Inequivalence of mimetic gravity with models of loop quantum gravity

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Certain versions of mimetic gravity have recently been claimed to present potential covariant theories of canonically modified spherically symmetric gravity, motivated by ingredients from loop quantum gravity. If such an equivalence were to hold, it would demonstrate general covariance of a large class of models considered in loop quantum gravity. However, the relationship with mimetic gravity as presented so far is incomplete because it has been proposed only in preferred space-time slicings of uniform scalar fields. Here, several independent arguments are used to show that neither an equivalence nor a covariance claim are correct for models of loop quantum gravity. The framework of emergent modified gravity is found to present a broad setting in which such questions can be analyzed efficiently. As an additional result, the discussion sheds light on the coexistence of different and mutually inequivalent approaches to an implementation of the gravitational dynamics within loop quantum gravity.

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I. INTRODUCTION

Canonical quantum gravity is expected to change the structure of space-time by incorporating general quantum effects such as fluctuations or discreteness. The dynamics of general relativity, no longer unfolding on a continuous Riemannian geometry, is then likely to be modified, giving rise to potential new phenomena in particular at high curvature, such as at the big bang or in black holes. However, the definition of physical effects relevant for the expanding Universe or horizons requires certain geometrical notions that work well in the classical continuum but become less clear, at least according to the present stage of knowledge, in fundamentally discrete theories. Investigations of potential physical effects of proposals for quantum gravity therefore aim to strike a balance between including sufficiently many interesting dynamical modifications while trying to stay close enough to the classical space-time structure in order to apply familiar notions of curvature scalars or geodesic properties. An important tool in such effective descriptions is given by space-time metric tensors or line elements, suitably modified by supposed quantum effects.

While effective line elements can be used to apply classical definitions to proposed quantum solutions, they are not always available because they implicitly assume that expressions used for modified metric tensors indeed enjoy proper tensor transformations under coordinate changes. This condition ensures that the line element is, as required, invariant under coordinate transformations and therefore presents an unambiguous description of space-time geometry. In models of canonical quantum gravity, however, the underlying equations for gauge transformations that classically are equivalent to coordinate changes are usually modified. It is then not guaranteed that modified metric components obey the condition required for a well-defined line element. In more general terms, it is far from clear whether canonical quantum gravity or its effective models can be consistent with general covariance at least in regimes in which one tries to apply the usual notions of curvature scalars or horizons that are based on the existence of an invariant line element.

In models of loop quantum gravity, the covariance question has been analyzed in some detail, resulting in several no-go results for previously proposed formulations [1–5]. It is therefore surprising that a recent analysis [6] concludes that some of the models that had already been ruled out as noncovariant or even anomalous may be described in an equivalent way by certain versions of mimetic gravity, which are generally covariant. However, the arguments presented there are incomplete because they demonstrate a relationship between canonical equations in models of loop quantum gravity and tensor equations derived from mimetic gravity only in a preferred set of slicings. This set of slicings is defined by uniform scalar fields that by definition occur in mimetic gravity, but are included by hand on the canonical side in order to perform simple deparametrizations of the constrained dynamics. The relationship can be extended to the general theories, then holding for any slicing, only if one of the following two assumptions is met: (i) Gauge transformations of the modified canonical theory are compatible

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with slicing changes or coordinate transformations in space-time, or (ii) the mimetic version reduced to alternative gauge choices with nonuniform scalar field in some way resembles Hamiltonians expected from the procedures of loop quantum gravity. The first assumption is precisely the property that had already been ruled out for some of the same models, for instance in [1], without using the connection to mimetic gravity. The second assumption has been made implicitly but not evaluated further in [6].

We will show that the claimed equivalence of mimetic gravity and models of loop quantum gravity, as well as general covariance of the latter, can quickly and easily be disproved because the same constructions could be applied to versions of Hořava–Lifshitz gravity [7] that are known not to be covariant and are physically inequivalent to mimetic gravity. Nevertheless, given the importance of the covariance question in models of canonical and loop quantum gravity, it is of interest to provide a detailed analysis in order to determine where exactly the equivalence fails, tracing it back to specific terms in the Hamiltonian constraint of spherically symmetric general relativity and its possible modifications. We will present these details in Sec. II. In Sec. III, we will perform a related analysis within the framework of emergent modified gravity [8,9] in which it is possible to derive all compatible covariant modifications up to a given order in derivatives. Before our Conclusions, Sec. IV places our results in the context of different approaches within loop quantum gravity.

II. COVARIANCE WITHIN MIMETIC GRAVITY

The constructions in [6] make use of the Lagrangian,

$$S[g,\phi,\lambda] = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-\det g}\big(R + L[\phi] + \lambda((\nabla_\mu\phi)(\nabla^\mu\phi)+1)\big), \tag{1}$$

where

$$L[\phi] = L\big(\nabla^\mu\nabla_\mu\phi, (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\big), \tag{2}$$

and λ is a Lagrange multiplier, as a special version of mimetic gravity [10–14]. Mimetic gravity had previously been used for models of loop quantum cosmology [15], where it had been found not to provide an equivalence [16,17] if more complicated features beyond purely isotropic models were included. It would therefore be surprising if mimetic gravity could be used as a complete equivalence for general spherically symmetric models, as claimed in [6].

