Einstein gravity as the thermal equilibrium state of a nonminimally coupled scalar field geometry

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We test ideas of the recently proposed first-order thermodynamics of scalar-tensor gravity using an exact geometry sourced by a conformally coupled scalar field. We report a nonmonotonic behavior of the effective "temperature of gravity" not observed before and due to a new term in the equation describing the relaxation of gravity toward its state of equilibrium, i.e., Einstein gravity, showing a richer range of thermal behaviors of modified gravity than previously thought.

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I. INTRODUCTION

Einstein's theory of gravity, general relativity (GR) has been very successful in the regimes in which it is tested [1–4] but there is little doubt that, ultimately, it has to be replaced by some other theory (of which there is no shortage [1,5–8]). First, GR contains spacetime singularities inside black holes and in cosmology, thus predicting its own failure. These spacetime singularities should presumably be cured by quantum mechanics, but virtually all attempts to introduce quantum corrections also introduce deviations from GR [9,10]. The low-energy limit of the simplest string theory, the bosonic string, does not reproduce GR but yields $\omega = -1$ Brans-Dicke gravity instead [11,12].

In addition to severe observational tensions [13,14], from the theoretical point of view the standard GR-based Λ-cold dark matter (ΛCDM) model of cosmology [15] is left wanting. Its main ingredient, i.e., the dark energy accounting for approximately 70% of the energy content of the universe, was introduced overnight to explain the current acceleration of the universe discovered in 1998 with type Ia supernovae. The nature of this dark energy is a mystery. It is believed that, if one explains it with the cosmological constant Λ, extreme fine-tuning arises. An alternative to dark energy consists of modifying gravity at large scales [16,17]. For this purpose, $f(R)$ theories of gravity are very popular (see [18–20] for reviews). The first scenario of inflation in the early universe, Starobinsky inflation [21], which is also the scenario favored by current observations [22], is based on quadratic corrections to GR, where $f(R) = R + \alpha R^2$ (here R denotes the Ricci scalar).

$f(R)$ gravity is a subclass of scalar-tensor gravity, the prototype of which is the original Brans-Dicke theory [23] later generalized by various authors [24–27]. These "old" or "first generation" scalar-tensor theories were further generalized by Horndeski [28]. The past decade has seen intense research activity on the rediscovered Horndeski theories, which were believed to be the most general scalar-tensor gravities with second order equations of motion, avoiding the notorious Ostrogradsky instability that plagues theories with higher order equations. This record now belongs to further generalizations, the so-called degenerate higher order scalar-tensor (DHOST) theories (see, e.g., [29–44] and the references therein).

From another point of view, the idea has been proposed that perhaps, unlike the other three known interactions, gravity is not fundamental but could instead be emergent, similar to the way macroscopic thermodynamics emerges from microscopic degrees of freedom. This idea has been pursued in various implementations, see, e.g., [45–50]. One remarkable piece of work is Jacobson's thermodynamics of spacetime, in which the Einstein equation of GR is derived with purely thermal considerations and plays a role analogous to that of a macroscopic equation of state [51]. Furthermore, quadratic gravity (which is an $f(R)$, therefore a scalar-tensor, gravity) can be obtained in a similar way, but its derivation requires the introduction of entropy-generation terms [52]. The implication is that GR is a thermal equilibrium state at zero temperature, while modified gravity corresponds to an excited state at higher temperature [52,53].[1] The problem is, the "temperature of gravity" and equations describing the relaxation to the GR

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[1]This fact is natural when extra propagating degrees of freedom, in addition to the two massless spin-two modes of GR, are excited.

equilibrium have never been found, in spite of a large literature on spacetime thermodynamics.

Recently, a more modest proposal was advanced in which this idea of modified gravity being an excitation of the GR equilibrium is reexamined in a context completely different from Jacobson's thermodynamics of spacetime. This new proposal, dubbed "first-order thermodynamics of scalar-tensor gravity," begins by noticing that the field equations of this class of theories can be written as effective Einstein equations with an effective stress-energy tensor in their right-hand side acting as an effective source, which has the structure of a dissipative fluid [54]. This fact is well known in special scalar-tensor theories or for special geometries (especially cosmological ones) [55] and it extends to generic "first-generation" scalar-tensor theories [56,57] and to "viable" Horndeski [58,59] gravity. Taking this dissipative structure seriously, one attempts to apply Eckart's first-order thermodynamics [60] to it. Rather unexpectedly, the main constitutive relation of Eckart's theory (a generalized Fourier law) is satisfied [57], which makes it possible to read off this equation the product $\mathcal{K}\mathcal{T}$ of a "thermal conductivity of spacetime" $\mathcal{K}$ and of the "temperature of gravity" $\mathcal{T}$ [57,59,61–65]. GR, obtained when the gravitational Brans-Dicke-like scalar field ψ is constant, corresponds to $\mathcal{K}\mathcal{T}=0$ while scalar-tensor gravity is a state at $\mathcal{K}\mathcal{T}>0$ [57,59,61,62].

Thus far, this formalism is the closest that one has come to defining a "temperature of gravity." An equation describing the approach to the GR thermal equilibrium, or the departure from it, is also provided in the formalism [61,62], which is still under development. The formalism has been extended to "viable" Horndeski gravity [59], applied to cosmology [64,66], to the Einstein frame description of these theories [67], and to multiscalar-tensor theories [68]. To make progress and gain insight into this new thermal description of gravity, one needs to test its basic ideas with special theories of modified gravity and their analytic solutions. While this work has begun [69–72], there are still many open questions and we continue this study here by applying the first-order thermodynamics to a special solution of nonminimally coupled scalar field theory (which is a scalar-tensor gravity) found recently by Sultana [73]. This spacetime is inhomogeneous, spherically symmetric, and time-dependent and is conformal to a GR geometry obtained by Sultana by generalizing a previous GR solution due to Wyman [74] to include a (positive) cosmological constant Λ (we refer to the latter as the Sultana-Wyman solution). Although this geometry (which contains a naked singularity) is devoid of direct physical meaning, it is used here for convenience since it allows us to glimpse new aspects of the thermal view of scalar-tensor gravity. The Sultana solution contains a scalar field that depends only on time and its gradient is timelike and future-oriented, therefore it can be used to define the four-velocity of an effective dissipative fluid on which the thermal view of scalar-tensor gravity is based (by contrast, more physical black hole solutions have spacelike scalar field gradient and the first-order formalism cannot be applied).

In order to fix the notation, the next section recalls the basics of scalar-tensor gravity and of nonminimally coupled scalar field theory and introduces Sultana's solution, whose geometry has already been studied in detail in [69]. Section III discusses the approach to the GR equilibrium identified by $\mathcal{K}\mathcal{T}=0$. This discussion includes a term in the relevant relaxation equation that was set to zero for simplicity in previous literature. Section IV makes a brief parallel with an exact solution of Brans-Dicke theory conformal to the Sultana-Wyman solution, discussing its thermal properties.

We follow the notation of Ref. [75]: the signature of the metric tensor g_{ab} is $-+++$, units are used in which the speed of light c and Newton's constant G are unity (but, for convenience, we restore G when discussing the nonminimally coupled scalar field), $\kappa\equiv 8\pi G$, R_{ab} is the Ricci tensor, $R\equiv R^a{}_a$, $G_{ab}\equiv R_{ab}-Rg_{ab}/2$ is the Einstein tensor, and $\Box\equiv g^{ab}\nabla_a\nabla_b$ is the curved space d'Alembertian.

