# Hot spaces with positive cosmological constant in the canonical ensemble: de Sitter solution, Schwarzschild–de Sitter black hole, and Nariai universe

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In a space with fixed positive cosmological constant $\Lambda$, we consider a system with a black hole surrounded by a heat reservoir at radius $R$ and fixed temperature $T$, i.e., we analyze the Schwarzschild–de Sitter black hole space in a cavity. We use results from the Euclidean path integral approach to quantum gravity to study, in a semiclassical approximation, the corresponding canonical ensemble and its thermodynamics. We give the action for the Schwarzschild–de Sitter black hole space and calculate expressions for the thermodynamic energy, entropy, temperature, and heat capacity. The reservoir radius $R$ gauges the other scales. Thus, the temperature $T$, the cosmological constant $\Lambda$, the black hole horizon radius $r_+$, and the cosmological horizon radius $r_c$, are gauged to $RT$, $\Lambda R^2$, $\frac{r_+}{R}$, and $\frac{r_c}{R}$. The whole extension of $\Lambda R^2$, $0 \leq \Lambda R^2 \leq 3$, can be split into three ranges. The first range, $0 \leq \Lambda R^2 < 1$, includes York's pure Schwarzschild black holes. The other values of $\Lambda R^2$ within this range also have black holes. The second range, $\Lambda R^2 = 1$, opens up a folder containing Nariai universes, rather than black holes. The third range, $1 < \Lambda R^2 \leq 3$, is unusual. One striking feature here is that it interchanges the cosmological horizon with the black hole horizon. The end of this range, $\Lambda R^2 = 3$, only existing for infinite temperature, represents a cavity filled with de Sitter space inside, except for a black hole with zero radius, i.e., a singularity, and with the cosmological horizon coinciding with the reservoir radius. For the three ranges, for sufficiently low temperatures, which for quantum systems involving gravitational fields can be very high when compared to normal scales, there are no black hole solutions and no Nariai universes, and the space inside the reservoir is hot de Sitter. The limiting value $RT$ that divides the nonexistence from existence of black holes or Nariai universes, depends on the value of $\Lambda R^2$. For each $\Lambda R^2$ different from one, for sufficiently high temperatures there are two black holes, one small and thermodynamically unstable, and one large and stable. For $\Lambda R^2 = 1$, for any sufficiently high temperature there is the small unstable black hole, and the neutrally stable hot Nariai universe. Phase transitions can be analyzed, the dominant phase has the least action. The transitions are between Schwarzschild-de Sitter black hole and hot de Sitter phases and between Nariai and hot de Sitter. For small cosmological constant, the action for the stable black hole equals the pure de Sitter action at a certain black hole radius and temperature, and so the phases coexist equally. For $0 < \Lambda R^2 < 1$ the equal action black hole radius is smaller than the Buchdahl radius, the radius for total collapse, and the corresponding Buchdahl temperature is greater than the equal action temperature. So above the Buchdahl temperature, the system collapses and the phase is constituted by a black hole. For $\Lambda R^2 \geq 1$ a phase analysis is also made.

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## I. INTRODUCTION

One of the most fascinating aspects of an event horizon consists in the fact that it possesses entropy as well as other quantum and thermodynamic properties. These understandings emerged through the initial works of Bekenstein [1] and Hawking [2], and were endorsed by Gibbons and

Hawking [3] within a Euclidean path integral approach that led to the statistical mechanics canonical ensemble formalism for black holes. The Euclidean path integral approach was extended by Gibbons and Hawking [4] and Ginsparg and Perry [5] to include cosmological horizons, in particular the de Sitter one, that further permitted the analysis of semiclassical effects in such spaces. The path integral and the ensemble theory were put on a firm basis by York [6] who realized that proper boundary conditions on the walls of a heat reservoir that encloses a cavity with a black hole inside is a well posed problem. This idea was implemented by Whiting and York [7] by advancing the correct manner to constraining the problem. Hayward [8] examined the approach in spaces with cosmological de Sitter horizons, Braden *et al.* [9] included electrically charged black holes in the formalism, Zaslavskii enlarged it further to ensembles with arbitrary configurations of self-gravitating systems [10], Lemos applied the approach to the two-dimensional black hole of the Teitelboim-Jackiw theory [11], Zaslavskii studied the extreme state of a charged black hole in a grand canonical ensemble [12], and also analyzed the geometry of nonextreme black holes near the extreme state [13]. Peça and Lemos implemented the formalism to the grand canonical ensemble of electrically charged black holes in anti–de Sitter spaces [14], André and Lemos extended the results to $d$ dimensions [15,16], Fernandes and Lemos constructed the grand canonical ensemble of the electric charged $d$-dimensional case [17], and Lemos and Zaslavskii studied the interaction between black holes and matter in the canonical ensemble [18]

The de Sitter space with its cosmological horizon is in itself fascinating, and its study is now of central importance since it is realized as the asymptotic solution for our real expanding universe, as well as forming the basis for the inflationary models of the early universe. It is thus of significance to learn not only the classical aspects related to it, but also its quantum and thermodynamic properties. A key feature of this space is that its cosmological horizon radiates through quantum processes, but since this radiation is due to a fixed cosmological constant, it radiates on and on, and so, unlike a black hole horizon, the de Sitter horizon does not evanesce. In this sense, the de Sitter cosmological horizon is quantum stable. One might want to add a central black hole to the de Sitter space, obtaining thus the Schwarzschild–de Sitter space, which is a solution of general relativity. The Schwarzschild–de Sitter solution has an appeal of its own since it has two horizons, namely, the black hole horizon and the cosmological horizon. These two horizons generically have different temperatures, and so there is no possible thermodynamic equilibrium solution. This means that in this setting the problem should be treated as a nonequilibrium case and no thermodynamics can be properly devised. However, some insights to bypass this obstacle to a thermodynamic formulation have been given. One way to have a proper thermodynamics is to apply a reservoir kept at some temperature and with boundary at some radius, and use York's Euclidean path integral formalism. Developments in this direction can be mentioned. Wang and Huang [19,20] made a study of the thermodynamics of the Schwarzschild–de Sitter space in York's formalism and also extended to the thermodynamics of Reissner-Nordström–de Sitter. Ghezelbash and Mann [21] analyzed the action and entropy of Schwarzschild–de Sitter black holes. Saida [22] treated some aspects of the Schwarzschild–de Sitter thermodynamics in the canonical ensemble, and Draper and Farkas [23] discussed de Sitter black holes in the Euclidean path integral approach. Banihashem and Jacobson [24] considered thermodynamic ensembles with cosmological horizons, Banihashemi *et al.* [25] explored further the minus sign that enters the thermodynamic energy in the first law of thermodynamics for de Sitter horizons, Jacobson and Visser [26,27] built the partition function for a volume of space as well as the partition function and the entropy of causal diamond ensembles, and Morvan *et al.*, examined the Euclidean action of de Sitter black holes [28].