The four-dimensional action is then reduced to a canonical formulation of spherically symmetric configurations by assuming a space-time metric according to the line element,

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + \frac{(E^\varphi)^2}{E^x}(\mathrm{d}x + M\mathrm{d}t)^2 + E^x(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2), \tag{3}$$

with the lapse function N, the shift vector M, and a spatial metric expressed in terms of densitized-triad components E^φ and E^x (assumed positive, fixing the spatial orientation) as used in models of loop quantum gravity [18–20]. The scalar field and Lagrange multiplier are assumed to depend only on t and x in order to respect spherical symmetry. The resulting scalar-tensor theory is covariant under coordinate transformations or slicing changes that preserve spherical symmetry.

A. Gauge fixing and Legendre transformation

In the next step of [6], gauge-fixing conditions are introduced. A preferred slicing is defined by uniform scalar fields, such that $\phi(t)$ is no longer allowed to depend on x. Locally, this choice is always possible in a covariant theory. The mimetic condition,

$$0 = g^{\mu\nu}(\nabla_\mu\phi)(\nabla_\nu\phi) + 1 = -\frac{\dot\phi^2}{N^2} + 1, \tag{4}$$

then determines the lapse function, which also depends only on t. Locally, one may choose $t = \phi$ such that $N = 1$. The scalar terms in the reduced action can then be evaluated explicitly in terms of derivatives of the triad components, for which we will provide more details in Sec. IV. The result is that a generic spherically symmetric gauge-fixed action takes the form,

$$S = \frac{1}{2G}\int \mathrm{d}t\mathrm{d}xNE^\varphi\sqrt{E^x}\left(-2XY + Y^2 + \tilde L(X,Y) + \frac{1}{2}R^{(3)}\right), \tag{5}$$

where $R^{(3)}$ is the spatial Ricci scalar,

$$R^{(3)} = \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^3} - \frac{((E^x)')^2}{4E^x(E^\varphi)^2} - \frac{(E^x)''}{(E^\varphi)^2} + \frac{1}{E^x}, \tag{6}$$

and we have

$$X = \frac{\dot E^\varphi - (ME^\varphi)'}{NE^\varphi}, \qquad Y = \frac{\dot E^x - M(E^x)'}{2NE^x}. \tag{7}$$

The function $\tilde L$ is determined by the original function L of mimetic gravity. The construction relies on a coincidence in the gauge used, which implies that extrinsic-curvature components of a spherically symmetric metric, which imply the terms $-2XY + Y^2$ in the action using Gauss-Codazzi relationships, can be expressed uniquely through the same two functions, X and Y, that determine the independent scalar contributions to a spherically symmetric mimetic theory with a dependence of the form (2). We will return to this observation in Sec. IV.

There is also a class of deformations in [6] parameterized by a real number η for purposes of realizing different models of dilaton gravity with actions,

$$S_\eta = \frac{1}{2G}\int \mathrm{d}t\mathrm{d}x N E^\varphi \sqrt{E^x}\bigg(-2XY + (1-\eta)Y^2 + \tilde{L}(X,Y) + \frac{1}{2}R_\eta^{(3)}\bigg), \quad (8)$$

where

$$R_\eta^{(3)} = \frac{(E^x)'(E^\varphi)'}{(E^\varphi)^3} - (1-\eta)\frac{((E^x)')^2}{4E^x(E^\varphi)^2} - \frac{(E^x)''}{(E^\varphi)^2} + \frac{\eta+1}{(E^x)^{1-\eta}}. \quad (9)$$

Although we will not be interested in dilaton gravity in the present paper, it is useful to keep the parameter η for our covariance analysis.

The action functional determines momenta,

$$\pi_\varphi = 2G\frac{\delta S_\eta}{\delta \dot{E}^\varphi} = \sqrt{E^x}(\partial_X \tilde{L} - 2Y) \quad (10)$$

and

$$\pi_x = 2G\frac{\delta S_\eta}{\delta \dot{E}^x} = \frac{E^\varphi}{2\sqrt{E^x}}\left(\partial_Y \tilde{L} - 2X + 2(1-\eta)Y\right) \quad (11)$$

canonically conjugate to E^φ and E^x, respectively. A Legendre transformation leads to the Hamiltonian,

$$H_\eta = -\frac{1}{2G}\int \mathrm{d}x N\sqrt{E^x}E^\varphi\bigg(\tilde{L} - X\partial_X\tilde{L} - Y\partial_Y\tilde{L} + 2XY - (1-\eta)Y^2 + \frac{1}{2}R_\eta^{(3)}\bigg) + \frac{1}{2G}\int \mathrm{d}x(M(E^x)'\pi_x + (ME^\varphi)'\pi_\varphi). \quad (12)$$

Varying by M, the second line implies the diffeomorphism constraint, and if N had not been gauge fixed, the first line would imply the Hamiltonian constraint.

The case of $\tilde{L} = 0$ should correspond to vacuum spherically symmetric gravity. A direct calculation indeed yields

$$Y_{\tilde{L}=0} = -\frac{\pi_\varphi}{2\sqrt{E^x}}, \qquad X_{\tilde{L}=0} = -\frac{\sqrt{E^x}}{E^\varphi}\pi_x - \frac{1-\eta}{2\sqrt{E^x}}\pi_\varphi, \quad (13)$$

and therefore,

$$2X_{\tilde{L}=0}Y_{\tilde{L}=0} - (1-\eta)Y^2_{\tilde{L}=0} = \frac{\pi_\varphi\pi_x}{E^\varphi} + \frac{1-\eta}{4E^x}\pi_\varphi^2, \quad (14)$$

which produces the correct momentum-dependent terms for the Hamiltonian constraint used, for instance, in [9] if $\eta = 0$.

