= h_{AB}(r(\kappa v + 1)), \qquad \mu^{AB}(r, v) = h^{AB}(r(\kappa v + 1)), \\ F_{vr}(r, v) &= F_{\tau\rho}(r(\kappa v + 1)), \qquad F_{AB}(r, v) = F_{AB}(r(\kappa v + 1)), \\ F_{rA}(r, v) &= (\kappa v + 1)F_{\rho A}(r(\kappa v + 1)), \\ F_{vA}(r, v) &= r\Big[\kappa F_{\rho A}(r(\kappa v + 1)) + f_A(r(\kappa v + 1))\Big], \\ \phi(r, v) &= \phi(r(\kappa v + 1)). \end{aligned} \tag{51}$$

The importance of this is that the v dependence of these quantities is severely restricted by the τ independence of the original Killing vector GNC quantities. The zero boost weight quantities α, β_A, μ_{AB}, μ^{AB}, F_{vr}, F_{AB}, and ϕ depend on v strictly through the combination rv. Therefore, taking a ∂_v of these quantities will produce an overall factor of r, which vanishes on the horizon. The positive boost weight quantity F_{vA} already has a prefactor of r and also depends on v strictly through rv. Finally,

$$\begin{aligned} \partial_v^2 F_{rA} = r[&2\kappa\partial_\rho F_{\rho A}(r(\kappa v + 1)) \\ &+ r\kappa(\kappa v + 1)\partial_\rho^2 F_{\rho A}(r(\kappa v + 1))]. \end{aligned} \tag{52}$$

Thus, we can write all of the quantities $\varphi \in \{F_{vA}, \partial_v\alpha, \partial_v\beta_A, \partial_v\mu_{AB}, \partial_v\mu^{AB}, \partial_v F_{vr}, \partial_v F_{AB}, \partial_v\phi, \partial_v^2 F_{rA}\}$ from (46) in the form

$$\varphi = r f_\varphi(r(\kappa v + 1)). \tag{53}$$

Taking a $(\partial_r\partial_v)$ derivative preserves this form, as does taking ∂_A derivatives. Thus, every positive boost weight building block satisfies

$$\begin{aligned} \partial_{A_1}\ldots\partial_{A_n}\partial_r^p\partial_v^q\varphi &= \partial_v^{q-p}[\partial_{A_1}\ldots\partial_{A_n}(\partial_r\partial_v)^p\varphi] \\ &= \partial_v^{q-p}\Big[r f_{\partial_{A_1}\ldots\partial_{A_n}(\partial_r\partial_v)^p\varphi}(r(\kappa v + 1))\Big] \\ &\propto r^{1+q-p}, \end{aligned} \tag{54}$$

[7] Note this is slightly different from the choice in [5] in that we have $(\kappa v + 1)$, where they have κv. We have added the 1 so that $v = 0$ corresponds to $\tau = 0$ and also to put it in such a form that if the black hole is extremal, i.e., $\kappa = 0$, then the transformation is the identity $\rho = r$, $\tau = v$.

with $q \geq p$. Therefore, all positive boost weight building blocks vanish on the horizon $r = 0$.

This proves that all positive boost weight quantities vanish on the horizon in the choice of affinely parametrized GNCs given by the transformation (49). However, as discussed in Sec. III A, there are infinitely many choices of such GNCs, all related by $v' = v/a(x^A)$ on $\mathcal{N}$ for some arbitrary function $a(x^A) > 0$. To prove that positive boost weight quantities vanish on the horizon in all choices of affinely parametrized GNCs, we use Lemma 2.1 of [9], which states that on the horizon a quantity of certain boost weight in (r, v, x^A) GNCs can be written as the sum of terms of the same boost weight in (r', v', x^A) GNCs. This means that if all positive boost weight quantities vanish on the horizon in one choice of affinely parametrized GNCs, then they vanish in all choices of affinely parametrized GNCs.

D. Completion of the inductive step

We will now use the statement about positive boost weight quantities on the horizon to complete our inductive step by proving $E^{[k]}_{\tau A}[\Phi^{[k-1]}_J]|_{\mathcal{N}} = 0$ and $E^{[k]}_{\tau}[\Phi^{[k-1]}_J]|_{\mathcal{N}} = 0$.

Our inductive hypothesis was that $\partial_A X^{(n)}|_{\rho=0} = 0$ and $F^{(n)}_{\tau A}\Big|_{\rho=0} = 0$ for $n < k$. Therefore, the fields $\Phi^{[k-1]}_I \equiv \Phi^{(0)}_I + l\Phi^{(1)}_I + \cdots + l^{k-1}\Phi^{(k-1)}_I$ satisfy the generalized zeroth law. In particular, $X^{[k-1]}|_{\rho=0} = 2\kappa^{[k-1]}$ is constant, and we can make the coordinate transformation

$$\rho = r(\kappa^{[k-1]}v + 1), \qquad \tau = \frac{1}{\kappa^{[k-1]}}\log\left(\kappa^{[k-1]}v + 1\right) \tag{55}$$

to bring the fields $\Phi^{[k-1]}_I = (g^{[k-1]}_{\mu\nu}, F^{[k-1]}_{\mu\nu}, \phi^{[k-1]})$ into the affinely parametrized form of (50). Then, by the above proof, any positive boost weight quantity made out of $\Phi^{[k-1]}_I$ will vanish on the horizon in these coordinates. In particular,

$$E^{[k]}_{vA}[\Phi^{[k-1]}_J]\Big|_{r=0} = 0 \quad \text{and} \quad E^{[k]}_{v}[\Phi^{[k-1]}_J]\Big|_{r=0} = 0 \tag{56}$$

because they have boost weight $+1$. But we also know how $E^{[k]}_{\mu\nu}[\Phi^{[k-1]}_J]$ and $E^{[k]}_{\mu}[\Phi^{[k-1]}_J]$ transform under the change of coordinates (55) because they are tensors. The inverse coordinate transformation is

$$r = \rho e^{-\kappa^{[k-1]}\tau} \qquad v = \frac{1}{\kappa^{[k-1]}}\left(e^{\kappa^{[k-1]}\tau} - 1\right). \tag{57}$$

So,

$$\begin{aligned} E^{[k]}_{\tau A}\left[\Phi^{[k-1]}_J\right]|_{\rho=0} &= \frac{\partial\tilde{x}^\mu}{\partial\tau}\frac{\partial\tilde{x}^\nu}{\partial x^A}E^{[k]}_{\mu\nu}\left[\Phi^{[k-1]}_J\right]|_{r=0} \\ &= e^{\kappa^{[k-1]}\tau}E^{[k]}_{vA}\left[\Phi^{[k-1]}_J\right]|_{r=0} \\ &= 0. \end{aligned} \tag{58}$$

Similarly,

$$\begin{aligned} E^{[k]}_{\tau}[\Phi^{[k-1]}_J]|_{\rho=0} &= \frac{\partial\tilde{x}^\mu}{\partial\tau}E^{[k]}_{\mu}\left[\Phi^{[k-1]}_J\right]|_{r=0} \\ &= e^{\kappa^{[k-1]}\tau}E^{[k]}_{v}\left[\Phi^{[k-1]}_J\right]|_{r=0} \\ &= 0. \end{aligned} \tag{59}$$

This completes the proof of the generalized zeroth law.

E. The generalized zeroth law for a charged scalar field

This proof of the generalized zeroth law can be modified to apply to the EFT of gravity, electromagnetism, and a *charged* scalar field. In this scenario, we assume we have a global gauge potential A_μ with $F = dA$. The dynamical fields are $\Phi_I = (g_{\mu\nu}, A_\mu, \phi)$, the scalar field ϕ is complex with some charge λ, and A_μ and ϕ transform under an electromagnetic gauge transformation as

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu\chi, \qquad \phi \to \tilde{\phi} = e^{i\lambda\chi}\phi, \tag{60}$$

with χ an arbitrary real-valued function. We generalize our leading order Lagrangian to

$$\begin{aligned} \mathcal{L}_2 = R - V(|\phi|^2) - g^{\alpha\beta}(\mathfrak{D}_\alpha\phi)^*\mathfrak{D}_\beta\phi \\ -\frac{1}{4}c_1(|\phi|^2)F_{\alpha\beta}F^{\alpha\beta} + c_2(|\phi|^2)F_{\alpha\beta}F_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}, \end{aligned} \tag{61}$$

where $\mathfrak{D}_\alpha$ is the gauge covariant derivative $\mathfrak{D}_\alpha = \nabla_\alpha - i\lambda A_\mu$. $\mathcal{L}_2$ is invariant under the gauge transform (60). Since the charge λ adds a new scale into the theory, the EFT series is now a joint series in derivatives and powers of λ. We assume $\lambda \leq 1/L$ where L is a typical length scale of the solution, so that λ is comparable to a one-derivative term. This is reasonable if we want the classical approximation to be valid. The EFT Lagrangian is

$$\mathcal{L} = \mathcal{L}_2 + \sum_{n=1}^{\infty} l^n\mathcal{L}_{n+2}, \tag{62}$$

where the $\mathcal{L}_{n+2}$ contains all gauge-independent terms with $n+2$ derivatives or powers of λ.

The equations of motion are

$$E_I[\Phi_J] \equiv E^{(0)}_I[\Phi_J] + \sum_{n=1}^{\infty} l^n E^{(n)}_I[\Phi_J], \tag{63}$$

where the parts arising from the leading order theory, $E^{(0)}_I[\Phi_J]$, are

$$\begin{aligned} E^{(0)}_{\alpha\beta} = R_{\alpha\beta} - (\mathfrak{D}_{(\alpha}\phi)^*\mathfrak{D}_{\beta)}\phi - \frac{1}{2}c_1(|\phi|^2)F_{\alpha\delta}F^{\delta}_{\beta} \\ -\frac{1}{2}g_{\alpha\beta}\bigg(R - V(|\phi|^2) - g^{\gamma\delta}(\mathfrak{D}_\gamma\phi)^*\mathfrak{D}_\delta\phi \\ -\frac{1}{4}c_1(|\phi|^2)F_{\gamma\delta}F^{\gamma\delta}\bigg), \end{aligned} \tag{64}$$

$$E^{(0)}_\alpha = i\lambda[\phi^* \mathfrak{D}_\alpha\phi - \phi(\mathfrak{D}_\alpha\phi)^*] + \nabla^\beta\Big[c_1(|\phi|^2)F_{\alpha\beta} - 4c_2(|\phi|^2)F^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}\Big], \tag{65}$$

$$E^{(0)} = g^{\alpha\beta}\mathfrak{D}_\alpha\mathfrak{D}_\beta\phi - \phi V'(|\phi|^2) - \frac{1}{4}\phi c_1'(|\phi|^2)F_{\alpha\beta}F^{\alpha\beta} + \phi c_2'(|\phi|^2)F_{\alpha\beta}F_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}. \tag{66}$$

We again assume that we have a stationary black hole solution to these equations, with a Killing horizon $\mathcal{N}$. What needs more subtlety is our assumption of the invariance of the matter fields on the symmetry corresponding to the Killing vector $\xi = \frac{\partial}{\partial\tau}$. The assumption that ϕ and A_μ are independent of τ is no longer appropriate here because the conditions $\partial_\tau\phi = 0$ and $\partial_\tau A_\mu = 0$ are not invariant under an electromagnetic gauge transformation. Instead, we need to modify our notion of a symmetry of the system.

For the metric, a Killing vector symmetry corresponds to invariance under the diffeomorphism $\tau \to \tau + s$ for all s, i.e., $g_{\mu\nu}(\tau + s) = g_{\mu\nu}(\tau)$. This diffeomorphism can be viewed as a one-parameter coordinate gauge transformation of the metric, labeled by s. A complete gauge transformation of our matter fields ϕ and A_μ would be a *combined* diffeomorphism and electromagnetic gauge transformation. Hence, we assume the notion of symmetry for ϕ and A_μ is their invariance under a one-parameter combined diffeomorphism and electromagnetic gauge transformation. More precisely, given any fixed gauge of A_μ and ϕ, we assume there exists some one-parameter family of functions θ_s with $\theta_0 = 0$ such that

$$A_\mu(\tau + s) + \partial_\mu\theta_s = A_\mu(\tau), \quad e^{i\lambda\theta_s}\phi(\tau + s) = \phi(\tau) \tag{67}$$

for all s. We can take the derivative of this with respect to s and set $s = 0$ to obtain the conditions

$$\partial_\tau A_\mu = -\partial_\mu\Theta, \qquad \partial_\tau\phi = -i\lambda\Theta\phi, \tag{68}$$

where $\Theta = \frac{d\theta_s}{ds}\Big|_{s=0}$. These are the conditions[8] which constrain the τ dependence of A_μ and ϕ. Note the first condition implies $\partial_\tau F_{\mu\nu} = 0$, which was the condition we assumed in the real scalar field case.

Let us now make an electromagnetic gauge transformation of the form (60). Using (68), the τ dependence of $\tilde{A}_\mu$ and $\tilde{\phi}$ can be found to be

$$\partial_\tau\tilde{A}_\mu = -\partial_\mu(\Theta - \partial_\tau\chi), \qquad \partial_\tau\tilde{\phi} = -i\lambda(\Theta - \partial_\tau\chi)\tilde{\phi}. \tag{69}$$

[8]These conditions can be proved to be equivalent to the conditions assumed in (2.9) of [16] (a recent paper discussing stationary black hole solutions with charged scalar hair). The formulation above avoids the need to define the phase of ϕ however.

From this we see that conditions (68) are preserved under an electromagnetic gauge transformation, so long as we relabel $\tilde{\Theta} = \Theta - \partial_\tau\chi$. In particular, we can take $\chi = \int^\tau \Theta(\tau')d\tau'$ to find a gauge in which

$$\partial_\tau\tilde{A}_\mu = 0, \qquad \partial_\tau\tilde{\phi} = 0. \tag{70}$$

We will drop the tildes and work in this gauge to prove the generalized zeroth law. It must then hold in all gauges because the statements $\partial_A X|_{\rho=0} = 0$ and $F_{\tau A}|_{\rho=0} = 0$ are gauge independent. Note that $F_{\tau A} = -\partial_A A_\tau$ in this gauge. To prove $F_{\tau A}|_{\rho=0} = 0$, we will actually show that $A_\tau|_{\rho=0} = 0$ in this gauge.

The proof follows in a similar fashion to that of Einstein-Maxwell-scalar EFT above, with modifications to deal with the fact that A_μ can now appear outside of the gauge-independent combination $F_{\mu\nu}$. The relevant parts of the equation of motion will be $E^{(0)}_{\tau\tau}[\Phi_I]|_{\rho=0}$, $E^{(0)}_{\tau A}[\Phi_I]|_{\rho=0}$, and $E^{(0)}_\tau[\Phi_I]|_{\rho=0}$ as before. In this gauge they can be found to be

$$E^{(0)}_{\tau\tau}[\Phi_I]|_{\rho=0} = -\lambda^2 A_\tau^2|\phi|^2 - \frac{1}{2}c_1(|\phi|^2)F_{\tau A}F_{\tau B}h^{AB}, \tag{71}$$

$$E^{(0)}_{\tau A}[\Phi_I]|_{\rho=0} = -\frac{1}{2}\partial_A X - \frac{1}{2}i\lambda A_\tau(\phi^*\partial_A\phi - \phi\partial_A\phi^* - 2i\lambda A_A|\phi|^2) - \frac{1}{2}c_1(|\phi|^2)(F_{AB}h^{BC} - F_{\tau\rho}\delta^C_A)F_{\tau C}, \tag{72}$$

$$E^{(0)}_\tau[\Phi_I]|_{\rho=0} = 2\lambda^2 A_\tau|\phi|^2 + h^{AB}\mathcal{D}_A\Big[c_1(|\phi|^2)F_{\tau B} - 8c_2(|\phi|^2)\epsilon_B{}^C F_{\tau C}\Big]. \tag{73}$$

For simplicity, we will assume $\phi^{(0)}$ is not identically 0 on $\mathcal{N}$ in the following proof. The case $\phi^{(0)}|_{\rho=0} \equiv 0$ adds a variety of technical difficulties that stray from the main argument and is dealt with in Appendix A 2.

Let us first look at the equations of motion at order l^0. From $E^{(0)}_{\tau\tau}[\Phi^{(0)}_I]|_{\rho=0} = 0$, we get $A^{(0)}_\tau\phi^{(0)}|_{\rho=0} = 0$ and $F^{(0)}_{\tau A}|_{\rho=0} = 0$. But since $F^{(0)}_{\tau A} = -\partial_A A^{(0)}_\tau$, the latter condition means $A^{(0)}_\tau$ is constant on the horizon ($\partial_\tau A^{(0)}_\tau = 0$ in this gauge). Therefore, we can extract $A^{(0)}_\tau|_{\rho=0} = 0$ from the former condition because we are assuming $\phi^{(0)}$ is not identically 0 on the horizon. Plugging $A^{(0)}_\tau|_{\rho=0} = 0$ into $E^{(0)}_{\tau A}[\Phi^{(0)}_I]|_{\rho=0} = 0$, we get $\partial_A X^{(0)}|_{\rho=0} = 0$, and so the generalized zeroth law is proved at order l^0.

We now take our inductive hypothesis to be $\partial_A X^{(n)}|_{\rho=0} = 0$ and $A_\tau^{(n)}|_{\rho=0} = 0$ for $n < k$. We again decompose the order l^k part of $E_I[\Phi_J] = 0$ into $E_I^{[k]}[\Phi_J^{[k-1]}]$ plus $E_I^{(0)}[\Phi_J^{(0)} + l^k \Phi_J^{(k)}]$ linearized around $\Phi_J^{(0)}$.