II. SCALAR-TENSOR GRAVITY AND THE SULTANA SOLUTION

The gravitational sector of "first generation" scalar-tensor gravity is described by the Jordan frame action [23–27]

$$S_{\mathrm{ST}}=\frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\psi R-\frac{\omega(\psi)}{\psi}\nabla^c\psi\nabla_c\psi-V(\psi)\right], \tag{2.1}$$

where ψ is the Brans-Dicke-like scalar field corresponding, approximately, to the inverse of the effective gravitational coupling $G_{\mathrm{eff}}=1/\psi$, $\omega(\psi)$ is the "Brans-Dicke coupling" (which was constant in the original Brans-Dicke theory [23]), $V(\psi)$ is a scalar field potential, and g is the determinant of the spacetime metric g_{ab}. The vacuum field equations obtained by varying the action (2.1) are

$$G_{ab}=\frac{\omega}{\psi^2}\left(\nabla_a\psi\nabla_b\psi-\frac{1}{2}g_{ab}\nabla^c\psi\nabla_c\psi\right)+\frac{1}{\psi}(\nabla_a\nabla_b\psi-g_{ab}\Box\psi)-\frac{V}{2\psi}g_{ab}, \tag{2.2}$$

$$\Box\psi=\frac{1}{2\omega+3}\left(\psi\frac{dV}{d\psi}-2V-\frac{d\omega}{d\psi}\nabla^c\psi\nabla_c\psi\right). \tag{2.3}$$

By conformally rescaling the metric and redefining the scalar field according to[2]

[2]Taking the absolute value of $(2\omega+3)$ in Eq. (2.5) guarantees that the Einstein frame scalar field $\tilde{\psi}$ is not a phantom.

$$g_{ab} \to \tilde{g}_{ab} = \Omega^2 g_{ab} = \psi g_{ab}, \tag{2.4}$$

$$\psi \to \tilde{\psi} = \sqrt{\frac{|2\omega+3|}{16\pi}} \ln\left(\frac{\psi}{\psi_*}\right) \tag{2.5}$$

(where ψ_* is a positive constant), the Brans-Dicke action is recast in its Einstein frame form (where quantities are denoted by a tilde)

$$S_{\mathrm{BD}} = \int d^4x \sqrt{-\tilde{g}} \left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\psi}\tilde{\nabla}_b\tilde{\psi} - U(\tilde{\psi})\right], \tag{2.6}$$

where

$$U(\tilde{\psi}) = \frac{V(\psi)}{\psi^2}\bigg|_{\psi=\psi(\tilde{\psi})}. \tag{2.7}$$

In the Einstein conformal frame, vacuum scalar-tensor gravity looks like GR sourced by a minimally coupled scalar field $\tilde{\psi}$. The conformal transformation is commonly used as a solution-generating technique using GR solutions sourced by minimally coupled scalar fields as seeds.

A. Jordan frame scalar-tensor gravity and its first-order thermodynamics

The first step leading to the first-order thermodynamics of scalar-tensor gravity consists of writing the (vacuum) field equations as effective Einstein equations $G_{ab} = T^{(\psi)}_{ab}$ by collecting all terms other than the Einstein tensor G_{ab} in the right-hand side (indeed, this is the way in which these field equations are usually presented). By assuming that $\nabla^a\psi$ is timelike and future-oriented (which is a fundamental limitation in the applicability of this formalism), $T^{(\psi)}_{ab}$ necessarily has the structure [57] of a dissipative fluid energy-momentum tensor [75]

$$T^{(\psi)}_{ab} = \rho^{(\psi)} u_a u_b + P^{(\psi)} h_{ab} + \pi^{(\psi)}_{ab} + q^{(\psi)}_a u_b + q^{(\psi)}_b u_a, \tag{2.8}$$

with four-velocity

$$u^a = \frac{\nabla^a\psi}{\sqrt{-\nabla^c\psi\nabla_c\psi}}, \tag{2.9}$$

spatial 3-metric

$$h_{ab} = u_a u_b + g_{ab}, \tag{2.10}$$

energy density

$$\rho^{(\psi)} = T^{(\psi)}_{ab} u^a u^b = -\frac{\omega}{2\psi^2}\nabla^e\psi\nabla_e\psi + \frac{V}{2\psi} + \frac{1}{\psi}\left(\Box\psi - \frac{\nabla^a\psi\nabla^b\psi\nabla_a\nabla_b\psi}{\nabla^e\psi\nabla_e\psi}\right), \tag{2.11}$$

isotropic pressure

$$P^{(\psi)} = \frac{1}{3}T^{(\psi)a}{}_a = -\frac{\omega}{2\psi^2}\nabla^e\psi\nabla_e\psi - \frac{V}{2\psi} - \frac{1}{3\psi}\left(2\Box\psi + \frac{\nabla^a\psi\nabla^b\psi\nabla_b\nabla_a\psi}{\nabla^e\psi\nabla_e\psi}\right), \tag{2.12}$$

heat flux density

$$q^{(\psi)}_a = -T^{(\psi)}_{cd} u^c h_a{}^d = \frac{\nabla^c\psi\nabla^d\psi}{\psi(-\nabla^e\psi\nabla_e\psi)^{3/2}} \times (\nabla_d\psi\nabla_c\nabla_a\psi - \nabla_a\psi\nabla_c\nabla_d\psi) \tag{2.13}$$

$$= -\frac{\nabla^c\psi\nabla_a\nabla_c\psi}{\psi(-\nabla^e\psi\nabla_e\psi)^{1/2}} - \frac{\nabla^c\psi\nabla^d\psi\nabla_c\nabla_d\psi}{\psi(-\nabla^e\psi\nabla_e\psi)^{3/2}}\nabla_a\psi, \tag{2.14}$$

and anisotropic stress tensor

$$\pi^{(\psi)}_{ab} = T^{(\psi)}_{cd} h_a{}^c h_b{}^d - P h_{ab} = \frac{1}{\psi\nabla^e\psi\nabla_e\psi} \times \left[\frac{1}{3}(\nabla_a\psi\nabla_b\psi - g_{ab}\nabla^c\psi\nabla_c\psi) \times \left(\Box\psi - \frac{\nabla^c\psi\nabla^d\psi\nabla_d\nabla_c\psi}{\nabla^e\psi\nabla_e\psi}\right) + \nabla^d\psi\left(\nabla_d\psi\nabla_a\nabla_b\psi - \nabla_b\psi\nabla_a\nabla_d\psi - \nabla_a\psi\nabla_d\nabla_b\psi + \frac{\nabla_a\psi\nabla_b\psi\nabla^c\psi\nabla_c\nabla_d\psi}{\nabla^e\psi\nabla_e\psi}\right)\right], \tag{2.15}$$

while

$$T^{(\psi)} \equiv g^{ab}T^{(\psi)}_{ab} = -\frac{\omega}{\psi^2}\nabla^c\psi\nabla_c\psi - \frac{3\Box\psi}{\psi} - \frac{2V}{\psi}. \tag{2.16}$$

As for the relevant kinematic quantities, the acceleration, expansion scalar, and shear tensor of the effective ψ-fluid are computed directly from the definition of the four-velocity (2.9) (and are, therefore, theory-independent). They read [57]

$$\dot{u}^a \equiv u^b\nabla_b u^a = (-\nabla^e\psi\nabla_e\psi)^{-2}\nabla^b\psi[(-\nabla^e\psi\nabla_e\psi)\nabla_a\nabla_b\psi + \nabla^c\psi\nabla_b\nabla_c\psi\nabla_a\psi], \tag{2.17}$$

$$\Theta = \nabla_a u^a = \frac{\Box\psi}{(-\nabla^e\psi\nabla_e\psi)^{1/2}} + \frac{\nabla_a\nabla_b\psi\nabla^a\psi\nabla^b\psi}{(-\nabla^e\psi\nabla_e\psi)^{3/2}}, \tag{2.18}$$

$$\sigma_{ab} = h_a{}^c h_b{}^d \nabla_{(c} u_{d)} = (-\nabla^e \psi \nabla_e \psi)^{-3/2} \times \left[-(\nabla^e \phi \nabla_e \psi) \nabla_a \nabla_b \psi - \frac{1}{3} \times (\nabla_a \psi \nabla_b \psi - g_{ab} \nabla^c \psi \nabla_c \psi) \Box \psi - \frac{1}{3} \left(g_{ab} + \frac{2 \nabla_a \psi \nabla_b \psi}{\nabla^e \psi \nabla_e \psi} \right) \nabla_c \nabla_d \psi \nabla^d \psi \nabla^c \psi + (\nabla_a \psi \nabla_c \nabla_b \psi + \nabla_b \psi \nabla_c \nabla_a \psi) \nabla^c \psi \right]. \tag{2.19}$$

The three-metric, heat flux density, and anisotropic stress-tensor are purely spatial with respect to the observers comoving with the effective ψ-fluid,

$$h_{ab} u^a = h_{ab} u^b = q_c u^c = \pi_{ab} u^a = \pi_{ab} u^b = 0, \tag{2.20}$$

and $\pi^a{}_a = 0$.