B. Modifications and violations of covariance

Continuing for now with $\tilde{L} = 0$, the reduced gauge-fixed theory is equivalent to vacuum spherically symmetric general relativity and therefore covariant. The procedure outlined in [6] then attempts to map modifications of the canonical theory, such as periodic dependences on the momenta as motivated by loop quantum gravity, to suitable nonzero $\tilde{L}$. In this way, models of loop quantum gravity are mapped to specific versions of gauge-fixed action principles for spherically symmetric configurations. Without further discussion, it is then claimed that the original theories must be covariant because they are strictly related to mimetic gravity, which is covariant.

However, taken at face value, the correspondence only shows that models of loop quantum gravity are equivalent to certain gauge-fixed action principles for spherically symmetric configurations. It does not follow that the gauge fixing can be relaxed while maintaining the correspondence. To be sure, there are scalar fields on both sides, used for deparametrization in models of loop quantum gravity and included as dynamical matter on the mimetic side. But the specific contributions of scalar fields to constraints or actions are never compared with each other. It is therefore unclear whether the specific dynamical dependence of the mimetic action on ϕ via L, whose tensorial nature is important for general covariance, is equivalent to the form in which the scalar field is implemented on the canonical side for models of loop quantum gravity. Without careful adjustments, which have not been made when deparametrizing the models, it is highly unlikely that the scalar terms match and complete all conditions for general covariance. In an alternative viewpoint, one may use the covariant mimetic theory in order to define what the scalar couplings of the canonical theory should look like in any gauge for it to be covariant. The main question, whether the required scalar terms have a good chance of resembling what is usually considered a model of loop quantum gravity, will be discussed in Sec. IV.

On the mimetic side, there are two versions of canonical spherically symmetric reductions, first the symmetry reduction by itself and then the reduction on a preferred slicing given by uniform ϕ. For simplicity, we will call the former the (E,π,ϕ)-theories and the latter the (E,π)-theories (both for various $\tilde{L}$) since in this case ϕ has been eliminated by gauge fixing. While the (E,π,ϕ)-theories are clearly covariant, as symmetry reductions of four-dimensional covariant theories, the correspondence with models of loop quantum cosmology envisaged in [6] only considers the (E,π)-theories. We will first see whether a covariance argument in the sense of slicing independence can be made for these theories with generic $\tilde{L}$, for which the equivalence to spherically symmetric gravity is not available.

Starting with a spatial slice Σ with initial values E_0, π_0, and ϕ_0 of an (E,π)-theory, it can always be embedded in a corresponding (E,π,ϕ)-theory with the same $\tilde{L}$ in which Σ is realized as a spatial slice $\Sigma'\colon\phi=\phi_0$ with ϕ_0 constant. On this slice, we may impose the same initial values E_0 and π_0 for the gravitational fields and choose the momentum of ϕ such that the Hamiltonian constraint is satisfied. (The diffeomorphism constraint does not depend on the momentum since $\phi=\phi_0$ is constant on the initial slice, such that $\phi'\pi_\phi=0$). Since all (E,π,ϕ)-theories are covariant, we know that its solutions determine space-time geometries, such that there are transformations to a different slice Σ'' on which we have new values E_1, π_1, ϕ_1 and scalar momenta obtained from the original values on Σ'. For a covariance argument, we would then like to restrict field values on Σ' to just the gravitational fields and conclude that the transformation could be interpreted as a slicing change in the (E,π)-theory. However, this is not possible because there is no guarantee that the new field ϕ_1 on Σ'' is constant, as required by definition of the gauge-fixed (E,π)-theory.

In fact, it can be shown explicitly that, in general, ϕ cannot be constant on a new slice obtained by a coordinate transformation. Consider a general coordinate transformation $(t,x)\to(\tilde{t},\tilde{x})$, with new coordinates indicated by a tilde. If we express the original coordinates (in which $\phi'=\partial\phi/\partial x=0$ on the given slice), as function of the new ones, $t(\tilde{t},\tilde{x})$ and $x(\tilde{t},\tilde{x})$, we have

$$\frac{\partial\phi}{\partial\tilde{x}}=\dot{\phi}\frac{\partial t}{\partial\tilde{x}}+\phi'\frac{\partial x}{\partial\tilde{x}}=\dot{\phi}\frac{\partial t}{\partial\tilde{x}}. \tag{15}$$

If $\partial t/\partial\tilde{x}\neq 0$ then in the new coordinate system we have $\widetilde{\phi'}\neq 0$ (taking the spatial derivative in the new coordinate system, as referred to by the tilde). Therefore, this gauge is not contained in the (E,π)-theories, in contrast to what is implicitly assumed by using covariance statements for the gauge-fixed mimetic theory. In the new coordinate system, the Hamiltonian (12) for a general function $\tilde{L}$ is no longer recovered because the latter descends from the $\nabla_\mu\phi$-dependent L in (1), and extra terms appear in the new gauge if $\tilde{L}\neq 0$. [See Eqs. (31) and (32) to be discussed in Sec. IV.]