$E_{\tau A}^{[k]}[\Phi_J^{[k-1]}]$ and $E_\tau^{[k]}[\Phi_J^{[k-1]}]$ can again be shown to vanish on the horizon by the statement that positive boost weight quantities vanish on the horizon if $\partial_A X|_{\rho=0} = 0$ and $A_\tau|_{\rho=0} = 0$ in this gauge. The proof of this statement follows exactly as in Sec. V C, except we must additionally show that positive boost weight quantities made from A_μ, such as A_v, $\partial_v A_A$, and $\partial_{vv} A_r$, vanish on the horizon. To do this, we again use the relation between Killing vector GNCs and affinely parametrized GNCs.

Using $\partial_\tau A_\mu = 0$, we can write A_μ in Killing vector GNCs as (suppressing x^C dependence)

$$A = A_\tau(\rho)\mathrm{d}\tau + A_\rho(\rho)\mathrm{d}\rho + A_A(\rho)\mathrm{d}x^A. \tag{74}$$

We transform to affinely parametrized GNCs, $\rho = r(\kappa v + 1)$, $\tau = \frac{1}{\kappa}\log(\kappa v + 1)$, to get

$$\begin{aligned} A_v(r, v) &= \frac{1}{\kappa v + 1} A_\tau(r(\kappa v + 1)) + \kappa r A_\rho(r(\kappa v + 1)), \\ A_r(r, v) &= (\kappa v + 1) A_\rho(r(\kappa v + 1)), \\ A_A(r, v) &= A_A(r(\kappa v + 1)). \end{aligned} \tag{75}$$

Positive boost weight quantities involving A_μ are given by $\partial_{A_1} \ldots \partial_{A_n} \partial_r^p \partial_v^q \varphi$ with $\varphi \in \{A_v, \partial_v A_A, \partial_v^2 A_r\}$ and $q \geq p$. It is easy to show that $\partial_v A_A$ and $\partial_v^2 A_r$ have the functional form $r f_\varphi(r(\kappa v + 1))$ which, as shown in Sec. V C, means their positive boost weight derivatives vanish on the horizon. $\partial_r \partial_v A_v$ also has this functional form, so if $q \geq p \geq 1$ then $\partial_{A_1} \ldots \partial_{A_n} \partial_r^p \partial_v^q A_v$ vanishes on the horizon. This leaves only terms with $p = 0$, however, one can show

$$\partial_v^q A_v|_{r=0} = \frac{(-\kappa)^q}{(\kappa v + 1)^{q+1}} A_\tau|_{\rho=0}. \tag{76}$$

Therefore, if $A_\tau|_{\rho=0} = 0$ then $\partial_{A_1} \ldots \partial_{A_n} \partial_v^q A_v$ also vanishes on the horizon.

Therefore, we can return to our inductive step and look at $E_I^{(0)}[\Phi_J^{(0)} + l^k \Phi_J^{(k)}]|_{\rho=0}$ linearized around $\Phi_J^{(0)}$ for $I = (\tau A)$ and $I = \tau$. For $E_{\tau A}^{(0)}[\Phi_I]|_{\rho=0}$, use $A_\tau^{(0)}|_{\rho=0} = 0$ to obtain,

$$\begin{aligned} \text{at order } l^k, \qquad E_{\tau A}[\Phi_J]\Big|_{\rho=0} = &-\frac{1}{2} l^k \partial_A X^{(k)} - \frac{1}{2} i l^k \lambda A_\tau^{(k)} \left(\phi^{(0)*}\partial_A \phi^{(0)} - \phi^{(0)} \partial_A \phi^{(0)*} - 2i\lambda A_A^{(0)} |\phi^{(0)}|^2\right) \\ &- \frac{1}{2} c_1(|\phi^{(0)}|^2) \left(F_{AB}^{(0)} h^{(0)BC} - F_{\tau\rho}^{(0)} \delta_A^C\right) l^k F_{\tau C}^{(k)} = 0, \end{aligned} \tag{77}$$

which implies that $\partial_A X^{(k)}|_{\rho=0} = 0$ if $A_\tau^{(k)}|_{\rho=0} = 0$.

Finally, for $E_\tau^{(0)}[\Phi_I]|_{\rho=0}$ use $A_\tau^{(0)}|_{\rho=0} = 0$ to get,

$$\text{at order } l^k, \qquad E_\tau[\Phi_J]\Big|_{\rho=0} = 2\lambda^2 l^k |\phi^{(0)}|^2 A_\tau^{(k)} + l^k h^{(0)AB} \mathcal{D}_A^{(0)} \left[c_1(|\phi^{(0)}|^2) F_{\tau B}^{(k)} - 8 c_2(|\phi^{(0)}|^2) \epsilon_B^{(0)C} F_{\tau C}^{(k)}\right] = 0. \tag{78}$$

Plugging in $F^{(k)} = -\partial_A A_\tau^{(k)}$ gives

$$2\lambda^2 |\phi^{(0)}|^2 A_\tau^{(k)} - h^{(0)AB} \mathcal{D}_A^{(0)} \left[c_1(|\phi^{(0)}|^2) \mathcal{D}_B^{(0)} A_\tau^{(k)} - 8 c_2(|\phi^{(0)}|^2) \epsilon_B^{(0)C} \mathcal{D}_C^{(0)} A_\tau^{(k)}\right] = 0. \tag{79}$$

Integrate this against $\sqrt{h^{(0)}} A_\tau^{(k)}$ over $C(\tau)$, integrate by parts, and again use the antisymmetry of $\epsilon^{(0)BC}$ to obtain

$$\int_{C(\tau)} \mathrm{d}^{d-2}x \sqrt{h^{(0)}} \left[2\lambda^2 l^k |\phi^{(0)}|^2 (A_\tau^{(k)})^2 + c_1(|\phi^{(0)}|^2) h^{(0)AB} \left(\partial_A A_\tau^{(k)}\right)\left(\partial_B A_\tau^{(k)}\right)\right] = 0. \tag{80}$$

Both terms in the integrand are manifestly non-negative, and hence $A_\tau^{(k)}|_{\rho=0} = 0$, once again using our assumption that $\phi^{(0)}$ is not identically 0 on the horizon.[9] This completes the induction.

Therefore, $\partial_A X|_{\rho=0} = 0$ and $F_{\tau A}|_{\rho=0} = 0$ in this gauge. But since these statements are gauge independent, the generalized zeroth law holds in all gauges for this charged scalar field EFT.

[9] $\partial_\tau \phi = 0$ in this gauge, so if $\phi^{(0)}$ is not identically 0 on the horizon, then it is also not identically 0 on any individual spatial cross section $C(\tau)$.

VI. THE SECOND LAW

The second law of black hole mechanics is the statement that the entropy of dynamical (i.e., nonstationary and therefore out of equilibrium) black hole solutions is nondecreasing in time. This is assumed to be the classical limit of the second law of thermodynamics that would say the thermodynamic entropy of the whole system is nondecreasing.

In standard two-derivative GR coupled to matter satisfying the null energy condition, it can be proved that the area $A(v)$ of a spacelike cross section of the horizon is always nondecreasing in v. This supports a natural interpretation of the entropy of a black hole as proportional to its area. However, when we include higher derivative terms in the metric or matter fields, $A(v)$ is no longer necessarily nondecreasing. Therefore, we need a generalization of the definition of entropy in order to satisfy a second law. While there has been no answer that applies to all situations, a fruitful avenue has been to study dynamical black holes that settle down to equilibrium at late times and that are in the regime of validity of EFT.

Here we extend the recent work of [8,9,12] to define an entropy that satisfies the second law up to an arbitrarily high order l^N in our EFT for Einstein-Maxwell-scalar theory.

A. Perturbations around stationary black holes

We consider the scenario of a black hole settling down to a stationary equilibrium. As such, we assume our dynamical black hole solution tends to some stationary black hole solution Φ_I^{st} at late times. In [8,9], the solution is assumed to be close to Φ_I^{st} and they consider perturbation theory around it. We will not need to do this and thus can include highly dynamical situations, so long as their horizons remain smooth for all future time (which is needed for us to define GNCs). Therefore, situations like the period after merger or gravitational collapse, or a black hole interacting with weak gravitational waves, are applicable.

However, the order of perturbation around Φ_I^{st} will still be an important concept to compare our definition to others. To make this concept precise, we use the statement proved in Sec. V C: positive boost weight quantities vanish on the horizon of a stationary black hole solution. Our construction of the entropy $S(v)$ will consist of manipulating affinely parametrized GNC quantities evaluated on the horizon, and so the number of factors of positive boost weight quantities determines the order of perturbation. For example, K_{AB} has boost weight $+1$ and so a term such as $K_{AB}K^{AB}$ is quadratic order.

To zeroth order, our entropy $S(v)$ will be the Wald entropy of the stationary solution Φ_I^{st}. This is constant in time, and so $\dot{S}(v) = 0$ to zeroth order.

To linear order, our entropy will be the one defined by Biswas *et al.* (BDK) in [8], where it is proved to be constant at linear order. Therefore, linearized around Φ_I^{st}, we have $\delta\dot{S}(v) = 0$.

This paper extends the BDK entropy by adding terms quadratic in positive boost weight quantities in a similar fashion to how Hollands *et al.* [9] and then Davies and Reall [12] extended the Iyer-Wald-Wall entropy [10,11]. We will show that such an entropy satisfies the second law *nonpertubatively*, i.e., $\dot{S}(v)$ is non-negative to all orders in perturbations around a black hole. However, this can only be done in the regime of validity of EFT.

B. Regime of validity of EFT

We shall not be interested in arbitrary black hole solutions of our Einstein-Maxwell-scalar EFT. In general, there will be pathological solutions that blow up in time or exhibit rapid oscillations and are considered unphysical. See Sec. IV of [17] for a discussion around the existence of such solutions, which should not be expected to satisfy the second law.

Instead, we shall consider only black hole solutions that lie within the "regime of validity of the EFT." This is defined in [9] as follows. We assume we have a one-parameter family of dynamical black hole solutions labeled by a length scale L (e.g., the size of the black hole or some other dynamical length/timescale) such that $\mathcal{N}$ is the event horizon for all members of the family (this is a gauge choice). We assume there exist affinely parametrized GNCs defined near $\mathcal{N}$ such that any quantity constructed from n derivatives of $\{\alpha, \beta_A, \mu_{AB}, \phi\}$ or $n-1$ derivatives of $F_{\mu\nu}$ is bounded by C_n/L^n for some constant C_n, and that $|V|L^2 \leq 1$. Then the solution lies within the regime of validity of EFT if $l/L << 1$. This definition captures the notion of a solution "varying over a length scale L" with L large compared to the UV scale l.

Note that we are no longer assuming the black hole solution is analytic in l, as we did in the proof of the zeroth law. This is not an applicable assumption in a dynamical situation because treating the solution as an expansion in l can typically lead to secular growth. See footnote 1 of [15] for an example of such a situation.

C. Order of the EFT

Our EFT action (4) is made up of potentially infinitely many higher derivative terms. In practice, we will only know finitely many of the coefficients of these terms, and so there will be some N for which we know all the terms with $N+1$ or fewer derivatives. In this case we only fully know part of the equations of motion, which satisfies

$$E_I^{[N-1]} \equiv E_I^{(0)} + \sum_{n=1}^{N-1} l^n E_I^{(n)} = O(l^N). \tag{81}$$

Since we only know our theory up to some accuracy of order l^N, it is reasonable to expect our second law to only be provable up to order l^N terms. This is indeed what we will show, i.e.,

$$\dot{S}(v) \geq -O(l^N), \tag{82}$$

where the rhs of the inequality signifies that $\dot{S}(v)$ might be negative but only by an $O(l^N)$ amount. This means that the better we know our EFT, the closer we can construct an entropy satisfying a complete second law. The entropy $S(v)$ will contain terms of up to $N-2$ derivatives.

D. Review of recent progress on the second law in vacuum gravity EFT

Before we jump into proving the second law for our Einstein-Maxwell-scalar EFT, we shall briefly review the recent progress in Einstein-scalar EFTs that we are building on. First, the Iyer-Wald-Wall entropy satisfies the second law to linear order in perturbations around a stationary black hole. Second, the extension made by HKR defines an entropy that satisfies the second law to quadratic order in perturbations, up to order l^N terms. Finally, the very recent work by Davies and Reall in the companion to this paper [12] adds extra terms to the HKR entropy, which result in the second law being satisfied nonperturbatively, up to order l^N terms.

The Iyer-Wald-Wall entropy was devised by Wall [10] as an improvement on the entropy defined by Iyer and Wald [18]. It was formalized by Bhattacharyya *et al.* [11]. It applies to any theory of gravity and a scalar field with diffeomorphism-invariant Lagrangian (under no EFT assumption). The approach is to use affinely parametrized GNCs and study the E_{vv} equation of motion. They prove that it can always be manipulated into the following form on the horizon:

$$-E_{vv}\Big|_{\mathcal{N}} = \partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\mathrm{IWW}}) + D_A s^A\right] + \cdots, \tag{83}$$

where the ellipsis denotes terms that are quadratic or higher order in positive boost weight quantities and hence quadratic order in perturbations around a stationary black hole. $(s^v_{\mathrm{IWW}}, s^A)$ is denoted the Iyer-Wald-Wall entropy current. They are only defined uniquely up to first order in positive boost weight quantities, as any higher order terms can be absorbed into the ellipsis. As proved in [9], the higher order terms can be fixed so that s^v_{IWW} is invariant under a change of GNCs.

The Iyer-Wald-Wall entropy of the horizon cross section $C(v)$ is then defined as

$$S_{\mathrm{IWW}}(v) = 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}s^v_{\mathrm{IWW}}. \tag{84}$$

Taking the v derivative of this gives

$$\begin{aligned}\dot{S}_{\mathrm{IWW}} &= 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\mathrm{IWW}}) + D_A s^A\right]\\ &= -4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}\\ &\quad\times\int_v^\infty \mathrm{d}v'\partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\mathrm{IWW}}) + D_A s^A\right](v',x),\end{aligned} \tag{85}$$

where in the first line we trivially added the total derivative $\sqrt{\mu}D_A s^A$ to the integrand, and in the second line we assumed the black hole settles to the stationary black hole solution, so positive boost weight quantities vanish on the horizon at late times. The integrand can then be swapped for terms that are quadratic or higher in positive boost weight quantities using (83) and the equation of motion $E_{vv} = 0$. Thus, $\dot{S}_{\mathrm{IWW}}$ is quadratic order in perturbations around a stationary black hole, and so $\dot{S}_{\mathrm{IWW}}|_{\Phi_I^{st}} = 0$ and the first variation $\delta\dot{S}_{\mathrm{IWW}} = 0$. Therefore, S_{IWW} satisfies the second law to linear order. Even stronger than that, its change in time vanishes to linear order rather than being non-negative.

To see a possible increase in the entropy, we must go to quadratic order, which is what the extension by Hollands *et al.* achieves in [9]. They show that if the theory and solution lie in the regime of EFT, the ellipsis in (83) can be manipulated into the following form:

$$\begin{aligned}-E_{vv}\Big|_{\mathcal{N}} &= \partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\mathrm{IWW}}) + D_A s^A\right]\\ &\quad+\partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}\varsigma^v)\right]\\ &\quad+(K_{AB}+X_{AB})(K^{AB}+X^{AB})\\ &\quad+\frac{1}{2}(\partial_v\phi + X)^2 + D_A Y^A + O(l^N).\end{aligned} \tag{86}$$

X^{AB} and X are linear in positive boost weight quantities, and Y^A and the $O(l^N)$ terms are quadratic. To do this, they go "on shell," meaning they use the equations of motion to swap out various terms. The entropy density is then defined by $s^v_{\mathrm{HKR}} = s^v_{\mathrm{IWW}} + \zeta^v$, and the Hollands-Kovács-Reall entropy is given by

$$S_{\mathrm{HKR}}(v) = 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}s^v_{\mathrm{HKR}}. \tag{87}$$

Just as in (85), we can take the v derivative of this and substitute in (86) to get

$$\begin{aligned}\dot{S}_{\mathrm{HKR}}(v) &= 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}\\ &\quad\times\int_v^\infty \mathrm{d}v'[W^2 + D_A Y^A + O(l^N)](v',x),\end{aligned} \tag{88}$$

where $W^2 = (K_{AB} + X_{AB})(K^{AB} + X^{AB}) + \frac{1}{2}(\partial_v\phi + X)^2$. Since W^2, Y^A, and the $O(l^N)$ terms are quadratic in positive boost weight, they vanish on the horizon along with their first variations, so once again we have $\dot{S}_{\rm HKR}|_{\Phi_I^{st}} = 0$ and $\delta\dot{S}_{\rm HKR} = 0$.

Turning to the second variation, $\delta^2 W^2 = (\delta W)^2$ is a positive definite form so must be non-negative. The second term is

$$\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}\Big|_{\Phi_I^{st}} \int_v^\infty \mathrm{d}v' D_A\Big|_{\Phi_I^{st}} \delta^2 Y^A. \tag{89}$$

The induced metric μ_{AB} is independent of v on the horizon for the stationary solution Φ_I^{st} [see (51) with $r = 0$]. Therefore, we can exchange the order of integrations and see the integrand is a total derivative on $C(v)$. Hence this integral vanishes and so $\dot{S}_{\rm HKR}$ is non-negative to quadratic order, modulo $O(l^N)$ terms. Thus, it satisfies the second law to quadratic order in the sense of $\delta^2\dot{S}_{\rm HKR} \geq -O(l^N)$.