The isotropic pressure is decomposed into nonviscous and viscous contributions,

$$P = \bar{P}^{(\psi)} + P^{(\psi)}_{\text{visc}}. \tag{2.21}$$

This dissipative fluid structure is common to all symmetric 2-index tensors and, *per se*, there is no physics in the decomposition (2.8) [64]. However, by taking seriously the dissipative fluid structure of $T^{(\psi)}_{ab}$, one is tempted to apply Eckart's first-order thermodynamics [60] to it. This dissipative ψ-fluid does not, in general, satisfy energy conditions and it is impossible to identify unambiguously all the thermodynamical quantities familiar from real fluids. However, one can limit oneself to considering the assumptions of Eckart's thermodynamics that relate heat flux density, anisotropic stresses, and viscous pressure to the kinematic quantities of the effective fluid, i.e., the assumed constitutive relations where

$$q_a = -\mathcal{K} h_{ab} (\nabla^b \mathcal{T} + \mathcal{T} \dot{u}^b), \tag{2.22}$$

$$\pi_{ab} = -2\eta \sigma_{ab}, \tag{2.23}$$

$$P_{\text{visc}} = -\zeta \Theta, \tag{2.24}$$

$\mathcal{K}$ is the thermal conductivity, while ζ and η are effective bulk and shear viscosity coefficients. The key point of the effective first-order thermodynamics is that, by comparing Eq. (2.13) (obtained directly from the scalar-tensor field equations) with (2.17) [obtained from the definition of four-velocity (2.9)], one obtains

$$q_a^{(\psi)} = -\frac{\sqrt{-\nabla^c \psi \nabla_c \psi}}{8\pi \psi} \dot{u}_a. \tag{2.25}$$

It is a miracle that the effective ψ-fluid satisfies Eckart's constitutive relation (2.22) (with $h_{ab} \nabla^b \mathcal{T} = 0$) [57]. This fact comes as a surprise and allows one to identify the product of a "thermal conductivity" $\mathcal{K}$ and a "temperature" $\mathcal{T}$ of gravity as [61,62]

$$\mathcal{K}\mathcal{T} = \frac{\sqrt{-\nabla^c \psi \nabla_c \psi}}{\kappa \psi}. \tag{2.26}$$

GR which, as is well known, is obtained for $\psi = \text{const}$ and constant gravitational coupling strength, corresponds to thermal equilibrium at $\mathcal{K}\mathcal{T} = 0$, while any state of scalar-tensor gravity in which the scalar field ψ is dynamical and propagates,[3]

The approach to (or departure from) the GR equilibrium is described by [62]

$$\frac{d(\mathcal{K}\mathcal{T})}{d\bar{\tau}} = \kappa (\mathcal{K}\mathcal{T})^2 - \Theta \mathcal{K}\mathcal{T} + \frac{\Box \psi}{\kappa \psi}, \tag{2.27}$$

where $\bar{\tau}$ is the proper time along the lines of the ψ-fluid and $d/d\bar{\tau} \equiv u^c \nabla_c$, and the expansion scalar is given by Eq. (2.18). This equation can be obtained simply by differentiating $\mathcal{K}\mathcal{T}$ along the effective fluid lines [62] and it is significant because, in spite of two decades of theoretical efforts, no equation describing the relaxation of modified gravity to GR was produced in the context of emergent gravity, or of the thermodynamics of spacetime. Although the first-order thermodynamics of scalar-tensor gravity is ultimately just an analogy (between the extra degree of freedom of gravity and a dissipative fluid), it identifies clearly an order parameter and describes quantitatively the approach to GR, or the departure from it. These phenomena are far from trivial, as we will see in the next two sections.

B. The Sultana solution with a nonminimally oupled scalar field

Let us come to nonminimally coupled scalar fields. The gravitational sector of nonminimally coupled scalar field

[3]Certain theories with nondynamical scalar field ψ correspond to $\mathcal{K}\mathcal{T} = 0$ as well, as it should be since they contain no extra degree of freedom in comparison to GR [69]. corresponds to $\mathcal{K}\mathcal{T} > 0$. From the physical point of view, this situation is rather natural because the excitation of the new degree of freedom ψ with respect to GR should correspond to an "excited state" of gravity and to positive temperature. The first order thermal view of scalar-tensor gravity arises from the dissipative structure of the effective stress-energy tensor of ψ plus the essential ingredient that Eckart's constitutive relation is satisfied, which is by no means to be taken for granted. The combination of these two properties provides the quantity $\mathcal{K}\mathcal{T}$ (unfortunately, not $\mathcal{K}$ and $\mathcal{T}$ separately). We have, therefore, a notion of "temperature of gravity" and a formalism in which the approach of modified gravity to GR is akin to thermal relaxation to zero temperature. This formalism [57,61–64] is still under development and, thus far, has been extended to "viable" Horndeski gravity [59] and to Nordström scalar gravity as a toy model [69], and is being tested for consistency against special scalar-tensor gravities and special solutions of these theories [65–72].

theory is described by the Jordan frame action

$$S_{\mathrm{NMC}} = \int d^4x\sqrt{-g}\left[\left(\frac{1}{\kappa} - \xi\phi^2\right)\frac{R}{2} - \frac{1}{2}\nabla^c\phi\nabla_c\phi - V(\phi)\right], \tag{2.28}$$

where the constant ξ describes the nonminimal coupling of the field ϕ to the Ricci scalar. The corresponding field equations read

$$G_{ab} = \kappa(1-\kappa\xi\phi^2)^{-1}\left(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}\nabla^c\phi\nabla_c\phi - Vg_{ab} + \xi[g_{ab}\Box(\phi^2) - \nabla_a\nabla_b(\phi^2)]\right), \tag{2.29}$$

$$\Box\phi - \xi R\phi - \frac{dV}{d\phi} = 0. \tag{2.30}$$

Nonminimal coupling (i.e., $\xi \neq 0$) seems to have been originally introduced in the context of radiation problems in curved space [76]. It is unavoidable when quantizing a scalar field on a curved background ([77], see also [78–82]). The value $\xi = 1/6$ (conformal coupling) makes Eq. (2.30) conformally invariant if $V(\phi)$ is quartic or identically zero. From a classical point of view, conformal coupling is necessary to avoid causal pathologies, i.e., the propagation of a massive scalar field ϕ strictly along the light cone [83]. $\xi = 1/6$ is also an infrared fixed point of the renormalization group in grand unified theories [84–88,90,91].

The field equations for the conformally coupled ϕ can be written as

$$G_{ab} = G_{\mathrm{eff}} T^{(\psi)}_{ab}, \tag{2.31}$$

where

$$G_{\mathrm{eff}} = \frac{G}{1-\alpha^2\phi^2}, \qquad \alpha \equiv \sqrt{\frac{\kappa}{6}} \tag{2.32}$$

is the effective gravitational coupling strength. It seems that G_{eff} could change sign at the critical scalar field values $\pm 1/\sqrt{\alpha}$: however, in order to do this, G_{eff} must go through a divergence at the critical scalar field values. There are usually instabilities associated with this divergence: for example, in anisotropic universes sourced by a nonminimally coupled scalar field, the shear diverges when the singularities of G_{eff} are approached and, in practice, G_{eff} cannot change sign [89].