In this work, we want to further understand the thermodynamics of the Schwarzschild–de Sitter black hole space in the canonical ensemble within York's formalism. In using it one has to choose whether the heat reservoir, put in-between the two horizons, is a reservoir for the inside, i.e., is a reservoir for the black hole horizon region, or is a reservoir for the outer universe, i.e., for the region that includes the cosmological horizon. Here we are interested in the first situation, and will study the thermodynamics of the black hole region in a cavity with a heat reservoir outside. The black hole horizon and its temperature together with the reservoir and its temperature play a principal role in the thermodynamic analysis, indeed the equilibrium situations are established by them. The cosmological horizon has no major role in this setting, actually its place is a function of the black hole horizon location, and it is directly determined once the location off this latter has been found through thermodynamic computation. In this setup there are three main scales, the scale set by the size of the reservoir, the scale set by the temperature of the reservoir, and the scale set by the cosmological constant, which in turn yield the scale set by the size of the black hole horizon and the scale set by the size of the cosmological horizon. It is thus expected that the existence of these various scales yields new, interesting, and important properties of the system. One that we can advance now, is that the set of ensembles is comprised not only of the Schwarzschild–de Sitter black hole but also of the Nariai universe, which arises naturally when the cosmological length scale and the reservoir length scale are equal. This intermediate case divides the ensembles into a set of ensembles with low cosmological constant that has familiar properties, and another set with high cosmological constant that is new and in which the black hole and cosmological horizons exchange roles. Other thermodynamic properties become quite unusual as compared with the no cosmological constant black hole case in the canonical ensemble.

We would like to mention several other different attempts devised to understand the quantum and thermodynamic nature of black holes in de Sitter space that are interesting on themselves but that do not have a direct bearing with our work here. Davies [29] studied the black hole mechanics and thermodynamics of the Kerr-Newman-de Sitter family of solutions, and Romans [30] performed an important analysis of the de Sitter black holes, classifying them as temperature goes in cold, lukewarm, and warm. Bousso and Hawking [31] put forward the interesting possibility of having evaporation and anti-evaporation of Schwarzschild–de Sitter black holes, Maeda *et al.* [32] found an upper bound for the entropy of an asymptotically de Sitter spacetime, Wu [33] worked out the entropy of black holes with different surface gravities with applications to Schwarzschild–de Sitter black holes, Yueqin *et al.* [34] used the brick wall method of 't Hooft to calculate the entropy of Schwarzschild–de Sitter black holes, Bousso [35] has defined causal diamonds in de Sitter black hole spaces and inspected their entropy, Cai [36] considered the Cardy-Verlinde formula in connection to thermodynamics of de Sitter black holes, Shankaranarayanan [37] attempted to set a scheme where the different temperatures of the two Schwarzschild–de Sitter horizons could be made consistent, and Teitelboim and Gomberoff [38,39] examined the de Sitter black holes with either one of the two horizons working as a thermodynamic boundary. Dias and Lemos [40] analyzed pair creation of de Sitter black holes on a cosmic string background and the associated entropy, Cardoso *et al.* [41] displayed the Schwarzschild–de Sitter, Nariai, Bertotti-Robinson, and anti-Nariai solutions in higher dimensions, in particular their temperatures including the lukewarm cases, Sekiwa [42] investigated Schwarzschild–de Sitter spaces with the cosmological constant as a thermodynamic variable, Choudhury and Padmanabhan [43] invoked a new concept of temperature in spaces with several horizons, Myung [44] inspected the thermodynamics of the Schwarzschild–de Sitter black hole and the Nariai solution in five dimensions, Pappas and Kanti [45] treated Schwarzschild–de Sitter spaces and the role of temperature in the emission of Hawking radiation, Simovic and Mann [46] exhibited critical phenomena of certain types of de Sitter black holes in cavities, Qiu and Traschen [47] obtained new results related to thermodynamics of black pair production in Schwarzschild–de Sitter spaces, Singha [48] developed further the thermodynamics of spaces with several horizons, Volovik [49] suggested a double Hawking temperature ansatz to explain the thermodynamics of a black hole in de Sitter space, and Akhmedov and Bazarov [50] made an analysis of the backreaction issue for a black hole in de Sitter space.

An important concept for finite self-gravitating systems, as the ones we want to consider, is the Buchdahl bound that sets a maximum mass or a maximum gravitational radius for the energy that can be enclosed in a cavity before the system turns singular and presumably suffers total gravitational collapse. Usually, the Buchdahl radius concerns the mechanical structure of balls or stars in general relativity, but it should also appear somehow in connection with thermodynamics and thermodynamic phases, since when, in a cavity, there is energy in the form of matter or radiation with gravitational radius larger than the gravitational radius permitted by the Buchdahl bound for a given cavity size, that energy should collapse. Thus, a thermodynamic system that has too much thermodynamic energy for a given cavity size must collapse. Since here, we are interested in self-gravitating systems in a positive cosmological constant background in a general relativistic context, the Buchdahl bound of interest is the one found by Andréasson and Böhmer [51].