Finally, the recent work by Davies and Reall [12] showed that for vacuum gravity EFTs (i.e., with no scalar field ϕ) we can manipulate the terms in the rhs of (88) further into the form

$$\begin{aligned}\dot{S}_{\rm HKR}(v) = &-\frac{d}{dv}\left(4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu(v)}\sigma^v(v)\right)\\ &+ 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}\int_v^\infty \mathrm{d}v'[(K_{AB} + Z_{AB})\\ &\times (K^{AB} + Z^{AB}) + O(l^N)](v, v', x),\end{aligned} \tag{90}$$

where $Z_{AB}(v, v')$ is made up of so-called "bilocal" quantities, meaning they depend on both on v and the integration variable v'. The final integral is a positive definite form up to $O(l^N)$ terms, and hence the entropy defined by

$$S(v) = 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}s^v \tag{91}$$

with $s^v = s^v_{\rm HKR} + \sigma^v$ satisfies $\dot{S}(v) \geq -O(l^N)$; i.e., it satisfies the second law nonperturbatively up to $O(l^N)$ terms.

We shall now show how the above methods can be extended to define an entropy for Einstein-Maxwell-scalar EFT (with real, uncharged scalar field) that satisfies the second law in the same sense.

VII. PROOF OF THE SECOND LAW

A. The desired generalizations

Throughout the proof, we work exclusively in affinely parametrized GNCs. Rewriting here for convenience,

$$\begin{aligned}g = &\,2\mathrm{d}v\mathrm{d}r - r^2\alpha(r, v, x^C)\mathrm{d}v^2 - 2r\beta_A(r, v, x^C)\mathrm{d}v\mathrm{d}x^A\\ &+ \mu_{AB}(r, v, x^C)\mathrm{d}x^A\mathrm{d}x^B.\end{aligned} \tag{92}$$

We raise and lower $A, B, C, \ldots$ indices with μ_{AB} and denote the covariant derivative with respect to μ_{AB} by D_A. As well as $K_{AB} \equiv \frac{1}{2}\partial_v\mu_{AB}$, $\bar{K}_{AB} \equiv \frac{1}{2}\partial_r\mu_{AB}$ defined previously, it will be useful to define

$$K_A \equiv F_{vA}, \qquad \bar{K}_A \equiv F_{rA}, \qquad \psi = F_{vr}. \tag{93}$$

K_A has boost weight $+1$, $\bar{K}_A$ has boost weight -1, and ψ has boost weight 0.

The Iyer-Wald-Wall entropy has already been generalized to Einstein-Maxwell-scalar EFT with real scalar field by Biswas *et al.* in [8], as discussed in Sec. VII C.

The main body of the proof consists of generalizing the HKR entropy by studying the E_{vv} component of the equations of motion in affinely parametrized GNCs on the horizon. Our generalization of Eq. (86) is as follows. We will show that on shell [i.e., by using the known part of the equations of motion $E_I^{[N-1]} = O(l^N)$] we can bring $E_{vv}|_{\mathcal{N}}$ into the form

$$\begin{aligned}-E_{vv}\Big|_{\mathcal{N}} = &\,\partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\rm HKR}) + D_A s^A\right]\\ &+ (K_{AB} + X_{AB})(K^{AB} + X^{AB})\\ &+ \frac{1}{2}c_1(\phi)(K_A + X_A)(K^A + X^A)\\ &+ \frac{1}{2}(\partial_v\phi + X)^2 + D_A Y^A + O(l^N),\end{aligned} \tag{94}$$

where $X = \sum_{n=1}^{N-1} l^n X^{(n)}$, $X_A = \sum_{n=1}^{N-1} l^n X_A^{(n)}$, $X_{AB} = \sum_{n=1}^{N-1} l^n X_{AB}^{(n)}$ (boost weights $+1$) are linear or higher in positive boost weight quantities, and $Y^A = \sum_{n=1}^{N-1} l^n Y^{(n)A}$ (boost weight $+2$) and the $O(l^N)$ terms are quadratic. $s^v_{\rm HKR} = \sum_{n=0}^{\infty} l^n s^{(n)v}_{\rm HKR}$ has boost weight 0 and $s^A = \sum_{n=1}^{\infty} l^n s^{(n)A}$ has boost weight $+1$. They will be invariant upon change of electromagnetic gauge.

The generalization of the HKR entropy of the spacelike cross section $C(v)$ is then defined to be

$$S_{\rm HKR}(v) = 4\pi\int_{C(v)} \mathrm{d}^{d-2}x\sqrt{\mu}s^v_{\rm HKR}. \tag{95}$$

The proof that $\delta^2\dot{S} \geq -O(l^N)$ follows in the same way as for $S_{\rm HKR}$ detailed above, with the only change being $W^2 = (K_{AB} + X_{AB})(K^{AB} + X^{AB}) + \frac{1}{2}(\partial_v\phi + X)^2 + \frac{1}{2}c_1(\phi)\times (K_A + X_A)(K^A + X^A)$. The additional term is still a positive definite form and so the same method holds.

The algorithm to write E_{vv} in the form (94) is very similar to the algorithm devised by Hollands *et al.* [9] (and further detailed in [15]) for Einstein-scalar EFT. We will emphasize where we need to extend the HKR algorithm to apply to our Einstein-Maxwell-scalar EFT.

Finally, we generalize the entropy defined by Davies and Reall by proving we can write $\dot{S}_{\mathrm{HKR}}(v)$ as

$$\begin{aligned}\dot{S}_{\mathrm{HKR}}(v) = &-\frac{d}{dv}\left(4\pi\int_{C(v)}\mathrm{d}^{d-2}x\sqrt{\mu(v)}\sigma^v(v)\right)\\&+4\pi\int_{C(v)}\mathrm{d}^{d-2}x\sqrt{\mu}\int_v^\infty \mathrm{d}v'\Big[(K_{AB}+Z_{AB})\\&\times(K^{AB}+Z^{AB})+\frac{1}{2}c_1(\phi)(K_A+Z_A)(K^A+Z^A)\\&+\frac{1}{2}(\partial_v\phi+Z)^2+O(l^N)\Big](v,v',x)\end{aligned}\tag{96}$$

for bilocal Z_{AB}, Z_A, and Z. Thus, we can define an entropy

$$S(v) = 4\pi\int_{C(v)}\mathrm{d}^{d-2}x\sqrt{\mu}s^v \tag{97}$$

with $s^v = s^v_{\mathrm{HKR}} + \sigma^v$ that satisfies $\dot{S}_{\mathrm{HKR}}(v) \geq -O(l^N)$ as desired.

B. Leading order Einstein-Maxwell-scalar theory

Let us look at how this works for the leading order Einstein-Maxwell-scalar terms arising from $\mathcal{L}_2$. The leading order part of the $(\alpha\beta)$ equation of motion is

$$\begin{aligned}E^{(0)}_{\alpha\beta} = &R_{\alpha\beta} - \frac{1}{2}\nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}c_1(\phi)F_{\alpha\delta}F_\beta^\delta\\&-\frac{1}{2}g_{\alpha\beta}\left(R - V(\phi) - \frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi - \frac{1}{4}c_1(\phi)F_{\gamma\delta}F^{\gamma\delta}\right).\end{aligned}\tag{98}$$

In affinely parametrized GNCs on the horizon, we have

$$E^{(0)}_{vv}|_{\mathcal{N}} = R_{vv} - \frac{1}{2}(\partial_v\phi)^2 - \frac{1}{2}c_1(\phi)K_AK^A. \tag{99}$$

Using $R_{vv}|_{\mathcal{N}} = -\mu^{AB}\partial_v K_{AB} + K_{AB}K^{AB}$ and $\partial_v\sqrt{\mu} = \sqrt{\mu}\mu^{AB}K_{AB}$, we can write this as

$$\begin{aligned}-E^{(0)}_{vv}|_{\mathcal{N}} = &\partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu})\right] + K_{AB}K^{AB}\\&+\frac{1}{2}c_1(\phi)K_AK^A + \frac{1}{2}(\partial_v\phi)^2.\end{aligned}\tag{100}$$

This is of the form (94) with $s^{(0)v}_{\mathrm{HKR}} = 1$ and $s^{(0)A} = X^{(0)} = X^{(0)}_A = X^{(0)}_{AB} = Y^{(0)}_A = 0$. Because there is no total derivative term $D_AY^{(0)A}$, there are no further manipulations needed to get to (96) with $\sigma^v = 0$, $Z^{(0)} = X^{(0)}$, $Z^{(0)}_A = X^{(0)}_A$, and $Z^{(0)}_{AB} = X^{(0)}_{AB}$. Thus, we have proved our theory satisfies the second law nonperturbatively at leading order l^0 (which of course can be proved by the usual proof of the second law on the two-derivative theory).

We will ultimately work through the higher order terms order by order to mold them into the correct form. To get to that point however, we must start with the Biswas-Dhivakar-Kundu entropy.

C. The Biswas-Dhivakar-Kundu entropy

Our starting point is the generalization of the Iyer-Wald-Wall entropy by Biswas *et al.* defined in [8]. They prove that for any theory of gravity, electromagnetism and a real (uncharged) scalar field with diffeomorphism-invariant and electromagnetic gauge-independent Lagrangian, the E_{vv} component of the equations of motion can be brought into the following form on the horizon[10]:

$$-E_{vv}|_{\mathcal{N}} = \partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\mathrm{BDK}}) + D_As^A\right] + \cdots, \tag{101}$$

where the ellipsis denotes terms at least quadratic in positive boost weight quantities. We will call the quantity $(s^v_{\mathrm{BDK}}, s^A)$ the BDK entropy current. It is proved to only depend on the electromagnetic potential through $F_{\mu\nu}$ and is thus invariant upon a gauge transformation $A_\mu \to A_\mu + \partial_\mu\chi$. s^A is a vector in $A, B, \ldots$ indices on $C(v)$, while s^v is a scalar. They are only defined uniquely up to linear order in positive boost weight quantities, as any higher order terms can be absorbed into the ellipsis. As discussed in Sec. VII G, we assume we can fix the higher order terms so that s^v_{BDK} is invariant under a change of GNCs.

The BDK entropy is then defined by

$$S_{\mathrm{BDK}}(v) = 4\pi\int_{C(v)}\mathrm{d}^{d-2}x\sqrt{\mu}s^v_{\mathrm{BDK}}. \tag{102}$$

This can be proved to satisfy $\delta\dot{S}_{\mathrm{BDK}} = 0$ in the same way as above, since the ellipsis is quadratic in positive boost weight quantities.

For our Einstein-Maxwell-scalar EFT, we can calculate s^v_{BDK} to all orders in l and take (101) as our starting point. We group all the remaining terms in the ellipsis and define

$$H \equiv -E_{vv}\Big|_{\mathcal{N}} - \partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v(\sqrt{\mu}s^v_{\mathrm{BDK}}) + D_As^A\right]. \tag{103}$$

We will use the fact that H is quadratic in positive boost weight terms to show we can manipulate it so that (101)

[10] In [8], they have an additional T_{vv} in this defining equation, which is the part of the energy-momentum tensor arising from the minimal coupling part of the matter sector Lagrangian. However, they also show that T_{vv} is quadratic in positive boost weight quantities if $T_{\mu\nu}$ satisfies the NEC, which is the case for our two-derivative Einstein-Maxwell-scalar theory and hence we can absorb T_{vv} into the ellipsis.

becomes (94). The resulting generalization of the HKR entropy density s^v_{HKR} will be

$$s^v_{\mathrm{HKR}} = s^v_{\mathrm{BDK}} + \sum_{n=0}^{N-1} l^n \varsigma^{(n)v}, \tag{104}$$

where the $\varsigma^{(n)v}$ are quadratic in positive boost weight quantities. We will not need to add any terms to s^A.

From (100), we can see that, for the leading order theory $\mathcal{L}_2$, the BDK entropy density is $s^{(0)v}_{\mathrm{BDK}} = 1$ and we need no correction, $\varsigma^{(0)v} = 0$.

D. Reducing to allowed terms

To generalize the HKR entropy we study the possible quantities out of which H is made. H comes from the equations of motion and is gauge invariant and so is made from the fields $g_{\mu\nu}, F_{\mu\nu}$, and ϕ and their derivatives. It is also evaluated in affinely parametrized GNCs on the horizon, and H is a scalar with respect to $A, B, \ldots$ indices. Therefore, it is made from gauge-invariant affinely parametrized GNC quantities of the metric and matter fields that are covariant in $A, B, \ldots$ indices, namely,

$$D^k \partial_v^p \partial_r^q \varphi \quad \text{for}$$

$$\varphi \in \{\alpha, \beta_A, \mu_{AB}, R_{ABCD}[\mu], \epsilon_{A_1\ldots A_{d-2}}, \phi, F_{AB}, K_A, \overline{K}_A, \psi\}, \tag{105}$$

where $k,\ p,\ q \geq 0$ and we have suppressed the indices $D^k = D_{A_1}\ldots D_{A_k}$. $K_A, \bar{K}_A$, and ψ are defined in (93). Section 3.3 of [15] gives commutation rules for commuting D_A derivatives past ∂_v and ∂_r derivatives, which allow us to have all D_A derivatives on the left. $R_{ABCD}[\mu]$ is the induced Riemann tensor with respect to μ_{AB}.

We will now show that we can reduce this set of possible objects that can appear on the horizon by using the equations of motion. It is worth emphasizing, this reduction holds in an EFT sense, meaning it is *only* done up to $O(l^N)$ terms. In the HKR procedure of [9] for Einstein-scalar EFT, they show how to reduce the metric and scalar field terms to the set μ_{AB}, $\epsilon_{A_1\ldots A_{d-2}}$, $D^k R_{ABCD}[\mu]$, $D^k\beta_A$, $D^k\partial_v^p K_{AB}$, $D^k\partial_r^p \bar{K}_{AB}$, $D^k\partial_v^p\phi$, $D^k\partial_r^p\phi$ with $p \geq 0$. This procedure still holds in our Einstein-Maxwell-scalar EFT. To focus on where we need to generalize the HKR procedure, we only detail how to reduce the Maxwell terms.

We aim to reduce the set of Maxwell terms on the horizon to

$$D^k\psi, \qquad D^k F_{AB}, \qquad D^k\partial_v^p K_A, \qquad D^k\partial_r^q \bar{K}_A \tag{106}$$

To do this, we must eliminate any ∂_v and ∂_r derivative of both ψ and F_{AB}. We must also eliminate any ∂_r derivative of K_A and any ∂_v derivative of $\bar{K}_A$.

To begin we use the fact that

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0, \tag{107}$$

which follows from $F = \mathrm{d}A$. Taking $\alpha = v, \beta = A, \gamma = B$, we can rearrange this to[11]

$$\partial_v F_{AB} = D_A K_B - D_B K_A. \tag{108}$$

Similarly, taking $\alpha = v, \beta = A, \gamma = B$ gives

$$\partial_r F_{AB} = D_A \bar{K}_B - D_B \bar{K}_A. \tag{109}$$

These two relations allow us to eliminate all ∂_v and ∂_r derivatives of F_{AB} in favor of other Maxwell and metric terms.

Furthermore, taking $\alpha = v, \beta = r, \gamma = A$, we get

$$\partial_v \bar{K}_A = \partial_r K_A - D_A\psi, \tag{110}$$

which allows us to eliminate any ∂_v derivative or mixed ∂_v and ∂_r derivative of $\bar{K}_A$.

To go further, we will have to use the equations of motion for the Maxwell field. In particular, we inspect the leading order part $E^{(0)}_\alpha = O(l)$,

$$\nabla^\beta\Big[c_1(\phi)F_{\alpha\beta} - 4c_2(\phi)F^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}\Big] = O(l), \tag{111}$$

where in theory we know all the terms on the right-hand side up to $O(l^N)$. We can use $\epsilon_{\alpha\beta\gamma\delta}\nabla^\beta F^{\gamma\delta} = 0$ [which follows from (107)] to rewrite this as

$$\nabla^\beta F_{\alpha\beta} = \frac{1}{c_1(\phi)}[4c_2'(\phi)\nabla^\beta\phi F^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} - c_1'(\phi)\nabla^\beta\phi F_{\alpha\beta}] + O(l). \tag{112}$$

The order l^0 terms on the right-hand only involve Maxwell terms that we are not trying to eliminate. Let us now evaluate the v component of $\nabla^\beta F_{\alpha\beta}$ in affinely parametrized GNCs,

$$\nabla^\beta F_{v\beta} = \partial_v\psi + D^A K_A + \psi K + \cdots, \tag{113}$$

where the ellipsis denotes terms that vanish on $\mathcal{N}$. We can substitute this into (112) to get an expression for $\partial_v\psi$ on the horizon up to terms higher order in l,

[11] If we had explicitly picked a gauge A_μ, then this relation would be trivially true and we would have fewer terms to eliminate. However, we would like to keep the entropy current manifestly gauge invariant, and hence we do not pick a gauge.

$$\partial_v\psi|_{\mathcal{N}} = -D^A K_A - \psi K + \frac{1}{c_1(\phi)}[4c_2'(\phi)\epsilon^{AB}(2D_A\phi K_B - \partial_v\phi F_{AB}) - c_1'(\phi)(\psi\partial_v\phi + K_A D^A\phi)] + O(l). \tag{114}$$

Therefore, wherever we find a $\partial_v\psi$ in H, we can swap it out order by order in l, pushing it to higher order with each step. Eventually, it will only appear at $O(l^N)$, at which point it is not relevant to our analysis since we do not know the equations of motion at that order.