We conclude that slicing independence or covariance of mimetic gravity and its spherically symmetric reduction does not imply slicing independence of the gauge-fixed theory which is formulated only for the gravitational fields. For some $\tilde{L}$, the gauge-fixed theory may be covariant as in the case of $\tilde{L}=0$, but checking covariance for nonzero $\tilde{L}$ requires additional conditions that have not been considered in [6] for models of loop quantum gravity. We will fill this gap in the remainder of this paper, showing that covariance is, in fact, violated.

C. Quadratic theories and Hořava-Lifshitz gravity

A function $\tilde{L}(X,Y)=aX^2+bXY+cY^2$ can be used to generate an arbitrary quadratic dependence of the Hamiltonian on momenta. To see this, we simply evaluate the contribution,

$$\begin{aligned}
&\tilde{L}-X\partial_X\tilde{L}-Y\partial_Y\tilde{L}+2XY-(1-\eta)Y^2\\
&=-aX^2+(2-b)XY-(1-\eta+c)Y^2\\
&=-a\frac{E^x}{(E^\varphi)^2}\pi_x^2+\big(1-b/2-a(1-\eta)\big)\frac{\pi_x\pi_\varphi}{E^\varphi}\\
&\quad+\big((1-b)(1-\eta)-c-a(1-\eta)^2\big)\frac{\pi_\varphi^2}{4E^x}.
\end{aligned} \tag{16}$$

For suitable a, b and c (irrespective of η), any quadratic dependence on the momenta can be generated.

However, it is well known that the dependence of the Hamiltonian on momenta or of the action on extrinsic curvature is restricted by covariance. Quadratic momenta without higher time derivatives imply classical theories. Without symmetry reduction, the quadratic dependence of the action on extrinsic curvature K_{ab} is completely determined as $K^{ab}K_{ab}-(K_a^a)^2$, which in a spherically symmetric reduction implies the specific coefficients of π_φ^2, $\pi_\varphi\pi_x$, and π_x^2 seen in (14): In a triad basis, K_{ab} has the components K_x and K_φ where $K_x=\pi_x$ and $2K_\varphi=\pi_\varphi$ [19]. Therefore, supplying density weights by using the spatial metric q_{ab},

$$\sqrt{\det q}K_a^a=K_a^iE_i^a=K_xE^x+2K_\varphi E^\varphi, \tag{17}$$

and

$$\begin{aligned}
\sqrt{\det q}K^{ab}K_{ab}&=\frac{1}{\sqrt{\det q}}K_a^iE_i^bK_b^jE_j^a\\
&=\frac{(E^x)^{3/2}}{E^\varphi}K_x^2+2\frac{E^\varphi}{\sqrt{E^x}}K_\varphi^2.
\end{aligned} \tag{18}$$

The combination,

$$\begin{aligned}
&\sqrt{\det q}\big(K^{ab}K_{ab}-(K_a^a)^2\big)\\
&=-4\sqrt{E^x}K_xK_\varphi-2\frac{E^\varphi}{\sqrt{E^x}}K_\varphi^2,
\end{aligned} \tag{19}$$

is indeed proportional to (14). The possibility of using an unrestricted quadratic $\tilde{L}$ in gauge-fixed reduced theories of mimetic gravity does not respect the required conditions on coefficients.

It is in fact possible to generalize spherically symmetric theories while keeping them covariant, as seen in two-dimensional dilaton gravity. The coefficients in (14) are unique for the reduction of general relativity to spherical symmetry. They may be changed within two-dimensional covariant theories, but still not arbitrarily so, as it would be suggested by a general quadratic $\tilde{L}$. We will discuss restrictions on covariant two-dimensional theories in the next section and now show that the generality of quadratic $\tilde{L}$ can be used to construct a correspondence between gauge-fixed reduced mimetic gravity and Hařava–Lifshitz theories that are known not to be covariant.

In four-dimensional space-time, a class of Hořava–Lifshitz gravity theories [7] is defined by the action,

$$S_{\mathrm{HL}}^{\lambda} = \frac{1}{16\pi G}\int \mathrm{d}^4 x N\sqrt{\det q}\left(K^{ab}K_{ab} - \lambda (K_a^a)^2 + \frac{1}{1-3\lambda}R^{(3)}\right), \tag{20}$$

where q_{ab} and K_{ab} are metric and extrinsic-curvature tensors defined on the slices of a foliation. For $\lambda = 1$, the theory is a restriction of general relativity to the foliation, but for $\lambda \neq 1$ there is no such correspondence to a covariant theory. If we repeat our reduction of the K-terms to spherical symmetry, we obtain

$$\begin{aligned}\sqrt{\det q}\left(K^{ab}K_{ab} - \lambda(K_a^a)^2\right) &= (1-\lambda)\frac{(E^x)^{3/2}}{E^\varphi}K_x^2 - 4\lambda\sqrt{E^x}K_x K_\varphi \\ &\quad + 2(1-2\lambda)\frac{E^\varphi}{\sqrt{E^x}}K_\varphi^2.\end{aligned} \tag{21}$$

For every λ, there is a $\tilde{L}$ such that this combination of K_x and K_φ has a correspondence with gauge-fixed reduced mimetic gravity according to [6]. Since generic Hořava-Lifshitz theories are not covariant, this correspondence cannot be used to prove covariance in other cases, such as models of loop quantum gravity. Moreover, the correspondence does not imply physical equivalence of the formally related theories.