The effective fluid quantities derived from $T^{(\phi)}_{ab}$ are

$$\rho^{(\psi)} = T^{(\psi)}_{ab}u^a u^b = \left(1 - \frac{4\pi\phi^2}{3}\right)^{-1}\left\{-\frac{1}{2}\nabla^e\phi\nabla_e\phi + \frac{V(\phi)}{2} + \frac{1}{6}\left[\frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b(\phi^2)}{\nabla^e\phi\nabla_e\phi} - \Box(\phi^2)\right]\right\}, \tag{2.33}$$

$$P^{(\psi)} = \frac{1}{3}T^{(\psi)a}{}_a = \left(1 - \frac{4\pi\phi^2}{3}\right)^{-1}\left\{-\frac{1}{2}\nabla^e\phi\nabla_e\phi - \frac{V(\phi)}{2} + \frac{1}{18}\left[\frac{\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b(\phi^2)}{\nabla^e\phi\nabla_e\phi} + 2\Box(\phi^2)\right]\right\}, \tag{2.34}$$

$$q^{(\psi)}_a = -T^{(\psi)}_{cd}u^c h_a{}^d = \frac{(1-4\pi\phi^2/3)^{-1}\nabla^c\phi\nabla^d\phi}{6(-\nabla^e\phi\nabla_e\phi)^{3/2}} \times [\nabla_d\phi\nabla_a\nabla_c(\phi^2) - \nabla_a\phi\nabla_c\nabla_d(\phi^2)], \tag{2.35}$$

and

$$\begin{aligned}\pi^{(\psi)}_{ab} = T^{(\psi)}_{cd}h_a{}^c h_b{}^d - Ph_{ab} = &-\frac{(1-4\pi\phi^2/3)^{-1}}{6\nabla^e\phi\nabla_e\phi} \\ &\times\left\{\frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^e\phi\nabla_e\phi)\right. \\ &\times\left[\Box(\phi^2) - \frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right] + \nabla^d\phi \\ &\times\left[\nabla_d\phi\nabla_a\nabla_b(\phi^2) - \nabla_b\phi\nabla_a\nabla_d(\phi^2)\right. \\ &\left.\left. - \nabla_a\phi\nabla_b\nabla_d(\phi^2) + \frac{\nabla^c\phi\nabla_a\phi\nabla_b\phi\nabla_c\nabla_d(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right]\right\}.\end{aligned} \tag{2.36}$$

Here we are interested in a particular solution for a conformally coupled scalar field to elucidate features of the first-order thermodynamics of scalar-tensor gravity. The starting point is a solution of GR with a minimally coupled (i.e., $\xi = 0$) scalar field found by Wyman[4] [74]

$$d\tilde{s}^2 = -\kappa r^2 dt^2 + 2dr^2 + r^2 d\Omega^2_{(2)}, \tag{2.37}$$

$$\tilde{\phi}(t) = \tilde{\phi}_0 t, \tag{2.38}$$

where $d\Omega^2_{(2)} \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere and $\tilde{\phi}_0$ is a constant. Wyman's "other"

[4]This is sometimes called Wyman's "other" solution to distinguish it from the better-known solution discovered by Fisher [92] and rediscovered over and over again [74,93–95]. The latter is the general solution of the $\Lambda = 0$ Einstein equations which is static, spherically symmetric, asymptotically flat, and with a free scalar field as the source (see the recent review [96]).

solution coincides with a special case of interior solutions for relativistic stars with a stiff fluid found by Ibañez and Sanz [97] in 1982, previously studied by Buchdahl and Land [98]. This is, in turn, a special case of the Tolman IV class of solutions of the Einstein equations discovered in 1939 [99–101]. The matching with an exterior GR solution was studied in [102].

The Sultana geometry with $\Lambda > 0$ [73] coincides with another special case of a class of perfect fluid solutions given by Ibañez and Sanz. It should probably be called Buchdahl-Land-Sultana-Wyman-Ibañez-Sanz (BLSWIS) solution. More precisely, this BLSWIS metric is a special limit of Buchdahl and Land's [98] stiff fluid solution of the Einstein equations with vanishing cosmological constant but pressure

$$P = \rho - \rho_0 \tag{2.39}$$

(where ρ is the fluid energy density and ρ_0 is a constant), which is supposed to describe an incompressible fluid but, in practice, a cosmological constant is reintroduced.

The more general Buchdahl-Land solution constitutes a special case of the Tolman IV class of solutions [100] describing the interior of a perfect fluid ball with cosmological constant [99].

Sultana [73] generalized the Wyman solution to the case in which there is a positive cosmological constant Λ. This geometry (that we refer to as the Sultana-Wyman solution of GR) can be regarded as the Einstein frame version of a Jordan frame solution of conformally coupled scalar field theory. Then, the inverse conformal map from the Einstein to the Jordan frame produces a new vacuum solution of this theory (which is, of course, conformal to the Sultana-Wyman solution of GR), here referred to as the Sultana solution.

The inhomogeneous, spherically symmetric, and time-dependent Sultana solution of conformally coupled scalar field theory is [73]

$$ds^2 = \cosh^2(\alpha t)\left[-\kappa r^2 dt^2 + \frac{2dr^2}{1-2\Lambda r^2/3} + r^2 d\Omega^2_{(2)}\right], \tag{2.40}$$

$$\phi(t) = \pm\frac{1}{\alpha}\tanh(\alpha t), \tag{2.41}$$

where $-\infty < t < +\infty$ and $0 < r < \sqrt{\frac{3}{2\Lambda}}$. The scalar field potential obtained by mapping back the cosmological constant Λ from the Einstein to the Jordan conformal frame is the Higgs potential [73]

$$V(\phi) = \frac{\Lambda}{\kappa}(1-\alpha^2\phi^2)^2. \tag{2.42}$$

In general, the conformal map between Jordan and Einstein frames produces scalar field potentials that are not physically motivated, but this is not the case here. $V(\phi)$ is non-negative since (following [73]) we assume $\Lambda \geq 0$. The effective gravitational coupling strength (2.32) is kept positive by requiring that $-\phi_c < \phi < \phi_c$, where $\phi_c \equiv \alpha^{-1} = \sqrt{6/\kappa}$ is a critical scalar field value. With this potential, the action (2.28) is invariant under the exchange $\phi \to -\phi$.

The Sultana geometry exhibits a spacetime singularity ar $r = 0$, where the Kretschmann scalar

$$R_{abcd}R^{abcd} = \frac{3 + 2\cosh(\alpha t) + \cosh(4\alpha t)}{3r^4\cosh^8(\alpha t)} \tag{2.43}$$

diverges [73].

The nature of the Sultana-Wyman solution of GR was studied in detail in Ref. [103], which analyzes its radial null geodesics. Since the Sultana geometry (2.40), (2.41) is conformal to the Sultana-Wyman geometry of GR its causal structure, which is conformally invariant, is the same. In particular, the singularity at $r = 0$, where the curvature invariants diverge, persists. This is a naked singularity because no apparent horizons cover it. In fact, the equation locating all the apparent horizons, when they exist, is $\nabla^c\mathcal{R}\nabla_c\mathcal{R} = 0$, where $\mathcal{R}$ is the areal radius [104] and this equation does not admit solutions in the Sultana spacetime [103].

The gradient of the scalar field (2.41)

$$\nabla^a\phi = \mp\frac{\delta_0{}^a}{\kappa r^2\cosh^4(\alpha t)} \tag{2.44}$$

is timelike,

$$\nabla^a\phi\nabla_a\phi = -\frac{1}{\kappa r^2\cosh^6(\alpha t)} < 0 \tag{2.45}$$

for any time t, but is not future-oriented (and, therefore, cannot be used to define an effective fluid four-velocity) unless the negative sign is adopted in Eq. (2.41), which we do here[5]: hence from now on $\phi(t) = -\alpha^{-1}\tanh(\alpha t)$ and $\psi = 1 - \alpha^2\phi^2$. With this choice, the four-velocity of the effective ϕ-fluid

$$u^a = \frac{\nabla^a\psi}{\sqrt{-\nabla^e\psi\nabla_e\psi}} = \frac{\delta_0{}^a}{\sqrt{\kappa}r\cosh(\alpha t)} \tag{2.46}$$

is timelike, future-oriented, and normalized to $u^c u_c = -1$ at all times t and coincides with the time direction of the observers comoving with the effective fluid. The three-dimensional metric on the 3-space orthogonal to u^a is h_{ab}, described in Eq. (2.10), where $h_{ab}u^a = h_{ab}u^b = 0$.