The paper is organized as follows. In Sec. II we state the main general thermodynamic results, derived from the canonical ensemble set by the Euclidean path integral approach, for a cavity containing the black hole horizon region of the Schwarzschild–de Sitter space inside a heat reservoir. In Sec. III we give specific results for the thermodynamics of Schwarzschild–de Sitter black holes with small values of the cosmological constant, $\Lambda R^2 < 1$, also studying thermodynamic phases and phase transitions. In Sec. IV we give specific results for the thermodynamics of Schwarzschild–de Sitter black holes with the intermediate value of the cosmological constant, $\Lambda R^2 = 1$, which is found to be the Nariai universe, and also studying thermodynamic phases and phase transitions. In Sec. V we give specific results for the thermodynamics of Schwarzschild–de Sitter black holes with large values of the cosmological constant, $\Lambda R^2 > 1$, and also studying thermodynamic phases and phase transitions. In Sec. VI we present important plots and make a thorough analytic study of all the cases. In Sec. VII we draw our conclusions. In the Appendix A we state the basic geometric elements of the Schwarzschild–de Sitter and Nariai spaces. In the Appendix B the Nariai limit from the Schwarzschild–de Sitter space in a cavity in a thermodynamic setting is presented in all detail. In the Appendix C we derive explicitly some expressions of the main text.

## II. THERMODYNAMICS OF THE SCHWARZSCHILD–DE SITTER SPACE IN THE CANONICAL ENSEMBLE: GENERAL RESULTS FOR THE BLACK HOLE HORIZON REGION INSIDE A HEAT RESERVOIR

### A. Setup and Euclidean metric

Put, at some radius $R$, the boundary of a spherical cavity with a black hole in a positive cosmological constant background inside a heat reservoir. At this boundary, one

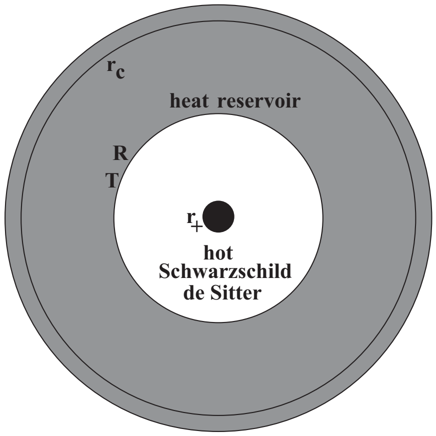


FIG. 1. A drawing of a black hole in a cavity within a heat reservoir at temperature $T$ and radius $R$ in a space with positive cosmological constant. Outside the black hole radius $r_+$ the geometry is a Schwarzschild–de Sitter geometry. The cosmological radius $r_c$ is beyond the heat reservoir. The Euclideanized space and its boundary have $R^2 \times S^2$ and $S^1 \times S^2$ topologies, respectively, where the $S^1$ subspace with proper length $\beta = \frac{1}{T}$ is not displayed. See text for more details.

specifies the data that determine the ensemble, see Fig. 1. We fix the temperature at the boundary at $R$, so we are working with the canonical ensemble. Furthermore, we consider that the heat reservoir is a heat reservoir for the inner region of the Schwarzschild–de Sitter space and thermodynamically discard the region $r > R$. This is the approach first suggested by York for the Schwarzschild metric [6–9], that was generalized further to include matter [10].

In order to implement the Euclidean path integral approach and work out canonical ensemble results for a black hole in a positive constant background we use the Schwarzschild–de Sitter solution in general relativity. Then, for the Schwarzschild–de Sitter black hole metric, one Euclideanizes time and fixes its period $\beta$ at radius $R$ to be the heat reservoir inverse temperature, $\beta \equiv \frac{1}{T}$. The Schwarzschild–de Sitter space is characterized by two parameters, namely, the mass $m$ and the positive cosmological constant $\Lambda$. Instead of working with $m$ and $\Lambda$, it is sometimes preferable to use the black hole horizon radius $r_+$ and the cosmological horizon radius $r_c$, the two sets of two parameters are interchangeable by precise formulas. In the case we are working, the reservoir is a reservoir for the inner region that contains a black hole, so that $r_+$ is inside $R$ and $r_c$ is outside $R$. The Euclidean line element of the Schwarzschild–de Sitter space in spherical coordinates $(t, r, \theta, \phi)$ is obtained by Euclideanizing time, $t \to it$, to get the Euclidean Schwarzschild–de Sitter space,

$$ds^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$0 \le t < \beta_+^{\rm H}, r_+ \le r \le R, \tag{1}$$

where the metric potential $V(r)$ has the form

$$V(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \tag{2}$$

and $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$. The range of coordinates of Euclidean time $t$ is $0 \le t < \beta_+^{\rm H}$, where $\beta_+^{\rm H}$ is the period of the time coordinate such that the line element given by Eqs. (1) and (2) has no conical singularities. The relation with the Hawking temperature $T_+^{\rm H}$ is $\beta_+^{\rm H} = \frac{1}{T_+^{\rm H}}$. Due to the reservoir at radius $R$ the range of the radial coordinate is $r_+ \le r \le R$. The line element provided in Eqs. (1) and (2) is the Euclideanized form of the Schwarzschild–de Sitter spacetime, see Appendix A, and its topology is $R^2 \times S^2$.

The black hole horizon radius $r_+$ is one of the two real zeros of $V(r)$ in Eq. (2) and so obeys

$$\frac{r_+}{2}\left(1 - \frac{\Lambda r_+^2}{3}\right) = m. \tag{3}$$

Then $m$ can be traded for $r_+$ to give $V(r)$ of Eq. (2) as $V(r) = 1 - \frac{r_+}{r} - \frac{\Lambda r^2}{3}(1 - (\frac{r_+}{r})^3)$, i.e.,

$$V(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{\Lambda r^2}{3}\left(1 + \frac{r_+}{r} + \left(\frac{r_+}{r}\right)^2\right)\right). \tag{4}$$

The cosmological horizon radius is the other zero of $V(r)$ in Eq. (2). It can be found once $r_+$ is known through the equation

$$r_c = -\frac{r_+}{2} + \frac{r_+}{2}\sqrt{\frac{12 - 3\Lambda r_+^2}{\Lambda r_+^2}}, \tag{5}$$

or $r_c = -\frac{r_+}{2} + \frac{1}{2}\sqrt{\frac{3}{\Lambda}}\sqrt{4 - \Lambda r_+^2}$. Since the heat reservoir envelopes the inside, the primary horizon radius is the black hole horizon $r_+$, which has to be found from thermodynamic considerations. The cosmological horizon radius $r_c$ has a secondary role, being determined once $r_+$ is known.