If we consider only two-dimensional theories of Hořava–Lifshitz type, there is no need to use the reduction (21). A general two-dimensional theory of this form can then be defined with K-terms proportional to

$$K_\varphi^2 + \lambda_1 \frac{K_x K_\varphi}{E^\varphi} + \lambda_2 \frac{K_x^2}{(E^\varphi)^2}, \tag{22}$$

with two parameters, λ_1 and λ_2. In spherical symmetry, there are three independent spatial scalars quadratic in the momenta, constructed by referring to the density weight zero for K_φ and E^x and density weight one for K_x and E^φ. Gauge-fixed reduced mimetic theories with quadratic $\tilde{L}$ are sufficiently general to include even these theories. There are no restrictions that could suggest any potential implementation of conditions for general covariance through the correspondence.

III. COVARIANCE FROM EMERGENT MODIFIED GRAVITY

We have seen that the gauge-fixed reduced theory of mimetic gravity with action (5) is covariant if $\tilde{L} = 0$ because it then equals a gauge fixing of vacuum spherically symmetric gravity, which clearly has a covariant extension to arbitrary slicings. This conclusion remains true if we use (8) with arbitrary η because $\tilde{L} = 0$ implies that the scalar field does not couple to gravity at all; it is merely used to define a slicing. If the scalar field does not couple to gravity, the generic hypersurface deformations discussed in Sec. II B can always be performed such that ϕ remains unchanged, and in particular constant, when transforming to a new slicing. Slicings of an (E,π)-theory can then always be embedded in slicings of an (E,π,ϕ)-theory, which are covariant. The action (8) with $\tilde{L} = 0$ therefore defines a covariant two-dimensional theory even though, for non-zero η, it modifies the classical relationship between coefficients of the momentum terms (14) as well as the spatial Ricci scalar (9).

We are again seeing an example of the fact that two-dimensional covariant models allow for more freedom than spherically symmetric reductions of four-dimensional covariant theories. According to [21], any Hamiltonian constraint of the form,

$$H = -\frac{1}{2G}\int \mathrm{d}x N\left(\alpha \frac{E^\varphi}{\sqrt{E^x}}K_\varphi^2 + 2\bar{\alpha}\sqrt{E^x}K_x K_\varphi + \alpha_\Gamma \frac{E^\varphi}{\sqrt{E^x}}(1-\Gamma_\varphi)^2 + 2\bar{\alpha}_\Gamma\sqrt{E^x}\Gamma_\varphi'\right) \tag{23}$$

with

$$\Gamma_\varphi = -\frac{(E^x)'}{2E^\varphi}, \tag{24}$$

defines an anomaly-free canonical theory in two dimensions, provided the E^x-dependent functions α, $\bar{\alpha}$, α_Γ and $\bar{\alpha}_\Gamma$ obey

$$(\bar{\alpha}\alpha_\Gamma - \alpha\bar{\alpha}_\Gamma)(E^x)' + 2(\bar{\alpha}'\bar{\alpha}_\Gamma - \bar{\alpha}\bar{\alpha}_\Gamma')E^x = 0. \tag{25}$$

This equation can be solved by the parametrization,

$$\bar{\alpha} = \sqrt{|\beta|}b_1 \tag{26}$$

$$\alpha = \sqrt{|\beta|}b_1 b_2 \tag{27}$$

$$\bar{\alpha}_\Gamma = \mathrm{sgn}(\beta)\frac{\sqrt{|\beta|}}{b_1} \tag{28}$$

$$\alpha_\Gamma = \mathrm{sgn}(\beta)\frac{\sqrt{|\beta|}}{b_1}\left(b_2 - 4\frac{\mathrm{d}\log b_1}{\mathrm{d}\log E^x}\right), \tag{29}$$

in terms of three functions, b_1, b_2 and β. The structure function in the Poisson bracket of two Hamiltonian constraints is then multiplied by β. Recently, these theories have been shown to be not only anomaly-free but also generally covariant [9] provided the radial component of the spatial metric of a compatible space-time line element is given by $|\beta|^{-1}(E^\varphi)^2/E^x$ rather than the classical $(E^\varphi)^2/E^x$. The signature of the space-time metric is determined by $\mathrm{sgn}(\beta)$, being Lorentzian if $\beta > 0$ and Euclidean if $\beta < 0$ [22]. Moreover, a term of the form $NE^\varphi V(E^x)$ with an arbitrary dilaton potential $V(E^x)$ can be added to the integrand of the Hamiltonian constraint (23) without introducing anomalies or violating covariance.

There is therefore a large class of covariant two-dimensional theories that do not have extensions to covariant four-dimensional theories, all quadratic in momenta. The canonical theory defined by (8) with (9) is a simple example of these models in which $\bar{\alpha} = 1 = \bar{\alpha}_\Gamma$ and $\alpha = \alpha_\gamma = 1 - \eta$ are constant, while the dilaton potential $V(E^x)$ can be used to account for the power $(E^x)^{\eta-1}$ in the last term of (9). This result confirms that any gauge-fixed reduction of mimetic gravity with $\tilde{L} = 0$, eliminating scalar couplings to gravity, defines a covariant two-dimensional theory. In terms of the parametrization by β, b_1 and b_2, the condition $\bar{\alpha} = \bar{\alpha}_\Gamma$ implies that $\mathrm{sgn}(\beta) = b_1^2$, or $\beta > 0$ and $b_1 = 1$. Since both $\bar{\alpha}$ and $\bar{\alpha}_\Gamma$ equal one in this case, we obtain $\beta = 1$. The second condition, $\alpha = \alpha_\gamma = 1 - \eta$, then equates b_2 with the constant $1 - \eta$. With $\beta = 1$, there is no signature change and the classical expression $(E^\varphi)^2/E^x$ can be used as the radial metric component for all η. It is important to note that this result must be derived from the constraint brackets and covariance conditions. In general, it cannot be taken for granted.