[5]A possible alternative consists of defining $u^a \equiv -(-\nabla^c\phi\nabla_c\phi)^{-1/2}\nabla^a\phi$ [63].

For convenience, we report below the kinematic quantities defined by u^a.

By using the Christoffel symbols

$$\Gamma^0_{00} = \alpha \tanh(\alpha t), \tag{2.47}$$

$$\Gamma^1_{00} = \frac{\kappa r}{2}\left(1 - \frac{2\Lambda r^2}{3}\right), \tag{2.48}$$

$$\Gamma^2_{00} = \Gamma^3_{00} = 0, \tag{2.49}$$

the effective fluid four-acceleration

$$\dot{u}^a = u^c \nabla_c u^a = u^0(\partial_0 u^a + \Gamma^a_{00} u^0) = \delta_1{}^a \frac{(1 - 2\Lambda r^2/3)}{2r\cosh^2(\alpha t)}, \tag{2.50}$$

turns out to be purely radial as expected. The velocity gradient (twice) projected onto the three-space orthogonal to u^a decomposes as

$$V_{ab} \equiv h_a{}^c h_b{}^d \nabla_d u_c = \sigma_{ab} + \frac{\Theta}{3} h_{ab}, \tag{2.51}$$

where Θ is the expansion scalar, σ_{ab} is the trace-free shear tensor, and the antisymmetric part of V_{ab}, the vorticity (ω_{ab}), vanishes because u^a is derived from a scalar and is orthogonal to the hypersurfaces of constant ϕ and constant time t. Four-acceleration and shear are purely spatial, $\dot{u}_c u^c = \sigma_{ab} u^a = \sigma_{ab} u^b = 0$ and $\pi^a{}_a = 0$.

The shear tensor and expansion scalar in coordinates $(t, r, \vartheta, \varphi)$ are (see Appendix A)

$$\sigma_{ab} = 0, \tag{2.52}$$

$$\Theta = \frac{3\tanh(\alpha t)}{\sqrt{6} r \cosh(\alpha t)}. \tag{2.53}$$

Next, one computes the effective fluid quantities composing the stress-energy tensor (2.8), obtaining (cf. Appendix A)

$$\rho^{(\psi)} = \frac{\cosh(2\alpha t)\mathrm{sech}^4(\alpha t)}{2\kappa r^2} + \frac{\Lambda \mathrm{sech}^2(\alpha t)}{2\kappa}, \tag{2.54}$$

$$P^{(\psi)} = \frac{\cosh(2\alpha t)\mathrm{sech}^4(\alpha t)}{6\kappa r^2} - \frac{\Lambda \mathrm{sech}^2(\alpha t)}{2\kappa}, \tag{2.55}$$

$$q_a^{(\psi)} = \delta_a{}^1 \frac{2\tanh(\alpha t)}{\sqrt{6}\kappa r^2 \cosh(\alpha t)}, \tag{2.56}$$

$$\pi_{ab}^{(\psi)} = 0. \tag{2.57}$$

Furthermore, the viscous pressure reads

$$P_{\mathrm{visc}}^{(\psi)} = \frac{\sinh^2(\alpha t) - 1}{3\kappa r^2 \cosh^4(\alpha t)}. \tag{2.58}$$

III. FIRST-ORDER THERMODYNAMICS AND APPROACH TO THE GR EQUILIBRIUM

As for any first-generation scalar-tensor theory, Eckart's generalization (2.22) of the Fourier law is satisfied, giving

$$q^a_{(\psi)} = -\mathcal{K} h_{ab}(\nabla^b \mathcal{T} + \mathcal{T}\dot{u}^b) = \frac{2\sinh(\alpha t)}{\sqrt{6}\kappa r \cosh^2(\alpha t)} \dot{u}^a, \tag{3.1}$$

which yields

$$\mathcal{K}\mathcal{T} = -\frac{2\sinh(\alpha t)}{\sqrt{6}\kappa r \cosh^2(\alpha t)}. \tag{3.2}$$

$\mathcal{K}\mathcal{T}$ is positive-definite for $t < 0$, therefore we restrict our consideration to negative times in the following. (Although there is nothing wrong *per se* with the analytical solution of the field equations for $\tau > 0$, the applicability of the first-order thermodynamics of scalar-tensor gravity requires $\mathcal{K}\mathcal{T} > 0$ and we restrict to this situation.)

It is convenient to introduce a new time coordinate τ defined by $d\tau \equiv \cosh(\alpha t) dt$, or

$$\tau(t) = \frac{\sinh(\alpha t)}{\alpha}, \tag{3.3}$$

which is always well defined since $d\tau/dt > 0$ at all times. In terms of τ, the Sultana solution reads

$$ds^2 = -\kappa r^2 d\tau^2 + (1 + \alpha^2\tau^2)\left(\frac{2dr^2}{1 - 2\Lambda r^2/3} + r^2 d\Omega^2_{(2)}\right), \tag{3.4}$$

$$\phi(\tau) = -\frac{\tau}{\sqrt{1 + \alpha^2\tau^2}}, \tag{3.5}$$

while

$$\mathcal{K}\mathcal{T} = \frac{-2\alpha\tau}{\sqrt{6}\kappa r(1 + \alpha^2\tau^2)}. \tag{3.6}$$

Using $\psi = 1 - \alpha^2\phi^2 = (1 + \alpha^2\tau^2)^{-1}$, it is straightforward to see that Eq. (3.6) matches the general expression (2.26) of $\mathcal{K}\mathcal{T}$ in first-generation scalar-tensor gravity. Furthermore, one has

$$\Box\phi = 0 \tag{3.7}$$

and

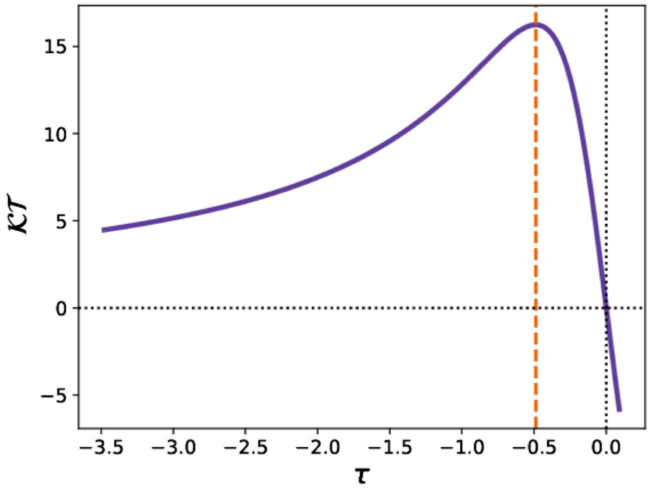

FIG. 1. Evolution of $\mathcal{KT}$ with the time τ (in units in which $r = 1$ and $G = 1$). The solid (purple) curve represents $\mathcal{KT}(\tau, 1)$. The dashed vertical (orange) line marks the maximum value of $\mathcal{KT}(\tau, 1)$.

$$R = \frac{-4\kappa V}{1 - \alpha^2\phi^2} = -4\Lambda(1 - \alpha^2\phi^2) = -\frac{4\Lambda}{\cosh^2(\alpha t)}, \qquad (3.8)$$

consistent with the equation of motion of ϕ. However, $\Box\psi \neq 0$ and the last term on the right-hand side of Eq. (2.27) does not vanish, allowing for a more generic test of the basic ideas of Refs. [57,59,61–65].