Now, the radius of the heat reservoir $R$ sets a scale for our problem. It is then meaningful to gauge all the length scales involved in the problem to $R$. Thus, the heat reservoir temperature $T$, the cosmological constant $\Lambda$, the black hole horizon radius $r_+$, and the cosmological horizon radius $r_c$, are gauged to quantities without units as $RT$, $\Lambda R^2$, $\frac{r_+}{R}$, and $\frac{r_c}{R}$. The extensions of these quantities are important. They are: $0 \le RT < \infty$, $0 \le \Lambda R^2 \le 3$, $0 \le \frac{r_+}{R} \le 1$, and $1 \le \frac{r_c}{R} < \infty$. It is advisable to separate the whole extension of $\Lambda R^2$, $0 \le \Lambda R^2 \le 3$, into three cases, namely, small values of the cosmological constant which means $\Lambda R^2 < 1$, more exactly $0 \le \Lambda R^2 < 1$, the intermediate value of the cosmological constant which means $\Lambda R^2 = 1$, and large values of the cosmological constant which means $\Lambda R^2 > 1$, more exactly $1 < \Lambda R^2 \le 3$. Note that the cosmological constant, which has units of inverse length square, has associated to it a natural cosmological length scale $\ell$ given by $\ell^2 = \frac{1}{\Lambda}$.

### B. Action, energy, entropy

We list here the most relevant general formulas that come out of the path integral approach, one can look elsewhere to pick up these formulas, see, e.g., [6–18]. The Euclidean action $I$ is

$$I = \beta R(1 - \sqrt{V(R)}) - \pi r_+^2, \tag{6}$$

where $\beta = \frac{1}{T}$ is the inverse temperature of the ensemble, i.e., at the boundary of the heat reservoir, and where $V(R)$ is given by Eq. (2) at $r = R$, i.e., $V(R) = 1 - \frac{2m}{R} - \frac{\Lambda R^2}{3}$ or

$$V(R) = \left(1 - \frac{r_+}{R}\right)\left(1 - \frac{\Lambda R^2}{3}\left(1 + \frac{r_+}{R} + \left(\frac{r_+}{R}\right)^2\right)\right). \tag{7}$$

Note that $I$ in Eq. (6) is $I = I(\beta, R, \Lambda; r_+)$. The statistical mechanics ensemble is characterized by $\Lambda$ which is fixed for each space, by $T = \frac{1}{\beta}$ and $R$ which are fixed for each ensemble, and by $r_+$ which can vary, there are particular $r_+$ solutions for which $I$ is stationary, $\frac{dI}{dr_+} = 0$, yielding the thermodynamic solutions of the problem.

The Euclidean action and the free energy are related by $I = \beta F$, so that $F = R(1 - \sqrt{V(R)}) - T\pi r_+^2$. Now, $F = E - TS$, so the thermodynamic or quasilocal energy here also the thermal energy at $R$ at this order can be found to be

$$E = R(1 - \sqrt{V(R)}). \tag{8}$$

The entropy of the system is

$$S = \pi r_+^2, \tag{9}$$

which is the Bekenstein-Hawking entropy.

To find the thermodynamic stability one has to compute the heat capacity at constant reservoir area $A$, $C_A$. This is given by

$$C_A = \left(\frac{dE}{dT}\right)_A. \tag{10}$$

If $C_A < 0$ the system is thermodynamically unstable, if $C_A \geq 0$ the system is thermodynamically stable, with the equality giving the marginal case. Since $A = 4\pi R^2$, Eq. (10) is equivalent to $C_R = (\frac{dE}{dT})_R$.

### C. Temperature and solutions

In order that the whole formalism be meaningful, the line element of Eqs. (1) and (2) should have no conical singularities in the Euclidean $r \times t$ plane. This implies that the time coordinate $t$ has to have period $\beta_+^{\mathrm{H}}$ given by $\beta_+^{\mathrm{H}} = \frac{4\pi}{\left(\frac{dV(r)}{dr}\right)_{r_+}}$. It turns out that this period is related to the Hawking temperature $T_+^{\mathrm{H}}$ through $\beta_+^{\mathrm{H}} = \frac{1}{T_+^{\mathrm{H}}}$, so $T_+^{\mathrm{H}} = \frac{\left(\frac{dV(r)}{dr}\right)_{r_+}}{4\pi}$. From Eq. (2) we obtain

$$T_+^{\mathrm{H}} = \frac{1}{4\pi r_+}(1 - \Lambda r_+^2). \tag{11}$$

Using Eq. (3) this can be also put in the form $T_+^{\mathrm{H}} = \frac{1}{2\pi r_+}(\frac{3m}{r_+} - 1)$.

The stationary points of the action Eq. (6) are found through $\frac{dI}{dr_+} = 0$, which yields the equation

$$T = \frac{T_+^{\mathrm{H}}}{\sqrt{V(R)}}, \tag{12}$$

with $T_+^{\mathrm{H}}$ being the Hawking temperature given in Eq. (11) and $V(R)$ is given in Eq. (7). This is the Tolman relation for the temperature at the reservoir and the Hawking temperature. Since $T$ has the expression given in Eq. (12) one finds explicitly for this case, using Eqs. (7) and (11), that $T = \frac{\frac{1}{4\pi r_+}(1-\Lambda r_+^2)}{\sqrt{1-\frac{2m}{R}-\frac{\Lambda R^2}{3}}}$, i.e., $T = \frac{\frac{1}{4\pi r_+}(1-\Lambda r_+^2)}{\sqrt{1-\frac{r_+}{R}-\frac{\Lambda}{3R}(R^3-r_+^3)}}$. Thus, $4\pi RT = \frac{1}{\frac{r_+}{R}}\frac{1-\Lambda R^2(\frac{r_+}{R})^2}{\sqrt{1-\frac{r_+}{R}-\frac{\Lambda R^2}{3}(1-(\frac{r_+}{R})^3)}}$, which can be put in the form

$$4\pi RT = \frac{1}{\frac{r_+}{R}}\frac{1 - \Lambda R^2(\frac{r_+}{R})^2}{\sqrt{1 - \frac{r_+}{R}}\sqrt{(1 - \frac{\Lambda R^2}{3}(1 + \frac{r_+}{R} + (\frac{r_+}{R})^2))}}. \tag{13}$$

We want to find $r_+$ that obey Eq. (13), and so generically one has $\frac{r_+}{R}(\Lambda R^2, RT)$. For fixed $RT$ one has $\frac{r_+}{R}(\Lambda R^2)$, and for fixed $\Lambda R^2$ one has $\frac{r_+}{R}(RT)$.