For nonlinear $\tilde{L}$, the Hamiltonian contains a term of the form K_x^2 or higher orders, which is not included in (23) and cannot be made compatible with general covariance as shown by the results of emergent modified gravity. This term in (22) is therefore the crucial difference between symmetry properties of Hořava-Lifshitz gravity and dilaton gravity in two dimensions. The attempted relationship with models of loop quantum gravity constructed in [6] considers terms of the form $\sin(\Delta_1 K_x)$ where Δ_1 may depend on E^x and E^φ. In a Taylor expansion, this function includes K_x^2-terms as well as higher orders in K_x, violating covariance. Terms of the form $\sin(\Delta_2 K_\varphi)$, again with a possible dependence of Δ_2 on E^x (but not on E^φ) can be made covariant, but, as shown in [9], only by a canonical transformation that introduces the sine function by hand. Specific properties of this function, often related to fundamental properties of loop quantum gravity, therefore cannot have physical implications on the level of effective space-time geometries. Finally, covariance of these modifications requires the presence of K'_φ-terms in the Hamiltonian constraint or coupling terms of K_φ to $(E^x)'$, which have not been included in [6] and do not conform with the mimetic correspondence.

IV. APPROACHES WITHIN LOOP QUANTUM GRAVITY

Given the challenging nature of implementing a consistent quantization of the gravitational constraints and finding physically relevant solutions, different approaches have been developed within loop quantum gravity in order to address this problem. Since the intention is to find more tractable procedures, some of these methods lead to potential shortcuts toward a physical solution space. However, since they forgo a full implementation of the constraints, their brackets, and geometrical conditions, such methods are not guaranteed to imply viable solutions unless their consistency can be checked by independent means.

The prime example, referred to also in [6], is deparametrization, in which one reformulates the constraints classically, often based on special matter ingredients, and quantizes only a Hamiltonian operator with respect to a fixed matter clock, rather than a complete Hamiltonian constraint that could be applied with different gauge or clock choices. In the full theory of loop quantum gravity, this viewpoint has been espoused for instance in [23,24]. A necessary consistency question is whether this approach, which builds on a fixed gauge, can be compatible with gauge-independent properties. Evaluations of this question are hard because they require a partial or complete undoing of the fixed gauge or clock choice, but they can be performed at least in spherically symmetric models.

The results of the present paper can be interpreted as ingredients of just such an evaluation. General covariance, one of the main topics studied here, is by definition a question about how different gauge choices can be related to one another in a specific way that resembles coordinate transformations applied to a metric tensor. This question cannot be addressed on a reduced phase space or with a fixed gauge, but it is possible to ask whether reduced or gauge-fixed models have extensions within a specific framework that are compatible with general covariance.

The answer to this questions depends on which specific framework is used to formulate possible extensions of a deparametrized or gauge-fixed model. In our analysis, we used the framework of models of loop quantum gravity, which can be defined as modified spherically symmetric theories in which the classical quadratic dependence on the gravitational momenta has been replaced by nonpolynomial, and usually trigonometric functions. This definition is motivated by the ubiquitous and, in fact, eponymous use of holonomies in loop quantizations. It aims to model intricate quantum constructions as performed for instance in [25–33] in a simpler setting of modified classical expressions, using some of the same functional ingredients in definitions of the constraints. As we have seen, this approach is subject to strong covariance conditions that, in particular, rule out the possibility of embedding the modified constraints of [6] in this framework.

The covariance claims made in [6] were based on a different strategy: Terms inspired by loop quantum gravity were used only in the preferred gauge based on deparametrization of the canonical theory. The corresponding Hamiltonian can be related to the Hamiltonian of a mimetic theory in the preferred slicing defined by spatially constant ϕ used as time, $\phi = t$. Since the mimetic side of this correspondence by construction has a covariant extension in which the gauge fixing is undone, it may be used as a covariant theory. In this viewpoint, therefore, models of loop quantum gravity as a framework are replaced by models of mimetic gravity. Since these are two different frameworks, it may well be that a covariant extension exists only in one version, here using mimetic gravity. However, since mimetic theories then replace models of loop quantum gravity, it remains to be analyzed to what degree these extensions can be interpreted as related to loop quantum gravity in slicings or gauges other than the preferred one. Another important question is whether such an extension is unique, which would be required for an equivalence claim.