As is clear from Eq. (3.6), $\mathcal{KT} \to +\infty$ as $r \to 0^+$: the naked spacetime singularity at $r = 0$ is "hot," in the sense that gravity departs from GR there, and the deviation is infinite. In the infinite past, $\tau \to -\infty$ (also $t \to -\infty$), $\phi(\tau)$ approaches a constant and $\mathcal{KT}$ tends to zero, thus gravity approaches GR asymptotically (Fig. 1). This fact is consistent with the idea that gravity "cools" as 3-space expands. In fact, the finite volume of 3-space is

$$\begin{aligned}V^{(3)}(\tau) &= \int d^3x\sqrt{g^{(3)}(\tau)} \\ &= \int_0^{\sqrt{\frac{3}{2\Lambda}}} dr \int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi \sqrt{\frac{2(1+\alpha^2\tau^2)^3}{1-2\Lambda r^2/3}} r^2 \sin\vartheta \\ &= 4\pi\sqrt{2}(1+\alpha^2\tau^2)^{3/2} \int_0^{\sqrt{\frac{3}{2\Lambda}}} dr \frac{r^2}{\sqrt{1-2\Lambda r^2/3}} \\ &= \frac{3\sqrt{3}\pi^2}{2\Lambda^{3/2}}(1+\alpha^2\tau^2)^{3/2} \to +\infty \quad \text{as } \tau \to -\infty.\end{aligned} \qquad (3.9)$$

The 3-volume $V^{(3)}$ is infinite in the infinite past $\tau \to -\infty$, decreases monotonically for $\tau < 0$, and reaches its absolute minimum $V_{\min} = \frac{3\sqrt{3}\pi^2}{2\Lambda^{3/2}}$ at $\tau = 0$, then increases monotonically, diverging again as $\tau \to +\infty$.

Since $\mathcal{KT} \to 0^+$ and $V^{(3)} \to +\infty$ as $\tau \to -\infty$, the Sultana solution corroborates the idea that the expansion of space "cools" gravity, even when the third term $\frac{\Box\psi}{\kappa\psi}$ in Eq. (2.27) does not vanish and for noncosmological spacetimes in which 3-space still expands. The effective gravitational coupling $G_{\text{eff}} = \frac{G}{1-(\phi/\phi_c)^2} \to +\infty$ as $\tau \to -\infty$ and $\phi \to -\phi_c$. Thus far, based on previous exact solution examples, singularities of G_{eff} have been regarded on par with spacetime singularities where gravity becomes "hot." As the Sultana solution shows, this picture is at least incomplete because here G_{eff}, which depends only on time, diverges where 3-space expands without limit, but gravity "cools" instead. The effect of 3-space expansion dominates over divergences of the gravitational coupling. In the end, it is the product $G_{\text{eff}} T_{ab}^{(\phi)}$ that enters the right-hand side of the field equations, so whether gravity approaches GR or departs from it is determined by the vanishing of this product and not by the individual factors G_{eff} and $T_{ab}^{(\phi)}$.

Increasing $\mathcal{KT}$ means increasing deviation from GR: we have

$$\frac{d(\mathcal{KT})}{d\bar{\tau}} \equiv u^c\nabla_c(\mathcal{KT}) = u^\tau \frac{d(\mathcal{KT})}{d\tau}, \qquad (3.10)$$

where $\bar{\tau}$ is the proper time along the fluid lines of the effective ϕ-fluid. The normalization $u^c u_c = -1$ (or, alternatively, the transformation property $u^\tau = \frac{\partial\tau}{\partial t} u^t$) gives $u^\tau = 1/(\sqrt{\kappa r})$ and

$$\frac{d(\mathcal{KT})}{d\bar{\tau}} = \frac{1}{\sqrt{\kappa r}} \frac{d(\mathcal{KT})}{d\tau} = \frac{\alpha^2\tau^2 - 1}{3\sqrt{\kappa r}(1+\alpha^2\tau^2)^2}. \qquad (3.11)$$

$\mathcal{KT}(\tau, r)$ increases for $-\infty < \tau < -\alpha^{-1}$, where it has a maximum $\mathcal{KT}_{\max} = \frac{1}{\sqrt{6\kappa r}}$ and decreases for $-\alpha^{-1} < \tau < 0$, vanishing as $\tau \to 0^-$ and changing concavity at $\tau = -\sqrt{3}\alpha$. Therefore, gravity is extremely close to GR in the far past, then it gradually departs from it but only to a finite extent ($\mathcal{KT}$ remains finite at all radii $r > 0$), then approaches GR again, coinciding with it at $\tau = 0$. To the best of our knowledge, all analytic solutions of scalar-tensor gravity studied thus far exhibit instead a monotonic approach to, or departure from, the GR equilibrium state.

The third term $\frac{\Box\psi}{\kappa\psi}$ in the right-hand side of Eq. (2.27) is responsible for the nonmonotonic behavior of $\mathcal{KT}$ and dominates near $\tau = 0$. It is possible for gravity to depart from GR and return to it after a temporary deviation, a behavior that was not observed before in the literature, which was limited to examples in which $\Box\psi = 0$.

IV. A RELATED SOLUTION OF BRANS-DICKE AND $f(R)$ GRAVITY

It is useful to contrast the thermal behavior of gravity in the Sultana geometry with the one of other solutions in which $\Box\psi = 0$. To this end, we choose a recent solution of

Brans-Dicke theory closely related to the Sultana spacetime of the previous sections, for which the thermal evolution of gravity has not been discussed.

A family of solutions of Brans-Dicke gravity was generated in Ref. [103] using the conformal transformation from Einstein to Jordan frame and the Sultana-Wyman geometry as a seed, obtaining

$$ds^2 = -\kappa r^2 d\tau^2 + \left(1 - \frac{\tau}{\tau_*}\right)^2 \left(\frac{2dr^2}{1 - 2\Lambda r^2/3} + r^2 d\Omega^2_{(2)}\right), \tag{4.1}$$

$$\psi(\tau) = \frac{\psi_*}{(1 - \tau/\tau_*)^2}, \tag{4.2}$$

where τ_* and ψ_* are constants related to the initial conditions. The Jordan frame scalar field potential is the simple mass term

$$V(\psi) = \frac{m^2\psi^2}{2}, \qquad m^2 = \frac{2\Lambda}{\kappa}. \tag{4.3}$$

This geometry is also a solution of pure R^2 gravity, which is given by the action

$$S_{f(R)} = \int d^4x \sqrt{-g} f(R) \tag{4.4}$$

where [103]

$$f(R) = \frac{\kappa}{4\Lambda} R^2 \tag{4.5}$$

and the effective scalar field is $\psi = f'(R) = \kappa R/(2\Lambda)$. This theory does not have a Newtonian limit around a flat background [105] (although it admits one around de Sitter backgrounds [106]), but approximates Starobinsky inflation in $f(R) = R + \alpha R^2$ gravity at high curvatures.

Since

$$\nabla^c \psi = \frac{2\psi_* \delta_0{}^c}{\kappa \tau_* r^2 (\frac{\tau}{\tau_*} - 1)^3} \tag{4.6}$$

and $\nabla^c\psi\nabla_c\psi < 0$, the scalar field gradient is timelike but is future-oriented only for $\tau > \tau_*$, to which range we restrict.

The geometry was analyzed in [103]: the Ricci scalar and Ricci tensor squared are

$$R = \frac{1}{\kappa(1 - \tau/\tau_*)^2} \left(2\Lambda\psi_* - \frac{4\omega}{\tau_*^2 r^2}\right), \tag{4.7}$$

$$R_{ab}R^{ab} = \frac{1}{\kappa\tau_*^4 r^4 (1 - \tau/\tau_*)^4} \left(\frac{9}{\kappa\tau_*^2} - 4 + \frac{8\Lambda r^2}{3}\right) + \frac{\Lambda^2}{\kappa^2\psi_*^2}(1 - \tau/\tau_*)^4. \tag{4.8}$$

Both curvature scalars diverge as $r \to 0^+$ and as $\tau \to \tau_*^+$ which locate, respectively, timelike and spacelike spacetime singularities. The former is again a naked central singularity in a background created by the cosmological constant, now morphed into the Jordan frame $V(\psi)$. Moreover, $G_{\mathrm{eff}} = 1/\psi \to 0$ as $\tau \to \tau_*^+$.