Thus, for a given fixed $T$ at the boundary, for the ensemble, one can look for solutions $r_+$. Indeed, depending on the parameters $(T, R, \Lambda)$, Eq. (13) can have no solution, one solution, or two solutions $r_+$. When it has two solutions we denote these by

$$r_{+1} = r_{+1}(R, \Lambda, T), \tag{14}$$

and

$$r_{+2} = r_{+2}(R, \Lambda, T), \tag{15}$$

with $r_{+1} \leq r_{+2}$, say. The functions $r_{+1}$ and $r_{+2}$ have to be worked out in some way or another. The fact that for given $T$ and $R$ there exist two roots $r_{+1}$ and $r_{+2}$ is similar to that for the Schwarzschild space [6], here it is for a given $T$ and $R$, and for a given $\Lambda$. Moreover, given the solutions $r_{+1}$ of

Eq. (14) and $r_{+2}$ of Eq. (15), one finds from Eq. (5) the two corresponding cosmological horizon radii, namely,

$$r_{c1} = r_{c1}\big(r_{+1}(R,\Lambda,T)\big), \tag{16}$$

and

$$r_{c2} = r_{c2}\big(r_{+2}(R,\Lambda,T)\big), \tag{17}$$

respectively, with $r_{c1} \geq r_{c2}$.

In brief, we have already generic results, though not explicit. We now treat separately the three cases, $0 \leq \Lambda R^2 < 1$, $\Lambda R^2 = 1$, and $1 < \Lambda R^2 \leq 3$. In general there are no analytical solutions. In the first case, analytical expressions for particular ranges of $\Lambda R^2$ can be found, in the second case one finds that one is in the presence of a Nariai universe which has exact thermodynamic solutions, and in the third case expressions for particular ranges of $\Lambda R^2$ can also be found. The high temperature limit in the three cases yields analytical solutions. Thermodynamic phases and phase transitions can be analyzed in all the three cases.

## III. THERMODYNAMICS OF SCHWARZSCHILD–DE SITTER BLACK HOLES IN THE CANONICAL ENSEMBLE: SMALL VALUES OF THE COSMOLOGICAL CONSTANT, $\Lambda R^2 < 1$

### A. Solutions

We treat here the small positive cosmological constant, $\Lambda R^2 < 1$, problem. Again, we put the boundary of a spherical cavity with a black hole in a positive cosmological constant background inside a heat reservoir, at some radius $R$, where it is also specified a fixed temperature $T$, see Fig. 1 anew. Small positive cosmological constant means exactly in this context that

$$0 \leq \Lambda R^2 < 1. \tag{18}$$

Within this range, for fixed $RT$ and generic $\Lambda R^2$, it is hard to find solutions of Eq. (13) for black hole horizon radii $r_+$ analytically. However, for very small $\Lambda R^2$ or for $\Lambda R^2$ very near one, one can make some progress.

For very small $\Lambda R^2$, i.e., for $\Lambda R^2 \ll 1$, one finds from Eq. (13) that there are no black hole solutions for

$$RT < \frac{\sqrt{27}}{8\pi}\left(1 - \frac{5}{54}\Lambda R^2\right), \quad \Lambda R^2 \ll 1, \tag{19}$$

only hot de Sitter space is possible. Still for very small $\Lambda$, i.e., for $\Lambda R^2 \ll 1$, there are two black hole solutions for

$$RT \geq \frac{\sqrt{27}}{8\pi}\left(1 - \frac{5}{54}\Lambda R^2\right), \quad \Lambda R^2 \ll 1. \tag{20}$$

One of the two solutions is the small black hole $r_{+1}(R,\Lambda,T)$, and the other solution is the large black hole $r_{+2}(R,\Lambda,T)$. For zero cosmological constant, $\Lambda R^2 = 0$, one has a pure Schwarzschild black hole and one recovers York's result of $RT \geq \frac{\sqrt{27}}{8\pi}$ to have black hole solutions. The minus sign inside the parenthesis in Eq. (20) is what one expects really. The two solutions merge into one sole solution when the equality sign in Eq. (20) holds. In this case the coincident double solution has horizon radius given by

$$\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3}\left(1 + \frac{17}{81}\Lambda R^2\right), \quad \Lambda R^2 \ll 1. \tag{21}$$

The corresponding cosmological radius can then be found from Eq. (5) to be given by

$$\frac{r_{c1}}{R} = \frac{r_{c2}}{R} = \sqrt{\frac{3}{\Lambda R^2}}\left(1 - \frac{1}{3}\sqrt{\frac{\Lambda R^2}{3}}\right), \quad \Lambda R^2 \ll 1. \tag{22}$$

For $\Lambda R^2$ very near unity, i.e., for $(1 - \Lambda R^2) \ll 1$, one finds from Eq. (13) that there are no black hole solutions for

$$RT < \frac{1}{2\pi}\left(1 + \left(\frac{3}{8}(1 - \Lambda R^2)\right)^{\frac{1}{3}}\right), \quad (1 - \Lambda R^2) \ll 1, \tag{23}$$

only hot de Sitter space is possible. Still for small $\Lambda R^2 - 1$, i.e., for $(1 - \Lambda R^2) \ll 1$, there are two black hole solutions for

$$RT \geq \frac{1}{2\pi}\left(1 + \left(\frac{3}{8}(1 - \Lambda R^2)\right)^{\frac{1}{3}}\right), \quad (1 - \Lambda R^2) \ll 1. \tag{24}$$

When the equality holds the coincident double solution has horizon radius given by

$$\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 1 - \left(\frac{3}{8}(1 - \Lambda R^2)^2\right)^{\frac{1}{3}}, \quad (1 - \Lambda R^2) \ll 1. \tag{25}$$

The corresponding cosmological radius can then be found from Eq. (5) to be given by

$$\frac{r_{c1}}{R} = \frac{r_{c2}}{R} = 1 + \left(\frac{3}{8}(1 - \Lambda R^2)^2\right)^{\frac{1}{3}}, \quad (1 - \Lambda R^2) \ll 1. \tag{26}$$

One could work out in both regimes, i.e., $\Lambda R^2 \ll 1$ and $(1 - \Lambda R^2) \ll 1$, the action $I$, the thermodynamic energy $E$, the entropy $S$, and the heat capacity $C_A$, given through Eqs. (6)–(10). Apart from the entropy expression $S = 4\pi r_+^2$, valid for each of the two black hole solutions, the calculation of the other quantities is not practical and not be particularly illuminating. However, an instance where all quantities can be worked out, in particular the heat capacity $C_A$, is the hight temperature limit to which we now turn.