These questions can be addressed in a straightforward manner because they merely requires a Hamiltonian analysis of spherically symmetric mimetic gravity for unrestricted scalar fields. It should not come as a surprise that the new equations are much longer than the neat expressions found in the preferred slicing because they also contain terms with ϕ', ϕ'' as well as $\dot{\phi}$ and $\ddot{\phi}$. The mimetic condition no longer implies a direct relationship between N and $\dot{\phi}$ but instead amounts to a condition on N, M as well as $\dot{\phi}$ and ϕ'. Detailed properties of these equations can then be used to discuss whether they may be motivated by loop quantum gravity. If this is not the case, the appearance of loop-like Hamiltonians in the preferred slicing would merely be a coincidence but could not be considered a generic feature of the covariant extension.

This discussion requires the following expressions: The mimetic dependence of the action on the scalar field as used in [6] refers to two terms,

$$
\begin{aligned}
g_{(2)}^{\bar{\mu}\bar{\nu}}\nabla_{\bar{\mu}}\nabla_{\bar{\nu}}\phi = g_{(2)}^{\bar{\mu}\bar{\nu}}(\partial_{\bar{\mu}}\partial_{\bar{\nu}}\phi - \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\nu}}\partial_{\bar{\alpha}}\phi) = & -\frac{\dot{\phi} - M\phi'}{2N^2E^x}\left(2E^x\frac{\dot{E}^\varphi}{E^\varphi} - \dot{E}^x\right) \\
& - \frac{\dot{\phi}}{2N^2E^x}\left(M(E^x)' - \frac{2E^x}{E^\varphi}(ME^\varphi)' + \frac{2E^x}{N}(MN' - \dot{N})\right) - \frac{\ddot{\phi} - (N^2E^x/(E^\varphi)^2 - M^2)\phi'' - 2M\dot{\phi}'}{N^2} \\
& + \frac{\phi'}{2N^2E^x}\left(2E^x\left(\dot{M} - M\frac{\dot{N}}{N}\right) + 2E^x\frac{N'}{N}\left(\frac{E^x}{(E^\varphi)^2}N^2 + M^2\right) - 2E^x\frac{(E^\varphi)'}{E^\varphi}\left(\frac{E^x}{(E^\varphi)^2}N^2 + M^2\right)\right. \\
& \left. + (E^x)'\left(\frac{E^x}{(E^\varphi)^2}N^2 + M^2\right) - 4E^xMM'\right),
\end{aligned} \tag{30}
$$

and

$$
Y = -g_{(2)}^{\bar{\mu}\bar{\nu}}\frac{\partial_{\bar{\mu}}E^x}{2E^x}\partial_{\bar{\nu}}\phi = \frac{\dot{E}^x\dot{\phi} - M((E^x)'\dot{\phi} + \dot{E}^x\phi') - (N^2E^x/(E^\varphi)^2 - M^2)(E^x)'\phi'}{2N^2E^x}, \tag{31}
$$

where $g_{(2)}$ is used to denote the two-dimensional metric of the $t - x$ hypersurface and the barred indices correspond to only the (t, x)-components. The latter expression is directly used in [6] as one of the independent expressions in the function L (or L' after reduction to spherical symmetry), called Y in the spherically symmetric reduction. The second function in L' is given by

$$
\begin{aligned}
X = -g_{(2)}^{\bar{\mu}\bar{\nu}}\nabla_{\bar{\mu}}\nabla_{\bar{\nu}}\phi - g_{(2)}^{\bar{\mu}\bar{\nu}}\frac{\partial_{\bar{\mu}}E^x}{2E^x}\partial_{\bar{\nu}}\phi = & \frac{\dot{\phi} - M\phi'}{N^2}\frac{\dot{E}^\varphi}{E^\varphi} + \frac{\dot{\phi}}{N^2}\left(\frac{\dot{E}^x}{2E^x} - \frac{(ME^\varphi)'}{E^\varphi} + \frac{MN' - \dot{N}}{N}\right) \\
& + \frac{\ddot{\phi} - (N^2E^x/(E^\varphi)^2 - M^2)\phi'' - 2M\dot{\phi}'}{N^2} - \frac{\phi'}{2N^2E^x}\left(2E^x\left(\dot{M} - M\frac{\dot{N}}{N}\right) + 2E^x\frac{N'}{N}\left(\frac{E^x}{(E^\varphi)^2}N^2 + M^2\right)\right. \\
& \left. - 2E^x\frac{(E^\varphi)'}{E^\varphi}\left(\frac{E^x}{(E^\varphi)^2}N^2 + M^2\right) + (E^x)'\frac{2E^x}{(E^\varphi)^2}N^2 - 4E^xMM'\right).
\end{aligned} \tag{32}
$$

It is easy to see that (32) and (31) reduce to (7) in the preferred slicing of spatially constant ϕ. The mimetic condition can then be used to eliminate $\dot{\phi}$, and X and Y are directly related to the extrinsic-curvature components of a spherically symmetric metric, independent of the scalar field. In other slicings, however, the full X and Y remain scalar dependent, and the mimetic condition does not seem to imply noteworthy simplifications. As a consequence, X and Y in the mimetic function L no longer equal components of extrinsic curvature. Therefore, there is no choice of L in such a slicing that would turn the classical quadratic dependence on the gravitational momenta into a dependence through trigonometric functions, as seen for instance in the reduced action,

$$S[g,\phi,\lambda] = \frac{1}{2G}\int \mathrm{d}t\mathrm{d}x N E^{\varphi}\sqrt{E^x}\left(-2\left(\frac{\dot{E}^{\varphi}-(ME^{\varphi})'}{NE^{\varphi}}\right)\right. \times\left(\frac{\dot{E}^x-M(E^x)'}{2NE^x}\right)+\left(\frac{\dot{E}^x-M(E^x)'}{2NE^x}\right)^2+\frac{1}{2}R^{(3)} \left.+\frac{1}{2}\left(L'(X,Y)+\lambda\left(g_{(2)}^{\bar{\mu}\bar{\nu}}(\nabla_{\bar{\mu}}\phi)(\nabla_{\bar{\nu}}\phi)+1\right)\right)\right), \quad (33)$$

for a nonuniform slicing.