Similarly, we can evaluate $\nabla^\beta F_{r\beta}$ in affinely parametrized GNCs,

$$\nabla^\beta F_{r\beta} = -\partial_r\psi + D^A\bar{K}_A + \bar{K}^A\beta_A - \psi\bar{K} + \cdots, \tag{115}$$

where, again, the terms in the ellipsis vanish on the horizon. We can substitute this into (112) to get an expression for $\partial_r\psi$ on the horizon,

$$\partial_r\psi|_{\mathcal{N}} = D^A\bar{K}_A + \bar{K}^A\beta_A - \psi\bar{K} + \frac{1}{c_1(\phi)}[c_1'(\phi)(\bar{K}_A D^A\phi - \psi\partial_r\phi) - 4c_2'(\phi)\epsilon^{AB}(\partial_r\phi F_{AB} - 2D_A\phi\bar{K}_B)] + O(l). \tag{116}$$

This allows us to eliminate $\partial_r\psi$ up to $O(l^N)$ in a similar fashion.

We can take ∂_v derivatives of (114) and (116) in order to eliminate $\partial_v^p\partial_r^q\psi$ for $p \geq 1$ and $q = 0$, 1. However, we cannot naively take ∂_r derivatives because these expressions are evaluated on the horizon $r = 0$. Instead, we must take successive ∂_r derivatives of (112) and (115), and then evaluate them on $r = 0$, possibly using substitution rules already calculated for lower order derivatives. This will involve taking care of the terms in the ellipsis in (116), which are given in full in Appendix A 3. However, these only ever involve lower order derivatives, for which we already have substitution rules and hence do not cause an issue. Therefore, we can eliminate all ∂_v and ∂_r derivatives of ψ up to order $O(l^N)$.

This just leaves ∂_r derivatives of K_A to be eliminated, for which we look at $\nabla^\beta F_{A\beta}$,

$$\nabla^\beta F_{A\beta} = -2\partial_r K_A + D_A\psi + D^B F_{AB} + 2\bar{K}^B K_{AB} + 2K^B\bar{K}_{AB} - \psi\beta_A - \bar{K}_A K - K_A\bar{K} + F_{AB}\beta^B + \cdots. \tag{117}$$

Substituting this into (112) gives us an expression which we can use to eliminate $\partial_r K_A$ on the horizon. Taking ∂_r derivatives of (117) again allows us to eliminate higher ∂_r derivatives of K_A because the terms in the ellipsis only involve lower order derivatives. This completes the reduction of Maxwell terms.

Combining the Maxwell terms with the metric and scalar field terms already reduced through the HKR procedure, we are left with a small set of "allowed terms,"

$$\text{Allowed terms:}\ \mu_{AB},\quad \mu^{AB},\quad \epsilon_{A1\ldots A_{d-2}},\quad D^k R_{ABCD}[\mu],\quad D^k\beta_A,\quad D^k\partial_v^p K_{AB},\quad D^k\partial_r^q\bar{K}_{AB},$$
$$D^k\psi,\quad D^k F_{AB},\quad D^k\partial_v^p K_A,\quad D^k\partial_r^q K_A,\quad D^k\partial_v^p\phi,\quad D^k\partial_r^q\phi. \tag{118}$$

In particular, the only allowed positive boost weight terms are of the form $D^k\partial_v^p K_{AB}$ and $D^k\partial_v^p K_A$ with $p \geq 0$, and $D^k\partial_v^p\phi$ with $p \geq 1$. This will be the crucial fact that allows us to manipulate the terms in H.

E. Manipulating terms order by order

Let us return to H. We use the above procedures to eliminate any nonallowed terms up to $O(l^N)$. Once doing so, we can write it as a series in l,

$$H = H^{(0)} + \sum_{n=1}^{N-1} l^n H^{(n)} + O(l^N). \tag{119}$$

By construction, the $H^{(n)}$ are quadratic in positive boost weight terms. Furthermore, $H^{(0)}$ are the terms calculated from the leading order part of the equation of motion in (100),

$$H^{(0)} = K_{AB}K^{AB} + \frac{1}{2}c_1(\phi)K_A K^A + \frac{1}{2}(\partial_v\phi)^2. \tag{120}$$

We proceed by induction order by order in l. Our inductive hypothesis is that we have manipulated the terms in H up to $O(l^m)$ into the form

$$H = \partial_v \left[\frac{1}{\sqrt{\mu}} \partial_v \left(\sqrt{\mu} \sum_{n=0}^{m-1} l^n \varsigma^{(n)v} \right) \right] + \left(K_{AB} + \sum_{n=0}^{m-1} l^n X^{(n)}_{AB} \right) \left(K^{AB} + \sum_{n=0}^{m-1} l^n X^{(n)AB} \right)$$
$$+ \frac{1}{2} c_1(\phi) \left(K_A + \sum_{n=0}^{m-1} l^n X^{(n)}_A \right) \left(K^A + \sum_{n=0}^{m-1} l^n X^{(n)A} \right) + \frac{1}{2} \left(\partial_v \phi + \sum_{n=0}^{m-1} l^n X^{(n)} \right)^2$$
$$+ D_A \sum_{n=0}^{m-1} l^n Y^{(n)A} + \sum_{n=m}^{N-1} l^n H^{(n)} + O(l^N), \tag{121}$$

where the $H^{(n)}$ may have gained extra terms compared to (119) but are still quadratic in positive boost weight terms.

By (120), this is true for $m = 1$ with $\varsigma^{(0)v} = X^{(0)}_{AB} = X^{(0)}_A = X^{(0)} = Y^{(0)A} = 0$. So assume it is true for some $1 \leq m \leq N - 1$.

We now consider $H^{(m)}$. It is quadratic in positive boost weight quantities. However, we have reduced the set of allowed positive boost weight quantities. Therefore, we can write it as a sum

$$H^{(m)} = \sum_{k_1,k_2,p_1,p_2,P_1,P_2} (D^{k_1} \partial_v^{p_1} P_1)(D^{k_2} \partial_v^{p_2} P_2) Q_{k_1,k_2,p_1,p_2,P_1,P_2}, \tag{122}$$

where $P_1, P_2 \in \{K_{AB}, K_A, \partial_v \phi\}$ and $Q_{k_1,k_2,p_1,p_2,P_1,P_2}$ is some linear combination of allowed terms. Note that we have dropped $A, B, \ldots$ indices here for notational ease, and they can be contracted in any way.

We now move the $D_1^k \partial_v^{p_1}$ derivatives off the leading positive boost weight factor in each term in the sum. The method of doing so is identical to the HKR procedure detailed in Sec. 3.5 of [15] but with P_1, P_2 in the place of factors of K, so we shall not repeat it here. It produces extra total derivative terms, with the end result being

$$H^{(m)} = \sum_{k,p,P_1,P_2} P_1 (D^k \partial_v^p P_2) Q_{k,p,P_1,P_2} + \partial_v \left[\frac{1}{\sqrt{\mu}} \partial_v (\sqrt{\mu} \varsigma^{(m)v}) \right] + D_A Y^{(m)A}, \tag{123}$$

with $\varsigma^{(m)v}$ and $Y^{(m)A}$ quadratic in positive boost weight quantities. It also produces terms that are higher order in l. These are still quadratic in positive boost weight quantities so can be absorbed into $\sum_{n=m+1}^{N-1} l^n H^{(n)}$.

We now split the sum over $P_1 \in \{K_{AB}, K_A, \partial_v \phi\}$, write the remaining sums as $2X^{(m)AB}$, $c_1(\phi) X^{(m)A}$, and $X^{(m)}$, and substitute this into (121),

$$H = \partial_v \left[\frac{1}{\sqrt{\mu}} \partial_v \left(\sqrt{\mu} \sum_{n=0}^{m} l^n \varsigma^{(n)v} \right) \right] + \left(K_{AB} + \sum_{n=0}^{m-1} l^n X^{(n)}_{AB} \right) \left(K^{AB} + \sum_{n=0}^{m-1} l^n X^{(n)AB} \right) + 2 l^m K_{AB} X^{(m)AB}$$
$$+ \frac{1}{2} c_1(\phi) \left(K_A + \sum_{n=0}^{m-1} l^n X^{(n)}_A \right) \left(K^A + \sum_{n=0}^{m-1} l^n X^{(n)A} \right) + l^m c_1(\phi) K_A X^{(m)A}$$
$$+ \frac{1}{2} \left(\partial_v \phi + \sum_{n=0}^{m-1} l^n X^{(n)} \right)^2 + l^m \partial_v \phi X^{(m)} + D_A \sum_{n=0}^{m} l^n Y^{(n)A} + \sum_{n=m+1}^{N-1} l^n H^{(n)} + O(l^N). \tag{124}$$

We now complete the three squares to bring $l^m X^{(m)AB}$, $l^m X^{(m)A}$, and $l^m X^{(m)}$ into the sums. The extra terms produced are $O(l^{m+1})$ because $X^{(0)}_{AB} = X^{(0)}_A = X^{(0)} = 0$ and are quadratic in positive boost weight quantities, so can be absorbed into $\sum_{n=m+1}^{N-1} l^n H^{(n)}$. This completes the inductive step.

This can be repeated until all terms up to $O(l^N)$ are of the correct form. Substituting this back into the definition of H in (103), we can now write $E_{vv}|_{\mathcal{N}}$ in the desired form (94) with

$$s^v_{\mathrm{HKR}} = s^v_{\mathrm{BDK}} + \sum_{n=0}^{N-1} l^n \varsigma^{(n)v}. \tag{125}$$

This completes the generalization of the HKR entropy

$$S_{\mathrm{HKR}}(v) = 4\pi \int_{C(v)} \mathrm{d}^{d-2} x \sqrt{\mu} s^v_{\mathrm{HKR}}, \tag{126}$$

which satisfies $\delta^2 \dot{S}_{\mathrm{HKR}} \geq -O(l^N)$.

F. Further modification of the entropy

We now further modify this to generalize the entropy defined by Davies and Reall in our companion paper [12]. Performing on S_{HKR} the same steps used to get to (88), we have

$$\dot{S}_{\mathrm{HKR}}(v_0) = 4\pi \int_{C(v_0)} \mathrm{d}^{d-2}x\sqrt{\mu(v_0)} \times \int_{v_0}^{\infty} \mathrm{d}v[W^2 + D_A Y^A + O(l^N)](v), \tag{127}$$

where $W^2 = (K_{AB}+X_{AB})(K^{AB}+X^{AB}) + \frac{1}{2}c_1(\phi)\times (K_A+X_A)(K^A+X^A) + \frac{1}{2}(\partial_v\phi + X)^2$. We have suppressed all x dependence and switched notation to v_0 and v to match [12]. The obstruction to this integral being non-negative up to $O(l^N)$ is $D_A Y^A(v)$. Despite being a divergence term, it does not integrate to zero because it is evaluated at the integration variable v, whereas the area element $\sqrt{\mu(v_0)}$ is evaluated at v_0. Define

$$a(v_0, v) = \sqrt{\frac{\mu(v)}{\mu(v_0)}}, \tag{128}$$

which measures the change in the area element from v_0 to v. Then, if we try to integrate $D_A Y^A(v)$ by parts, we get[12]

$$\begin{aligned}
&\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}\int_{v_0}^{\infty}\mathrm{d}v D_A Y^A(v)\\
&= \int_{v_0}^{\infty}\mathrm{d}v\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v)}a^{-1}(v_0,v)D_A Y^A(v)\\
&= -\int_{v_0}^{\infty}\mathrm{d}v\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v)}Y^A(v)D_A a^{-1}(v_0,v)\\
&= \int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}\int_{v_0}^{\infty}\mathrm{d}v Y^A(v)D_A \log a(v_0,v).
\end{aligned} \tag{129}$$

Now, $Y^A(v)$ is quadratic in positive boost weight quantities and so is a sum of terms of the form $(D^{k_1}\partial_v^{p_1}P_1)\,(D^{k_2}\partial_v^{p_2}P_2)$ $Q(v)$ where, as before, $P_1, P_2 \in \{K_{AB}, K_A, \partial_v\phi\}$ and $Q(v)$ is some linear combination of allowed terms. Therefore, this integrand closely resembles the terms we manipulated in the previous section, except with factors of $D_A \log a(v_0, v)$. We will show that these terms can still be absorbed into the positive definite terms in (127). We will do this via a similar induction over powers of l.

Our inductive hypothesis is that we have manipulated $\dot{S}_{\mathrm{HKR}}(v_0)$ up to $O(l^m)$ into the form

$$\begin{aligned}
\dot{S}_{\mathrm{HKR}}(v_0) = &-\frac{d}{dv}\left(4\pi\int_{C(v)}\mathrm{d}^{d-2}x\sqrt{\mu(v)}\sigma_m^v(v)\right)\\
&+ 4\pi\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}\\
&\times\int_{v_0}^{\infty}\mathrm{d}v\Big[(K^{AB}+Z_m^{AB})(K_{AB}+Z_{mAB})\\
&+\frac{1}{2}c_1(\phi)(K_A+Z_{mA})(K^A+Z_m^A)\\
&+\frac{1}{2}(\partial_v\phi+Z_m)^2 + R_m + O(l^N)\Big](v_0,v),
\end{aligned} \tag{130}$$

where $Z_m^{AB}(v_0,v)$, $Z_m^A(v_0,v)$, and $Z_m(v_0,v)$ are $O(l)$ and at least linear in positive boost weight quantities, and $R_m(v_0,v)$ is of the form

$$R_m(v_0,v) = \sum_{n=m}^{N-1} l^n \sum_{k_1,k_2,p_1,p_2,P_1,P_2}(D^{k_1}\partial_v^{p_1}P_1) \times (D^{k_2}\partial_v^{p_2}P_2)Q_{k_1,k_2,p_1,p_2,P_1,P_2,m,n}(v_0,v), \tag{131}$$

and, in particular, $Z_m^{AB}(v_0,v)$, $Z_m^A(v_0,v)$, $Z_m(v_0,v)$, and $Q_{k_1,k_2,p_1,p_2,P_1,P_2,m,n}(v_0,v)$ is each a linear combination of terms, where each term is a product of factors of two possible types: (i) allowed terms evaluated at v and (ii) $D^q \log a(v_0,v)$ with $q \geq 1$ (D_A evaluated at time v). If a factor of type (ii) is present, then the term is bilocal; otherwise it is local. All covariant derivatives D are constructed from $\mu_{AB}(v)$, and all P_1, P_2 terms are evaluated at v.

By (127) and (129), the base case $m=0$ is satisfied with $Z_0^{AB} = X^{AB}$, $Z_0^A = X^A$, $Z_0 = X$, and $R_0 = Y^A D_A \log a$. Assuming true for m, the obstruction to proceeding is the order l^m terms in the sum in R_m, which are of the form $l^m(D^{k_1}\partial_v^{p_1}P_1)\,(D^{k_2}\partial_v^{p_2}P_2)\,Q(v_0,v)$. We aim to remove the $D^{k_1}\partial_v^{p_1}$ from each term and then complete the square.

We first reduce k_1 by 1 in each term via a spatial integration by parts,

$$\begin{aligned}
&\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}\int_{v_0}^{\infty}\mathrm{d}v(D^{k_1}\partial_v^{p_1}P_1)(D^{k_2}\partial_v^{p_2}P_2)Q\\
&= -\int_{v_0}^{\infty}\mathrm{d}v\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v)}(D^{k_1-1}\partial_v^{p_1}P_1)\\
&\quad\times D[a^{-1}(D^{k_2}\partial_v^{p_2}P_2)Q]\\
&= -\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}\int_{v_0}^{\infty}\mathrm{d}v(D^{k_1-1}\partial_v^{p_1}P_1)\\
&\quad\times\Big[(D^{k_2+1}\partial_v^{p_2}P_2)Q + (D^{k_2}\partial_v^{p_2}P_2)DQ\\
&\quad - (D^{k_2}\partial_v^{p_2}P_2)QD\log a\Big],
\end{aligned} \tag{132}$$

[12]All cross sections $C(v)$ are diffeomorphic to each other, and thus we write them all as C for this section.

where in the last step we used $aD(a^{-1}) = -D\log a$. We repeat to bring k_1 to 0 in all terms, leaving us with terms of the form $l^m(\partial_v^{p_1}P_1)(D^k\partial_v^{p_2}P_2)Q$, with Q still made exclusively from local allowed terms and factors of $D^q\log a(v_0, v)$.

We now aim to reduce p_1 to 0 by v integration by parts. However, to avoid surface terms we must treat local and bilocal terms separately.

1. Bilocal terms

Bilocal terms have at least one factor of $D^q\log a(v_0, v)$. Their v integration by parts follows simply

$$\begin{aligned}&\int_{v_0}^{\infty}\mathrm{d}v(\partial_v^{p_1}P_1)(D^k\partial_v^{p_2}P_2)QD^q\log a(v_0,v)\\&=[(\partial_v^{p_1-1}P_1)(D^k\partial_v^{p_2}P_2)QD^q\log a(v_0,v)]_{v_0}^{\infty}\\&\quad-\int_{v_0}^{\infty}\mathrm{d}v(\partial_v^{p_1-1}P_1)\partial_v[(D^k\partial_v^{p_2}P_2)QD^q\log a(v_0,v)].\end{aligned} \tag{133}$$

The boundary term vanishes at $v=\infty$ because we assume the black hole settles down to stationarity, and so positive boost weight quantities vanish. The boundary term also vanishes at $v=v_0$ because $a(v_0, v_0)\equiv 1$ and hence $D^q\log a = 0$. In the remaining v integral, we can commute the ∂_v past any D derivatives using the formula

$$\begin{aligned}[\partial_v, D_A]t_{B_1\ldots B_n} = \sum_{i=1}^{n}\mu^{CD}(D_D K_{AB_i} - D_A K_{DB_i} - D_{B_i}K_{AD})\\ \times t_{B_1\ldots B_{i-1}CB_{i+1}\ldots B_n},\end{aligned} \tag{134}$$

which will produce additional terms proportional to some $D^{k'}K$. Commuting ∂_v past D^q will leave $D^q\partial_v\log a$, which initially looks like a new type of bilocal term; however, one can calculate that

$$\partial_v\log a = \mu^{AB}K_{AB}, \tag{135}$$

and so this term is actually proportional to D^qK. Similarly, in $\partial_v Q$, any v derivative of $D^{q'}\log a$ can be dealt with by commuting and then using ([135]), and any nonallowed terms such as $\partial_v\beta$ or $\partial_{vr}\phi$ can be swapped out to $O(l^N)$ using the equations of motion, which will generate additional terms in R_m of $O(l^{m+1})$.