The effective temperature for this Brans-Dicke solution is given by

$$\mathcal{K}\mathcal{T} = \frac{1}{4\pi\sqrt{\kappa}\tau_* r|1 - \tau/\tau_*|}: \tag{4.9}$$

it diverges as $r \to 0^+$ and as $\tau \to \tau_*^+$, while it vanishes asymptotically as $\tau \to +\infty$ and is monotonically decreasing in τ-time. This behavior is not as interesting as that of the Sultana solution (3.4), (3.5) and Eq. (2.27) describing the approach to the GR equilibrium lacks the third term on its right-hand side since here $\Box\psi = 0$.

V. CONCLUSIONS

The first-order thermodynamics of scalar-tensor gravity à la Eckart is a useful analogy that finally unveils a concept of "temperature of gravity" and an equation describing the approach of alternative gravity to GR in a thermal description, but it is just an intermediate step toward a more realistic description of the effective dissipative fluid created by the scalar field. The reason is that Eckart's thermodynamics suffers from lack of causality and instability problems (e.g., [107]). When one attempts to describe the effective ψ-fluid with a more realistic thermodynamical formalism [108–115], there are many more variables and it is much more difficult to identify terms corresponding to the effective fluid quantities in the longer set of equations. Work is in progress in this direction, but in the meantime it is useful to explore aspects of the effective Eckart thermodynamics that are still hidden, as done in this article.

In previous literature on the first-order thermodynamics of scalar-tensor gravity [57,59,61–65], Eq. (2.27) was always studied in situations in which $\Box\psi = 0$ to simplify the analysis. Two key ideas of this formalism emerged in this context: (a) spacetime singularities are "hot"; (b) the expansion of 3-space "cools" gravity [57,59,61–65]. Are these ideas valid only in the limited context $\Box\psi = 0$ or are they generic? Thus far, there is no known example in which the third term on the right-hand side of Eq. (2.27), proportional to $\Box\psi/\psi$, is nonvanishing and is allowed to play a role. It is interesting to learn how it can affect the evolution of $\mathcal{K}\mathcal{T}$ and the Sultana solution (3.4), (3.5) for a conformally coupled scalar field in the Higgs potential (2.42) allowed us to do just that. In this geometry, gravity is asymptotically Einstein gravity in the far past, then deviates from it but only by a finite extent (i.e., by a finite $\mathcal{K}\mathcal{T}$), and returns to GR. This is a new behavior due to the term $\frac{\Box\psi}{\kappa\psi}$: all example solutions previously examined show a monotonic relaxation to GR or departure from it, as exemplified by the Brans-Dicke geometry of Sec. IV.[6]

In the general theory, it is not possible to predict *a priori* the sign of $\frac{\Box\psi}{\kappa\psi}$ and, therefore, its cooling or heating effect on gravity. Specifying the functions $V(\psi)$ and $\omega(\psi)$, assuming $\nabla_c\psi$ to be timelike, and restricting to $2\omega+3>0$ in order to avoid a phantom field ψ seems to help, for then Eq. (2.3) yields

$$\frac{\Box\psi}{\kappa\psi}=\frac{\psi V'-2V+\omega'|\nabla^c\psi\nabla_c\psi|}{\kappa\psi|2\omega+3|}, \tag{5.1}$$

where a prime denotes differentiation with respect to ψ and, of course, $\psi>0$ to ensure that $G_{\text{eff}}>0$ (we have inserted an absolute value to make explicit the sign of the term containing $\nabla^c\psi\nabla_c\psi<0$). For given $V(\phi)$ and $\omega(\phi)$ (often, ω is constant), it is possible to predict the sign of $\Box\psi/\psi$ but whether this term dominates or not, or whether it vanishes asymptotically cannot be decided *a priori*. To conclude, although the two key ideas of the first-order thermodynamics of scalar-tensor gravity are corroborated, the full range of possible behaviors of gravity is richer.

Unfortunately, at this stage the first-order thermodynamics of scalar-tensor gravity is only able to address situations in which the gradient of the scalar field is timelike, which allows one to introduce the effective fluid four-velocity (2.9). The Sultana solution was selected here for its convenience because, although it is spatially inhomogeneous, the scalar field ψ depends only on time and $\nabla^a\psi$ is timelike and future-oriented. The catalog of exact solutions of the Brans-Dicke field equations with these properties is essentially nonexistent and one has to settle for a solution which, although devoid of immediate physical meaning (it contains a naked singularity), allows us to gain physical insight into the mechanisms of the new thermodynamical formalism. Work is in progress to remove the restriction of timelike scalar field gradient (which would allow one to discuss black holes with spacelike $\nabla^a\psi$ and no-hair theorems), but it is not yet clear whether this is possible.

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APPENDIX A: EFFECTIVE FLUID COMPONENTS

In order to compute the various fluid quantities, it is useful to know that

$$\left(1-\frac{4\pi\phi^2}{3}\right)^{-1}=\cosh^2(\alpha t), \tag{A1}$$

$$\Box\phi=0, \tag{A2}$$

$$\Box(\phi^2)=\frac{-2}{\kappa r^2\cosh^6(\alpha t)}, \tag{A3}$$

$$\nabla^a\phi\nabla_a\phi=\frac{-1}{\kappa r^2\cosh^6(\alpha t)}, \tag{A4}$$

$$\begin{aligned}\nabla_a\nabla_b(\phi^2)=&\,\delta_a{}^0\delta_b{}^0\frac{[2-6\sinh^2(\alpha t)]}{\cosh^4(\alpha t)}\\&-(\delta_a{}^0\delta_b{}^1+\delta_a{}^1\delta_b{}^0)\frac{2\tanh(\alpha t)}{\alpha r\cosh^2(\alpha t)}\\&-(\delta_a{}^2\delta_b{}^2+\delta_a{}^3\delta_b{}^3\sin^2\theta)\frac{2\tanh^2(\alpha t)}{\kappa\cosh^2(\alpha t)},\end{aligned} \tag{A5}$$

$$\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b\phi=\frac{3\alpha\tanh(\alpha t)}{\kappa^2r^4\cosh^{10}(\alpha t)} \tag{A6}$$

$$\nabla^a\phi\nabla^b\phi\nabla_a\nabla_b(\phi^2)=\frac{2[1-3\sinh^2(\alpha t)]}{\kappa^2r^4\cosh^{12}(\alpha t)}. \tag{A7}$$

The energy density (2.33) for the Sultana metric (2.40) and scalar field (2.41) is computed using Eqs. (A1), (A3), (A4) and (A7) in Eq. (2.33), which yields

$$\begin{aligned}\rho^{(\psi)}=&\cosh^2(\alpha t)\left\{\frac{\operatorname{sech}^6(\alpha t)}{2\kappa r^2}+\frac{\Lambda\operatorname{sech}^4(\alpha t)}{2\kappa}\right.\\&\left.+\frac{1}{6}\left[\left(\frac{2-6\sinh^2(\alpha t)}{\kappa^2r^4\cosh^{12}(\alpha t)}\right)\left(\frac{-\kappa r^2}{\operatorname{sech}^6(\alpha t)}\right)+\frac{2\operatorname{sech}^6(\alpha t)}{\kappa r^2}\right]\right\}\\=&\frac{\cosh(2\alpha t)\operatorname{sech}^4(\alpha t)}{2\kappa r^2}+\frac{\Lambda\operatorname{sech}^2(\alpha t)}{2\kappa}\end{aligned} \tag{A8}$$

using $1+2\sinh^2x=\cosh(2x)$.