### B. High temperature limit: Analytical solutions

For the range of values of the cosmological constant considered in this section, $0 \leq \Lambda R^2 < 1$, one can find solutions in the limit in which $RT$ goes to infinity, see Eqs. (12) and (13). Since $R$ is the quantity that we consider as the gauge, $RT$ going to infinity is the same in this context as $T$ going to infinity. Let us then find explicit results by taking the limit of high temperature. In this case the equations can be solved.

For a given $T$ there are two black hole solutions, the small black hole solution $r_{+1}$ and the large black hole solution $r_{+2}$. We set the heat reservoir temperature $T$ fixed but very high, in the sense that $T \to \infty$. From Eq. (12) there are two possibilities. Either $T_+^{\mathrm{H}} \to \infty$ which corresponds to the small black hole solution having a very small $r_{+1}$, or $V(R) \to 0$ which corresponds to the large black hole solution $r_{+2}$ approaching the reservoir radius. Let us work one at a time for $T$ fixed and very high.

The first solution for a very high heat reservoir temperature, $T \to \infty$, is $r_+ = r_{+1} \to 0$ with $T_+^{\mathrm{H}} \to \infty$. It is clear from Eq. (11) together with Eq. (12), or directly from Eq. (13), that this requires that the black hole solution is of the form $r_+ = r_{+1} \to 0$. So, in this limit one has

$$T_{+1}^{\mathrm{H}} = \frac{1}{4\pi r_{+1}}, \tag{27}$$

where the equality sign is valid within the approximation taken. From Eq. (13) one finds the small black hole solution $r_{+1}$ to be of the form

$$\frac{r_{+1}}{R} = \frac{1}{4\pi RT\sqrt{1 - \frac{\Lambda R^2}{3}}}, \tag{28}$$

where the equality sign is valid within the approximation taken. The expression inside the square root of Eq. (28) is clearly positive. As a by-product, we also find from Eq. (3) that in this limit one has $m_1 = \frac{r_{+1}}{2}$. The corresponding cosmological radius $r_{c1}$ can then be found directly from Eq. (5), yielding a correspondingly far away value for $r_{c1}$, see Eq. (5). One could work out in this order, i.e., $T \to \infty$, the action $I$, the energy $E$, the entropy $S$, and the heat capacity $C_A$, given through Eqs. (6)–(10). The most interesting quantity is the heat capacity $C_A$, which yields the criterion for thermodynamic stability, indeed when $C_A < 0$ the solution is thermodynamically unstable, when $C_A \geq 0$ the solution is thermodynamically stable. We thus find an explicit expression for $C_A$. From Eq. (10), i.e., $C_A = (\frac{dE}{dT})_A$ or equivalently, $C_A = (\frac{dE}{dT})_R$, we find from Eq. (8) that $C_A = \frac{1}{2\sqrt{V(R)}}(\frac{dr_{+1}}{dT})_R$, which upon using Eq. (28) yields

$$C_{A_{+1}} = -\frac{1}{8\pi T^2(1 - \frac{\Lambda R^2}{3})} < 0, \tag{29}$$

so that $C_A$ for the small black hole $r_{+1}$ is negative. The small black hole $r_{+1}$ solution is thus unstable. Note that actually, the black hole is surrounded by quantum fields. We neglect their backreaction on the metric. However, if $T^{\mathrm{H}} \to \infty$, the corresponding energy density and other components of the stress-energy tensor diverge. To avoid this, we restrict $r_{+1}$ in the sense that is has to be larger than the Planck length scale $l_{\mathrm{pl}}$, i.e., $r_{+1} > l_{\mathrm{pl}}$.

The second solution for a very high heat reservoir temperature, $T \to \infty$ has $V(R) \to 0$. It is clear from Eqs. (12) or (13) that the condition $V(R) \to 0$, implies, in the case $0 \leq \Lambda R^2 < 1$, that $r_{+2} = R$ minus a small quantity. Now, from Eq. (11) one has in this limit

$$T_{+2}^{\mathrm{H}} = \frac{1 - \Lambda R^2}{4\pi R}, \tag{30}$$

where the equality sign is valid within the approximation taken. In first order, we can perform a Taylor expansion, and write $V(R) = (\frac{dV}{dr})_{r_{+2}}(R - r_{+2})$ plus higher order terms. Since $(\frac{dV}{dr})_{r_{+2}} = 4\pi T_{+2}^{\mathrm{H}}$, one can write $V(R) = 4\pi T_{+2}^{\mathrm{H}}(R - r_{+2})$. Using Eq. (12), or Eq. (13), we have

$$\frac{r_{+2}}{R} = 1 - \frac{1 - \Lambda R^2}{(4\pi RT)^2}, \tag{31}$$

where the equality is valid within the approximation taken. As a by-product, we also find from Eqs. (3) and (31) that in this limit one has $m_2 = \frac{R}{2}\left[1 - \frac{\Lambda R^2}{3} - \frac{(1-\Lambda R^2)^2}{(4\pi RT)^2}\right]$. The corresponding cosmological radius $r_{c2}$ can then be found directly from Eq. (5), we refrain from showing the explicit formula here, noting nevertheless that $\Lambda R^2$ can be small of order of zero in which case the cosmological horizon is very far away, or of order one in which case the cosmological horizon is very near the reservoir and the black hole horizon. We could work out in this order, i.e., $T \to \infty$, the action $I$, the energy $E$, the entropy $S$, and the heat capacity $C_A$, given through Eqs. (6)–(10). Again, the most interesting one is the heat capacity $C_A$. For the heat capacity $C_A$, given in Eq. (10), i.e., $C_A = (\frac{dE}{dT})_A$, equivalently, $C_A = (\frac{dE}{dT})_R$, we find from Eq. (8) that $C_A = \frac{1}{2\sqrt{V(R)}}(\frac{dm_2}{dT})_R$, where it was used the expression $V(R) = 1 - \frac{2m_2}{R} - \frac{\Lambda R^2}{3}$ given in Eq. (2). Thus, using the expression for $m_2$ just found above we have $C_A = \frac{1}{2\sqrt{V(R)}}\frac{1}{2}\frac{(1-\Lambda R^2)^2}{16\pi^2 T^3 R}$ and since $\sqrt{V(R)} = \frac{1-\Lambda R^2}{4\pi RT}$ it gives

$$C_{A_{+2}} = \frac{1 - \Lambda R^2}{4\pi T^2} > 0, \tag{32}$$

so that $C_{A+2}$ is small and positive. The large black hole $r_{+2}$ solution is thus stable.