Therefore, we are left with two types of terms at order l^m: (i) terms that retain their factor of $D^q\log a$, which will be of the form $(\partial_v^{p_1-1}P_1)(D^k\partial_v^{p_2}P_2)QD^q\log a(v_0, v)$ (with possibly changed k, p_2, and Q), and (ii) terms that had $D^q\log a$ hit by ∂_v, which will be of the form $(D^{k'}K)(\partial_v^{p_1-1}P_1)Q$ for some k' and Q. This second type of term can potentially be local.

The v integration by parts can be repeated on terms of type (i) until p_1 is reduced to 0, producing more terms of type (ii) along the way (which will have varying p_1's). To terms of type (ii) we move the $D^{k'}$ derivatives off of K via the same spatial integration by parts as in (132). This brings them proportional to K, and hence, after relabeling this K as P_1 and the old P_1 as P_2, they also effectively have p_1 reduced to 0.

2. Local terms

Local terms are of the form $(\partial_v^{p_1}P_1)(D^k\partial_v^{p_2}P_2)Q(v)$ with $Q(v)$ made exclusively from allowed terms evaluated at v. We can no longer simply do a v integration by parts on this because there is no $D^q\log a$ to make the boundary term vanish at $v = v_0$. However, we can manipulate these terms in the same fashion as in the HKR procedure, namely, by noting there exist unique numbers a_j such that

$$\begin{aligned}&(\partial_v^{p_1}P_1)(D^k\partial_v^{p_2}P_2)Q\\&=\partial_v\left\{\frac{1}{\sqrt{\mu}}\partial_v\left[\sqrt{\mu}\sum_{j=1}^{p_1+p_2-1}a_j(\partial_v^{p_1+p_2-1-j}P_1)(D^k\partial_v^{j-1}P_2)Q\right]\right\}\\&\quad+\cdots,\end{aligned} \tag{136}$$

where the ellipsis denotes terms of the form $(\partial_v^{\bar{p}_1}P_1)(D^{\bar{k}}\partial_v^{\bar{p}_2}P_2)\tilde{Q}$ with $\bar{p}_1+\bar{p}_2 < p_1+p_2$ or $\bar{p}_1 = 0$ or $\bar{p}_2 = 0$. The proof follows identically to Appendix A.2 of [15] but with P_1, P_2 in the place of factors of K. The new $\tilde{Q}$ are still local, but do include terms like $\partial_v Q$ which will involve nonallowed terms. However, these can be swapped out to $O(l^N)$ using the equations of motion, generating more $O(l^{m+1})$ in R_m.

We repeat this procedure on the terms in the ellipsis with $\bar{p}_1+\bar{p}_2 < p_1+p_2$ until eventually $\bar{p}_1 = 0$ or $\bar{p}_2 = 0$ for all terms. This must eventually happen because $\bar{p}_1+\bar{p}_2$ must decrease by at least 1 if the new $\bar{p}_1\neq 0$ and $\bar{p}_2\neq 0$, and hence $\bar{p}_1+\bar{p}_2$ eventually falls below 2, meaning one of $\bar{p}_1$ and $\bar{p}_2$ must be 0. Therefore, we can write all local terms as a sum of terms of the form (i) $\partial_v\left\{\frac{1}{\sqrt{\mu}}\partial_v[\sqrt{\mu}\sigma^v(v)]\right\}$ with σ^v local and quadratic in positive boost weight quantities, (ii) $P_1(D^{\bar{k}}\partial_v^{\bar{p}_2}P_2)\tilde{Q}(v)$, and (iii) $(\partial_v^{\bar{p}_1}P_1)(D^{\bar{k}}P_2)\tilde{Q}(v)$.

We have successfully reduced $k_1 = p_1 = 0$ to 0 in terms of type (ii). For terms of type (iii), we can relabel $P_1\leftrightarrow P_2$ and then remove the $D^{\bar{k}}$ derivatives from P_1 by using spatial integration by parts, as in (132). This will introduce bilocal factors of $D^q\log a$, but this is fine: the resulting terms will all be of the desired form $P_1(D^k\partial_v^{\tilde{p}}P_2)Q(v_0, v)$, i.e., they also have $k_1 = p_1 = 0$.

Let us look at what happens to terms of type (i) when they are placed in the integral,

$$\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}\int_{v_0}^{\infty}\mathrm{d}v\partial_v\left\{\frac{1}{\sqrt{\mu}}\partial_v[\sqrt{\mu}\sigma^v(v)]\right\}$$
$$= -\int_C \mathrm{d}^{d-2}x\partial_v(\sqrt{\mu(v)}\sigma^v(v))\Big|_{v=v_0}$$
$$= -\frac{d}{dv}\left(\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v)}\sigma^v(v)\right)\Big|_{v=v_0}, \tag{137}$$

where in the first equality we set the boundary term at $v=\infty$ to zero because we assume the black hole settles down to stationarity. These are the terms which will modify our definition of the entropy.

3. Completion of the induction

To summarize, we have rewritten all order l^m terms in R_m as

$$-\frac{d}{dv}\left(l^m\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v)}\sigma^v(v)\right)\Big|_{v=v_0} + l^m\int_C \mathrm{d}^{d-2}x\sqrt{\mu(v_0)}$$
$$\times\int_{v_0}^{\infty}\mathrm{d}v\sum_{k,p,P_1,P_2}P_1(D^k\partial_v^pP_2)Q_{k,p,P_1,P_2}(v_0,v), \tag{138}$$

where $Q_{k,p,P_1,P_2}(v_0,v)$ is a linear combination of allowed terms evaluated at v and factors of $D^q\log a(v_0,v)$ with $q\geq 1$.

Similar to Sec. VII E, the final step in the induction is to split the sum over $P_1\in\{K_{AB},K_A,\partial_v\phi\}$ and write the remaining sums as $2l^m\tilde{Z}^{AB}$, $l^mc_1(\phi)\tilde{Z}^A$, and $l^m\tilde{Z}$. We then absorb them into the positive definite terms in (130) by completing the squares and setting $Z^{AB}_{m+1}=Z^{AB}_m+l^m\tilde{Z}^{AB}$, $Z^A_{m+1}=Z^A_m+l^m\tilde{Z}^A$, and $Z_{m+1}=Z_m+l^m\tilde{Z}$. The remainder terms will be $O(l^{m+1})$ [because Z^{AB}_m, etc. are $O(l)$], quadratic in positive boost weight (because Z^{AB}_m, etc. are linear), and linear combinations of local allowed terms and factors of $D^q\log a$ (because Z^{AB}_m, etc. are such linear combinations). Furthermore, Z^{AB}_{m+1}, Z^A_{m+1}, and Z_{m+1} retain these properties. Finally, we label $\sigma^v_{m+1}=\sigma^v_m+l^m\sigma^v$. Thus, the induction proceeds.

We continue the induction until $m=N$, at which point R_N is $O(l^N)$. Therefore, the entropy defined by

$$S(v) \coloneqq \int_C \mathrm{d}^{d-2}x\sqrt{\mu(v)}s^v(v) \tag{139}$$

with $s^v=s^v_{\mathrm{HKR}}+\sigma^v_N$ satisfies a nonperturbative second law up to $O(l^N)$.

G. Gauge (non)invariance of entropy

Through the above procedure we have constructed an entropy $S(v)$ that depends on the local geometry of the "constant time" slice $C(v)$ and satisfies a nonperturbative second law for Einstein-Maxwell-scalar EFT. Furthermore, its entropy density s^v differs from the BDK entropy density s^v_{BDK} defined in Sec. VII C by terms that are quadratic in perturbations around a stationary black hole. Thus, the facts that the BDK entropy reduces to the Wald entropy in equilibrium and satisfies the first law [8] imply they also hold for $S(v)$. Therefore, $S(v)$ satisfies many of the properties we should expect in a definition of the entropy of a black hole.

However, we should ask, is this definition of the entropy gauge invariant? There are two types of gauges in our theory: the choice of electromagnetic gauge and our choice of coordinates.

By construction, s^v only depends on Maxwell quantities through $F_{\mu\nu}$, which is invariant under a change of electromagnetic gauge. Therefore, the entropy $S(v)$ is independent of electromagnetic gauge.

As for coordinate independence, our procedure was performed in affinely parametrized GNCs with $r=v=0$ on a given spacelike cross section C of $\mathcal{N}$ (the GNCs can be defined starting from any horizon cross section, so the restriction $r=v=0$ is not restricting the choice of cross section considered). However, as discussed in Sec. III A, such affinely parametrized GNCs are not unique: we can reparametrize the affine parameter on each horizon generator by $v'=v/a(x^A)$. This will produce a new foliation $C'(v')$ of the horizon. We should not expect $S'(v')=S(v)$ for all v, because $S'(v')$ and $S(v)$ measure the entropy of the different surfaces $C'(v')$ and $C(v)$. However, we should hope that $S'(0)=S(0)$ because $C'(0)=C(0)=C$. Therefore, we should investigate how our entropy density s^v transforms under such a gauge transformation at $r=v=0$.

By construction, s^v can be split into two parts: s^v_{BDK} and the modification terms that are quadratic or higher order in positive boost weight terms. A proof that s^v_{BDK} is gauge invariant on C is beyond the scope of this paper, and we will just assume it holds here. Why should we expect it to be gauge invariant? Well, it is the generalization of the Iyer-Wald-Wall entropy density from Einstein-scalar EFT to Einstein-Maxwell-scalar EFT. It is proved in [9] that the Iyer-Wald-Wall entropy density is gauge invariant on C to linear order and can be made gauge invariant nonperturbatively by adjusting the nonunique higher order terms. We expect the proof can be extended to the Einstein-Maxwell-scalar EFT case. However, to delve into the covariant phase space formalism of the proof would divert somewhat from the material here.

Thus, we will solely concern ourselves with the quadratic or higher order modification terms. The gauge invariance of these terms for the HKR entropy in the Einstein-scalar case was discussed in Sec. 4 of [15], which found they are gauge invariant on C up to and including order l^4. This was done by noting that, by the HKR construction, $\varsigma^{(n)v}$ consists of terms with n derivatives that are of the form $\partial_v^{p_1} P_1 (D^k \partial_v^{p_2} P_2) Q_{n,k,p_1,p_2}$ with $P_1, P_2 \in \{K_{AB}, \partial_v \phi\}$. Using that the overall boost weight is 0, we can classify the allowed terms that can appear up to four total derivatives. The result is that only $K_{AB}, \bar{K}_{AB}, \partial_r \bar{K}_{AB}, \partial_v \phi, \partial_r \phi, \partial_r^2 \phi, \mu^{AB}$, and $\epsilon^{A_1 \ldots A_{d-2}}$ can appear, all of which are gauge invariant on $r = v = 0$ using the transformation rules given in Sec. 2.1 of [9].

The same analysis follows in the Einstein-Maxwell-scalar EFT here, with the differences being $P_1, P_2 \in \{K_{AB}, K_A, \partial_v \phi\}$ and Q_{n,k,p_1,p_2} can additionally consist of allowed Maxwell terms. The result is that $K_A, \bar{K}_A$, and $\partial_r \bar{K}_A$ can appear up to and including order l^4, all of which are still gauge invariant on $r = v = 0$. Therefore, s^v is gauge invariant to the same order as in the Einstein-scalar EFT case. As in that case, there are nongauge-invariant terms like β_A, $D_A \partial_v \phi$, and $D_A K_B$ that can appear at higher orders in l.

H. Discussion of the second law for a charged scalar field

We can ask, can we generalize our proof of the second law to the EFT of gravity, electromagnetism, and a charged scalar field as defined in Sec. V E? Our starting point in the above was the BDK entropy defined in Sec. VII C, which satisfies a linearized second law. However, such an entropy is only defined for a real uncharged scalar, and its generalization to a charged scalar does not exist in the literature. Proving such a generalization exists is beyond the scope of this paper as it would involve delving into phase space formalism, and therefore this section is merely a discussion. However, it seems reasonable that such a generalization would exist, in which case the following completes the generalization of the proof of the second law.

In the analysis of the real scalar field EFT, we could use positive boost weight quantities as a proxy for order of perturbation around a stationary black hole because in Sec. V C we proved all such quantities vanish on the horizon in equilibrium. However, in the charged scalar case things are more subtle because while, e.g., $\partial_v \phi$ may vanish in one electromagnetic gauge, it does not in another.

In our proof of the generalized zeroth law for a charged scalar in Sec. V E, we were able to prove in a *particular* choice of gauge that all positive boost weight quantities vanish on the horizon in equilibrium. However, that gauge was defined by the Killing vector symmetry which is no longer present in the dynamical setting of the second law, so we cannot use it directly. What we can infer, however, is that positive boost weight quantities made from gauge-invariant quantities like $F_{\mu\nu}$ vanish on the horizon in equilibrium *in all gauges*. Similarly, positive boost weight components of the gauged derivatives

$$(\partial_{\mu_1} - i\lambda A_{\mu_1}) \ldots (\partial_{\mu_n} - i\lambda A_{\mu_n}) \phi \tag{140}$$

vanish because if they vanish in one gauge then they vanish in all gauges.

We can apply these facts to a choice of gauge particularly suited to our affinely parametrized GNCs. By a suitable gauge transformation, we can always achieve [9]

$$A = r\eta \mathrm{d}v + A_A \mathrm{d}x^A \tag{141}$$

for some function $\eta(r, v, x^A)$ regular on the horizon. η and A_A have boost weight 0. In this gauge

$$\partial_r^p \eta|_{\mathcal{N}} = \partial_r^p F_{rv}|_{\mathcal{N}}, \quad \partial_r^q A_A|_{\mathcal{N}} = \partial_r^{q-1} F_{rA}|_{\mathcal{N}}, \quad \partial_v A_A|_{\mathcal{N}} = F_{vA}|_{\mathcal{N}} \tag{142}$$

for $p \geq 0$, $q \geq 1$, and hence all positive boost weight derivatives of η and A_A can be written as positive boost weight derivatives of $F_{\mu\nu}$ on the horizon. Similarly,

$$\partial_v^p \partial_r^q \phi|_{\mathcal{N}} = (\partial_v - i\lambda A_v)^p (\partial_r - i\lambda A_r)^q \phi|_{\mathcal{N}}, \tag{143}$$

and hence all positive boost weight derivatives of ϕ can be written as positive boost weight components of (140) (or their ∂_A derivatives) on the horizon. Therefore, in this gauge all positive boost weight quantities still vanish on the horizon in equilibrium and hence can still be used as a proxy for perturbations around a stationary black hole.

In this gauge, the leading order two-derivative part of $E_{vv}|_{\mathcal{N}}$ can be written as

$$-E_{vv}^{(0)}|_{\mathcal{N}} = \partial_v \left[\frac{1}{\sqrt{\mu}} \partial_v (\sqrt{\mu}) \right] + K_{AB} K^{AB} + \frac{1}{2} c_1(|\phi|^2) h^{AB} \partial_v A_A \partial_v A_B + |\partial_v \phi|^2. \tag{144}$$

For the higher derivative terms, let us now assume that we can generalize the BDK entropy to the charged scalar case. That is, we assume we can write

$$-E_{vv}\Big|_{\mathcal{N}} = \partial_v \left[\frac{1}{\sqrt{\mu}} \partial_v (\sqrt{\mu} s^v_{\mathrm{BDK}}) + D_A s^A \right] + \cdots \tag{145}$$

for some real entropy current $(s^v_{\mathrm{BDK}}, s^A)$ and where the ellipsis denotes terms that are quadratic in positive boost weight quantities.

We can now generalize the HKR procedure as follows. We first reduce, up to $O(l^N)$, to a set of allowed terms given by the following:

$$\text{Allowed terms:}\ \mu_{AB},\quad \mu^{AB},\quad \epsilon_{A_1\ldots A_{d-2}},\quad D^k R_{ABCD}[\mu],\quad D^k\beta_A,\quad D^k\partial_v^p K_{AB},\quad D^k\partial_r^q \bar{K}_{AB},$$
$$D^k\eta,\quad D^k\partial_v^p A_A,\quad D^k\partial_r^q A_A,\quad D^k\partial_v^p\phi,\quad D^k\partial_r^q\phi,\quad D^k\partial_v^p\phi^*,\quad D^k\partial_r^q q\phi^*. \tag{146}$$

The reduction of the metric terms follows straightforwardly in the same way as vacuum gravity by using the $E^{(0)}_{\mu\nu} = O(l)$ equations of motion. We can eliminate mixed v and r derivatives of ϕ by using $E^{(0)} = O(l)$ and evaluating $E^{(0)}$ in affinely parametrized GNCs in this gauge,

$$E^{(0)} = 2\partial_r\partial_v\phi + D^A D_A\phi + K^A_A\partial_r\phi + \beta^A D_A\phi + \bar{K}^A_A\partial_v\phi - 2i\lambda A^A D_A\phi - i\lambda\phi D^A A_A - i\lambda\eta\phi - i\lambda\beta^A A_A\phi - \lambda^2 A_A A^A\phi + \cdots, \tag{147}$$

where the ellipsis denotes terms that vanish on the horizon. The reduction of the Maxwell terms is achieved by using the equation of motion

$$E^{(0)}_\mu = c_1(|\phi|^2)\nabla^\nu F_{\mu\nu} + F_{\mu\nu}\nabla^\nu[c_1(|\phi|^2)] - 4\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\nabla^\nu[c_2(|\phi|^2)] + i\lambda[\phi^*\mathfrak{D}_\mu\phi - \phi(\mathfrak{D}_\mu\phi)^*] = O(l) \tag{148}$$

and by substituting our choice of gauge $\psi = \eta + r\partial_r\eta$, $K_A = \partial_v A_A - rD_A\eta$, $\bar{K}_A = \partial_r A_A$ into our affinely parametrized GNC expressions for $\nabla^\nu F_{\mu\nu}$ given in Appendix A 3. These allow us to eliminate v and r derivatives of η and mixed v and r derivatives of A_A.