The isotropic pressure (2.34) of the Sultana solution (2.40), (2.41) is

$$\begin{aligned}P^{(\psi)}=&\cosh^2(\alpha t)\left\{\frac{\operatorname{sech}^6(\alpha t)}{2\kappa r^2}-\frac{\Lambda\operatorname{sech}^4(\alpha t)}{2\kappa}\right.\\&+\frac{1}{18}\left[\left(\frac{2-6\sinh^2(\alpha t)}{\kappa^2r^4\cosh^{12}(\alpha t)}\right)\left(\frac{-\kappa r^2}{\operatorname{sech}^6(\alpha t)}\right)\right.\\&\left.\left.-\frac{4\operatorname{sech}^6(\alpha t)}{\kappa r^2}\right]\right\}\\=&\frac{\cosh(2\alpha t)\operatorname{sech}^4(\alpha t)}{6\kappa r^2}-\frac{\Lambda\operatorname{sech}^2(\alpha t)}{2\kappa}.\end{aligned} \tag{A9}$$

The viscous pressure for a conformally coupled scalar field can be found by subtracting the isotropic pressure of a minimally coupled scalar field, $P^{(\psi)}_{\xi=0}$, from the isotropic pressure of a conformally coupled scalar field, $P^{(\psi)}_{\xi=1/6}$,

[6]It can be regarded as a Brans-Dicke cousin of the Sultana solution since, like the latter, it is conformal to the Sultana-Wyman geometry.

which is equivalent to the result of Eq. (A9):

$$P_{\mathrm{visc}}^{(\psi)} = P_{\xi=1/6}^{(\psi)} - P_{\xi=0}^{(\psi)} = \frac{\sinh^2(\alpha t) - 1}{3\kappa r^2 \cosh^4(\alpha t)}. \quad \text{(A10)}$$

Moving to the heat flux for the conformally coupled scalar field, Eqs. (2.35), (A1), and (A4) give

$$q_a^{(\psi)} = \frac{\cosh(\alpha t)}{6\sqrt{\kappa} r}[-\nabla_a \nabla_0(\phi^2) + \delta_a{}^0 \nabla_0 \nabla_0(\phi^2)]. \quad \text{(A11)}$$

The expression inside the square brackets of Eq. (A11) is zero when $a = 0$, as it should be since $q_a^{(\psi)}$ is purely spatial. The second term inside these square brackets is zero for any other value of a, and only when $a = 0$, 1 is the first term nonzero. The use of Eq. (A5) results in

$$q_a^{(\psi)} = \delta_a{}^1 \frac{\cosh(\alpha t)}{6\sqrt{\kappa} r}[-\nabla_1 \nabla_0(\phi^2)] = \frac{2\delta_a{}^1 \tanh(\alpha t)}{\sqrt{6}\kappa r^2 \cosh(\alpha t)}. \quad \text{(A12)}$$

Since the anisotropic stress tensor $\pi_{ab}^{(\psi)}$ is purely spatial, we only compute its spatial components π_{ij} ($i, j = 1, 2, 3$). We have

$$\pi_{ij} = \frac{(1-\alpha^2\phi^2)}{-6\nabla^e\phi\nabla_e\phi}\left\{-\frac{h_{ij}}{3}\nabla^e\phi\nabla_e\phi\left[\Box(\phi^2) - \frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right] + (\nabla^d\phi\nabla_d\phi)\nabla_i\nabla_j(\phi^2) - \nabla_j\phi\nabla^d\phi\nabla_i\nabla_d(\phi^2) \right.$$
$$\left. - \nabla_i\phi\nabla^d\phi\nabla_j\nabla_d(\phi^2) + \nabla_i\phi\nabla_j\phi\frac{\nabla^c\phi\nabla^d\phi\nabla_c\nabla_d(\phi^2)}{\nabla^e\phi\nabla_e\phi}\right\}. \quad \text{(A13)}$$

Using the fact that $\nabla_i\phi = 0$ and computing

$$\Gamma^0_{ij} = \frac{\alpha\tanh(\alpha t)}{\kappa r^2}\left(\frac{2\delta_i{}^1\delta_j{}^1}{1-2\Lambda r^2/3} + \delta_i{}^2\delta_j{}^2 r^2 + \delta_i{}^3\delta_j{}^3 r^2\sin^2\vartheta\right), \quad \text{(A14)}$$

$$\nabla_i\nabla_j(\phi^2) = 2\phi\nabla_i\nabla_j\phi = -2\phi\Gamma^0_{ij}\dot{\phi} = -\frac{2\tanh^2(\alpha t)}{\kappa r^2\cosh^2(\alpha t)}\left(\frac{2\delta_i{}^1\delta_j{}^1}{1-2\Lambda r^2/3} + \delta_i{}^2\delta_j{}^2 r^2 + \delta_i{}^3\delta_j{}^3 r^2\sin^2\vartheta\right), \quad \text{(A15)}$$

$$(\nabla^d\phi\nabla_d\phi)\nabla_i\nabla_j(\phi^2) = \frac{-1}{\kappa r^2\cosh^6(\alpha t)}\frac{-2\tanh^2(\alpha t)}{\kappa r^2\cosh^2(\alpha t)}\left(\frac{2\delta_i{}^1\delta_j{}^1}{1-2\Lambda r^2/3} + \delta_i{}^2\delta_j{}^2 r^2 + \delta_i{}^3\delta_j{}^3 r^2\sin^2\vartheta\right) = \frac{2\tanh^2(\alpha t)}{\kappa^2 r^2\cosh^{10}(\alpha t)}h_{ij}, \quad \text{(A16)}$$

$$\nabla^d\phi\nabla_i\nabla_d(\phi^2) = \frac{\delta_i{}^1}{\kappa r^3\cosh^6(\alpha t)} \quad \text{(A17)}$$

and, putting everything together, we obtain

$$\pi_{ij} = \frac{\kappa r^2\cosh^6(\alpha t)}{6(1-\tanh^2(\alpha t))}\left\{\frac{h_{ij}}{3\kappa r^2\cosh^6(\alpha t)}\left[\frac{-2}{\kappa r^2\cosh^2(\alpha t)} + \frac{2(1-3\sinh^2(\alpha t))}{\kappa r^2\cosh^6(\alpha t)}\right] + \frac{2\tanh^2(\alpha t)}{\kappa^2 r^4\cosh^{10}(\alpha t)}h_{ij}\right\}$$
$$= \frac{\kappa r^2\cosh^8(\alpha t)}{6}\left[\frac{-2\tanh^2(\alpha t)}{\kappa^2 r^4\cosh^{10}(\alpha t)}h_{ij} + \frac{2\tanh^2(\alpha t)}{\kappa^2 r^4\cosh^{10}(\alpha t)}h_{ij}\right] = 0. \quad \text{(A18)}$$

The shear tensor is [57]

$$\sigma_{ab} = (-\nabla^e\phi\nabla_e\phi)^{-3/2}\left[-(\nabla^e\phi\nabla_e\phi)\nabla_a\nabla_b\phi - \frac{1}{3}(\nabla_a\phi\nabla_b\phi - g_{ab}\nabla^c\phi\nabla_c\phi)\Box\phi - \frac{1}{3}\left(g_{ab} + \frac{2\nabla_a\phi\nabla_b\phi}{\nabla^e\phi\nabla_e\phi}\right)\nabla_c\nabla_d\phi\nabla^d\phi\nabla^c\phi \right.$$
$$\left. + (\nabla_a\phi\nabla_c\nabla_b\phi + \nabla_b\phi\nabla_c\nabla_a\phi)\nabla^c\phi\right]. \quad \text{(A19)}$$

Since σ_{ab} is purely spatial, one only needs to compute the spatial components, which straightforwardly gives $\sigma_{ab} = 0$. Using now the four-velocity and Eq. (A4), the expansion scalar (2.18) becomes

$$\Theta = \nabla_a \left(\frac{\nabla^a \phi}{\sqrt{-\nabla^b \phi \nabla_b \phi}} \right) = \partial_a \left(\frac{\delta_0{}^a \mathrm{sech}(\alpha t)}{\sqrt{\kappa} r} \right) + \Gamma^a_{ab} \left(\frac{\delta_0{}^b \mathrm{sech}(\alpha t)}{\sqrt{\kappa} r} \right) = \frac{3 \tanh(\alpha t)}{\sqrt{6} r \cosh(\alpha t)}. \tag{A20}$$

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