### C. Thermodynamic phases and phase transitions between hot Schwarzschild–de Sitter and hot de Sitter in the $\Lambda R^2 < 1$ case

We now work out the thermodynamic phases and phase transitions for the $\Lambda R^2 < 1$ case. The discussion is valid for the thermodynamically stable black hole, the black hole $r_{+2}$, since the unstable one $r_{+1}$ has at most a fleeting existence and could not count for a phase. From Eq. (6) we get that the action $I$ for a hot Schwarzschild–de Sitter $r_{+2}$ phase is $I_{\rm SdS} = \beta R(1-\sqrt{V(R)}) - \pi r_{+2}^2$, where from Eq. (7) we have $V(R) = (1-\frac{r_{+2}}{R})\left(1-\frac{\Lambda R^2}{3}\left(1+\frac{r_{+2}}{R}+(\frac{r_{+2}}{R})^2\right)\right)$, with $0 \le \Lambda R^2 < 1$ here. The free energy $F = \frac{I}{\beta} = IT$ for hot Schwarzschild–de Sitter is then

$$F_{\rm SdS} = R(1-\sqrt{V(R)}) - \pi T r_{+2}^2, \quad 0 \le \Lambda R^2 < 1. \tag{33}$$

Another phase that might exist is hot de Sitter, in which case $r_+ = 0$, $V(R) = 1 - \frac{\Lambda R^2}{3}$ and the action is $I_{\rm HdS} = \beta R\left(1-\sqrt{1-\frac{\Lambda R^2}{3}}\right)$. The free energy is then

$$F_{\rm HdS} = \left(1-\sqrt{1-\frac{\Lambda R^2}{3}}\right)R, \qquad 0 \le \Lambda R^2 < 1. \tag{34}$$

In the canonical ensemble, for systems characterized by the size and the temperature of the heat reservoir, the phase that has lowest $F$ is the phase that dominates. So, the hot Schwarzschild–de Sitter black hole phase dominates over hot de Sitter, or the two phases coexist equally, when

$$F_{\rm SdS} \le F_{\rm HdS}, \quad 0 \le \Lambda R^2 < 1, \tag{35}$$

i.e., $\left(1-\sqrt{V(R)}\right)R - \pi T r_{+2}^2 \le \left(1-\sqrt{1-\frac{\Lambda R^2}{3}}\right)R$, i.e.,

$$RT \ge \frac{\sqrt{1-\frac{\Lambda R^2}{3}} - \sqrt{V(R)}}{\pi \frac{r_{+2}^2}{R^2}}, \quad 0 \le \Lambda R^2 < 1. \tag{36}$$

We see that Eq. (35) is an implicit equation, because $r_{+2} = r_{+2}(R, T)$. For each $\Lambda R^2$ in the interval above, and for each $RT$ one gets an $r_{+2}$, which can then be put into Eq. (36) to see whether the Schwarzschild–de Sitter phase dominates over the de Sitter phase or not. In the case it dominates then a black hole can nucleate thermodynamically from hot de Sitter space.

For $\Lambda R^2 \ll 1$ we can find some interesting numbers. One finds equality between the actions, i.e., that $F_{\rm SdS} = F_{\rm HdS}$, see Eqs. (35) and (36), when $RT = (RT)_{\rm eq}$ with

$$(RT)_{\rm eq} = \frac{27}{32\pi}\left(1-\frac{13}{486}\Lambda R^2\right), \quad \Lambda R^2 \ll 1, \tag{37}$$

valid in first order, as all equations in the discussion below will be valid in this order. It is of interest to put this value in decimal notation, i.e., $(RT)_{\rm eq} = 0.269(1-0.027\Lambda R^2)$, approximately. For $\frac{r_{+2\rm eq}}{R}$ one has in this case that

$$\frac{r_{+2\rm eq}}{R} = \frac{8}{9}\left(1+\frac{77}{729}\Lambda R^2\right), \quad \Lambda R^2 \ll 1. \tag{38}$$

In decimal notation, this can be put as $\frac{r_{+2\rm eq}}{R} = 0.889(1+0.106\Lambda R^2)$, approximately. Thus, $F_{\rm SdS} \le F_{\rm HdS}$, see Eqs. (35) and (36), when

$$RT \ge (RT)_{\rm eq},, \tag{39}$$

and when

$$\frac{r_{+2}}{R} \ge \frac{r_{+2\rm eq}}{R}. \tag{40}$$

So, when the inequalities given in Eqs. (39) and (40) hold, then the black hole phase dominates, but nevertheless the hot de Sitter phase has some probability of turning up.