In particular, the only positive boost weight allowed terms are $D^k\partial_v^p K_{AB}$ with $p \geq 0$ and $D^k\partial_v^p A_A$, $D^k\partial_v^p\phi$, and $D^k\partial_v^p\phi^*$ with $p \geq 1$. Therefore, we can rewrite all the terms in the ellipsis in (145), which we label H, up to $O(l^N)$ as a sum of terms of the form

$$(D^{k_1}\partial_v^{p_1}P_1)(D^{k_2}\partial_v^{p_2}P_2)Q \tag{149}$$

with $P_1, P_2 \in \{K_{AB}, \partial_v A_A, \partial_v\phi, \partial_v\phi^*\}$, and where Q is made from allowed terms.

Now, since $(s^v_{\mathrm{BDK}}, s^A)$ is real, the overall sum of these terms H is real. Hence we can pair each of these terms up with its complex conjugate (or itself if it is real) and write H as

$$H = \sum_{k_1,k_2,p_1,p_2,P_1,P_2}[(D^{k_1}\partial_v^{p_1}P_1)(D^{k_2}\partial_v^{p_2}P_2)Q_{k_1,k_2,p_1,p_2,P_1,P_2} + (D^{k_1}\partial_v^{p_1}P_1^*)(D^{k_2}\partial_v^{p_2}P_2^*)Q^*_{k_1,k_2,p_1,p_2,P_1,P_2}] \tag{150}$$

with $P_1, P_2 \in \{K_{AB}, \partial_v A_A, \partial_v\phi\}$.

We now generalize our inductive hypothesis (121) to

$$\begin{aligned} H &= \partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v\left(\sqrt{\mu}\sum_{n=0}^{m-1}l^n\varsigma^{(n)v}\right)\right] + \left(K_{AB} + \sum_{n=0}^{m-1}l^n X^{(n)}_{AB}\right)\left(K^{AB} + \sum_{n=0}^{m-1}l^n X^{(n)AB}\right) \\ &\quad + \frac{1}{2}c_1(|\phi|^2)\left(\partial_v A_A + \sum_{n=0}^{m-1}l^n X^{(n)}_A\right)\left(\partial_v A^A + \sum_{n=0}^{m-1}l^n X^{(n)A}\right) + \left(\partial_v\phi + \sum_{n=0}^{m-1}l^n X^{(n)}\right)\left(\partial_v\phi + \sum_{n=0}^{m-1}l^n X^{(n)}\right)^* \\ &\quad + D_A\sum_{n=0}^{m-1}l^n Y^{(n)A} + \sum_{n=m}^{N-1}l^n H^{(n)} + O(l^N), \end{aligned} \tag{151}$$

where the $H^{(n)}$, $X^{(n)}_{AB}$, etc. are real. To proceed the induction we manipulate the terms in H^m exactly as in Sec. VII E, except we always keep complex conjugates paired up and perform identical operations on them. This will ensure that when we get to the equivalent of (124) we can split the sum over $P_1 \in \{K_{AB}, \partial_v A_A, \partial_v\phi\}$ and get

$$\begin{aligned} H &= \partial_v\left[\frac{1}{\sqrt{\mu}}\partial_v\left(\sqrt{\mu}\sum_{n=0}^{m-1}l^n\varsigma^{(n)v}\right)\right] + \left(K_{AB} + \sum_{n=0}^{m-1}l^n X^{(n)}_{AB}\right)\left(K^{AB} + \sum_{n=0}^{m-1}l^n X^{(n)AB}\right) + 2l^m K_{AB}X^{(m)AB} \\ &\quad + \frac{1}{2}c_1(|\phi|^2)\left(\partial_v A_A + \sum_{n=0}^{m-1}l^n X^{(n)}_A\right)\left(\partial_v A^A + \sum_{n=0}^{m-1}l^n X^{(n)A}\right) + l^m c_1(|\phi|^2)\partial_v A_A X^{(m)A} + \left(\partial_v\phi + \sum_{n=0}^{m-1}l^n X^{(n)}\right) \\ &\quad \times\left(\partial_v\phi + \sum_{n=0}^{m-1}l^n X^{(n)}\right)^* + l^m\partial_v\phi X^{(m)*} + l^m\partial_v\phi^* X^{(m)} + D_A\sum_{n=0}^{m}l^n Y^{(n)A} + \sum_{n=m+1}^{N-1}l^n H^{(n)} + O(l^N), \end{aligned} \tag{152}$$

and thus we can still absorb the order l^m terms into the positive definite terms by completing the squares. The remainder terms are real, and thus the induction can proceed.

Generalizing the further modifications of Sec. VII F would follow similarly.

Thus, we can get a nonperturbative second law for a charged scalar field if we assume a BDK entropy exists in such a scenario. The procedure outlined here does not produce an entropy that is manifestly electromagnetic gauge-independent like in the real scalar field case. However, it seems reasonable this could be achieved if the hypothesized BDK entropy was gauge invariant. One could take a more careful approach to the gauge field, for example, by keeping derivatives of ϕ in terms of gauged derivatives $(D_A - i\lambda A_A)$, $(\partial_v - i\lambda A_v)$, etc.

VIII. DISCUSSION

This paper adds another brick in the wall of proving the laws of black hole mechanics for higher derivative theories of gravity. To summarize where it leaves us, we have a zeroth law, first law, and second law for the EFT regime of higher derivative theories of gravity, electromagnetism, and a real scalar field. The dynamical black hole entropy which is constructed along the way is independent of electromagnetic gauge for theories with any number of derivatives and is purely geometric for theories with up to six derivatives (order l^4). It reduces to the standard factor of the area in two-derivative GR and reduces to the Wald entropy in equilibrium for any number of derivatives. In addition, we have shown the zeroth law continues to hold if the scalar is charged, and there is strong motivation to think the second law would hold. This suggests a more general result involving theories of gravity with any matter fields that satisfy the NEC at two-derivative level may be provable. For example, it would be interesting to extend the result to Yang-Mills fields.

Our proofs of the zeroth and second laws are perhaps not as general as we would like them to be. For the zeroth law we required our solution to be analytic in l and excluded certain horizon topologies. For the second law we required our horizon to be smooth. Recent work [19] has considered the case of nonsmooth horizons and suggested there may be additional contributions to black hole entropy motivated by quantum entanglement entropy. They also demonstrate that certain terms in the entropy current defined above can diverge when integrated over nonsmooth features on the horizon. Furthermore, our definition of the entropy is dependent on our choice of GNCs above order l^4, which raises question about the uniqueness of black hole entropy. Therefore, there is still work to be done in this area.

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APPENDIX

1. Evaluation of $E_\tau^{(0)}$ on the horizon

We would like to evaluate

$$E_\tau^{(0)}[\Phi_J] = g^{\alpha\beta}\nabla_\alpha\left[c_1(\phi)F_{\tau\beta} - 4c_2(\phi)F^{\gamma\delta}\epsilon_{\tau\beta\gamma\delta}\right] \quad \text{(A1)}$$

in Killing vector GNCs on the horizon. The metric in Killing vector GNCs is given by

$$g = 2\mathrm{d}\tau\mathrm{d}\rho - \rho X(\rho, x^C)\mathrm{d}\tau^2 + 2\rho\omega_A(\rho, x^C)\mathrm{d}\tau\mathrm{d}x^A + h_{AB}(\rho, x^C)\mathrm{d}x^A\mathrm{d}x^B. \quad \text{(A2)}$$

On the horizon it is simply

$$g|_{\rho=0} = 2\mathrm{d}\tau\mathrm{d}\rho + h_{AB}\mathrm{d}x^A\mathrm{d}x^B. \quad \text{(A3)}$$

We can calculate the Christoffel symbols on the horizon in this metric. The nonzero components are

$$\Gamma^\tau_{\tau\tau} = \frac{1}{2}X, \qquad \Gamma^\tau_{\tau A} = -\frac{1}{2}\omega_A, \qquad \Gamma^\tau_{AB} = -\frac{1}{2}\partial_\rho h_{AB}, \qquad \Gamma^\rho_{\rho\tau} = -\frac{1}{2}X,$$
$$\Gamma^\rho_{\rho A} = \frac{1}{2}\omega_A, \qquad \Gamma^A_{\rho\tau} = \frac{1}{2}\omega_B h^{AB}, \qquad \Gamma^A_{\rho B} = \frac{1}{2}\partial_\rho h_{BC}h^{AC}, \qquad \Gamma^A_{BC} = \Gamma^A_{BC}[h], \quad \text{(A4)}$$

where $\Gamma^A_{BC}[h]$ is the Christoffel symbol built out of the induced metric h_{AB}.

Now, for notational convenience, let

$$H_{\alpha\beta} = c_1(\phi)F_{\alpha\beta} - 4c_2(\phi)F^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}. \quad \text{(A5)}$$

Note this is antisymmetric. Then we can evaluate

$$E_\tau^{(0)}[\Phi_J]\Big|_{\rho=0} = g^{\alpha\beta}\nabla_\alpha H_{\tau\beta}$$
$$= \nabla_\tau H_{\tau\rho} + h^{AB}\nabla_A H_{\tau B}$$
$$= -\Gamma^\mu_{\tau\tau}H_{\mu\rho} - \Gamma^\mu_{\tau\rho}H_{\tau\mu} + h^{AB}(\partial_A H_{\tau B} - \Gamma^\mu_{A\tau}H_{\mu B} - \Gamma^\mu_{AB}H_{\tau\mu}), \tag{A6}$$

where in the last line we used the fact that everything is independent of τ. Substituting in the Christoffel symbols computed above, we get some cancellations with the result being

$$E_\tau^{(0)}[\Phi_J]\Big|_{\rho=0} = h^{AB}(\partial_A H_{\tau B} - \Gamma^C_{AB}H_{\tau C})$$
$$= h^{AB}\mathcal{D}_A H_{\tau B}, \tag{A7}$$

where $\mathcal{D}_A$ is the covariant derivative with respect to h_{AB} and only acts on $A, B, \ldots$ indices. Finally,

$$H_{\tau A}\Big|_{\rho=0} = c_1(\phi)F_{\tau A} - 4c_2(\phi)F^{\gamma\delta}\epsilon_{\tau A\gamma\delta}$$
$$= c_1(\phi)F_{\tau A} - 8c_2(\phi)\epsilon_A^{\ B}F_{\tau B}, \tag{A8}$$

where we used our convention $\epsilon_{AB} = \epsilon_{\rho\tau AB}$. Therefore,

$$E_\tau^{(0)}[\Phi_J]\Big|_{\rho=0} = h^{AB}\mathcal{D}_A\Big[c_1(\phi)F_{\tau B} - 8c_2(\phi)\epsilon_B^{\ C}F_{\tau C}\Big]. \tag{A9}$$

2. Zeroth law for a charged scalar field in the case $\phi^{(0)}|_{\rho=0} \equiv 0$

We now deal with the case excluded in Sec. V E, in which $\phi^{(0)}$ vanishes identically on the horizon. Moreover, we assume ϕ vanishes on the horizon up to and including order l^m for some $m \geq 0$, i.e., $\phi^{[m]}|_{\rho=0} \equiv 0$.

The proof in Sec. V E breaks down for the following reasons: in the base case of our induction we can no longer extract $A_\tau^{(0)}|_{\rho=0} = 0$ from the equation $A_\tau^{(0)}\phi^{(0)}|_{\rho=0} = 0$, and similarly in the inductive step, we can no longer prove $A_\tau^{(k)}|_{\rho=0} = 0$. These were essential steps for the induction to proceed because they were used to prove the positive boost weight quantities $\partial_{A_1}\ldots\partial_{A_n}\partial_v^q A_v$ vanish on the horizon at each order [see Eq. (76)]. Without this fact, we cannot ignore the higher derivative parts of the equations of motion $E_{\tau A}^{[k]}[\Phi_J^{[k-1]}]$ and $E_\tau^{[k]}[\Phi_J^{[k-1]}]$ at each order in l.

The solution comes from the fact that $E_{\mu\nu}$ and E_μ are electromagnetic gauge invariant. This means any appearance of A_μ is either inside a $F_{\mu\nu}$ or arises from a term of schematic form $(\mathfrak{D}^p\phi)^*\mathfrak{D}^q\phi$, in which case it will appear in the combination $\partial^a\phi^*\partial^b A_\mu\partial^c\phi$. In the former case, we can use the methods from Sec. V C to show it vanishes at each order in the induction. In the latter case, we will show that the vanishing of $\phi^{[m]}$ on the horizon implies positive boost weight quantities involving $\partial^a\phi^*\partial^b A_\mu\partial^c\phi$ also vanish to sufficiently high order for the induction to proceed.

From $E_{\tau\tau}^{(0)}[\Phi_J^{(0)}]|_{\rho=0} = 0$ and $E_{\tau A}^{(0)}[\Phi_J^{(0)}]|_{\rho=0} = 0$ we can still deduce $F_{\tau A}^{(0)}|_{\rho=0} = 0$ and $\partial_A X^{(0)}|_{\rho=0} = 0$, respectively, and so we still have the generalized zeroth law holding at order l^0. This means $\kappa^{(0)}$ is constant. We split the analysis into two cases: (1) $\kappa^{(0)} \neq 0$ and (2) $\kappa^{(0)} = 0$.

a. Case 1: $\kappa^{(0)} \neq 0$

To proceed, we prove a lemma:

Lemma 1. If ϕ vanishes on the horizon up to and including order l^m, and $\kappa^{(0)} \neq 0$, then all derivatives of ϕ vanish on the horizon up to and including order l^m.

Proof. Clearly all tangential derivatives $\partial_\tau^p\partial^q_{A_1\ldots A_q}\phi$ vanish on the horizon up to and including order l^m. To investigate the remaining ρ derivatives, we will inspect the scalar field equation of motion, $E[\Phi_J] = 0$. At order l^0, this is $E^{(0)}[\Phi_J^{(0)}] = 0$. Equation (66) has the explicit form for $E^{(0)}[\Phi_J]$. We can evaluate it in Killing vector GNCs in our choice of electromagnetic gauge and find

$$E^{(0)}[\Phi_J] = (X - 2i\lambda A_\tau)\partial_\rho\phi + \rho X\partial_\rho^2\phi$$
$$+ \rho^2 h^{AB}\omega_A\omega_B\partial_\rho^2\phi + \cdots, \tag{A10}$$

where the ellipsis denotes terms that are proportional to ϕ or its spatial derivatives $\partial^q_{A_1\ldots A_q}\phi$. Therefore, plugging this and $\phi^{(0)}|_{\rho=0} \equiv 0$ into $E^{(0)}[\Phi_J^{(0)}]|_{\rho=0} = 0$ gives

$$(X^{(0)} - 2i\lambda A_\tau^{(0)})\partial_\rho\phi^{(0)}|_{\rho=0} = 0. \tag{A11}$$

We have that $X^{(0)}|_{\rho=0} = 2\kappa^{(0)}$, which is a constant we have assumed is nonzero. Therefore, $\partial_\rho\phi^{(0)}|_{\rho=0} \equiv 0$. Inductively assuming $\partial_\rho^k\phi^{(0)}|_{\rho=0} \equiv 0$ for all $k \leq s$ for some $s \geq 1$, we substitute (A10) into $\partial_\rho^s E^{(0)}[\Phi_J^{(0)}]|_{\rho=0} = 0$ to get

$$\Big[(s+1)X^{(0)} - 2i\lambda A_\tau^{(0)}\Big]\partial_\rho^{s+1}\phi^{(0)}|_{\rho=0} = 0, \tag{A12}$$

and so $\partial_\rho^{s+1}\phi^{(0)}|_{\rho=0} \equiv 0$. Hence all derivatives of $\phi^{(0)}$ vanish on $\mathcal{N}$, which proves the case $m = 0$. This means any appearance of ϕ is at least order l on the horizon.

Now assume $m \geq 1$. Inductively, let us assume all derivatives of ϕ vanish on the horizon up to and including order l^n for some n, $0 \leq n < m$. Therefore, any appearance of ϕ or its derivatives in $E_I[\Phi_J]|_{\rho=0}$ is at least order l^{n+1}.

The Lagrangian $\mathcal{L}$ is electromagnetic gauge invariant, therefore wherever a ϕ or its derivatives appears, it must be multiplied by ϕ^* or its derivatives, and vice versa. This means that every term in $E[\Phi_J]$, which is the equation of motion arising from varying ϕ^*, must be at least linear in ϕ or its derivatives.