The methods of loop quantum gravity can then be used to suggest modifications of the Hamiltonian only in a preferred slicing, which is tantamount to saying that in this viewpoint, loop quantum gravity is defined only on a preferred slicing of space-time. On other slices, the paradigm of loop quantum gravity is replaced by theories of mimetic gravity. Moreover, this definition of loop quantum gravity can only be made in the presence of a scalar field, which is used crucially in order to define the slicing by non-zero field values. There would be no vacuum theory of loop quantum gravity.

An independent concern is that the mimetic extension is far from being unique, starting with specific modifications of the deparameterized canonical theory. (See also [34].) In [6], a unique mimetic extension was derived for mimetic gravity in which L depends only on two local scalar invariants, $\nabla^{\mu}\nabla_{\mu}\phi$ and $(\nabla_{\mu}\nabla_{\nu}\phi)(\nabla^{\mu}\nabla^{\nu}\phi)$, reduced to X and Y in spherical symmetry. Mimetic gravity can be formulated with an infinite number of higher-order invariants with different contractions of n factors $\nabla_{\mu}\phi$, or $\nabla_{\bar{\mu}}\phi$ as well as $\nabla_{\bar{\nu}}E^x$ in a restriction to spherical symmetry. Any term that reduces to at most one derivative of E^x in spherical symmetry could play the role of Y in the preferred slicing, but it would imply an inequivalent mimetic extension of the deparametrized canonical theory. Moreover, the mimetic condition is not strictly required for these constructions because it is trivialized by the gauge choice $\phi = t$, and can therefore be replaced by this choice and then generalized to non-mimetic theories, such as those of Horndeski type. Since such covariant extensions of deparameterized canonical theories are not uniquely defined by the deparameterized theory, they cannot be considered equivalent descriptions.

V. CONCLUSIONS

We have demonstrated that the constructions given in [6] fail to give a faithful description of models of loop quantum gravity. By restricting the covariant theory of mimetic gravity to a preferred slicing, the resulting models not only lose reliable access to covariance properties within the paradigm of loop quantum gravity, they are also nonunique and obscure important dynamical features of the related theories. If a theory is known to be covariant, it may well be analyzed in a preferred slicing without losing any physical information. However, if a theory such as loop quantum gravity, whose covariance status is unclear, is related to a canonical theory in a preferred slicing, it is impossible to draw conclusions about any equivalence between the theories. This statement is clearly demonstrated by the example of Hořava-Lifshitz gravity. In this example, a specific form of the four-dimensional action principle replaces the framework of models of loop quantum gravity, but conceptually it gives rise to the same equivalence question.

Full equivalence between two general frameworks in which actions or Hamiltonians are specified by independent principles, which defines the desired kind of equivalence in the present context, can only be obtained if a comparison of equations in a preferred slicing is accompanied by a detailed analysis of gauge transformations. If gauge transformations on both sides of the correspondence are equivalent and equations of motion agree in a preferred gauge or slicing, the theories are equivalent in their dynamics as well as symmetry properties. For instance, models of emergent modified gravity can be related to mimetic gravity because they are covariant by construction, but the correspondence then requires using the correct emergent space-time metric that must be derived from gauge properties. [In spherical symmetry, the radial component of the emergent metric need not equal $(E^{\varphi})^2/E^x$.] Models of loop quantum gravity, by contrast, cannot be equivalent to covariant theories unless additional terms (such as K'_{φ}) are included as suggested by emergent modified gravity.

The mimetic correspondence proposed in [6] does not imply a strong equivalence because there is only one full, nongauge fixed theory in this case, given by a mimetic action. The proposal simply defines the non-gauge fixed version of a deparametrized model of loop quantum as being the same as the constructed mimetic theory. This procedure constitutes a definition but not the derivation of an equivalence. Moreover, as demonstrated here, the resulting theory is far from being unique even if a specific set of loop modifications is used in the deparametrized theory.

The constructions in [6] and other examples in loop quantum gravity attempt to evade a detailed discussion of anomaly freedom, covariance, and gauge transformations by using the canonical theory in deparametrized form, schematically replacing constraint equations $C = 0$ with equations $C = P$ where P represents terms linear in the momenta of matter fields. It is then argued that all gauge transformations of the complicated gravitational terms in C can be replaced by simple gauge transformations generated by expressions linear in momenta. However, terms linear in momenta do not reproduce the brackets of hypersurface deformations required for general covariance or slicing independence in a canonical theory. These brackets can be reproduced faithfully only by the full $C - P$ (if matter terms are desired). Consistency conditions then rule out arbitrary modifications such as spherically symmetric contributions to the Hamiltonian constraint that are not linear in K_x. As a corollary, the failed correspondence analyzed here therefore demonstrates that deparametrization is not a reliable way to construct generally covariant modified theories of gravity.

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