Therefore, $E[\Phi_J]|_{\rho=0}$ is already $O(l^{n+1})$, and its order l^{n+1} part is $E^{(0)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}, l^{n+1}\phi^{(n+1)}]|_{\rho=0}$. This is 0 by the equations of motion, and using (A10) and $\phi^{(n+1)}|_{\rho=0} \equiv 0$ similar to above, we get

$$(X^{(0)} - 2i\lambda A_\tau^{(0)})\partial_\rho \phi^{(n+1)}|_{\rho=0} = 0. \tag{A13}$$

This implies $\partial_\rho \phi^{(n+1)}|_{\rho=0} \equiv 0$, and similarly we can successively look at $\partial_\rho^s E^{(0)}[g_{\mu\nu}^{(0)}, A_\mu^{(0)}, l^{n+1}\phi^{(n+1)}]|_{\rho=0} = 0$ to deduce $\partial_\rho^{s+1}\phi^{(n+1)}|_{\rho=0} \equiv 0$. Therefore, all derivatives of ϕ vanish on $\mathcal{N}$ up to and including l^{n+1} and so the induction proceeds. ■

This lemma implies that if ϕ vanishes to *all* orders on the horizon, then all its derivatives vanish on the horizon. Therefore, a term of the form $\partial^a\phi^*\partial^b A_\mu \partial^c \phi$ would identically vanish on the horizon, and so A_μ could only appear inside an $F_{\mu\nu}$. But in this case there would be no new positive boost weight quantities to deal with over the real scalar field case, and the equations of motion would look identical. Hence, the generalized zeroth law would follow trivially from the real scalar field proof above.

Therefore, let us assume that $\phi^{(m+1)}$ is the lowest order at which ϕ does not identically vanish on the horizon, i.e., that $\phi^{[m]}|_{\rho=0} \equiv 0$ and $\phi^{(m+1)}(x^A)|_{\rho=0}$ is nonzero for some x^A. Then by the above lemma, all derivatives of ϕ vanish on the horizon up to and including order l^m, and so any appearance of ϕ is $O(l^{m+1})$ on the horizon. But this means the problematic terms of the form $\partial^a\phi^*\partial^b A_\mu\partial^c\phi$ are already at least order l^{2m+2}, and so will not appear in our induction until that order.

To make this precise, we take our inductive hypothesis to be $\partial_A X^{[k-1]}|_{\rho=0} = 0$, $F_{\tau A}^{[k-1]}|_{\rho=0} = 0$ and $A_\tau^{[k-2m-3]}|_{\rho=0} = 0$. This is true for the base case $k = 1$ because we proved above that $\partial_A X^{(0)}|_{\rho=0} = 0$ and $F_{\tau A}^{(0)}|_{\rho=0} = 0$, and trivially $A_\tau^{(n)}|_{\rho=0} = 0$ for $n < 0$ by analyticity in l.

Assuming the hypothesis holds, we would like to study $E_{\tau A}|_{\rho=0}$ and $E_\tau|_{\rho=0}$ at order l^k. By gauge invariance, any appearance of A_μ not inside an $F_{\mu\nu}$ will come multiplied by $\partial^a\phi^*\partial^b\phi$ and so can only involve $A_\mu^{[k-2m-2]}$. Therefore, separating out the dependence on A_μ and $F_{\mu\nu}$, we have the following:

$$\text{at order } l^k, \quad E_I[\Phi_J]\Big|_{\rho=0} = E_I^{[k]}[g_{\mu\nu}^{[k]}, F_{\mu\nu}^{[k]}, \phi^{[k]}, A_\mu^{[k-2m-2]}]|_{\rho=0} \tag{A14}$$

for $I = (\tau A)$ or $I = \tau$. Additionally, the highest order pieces $g_{\mu\nu}^{(k)}, F_{\mu\nu}^{(k)}, \phi^{(k)}, A_\mu^{(k-2m-2)}$ can only appear in $E_I^{(0)}$ because they will already come with l^k,

at order l^k,

$$E_I[\Phi_J]|_{\rho=0} = E_I^{(0)}[g_{\mu\nu}^{[k]}, F_{\mu\nu}^{[k]}, \phi^{[k]}, A_\mu^{[k-2m-2]}]|_{\rho=0} + \sum_{s=1}^{k} l^s E_I^{(s)}[g_{\mu\nu}^{[k-1]}, F_{\mu\nu}^{[k-1]}, \phi^{[k-1]}, A_\mu^{[k-2m-3]}]|_{\rho=0}. \tag{A15}$$

The first two inductive hypotheses $\partial_A X^{[k-1]}|_{\rho=0} = 0$ and $F_{\tau A}^{[k-1]}|_{\rho=0} = 0$ imply positive boost weight quantities involving $g_{\mu\nu}^{[k-1]}, F_{\mu\nu}^{[k-1]}$, and $\phi^{[k-1]}$ vanish on the horizon by Secs. V C and V D. Furthermore, as discussed around Eqs. (74)–(76), combining them with the third hypothesis $A_\tau^{[k-2m-3]}|_{\rho=0} = 0$ will imply all positive boost weight quantities involving $A_\mu^{[k-2m-3]}$ vanish on the horizon.

Therefore, for the components $I = (\tau A)$ and $I = \tau$ we see that the higher derivative parts still vanish on the horizon because they are proportional to positive boost weight components when we make the coordinate transformation $\rho = r(\kappa^{[k-1]}v + 1)$, $\tau = \frac{1}{\kappa^{[k-1]}}\log(\kappa^{[k-1]}v + 1)$ and only involve the fields $g_{\mu\nu}^{[k-1]}, F_{\mu\nu}^{[k-1]}, \phi^{[k-1]}, A_\mu^{[k-2m-3]}$. Thus, we need only look at $E_I^{(0)}|_{\rho=0}$ for these components.

First up, $E_\tau[\Phi_J]|_{\rho=0}$, is

at order l^k,

$$E_\tau[\Phi_J]|_{\rho=0} = 2\lambda^2 l^k |\phi^{(m+1)}|^2 A_\tau^{(k-2m-2)} + l^k h^{(0)AB}\mathcal{D}_A^{(0)}[c_1(0)F_{\tau B}^{(k)} - 8c_2(0)\epsilon_B^{(0)C}F_{\tau C}^{(k)}] = 0. \tag{A16}$$

Integrate this against $\sqrt{h^{(0)}}$ over $C(\tau)$ to get

$$2\lambda^2 l^k \int_{C(\tau)} \mathrm{d}^{d-2}x\sqrt{h^{(0)}}|\phi^{(m+1)}|^2 A_\tau^{(k-2m-2)} = 0, \tag{A17}$$

where the integral over the second term vanished because it was a total derivative. We have $\partial_A A_\tau^{(k-2m-2)}|_{\rho=0} = -F_{\tau A}^{(k-2m-2)}|_{\rho=0} = 0$ by our inductive hypothesis, so $A_\tau^{(k-2m-2)}|_{\rho=0}$ is a constant. Hence we can take it out of the integral to get

$$A_\tau^{(k-2m-2)}|_{\rho=0}\int_{C(\tau)} \mathrm{d}^{d-2}x\sqrt{h^{(0)}}|\phi^{(m+1)}|^2 = 0. \tag{A18}$$

However, $\phi^{(m+1)}$ is the lowest order piece of ϕ that does not identically vanish on the horizon, and so the integral is nonzero. Therefore, $A_\tau^{(k-2m-2)}|_{\rho=0} = 0$.

Plugging this back into (A16), let us now integrate it against $\sqrt{h^{(0)}}A_\tau^{(k)}$ over $C(\tau)$. In a similar fashion to the $\phi^{(0)}|_{\rho=0} \not\equiv 0$ case, we get

$$\int_{C(\tau)} \mathrm{d}^{d-2}x\sqrt{h^{(0)}}c_1(0)h^{(0)AB}\left(\partial_A A_\tau^{(k)}\right)\left(\partial_B A_\tau^{(k)}\right) = 0 \tag{A19}$$

and thus $F_{\tau A}^{(k)}|_{\rho=0} = 0$.

Finally, we look at $E_{\tau A}[\Phi_J]|_{\rho=0}$ at order l^k. This is at order l^k,

$$E_{\tau A}[\Phi_J]|_{\rho=0} = -\frac{1}{2}l^k\partial_A X^{(k)} - \frac{1}{2}il^k\lambda A_\tau^{(k-2m-2)}\left(\phi^{(m+1)*}\partial_A\phi^{(m+1)} - \phi^{(m+1)}\partial_A\phi^{(m+1)*} - 2i\lambda A_A^{(0)}|\phi^{(m+1)}|^2\right) - \frac{1}{2}c_1(0)\left(F_{AB}^{(0)}h^{(0)BC} - F_{\tau\rho}^{(0)}\delta_A^C\right)l^kF_{\tau C}^{(k)} = 0. \tag{A20}$$

Substituting in $A_\tau^{(k-2m-2)}|_{\rho=0} = 0$ and $F_{\tau A}^{(k)}|_{\rho=0} = 0$ we get $\partial_A X^{(k)}|_{\rho=0} = 0$, and thus the induction proceeds and we have proved the generalized zeroth law.

b. Case 2: $\kappa^{(0)} = 0$

Finally, we deal with the case $\kappa^{(0)} = 0$. Moreover, we assume $\kappa^{[n]} = 0$ for some $n \geq 0$. The proof now gets rather technical and mostly involves chasing powers of l. The physical relevance of this proof is questionable as we are heavily relying on analyticity in l, however we include it for completeness.

In this case we cannot apply Lemma 1. Our aim is still to prove $A_\tau|_{\rho=0} = 0$. Suppose we have an obstruction to this, in that $A_\tau^{[N]}|_{\rho=0} = 0$ but $A_\tau^{(N+1)}|_{\rho=0} \neq 0$ for some N. We will try to find a contradiction. In the previous cases we used the $2\lambda^2 A_\tau|\phi|^2$ term in $E_\tau^{(0)}|_{\rho=0}$ to prove A_τ vanished at each order. However, it is order l^{2m+N+3}, therefore we first need to be able to run the induction all the way up to that order before we can hope to conclude $A_\tau^{(N+1)}|_{\rho=0} = 0$. If $N < n$ it turns out we can prove a lemma that allows us to do this.

Before stating and proving the lemma, it is worth emphasizing some logic we will use repeatedly below. Suppose we have proved $\partial_A X^{[s]}|_{\rho=0} = 0$ and $F_{\tau A}^{[s]}|_{\rho=0} = 0$ for some $s \geq 0$. Then we can change to affinely parametrized coordinates via $\rho = r(\kappa^{[s]}v + 1)$, $\tau = \frac{1}{\kappa^{[s]}}\log(\kappa^{[s]}v + 1)$ in which all positive boost weight quantities involving $g_{\mu\nu}^{[s]}$, $F_{\mu\nu}^{[s]}$, $\phi^{[s]}$, and $A_\mu^{[s]}$ vanish on the horizon except from $\partial_{A_1}\ldots\partial_{A_n}\partial_v^q A_v^{[s]}$. But earlier we calculated

$$\partial_v^q A_v^{[s]}|_{r=0} = \frac{(-\kappa^{[s]})^q}{(\kappa^{[s]}v + 1)^{q+1}}A_\tau^{[s]}|_{\rho=0}. \tag{A21}$$

Therefore, if $\kappa^{[n]} = 0$ and $A_\tau^{[N]}|_{\rho=0} = 0$, then $\partial_v^q A_v^{[s]}|_{r=0}$ is at least order $l^{q(n+1)+N+1}$. Taking ∂_A derivatives does not change the order on the horizon, and so we will not explicitly mention them in the analysis going forward. Furthermore, we can calculate that the nonpositive boost weight quantities made from $\phi^{[s]}$, namely, $\partial_v^a\partial_r^b\phi^{[s]}$ with $b \geq a$ have the following form on the horizon:

$$\partial_v^a\partial_r^b\phi^{[s]}|_{r=0} = \frac{b!}{(b-a)!}\left(\kappa^{[q]}\right)^a\left(\kappa^{[q]}v + 1\right)^{b-a}\partial_\rho^b\phi^{[s]}|_{\rho=0}. \tag{A22}$$

Therefore, if we also happen to know that $\partial_\rho^b\phi^{[N_b]}|_{\rho=0} = 0$, then $\partial_v^a\partial_r^b\phi^{[s]}|_{r=0}$ is at least order $l^{a(n+1)+N_b+1}$.

Onto the lemma:

Lemma 2. If $\phi^{[m]}|_{\rho=0} = 0$, $\kappa^{[n]} = 0$, $A_\tau^{[N]}|_{\rho=0} = 0$, and $A_\tau^{(N+1)}|_{\rho=0} \neq 0$ with $N < n$ then

$$\partial_A X^{[2m+N+2]}|_{\rho=0} = 0, \qquad F_{\tau A}^{[2m+N+2]}|_{\rho=0} = 0, \qquad \text{and} \quad \forall\ p \geq 1 \partial_\rho^p\phi^{[m-p(N+1)]}|_{\rho=0} = 0. \tag{A23}$$

Proof. If $A_\tau^{(0)}|_{\rho=0} \neq 0$, i.e., $N = -1$, then the proof follows in the same way as Lemma 1. This is because Eq. (A11) becomes $-2i\lambda A_\tau^{(0)}\partial_\rho\phi^{(0)}|_{\rho=0} = 0$, from which we can still conclude $\partial_\rho\phi^{(0)}|_{\rho=0} = 0$. Similarly in (A12) and (A13) we still find the ρ derivatives of ϕ vanish, and so can conclude $\partial_\rho^p\phi^{[m]}|_{\rho=0} = 0\ \forall\ p$ as (A23) requires. We then trivially get $\partial_A X^{[2m+2]}|_{\rho=0} = 0$ and $F_{\tau A}^{[2m+2]}|_{\rho=0} = 0$ from the induction detailed in the rest of case 1 (and we get our contradiction that $A_\tau^{(0)}|_{\rho=0} = 0$ if there is some m such that $\phi^{(m+1)} \not\equiv 0$).

Therefore, let $N \geq 0$. We now proceed by induction on $0 \leq k < m$ with hypothesis

$$\partial_A X^{[2k+N+1]}|_{\rho=0} = 0, \qquad F_{\tau A}^{[2k+N+1]}|_{\rho=0} = 0, \qquad \text{and} \quad \forall\ p \geq 1 \partial_\rho^p\phi^{[k-p(N+1)]}|_{\rho=0} = 0. \tag{A24}$$

For the base case $k = 0$, we note that $A_\tau^{[N]}|_{\rho=0} = 0$ and $\partial_A X^{[N]}|_{\rho=0} = 0$ because $\kappa^{[n]} = 0$ and $N < n$. This implies all positive boost weight quantities made from $\Phi^{[N]}$ vanish on the horizon. Therefore, at order l^{N+1}, the higher derivative pieces in $E_\tau|_{\rho=0}$ and $E_{\tau A}|_{\rho=0}$ vanish, and using the same method as the real scalar field case we can prove $\partial_A X^{[N+1]}|_{\rho=0} = 0$ and $F_{\tau A}^{[N+1]}|_{\rho=0} = 0$. Furthermore, $\partial_\rho^p \phi^{[-p(N+1)]}|_{\rho=0} = 0 \ \forall \ p \geq 1$ trivially.

So let us assume the hypothesis holds for k. For $I = (\tau A)$ and $I = \tau$ we study

$$\text{at order } l^{2k+N+2},$$
$$E_I[\Phi_J]|_{\rho=0} = E_I^{(0)}[\Phi_J^{[2k+N+2]}]|_{\rho=0} + l \sum_{s=1}^{2k+N+2} l^{s-1} E_I^{(s)}[\Phi_J^{[2k+N+1]}]|_{\rho=0}. \tag{A25}$$

We again change coordinates via $\rho = r(\kappa^{[2k+N+1]} v + 1)$, $\tau = \frac{1}{\kappa^{[2k+N+1]}} \log(\kappa^{[2k+N+1]} v + 1)$, in which $I = (\tau A)$ and $I = \tau$ are proportional to positive boost weight components. As discussed above, the only positive boost weight quantity made from $\Phi_J^{[2k+N+1]}$ that does not vanish on the horizon by the induction hypotheses is $\partial_v^q A_v^{[2k+N+1]}$. By gauge invariance, if an A_μ appears, so must a $\partial^a \phi^* \partial^b \phi$. Therefore, the only positive boost weight terms left must have the combination

$$(\partial_v^{a_1} \partial_r^{b_1} \phi^*)(\partial_v^{q_1} A_v)\ldots(\partial_v^{q_M} A_v)(\partial_v^{a_2} \partial_r^{b_2} \phi) \tag{A26}$$

for some $M \geq 1$, with $b_1 \geq a_1$, $b_2 \geq a_2$, and overall boost weight $a_1 + a_2 - b_1 - b_2 + M + \sum q_i \geq 1$. But by (A21), (A22), and the inductive hypothesis [which implies $N_b = k - b(N+1)$], on the horizon this is of order

$$\begin{aligned} &2k+2+\left(a_1+a_2+\sum q_i\right)(n+1)+(M-b_1-b_2)(N+1) \\ &\geq 2k+2+\left(a_1+a_2-b_1-b_2+M+\sum q_i\right)(N+1) \\ &\geq 2k+2+N+1. \end{aligned} \tag{A27}$$

Therefore, since the higher derivative terms in (A25) come with at least one extra power of l, the remaining positive boost weight quantities (A26) cannot appear until order l^{2k+N+4}. Hence the higher derivative terms can be safely ignored at order l^{2k+N+2} and we can get $\partial_A X^{[2k+N+2]}|_{\rho=0} = 0$ and $F_{\tau A}^{[2k+N+2]}|_{\rho=0} = 0$ using the same method as the real scalar field case ($k < m$ so the $A_\tau \phi^* \phi$ terms do not appear in $E_I^{(0)}$ at this order). Furthermore, we can repeat this at order l^{2k+N+3} to get $\partial_A X^{[2k+N+3]}|_{\rho=0} = 0$ and $F_{\tau A}^{[2k+N+3]}|_{\rho=0} = 0$, which lets the first two inductive hypotheses proceed.

Turning to the third hypothesis, we start with $p = 1$, i.e., we show $\partial_\rho \phi^{(k+1-(N+1))}|_{\rho=0} = 0$. To do this, we study

$$\text{at order } l^{k+1},$$
$$E[\Phi_J]|_{\rho=0} = E^{(0)}[\Phi_J^{[k+1]}]|_{\rho=0} + l \sum_{s=1}^{k+1} l^{s-1} E^{(s)}[\Phi_J^{[k]}]|_{\rho=0}. \tag{A28}$$

As ever, make the coordinate change $\rho = r(\kappa^{[k]} v + 1)$, $\tau = \frac{1}{\kappa^{[k]}} \log(\kappa^{[k]} v + 1)$ in the higher derivative terms. As discussed in Lemma 1, every term in $E[\Phi_J]$ is at least linear in ϕ or its derivatives. $\phi^{[m]}|_{\rho=0} = 0$ so it cannot appear undifferentiated in the above. By (A22) and the inductive hypothesis, zero boost weight derivatives $(\partial_v \partial_r)^a \phi^{[k]}$ are at least order $a(n+1) + k - a(N+1) + 1 \geq k+1$ on the horizon, and so cannot appear in the higher derivative terms due to the extra factor of l. Furthermore, all positive boost weight derivatives of $\phi^{[k]}$ also vanish on the horizon because $\partial_A X^{[k]}|_{\rho=0} = 0$. This leaves negative boost weight derivatives of ϕ, however, since $E[\Phi_J]$ is overall zero boost weight, these must come multiplied by positive boost weight factors. The only $\Phi_J^{[k]}$ positive boost weight quantities that are nonvanishing on the horizon are $\partial_v^q A_v^{[k]}$, hence we are left with combinations of the form

$$(\partial_v^a \partial_r^b \phi)(\partial_v^{q_1} A_v)\ldots(\partial_v^{q_M} A_v) \tag{A29}$$

with $b > a$ and $M \geq 1$ and overall non-negative boost weight $a - b + M + \sum q_i \geq 0$. But on the horizon this is of order

$$\begin{aligned} &k+1+\left(a+\sum q_i\right)(n+1)+(M-b)(N+1) \\ &\geq k+1+\left(a-b+M+\sum q_i\right)(N+1) \\ &\geq k+1, \end{aligned} \tag{A30}$$

and so once again cannot appear in the higher derivative terms at order l^{k+1} due to the extra factor of l. Therefore, we can safely ignore these terms and just look at $E^{(0)}[\Phi_J^{[k+1]}]|_{\rho=0}$, which gives

$$\text{at order } l^{k+1},$$
$$E[\Phi_J]|_{\rho=0} = -2i\lambda l^{k+1} A_\tau^{(N+1)} \partial_\rho \phi^{(k+1-(N+1))}|_{\rho=0} = 0, \tag{A31}$$

therefore $\partial_\rho \phi^{(k+1-(N+1))}|_{\rho=0} = 0$ as desired. Now induct on the number of ρ derivatives, i.e., assume $\partial_\rho^p \phi^{[k+1-p(N+1)]}|_{\rho=0} = 0$ for all $p \leq s$ for some $s \geq 1$.

Then look at $(\nabla_\rho)^s E[\Phi_J]=0$ on the horizon at order $l^{k+1-s(N+1)}$. Change coordinates and note that the result has overall boost weight $-s$. Once again, it can be shown that the higher derivative terms can be ignored by calculating the order of the terms like (A29) that can appear and using the inductive hypotheses. We end up with

$$\text{at order} l^{k+1-s(N+1)},\quad (\nabla_\rho)^s E[\Phi_J]|_{\rho=0} = -2i\lambda l^{k+1-s(N+1)} A_\tau^{(N+1)} \partial_\rho^{s+1} \times \phi^{(k+1-(s+1)(N+1))}|_{\rho=0} = 0, \tag{A32}$$

and therefore $\partial_\rho^{s+1}\phi^{(k+1-(s+1)(N+1))}|_{\rho=0}=0$. This completes the induction over ρ derivatives, which completes the overall induction.

We can run the induction until $k=m$ to get $\partial_A X^{[2m+N+1]}|_{\rho=0}=0$, $F_{\tau A}^{[2m+N+1]}|_{\rho=0}=0$, and $\forall\ p\geq 1 \partial_\rho^p\phi^{[m-p(N+1)]}|_{\rho=0}=0$. We cannot go a full step further because $\phi^{(m+1)}|_{\rho=0}$ is not necessarily vanishing in $E^{(0)}[\Phi_J^{[m+1]}]|_{\rho=0}$ at order l^{m+1} and so Eq. (A31) would be much more complicated. However, we can go one order further in $E_I[\Phi_J]|_{\rho=0}$ for $I=\tau$ and $I=(\tau A)$ because the problem terms (A26) now do not appear until order l^{2m+N+4} in the higher derivative terms, and the $A_\tau\phi^*\phi$ terms do not appear in $E_I^{(0)}$ until order l^{2m+N+3}. Hence we can prove $\partial_A X^{[2m+N+2]}|_{\rho=0}=0$, $F_{\tau A}^{[2m+N+2]}|_{\rho=0}=0$, which completes the lemma. ∎

If ϕ vanishes to all orders on the horizon (i.e., we can take $m\to\infty$), then we see $\partial_A X|_{\rho=0}$ and $F_{\tau A}|_{\rho=0}$ also vanish to all orders and so the generalized zeroth law holds. Therefore, let us assume that $\phi^{(m+1)}(x^A)|_{\rho=0}$ is nonzero for some x^A. Let us try to go one step further in the induction and look at $E_\tau[\Phi_J]|_{\rho=0}$ at order l^{2m+N+3}. The higher derivative terms still vanish because terms of the form (A26) do not appear until order l^{2m+N+4} as discussed above. However, $A_\tau|\phi|^2|_{\rho=0}=l^{2m+N+3}A_\tau^{(N+1)}|\phi^{(m+1)}|^2|_{\rho=0}$ and so appears in $E_\tau^{(0)}[\Phi_J]|_{\rho=0}$,

$$\text{at order } l^{2m+N+3},\qquad E_\tau[\Phi_J]\Big|_{\rho=0} = 2\lambda^2 l^{2m+N+3}A_\tau^{(N+1)}|\phi^{(m+1)}|^2 + l^{2m+N+3}h^{(0)AB}\mathcal{D}_A^{(0)}\Big[c_1(0)F_{\tau B}^{(2m+N+3)} - 8c_2(0)\epsilon_B^{(0)C}F_{\tau C}^{(2m+N+3)}\Big]=0. \tag{A33}$$

However, we can now repeat the same analysis as case 1 by integrating against $\sqrt{h^{(0)}}$ over $C(\tau)$ to get $A_\tau^{(N+1)}|_{\rho=0}=0$, which contradicts our assumption that $A_\tau^{(N+1)}|_{\rho=0}\neq 0$.

Lemma 2 only holds for $N<n$, and hence this contradiction only applies up to $N=n-1$. Therefore, we can conclude one of the following must be true from the logic so far: (a) $A_\tau^{[n]}|_{\rho=0}=0$ or (b) $A_\tau^{[n]}|_{\rho=0}\neq 0$ and $\phi|_{\rho=0}\equiv 0$ to all orders in which case the generalized zeroth law holds.

Taking forward (a), if κ vanishes to all orders (i.e., we can take $n\to\infty$) then so does A_τ, which would prove the generalized zeroth law. Therefore, we assume $\kappa^{(n+1)}\neq 0$, in which case we can prove another lemma:

Lemma 3. If $\phi^{[m]}|_{\rho=0}=0$, $A_\tau^{[n]}|_{\rho=0}=0$, $\kappa^{[n]}=0$, and $\kappa^{(n+1)}|_{\rho=0}\neq 0$, then

$$\partial_A X^{[2m+n+2]}|_{\rho=0}=0,\qquad F_{\tau A}^{[2m+n+2]}|_{\rho=0}=0,\quad \text{and}\quad \forall\ p\geq 1\partial_\rho^p\phi^{[m-p(n+1)]}|_{\rho=0}=0. \tag{A34}$$

Proof. This proof follows using the same steps as the proof of Lemma 2 with $N=n$ except we use the nonvanishing of $\kappa^{(n+1)}$ rather than $A^{(N+1)}$ to conclude the vanishing of ρ derivatives of ϕ. For example, (A31) becomes

$$\text{at order } l^{k+1},\qquad E[\Phi_J]|_{\rho=0} = l^{k+1}(X^{(n+1)}-2i\lambda A_\tau^{(n+1)})\partial_\rho\phi^{(k+1-(n+1))}|_{\rho=0}=0. \tag{A35}$$

$X^{(n+1)}|_{\rho=0}$ is proved to be constant in the base case of the main induction and is nonzero because $\kappa^{(n+1)}|_{\rho=0}\neq 0$. Therefore, we can conclude $\partial_\rho\phi^{(k+1-(n+1))}|_{\rho=0}=0$. Higher ρ derivatives follow similarly. ∎

We see once again that if ϕ vanishes to all orders on the horizon then the generalized zeroth law holds. Therefore, we are left with the final case to deal with: $\phi^{[m]}|_{\rho=0}=0$, $\phi^{(m+1)}(x^A)|_{\rho=0}$ is nonzero for some x^A, $\kappa^{[n]}=0$, $\kappa^{(n+1)}|_{\rho=0}\neq 0$, and $A_\tau^{[n]}|_{\rho=0}=0$. We perform our final induction on this case, which has hypothesis

$$\partial_A X^{[k+2m+1]}|_{\rho=0}=0,\qquad F_{\tau A}^{[k+2m+1]}|_{\rho=0}=0,\qquad A_\tau^{[k-1]}|_{\rho=0}=0 \tag{A36}$$

for $k\geq n+1$. The base case $k=n+1$ follows from Lemma 3, from which we also have $\forall\ \ p\geq 1, \partial_\rho^p\phi^{[m-p(n+1)]}|_{\rho=0}=0$.

We now assume the hypothesis holds for k. For $I = (\tau A)$ and $I = \tau$, we study

$$\text{at order } l^{k+2m+2}, \qquad E_I[\Phi_J]|_{\rho=0} = E_I^{(0)}[\Phi_J^{[k+2m+2]}]|_{\rho=0} + l \sum_{s=1}^{k+2m+2} l^{s-1} E_I^{(s)}[\Phi_J^{[k+2m+1]}]|_{\rho=0.} \tag{A37}$$

Change coordinates via $\rho = r(\kappa^{[k+2m+1]} v + 1)$, $\tau = \frac{1}{\kappa^{[k+2m+1]}} \log(\kappa^{[k+2m+1]} v + 1)$, in which $I = (\tau A)$ and $I = \tau$ are proportional to positive boost weight components. As in Lemma 2, the only positive boost weight quantities made from $\Phi_J^{[k+2m+1]}$ that do not necessarily vanish on the horizon by the induction hypotheses are in the combination

$$(\partial_v^{a_1} \partial_r^{b_1} \phi^*)(\partial_v^{q_1} A_v) \ldots (\partial_v^{q_M} A_v)(\partial_v^{a_2} \partial_r^{b_2} \phi) \tag{A38}$$

for some $M \geq 1$, with $b_1 \geq a_1$, $b_2 \geq a_2$, and overall boost weight $a_1 + a_2 - b_1 - b_2 + M + \sum q_i \geq 1$. But by (A21), (A22), and the inductive hypotheses, on the horizon this is of order

$$2m + 2 + \left(a_1 + a_2 - b_1 - b_2 + \sum q_i\right)(n+1) + Mk \geq 2m + 2 + n + 1 + M(k - n - 1) \geq 2m + 2 + k, \tag{A39}$$

where in the last step we used $M \geq 1$ and $k \geq n + 1$. Therefore, we can once again ignore the higher derivative terms in (A37) because they come with an additional power of l.

Hence,

$$\begin{aligned} \text{at order } l^{k+2m+2}, \qquad E_\tau[\Phi_J]\Big|_{\rho=0} &= 2\lambda^2 l^{k+2m+2} A_\tau^{(k)} |\phi^{(m+1)}|^2 \\ &\quad + l^{k+2m+2} h^{(0)AB} \mathcal{D}_A^{(0)} \left[c_1(0) F_{\tau B}^{(k+2m+2)} - 8 c_2(0) \epsilon_B^{(0)C} F_{\tau C}^{(k+2m+2)}\right] = 0. \end{aligned} \tag{A40}$$

Identical to case 1, we integrate this against $\sqrt{h^{(0)}}$ and then against $\sqrt{h^{(0)}} A_\tau^{(k)}$ to get $A_\tau^{(k)}|_{\rho=0} = 0$ and $F_{\tau A}^{(k+2m+2)}|_{\rho=0} = 0$, respectively.

Finally,

$$\begin{aligned} \text{at order } l^{k+2m+2}, \qquad E_{\tau A}[\Phi_J]|_{\rho=0} &= -\frac{1}{2} l^{k+2m+2} \partial_A X^{(k+2m+2)} \\ &\quad - \frac{1}{2} i l^{k+2m+2} \lambda A_\tau^{(k)} \left(\phi^{(m+1)*} \partial_A \phi^{(m+1)} - \phi^{(m+1)} \partial_A \phi^{(m+1)*} - 2i\lambda A_A^{(0)} |\phi^{(m+1)}|^2\right) \\ &\quad - \frac{1}{2} c_1(0) \left(F_{AB}^{(0)} h^{(0)BC} - F_{\tau\rho}^{(0)} \delta_A^C\right) l^{k+2m+2} F_{\tau C}^{(k+2m+2)} = 0 \end{aligned} \tag{A41}$$

into which we substitute $A_\tau^{(k)}|_{\rho=0} = 0$ and $F_{\tau A}^{(k+2m+2)}|_{\rho=0} = 0$ to get $\partial_A X^{(k+2m+2)}|_{\rho=0} = 0$. Hence, the induction proceeds and we have proved the generalized zeroth law.

3. Full expressions for $\nabla^\beta F_{\alpha\beta}$ in affinely parametrized GNCs

The components of $\nabla^\beta F_{\alpha\beta}$ in affinely parametrized GNCs are as follows:

$$\begin{aligned} \nabla^\beta F_{v\beta} &= \partial_v \psi + D^A K_A + K\psi + r(-D^A \psi \beta_A + \beta^A \partial_r K_A - D^A \beta^B F_{AB} - \beta^A \beta_A \psi - \bar{K}^A \partial_v \beta_A \\ &\quad + K^A \bar{K} \beta_A - D^A \beta_A \psi - 2K^A \bar{K}_A{}^B \beta_B) + r^2(-\alpha \partial_r \psi - \beta^A \beta_A \partial_r \psi - F^{AB} \beta_A \partial_r \beta_B \\ &\quad + D^A \beta^B \bar{K}_A \beta_B + D^A \alpha \bar{K}_A - D^A \beta^B \bar{K}_B \beta_A + \bar{K}^A \alpha \beta_A - \beta^A \partial_r \beta_A \psi - \bar{K} \alpha \psi - \bar{K} \beta^A \beta_A \psi \\ &\quad + 2\bar{K}^{AB} \beta_A \beta_B \psi) + r^3(-\bar{K}^A \alpha \partial_r \beta_A - \bar{K}^A \beta^B \beta_B \partial_r \beta_A + \bar{K}^A \beta_A \beta^B \partial_r \beta_B + \bar{K}^A \beta_A \partial_r \alpha), \end{aligned} \tag{A42}$$

$$\nabla^\beta F_{r\beta} = -\partial_r \psi + D^A \bar{K}_A + \bar{K}^A \beta_A - \bar{K}\psi + r(\beta^A \partial_r \bar{K}_A - 2\bar{K}^A \bar{K}_A{}^B \beta_B + \bar{K}^A \partial_r \beta_A + \bar{K} \bar{K}^A \beta_A), \tag{A43}$$

$$\begin{aligned}\nabla^\beta F_{A\beta} = & -\partial_r K_A - \partial_v \bar{K}_A + D^B F_{AB} + 2K_{AB}\bar{K}^B + 2K^B\bar{K}_{AB} - \beta_A\psi - K\bar{K}_A - K_A\bar{K} + F_A{}^B\beta_B \\ & + r\bigg(-F^{BC}\bar{K}_{AB}\beta_C + D^B\beta_A\bar{K}_B + \bar{K}_A{}^B\beta_B\psi - \partial_r\beta_A\psi + \frac{1}{2}\bar{K}^B\beta_A\beta_B + F_A{}^B\bar{K}\beta_B \\ & - D^B\beta_B\bar{K}_A + F_A{}^B\partial_r\beta_B - \bar{K}_A\beta^B\beta_B - 2\bar{K}_A\alpha\bigg) + r^2\bigg(\bar{K}^B\bar{K}_{AB}\alpha + \bar{K}^B\bar{K}_{AB}\beta^C\beta_C \\ & + \frac{1}{2}\bar{K}^B\beta_B\partial_r\beta_A - \bar{K}\bar{K}_A\alpha - \bar{K}\bar{K}_A\beta^B\beta_B - \bar{K}_A\beta^B\partial_r\beta_B - \bar{K}_A\partial_r\alpha\bigg). \end{aligned} \tag{A44}$$

These were calculated using the symbolic algebra program CADABRA [20–22]. To get Eq. (117), we use (110) to eliminate $\partial_v\bar{K}_A$ in (A44).

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