There is another radius of interest here, which although not strictly thermodynamic, it appears through dynamical arguments, and is important in this discussion of phases and phase transitions. For matter or energy enclosed in a box, which one can consider that it configures a star, there is a mass, or energy, above which the star cannot support its self gravity and tends to collapse. This is called the Buchdahl limit which for spaces with positive cosmological constant has been calculated in [51]. Here one should envisage this limit as giving, for a given $R$ fixed, the mass $m_{\rm Buch}$ above which the energy within the system is so large that the system collapses. For a given $R$ and $\Lambda$, $m_{\rm Buch}$ is $\frac{m_{\rm Buch}}{R} = \frac{2}{9} + \frac{2}{9}\sqrt{1+3\Lambda R^2} - \frac{\Lambda R^2}{3}$. Since $m = \frac{r_+}{2}(1-\frac{\Lambda r_+^2}{3})$, see Eq. (3) one has the Buchdahl limit is given by the equation $\frac{r_{+\rm Buch}}{R}(1-(\frac{r_{+\rm Buch}}{R})^2\frac{\Lambda R^2}{3}) = \frac{4}{9} + \frac{4}{9}\sqrt{1+3\Lambda R^2} - \frac{2\Lambda R^2}{3}$, which is a cubic equation for $\frac{r_{+\rm Buch}}{R}$ that can in principle be solved. Given $\frac{r_{+\rm Buch}}{R}$ one can then work out what is the temperature $(RT)_{\rm Buch}$ that yields the related black hole with radius $\frac{r_{+2}}{R}$. Here we are dealing with small $\Lambda R^2$. in this case one gets

$$(RT)_{\rm Buch} = \frac{27}{32\pi}\left(1+\frac{985}{486}\Lambda R^2\right), \quad \Lambda R^2 \ll 1. \tag{41}$$

One further has

$$\frac{r_{+\rm Buch}}{R} = \frac{8}{9}\left(1+\frac{64}{81}\Lambda R^2\right), \qquad \Lambda R^2 \ll 1. \tag{42}$$

These two values can be put in decimal notation as $(RT)_{\rm Buch} = 0.269(1 + 2.027\Lambda R^2)$ and $\frac{r_{+\rm Buch}}{R} = 0.889(1 + 0.790\Lambda R^2_{\rm Buch})$, approximately. There is collapse when for a given $\Lambda$ and a given $R$ and $T$ one has

$$RT \geq (RT)_{\rm Buch}. \tag{43}$$

and

$$\frac{r_{+2}}{R} \geq \frac{r_{+\rm Buch}}{R} \tag{44}$$

So, there is collapse if for a given $\Lambda$, $R$ and $T$ Eqs. (43) and (44) are obeyed. Now, interestingly enough, comparing Eq. (37) with (41) and Eq. (38) with (42) we see that

$$(RT)_{\rm Buch} > (RT)_{\rm eq}. \tag{45}$$

and

$$\frac{r_{+\rm Buch}}{R} > \frac{r_{+\rm eq}}{R}. \tag{46}$$

Thus, one has that for sufficiently high temperatures the black hole is a dominant phase but not the unique, hot de Sitter might pop up, and for even higher temperatures then the black hole is the unique phase as the system tends to collapse. A comment is in order. The Buchdahl bound applies to a self-gravitating mechanical system consisting of a ball of radius $R$ containing matter. For a fixed $R$, the bound determines the maximum value of the gravitational radius $r_{+2}$, i.e., the maximum mass or energy within $R$, above which the system collapses. The system we are working with is a thermodynamic system, with a boundary that has radius $R$ and temperature $T$ fixed. In the approximation we are using, the system contains no matter with a black hole appearing as a result of thermodynamically imposed data in the ensemble, not as a result of a dynamic process. Nevertheless, one can think in going to the next order of approximation, where now the system contains lumps of energy or particles. In this case, for fixed $R$, there is a maximum value for the energy within $R$, above which the gravitational radius $r_{+2}$ is higher than the value permitted by the Buchdahl bound, and one can infer that the system must collapse. This reasoning is plausible, however it comes from dynamical arguments, and as such is outside the thermodynamic approach we are using.

So, we have the following picture for fixed and tiny $\Lambda R^2$. For $0 \leq RT < \frac{\sqrt{27}}{8\pi}(1 - \frac{5}{54}\Lambda R^2)$, see Eq. (19) there is only hot de Sitter space. For $\frac{\sqrt{27}}{8\pi}(1 - \frac{5}{54}\Lambda R^2) \leq RT < \frac{27}{32\pi}(1 - \frac{13}{486}\Lambda R^2)$, see Eqs. (20) and (37), hot de Sitter space is a phase that dominates over the Schwarzschild–de Sitter black hole phase, where the black hole is the large one, the small one being unstable is of no interest in this context. For $\frac{27}{32\pi}(1 - \frac{13}{486}\Lambda R^2) = RT$, the Schwarzschild–de Sitter black hole and the pure de Sitter phases coexist equally, see Eq. (37). For $\frac{27}{32\pi}(1 - \frac{13}{486}\Lambda R^2) < RT < \frac{27}{32\pi}(1 + \frac{985}{486}\Lambda R^2)$, the Schwarzschild–de Sitter black hole phase dominates over the pure de Sitter phase. For $\frac{27}{32\pi}(1 + \frac{985}{486}\Lambda R^2) \leq RT < \infty$, see Eqs. (41) and (45), there is only the Schwarzschild–de Sitter black hole phase, the system suffers total gravitational collapse. Note that in the phase transition from hot de Sitter to the Schwarzschild–de Sitter black hole phase there is topology change, since here the Euclidean topology of hot de Sitter is $S^1 \times R^3$, and the Euclidean topology of the Schwarzschild–de Sitter black hole is $R^2 \times S^2$.

The case $\Lambda R^2 = 0$ is a particular case of the $\Lambda R^2 \lll 1$ case just shown. Nevertheless, there are interesting aspects worth mentioning. In this case the thermodynamic phases are hot flat space and the Schwarzschild black hole. The Schwarzschild black hole phase dominates, or the two phases coexist equally, when $F_{\rm Schw} \leq F_{\rm HFS}$. We can take it directly to be $RT \geq \frac{1-\sqrt{V(R)}}{\pi \frac{r_{+2}^2}{R^2}}$, where here $V(R) = 1 - \frac{r_{+2}}{R}$. From the calculations above one finds that for $0 \leq RT < \frac{\sqrt{27}}{8\pi}$ there is only hot flat space. For $\frac{\sqrt{27}}{8\pi} \leq RT < \frac{27}{32\pi}$, hot de Sitter space is a phase that dominates over the Schwarzschild–de Sitter black hole phase. For $\frac{27}{32\pi} = RT$, the Schwarzschild–de Sitter black hole and the pure de Sitter phases coexist equally. Now, note that $\frac{r_{+2}}{R} = \frac{8}{9}$ is the radius where the two actions, for hot flat and Schwarzschild spaces, have the same value, which is zero in this case [6]. But this value is in fact equal to the Buchdahl limiting radius in general relativity, as was found for dimensions $d \geq 4$ in [15–17]. The fact that the thermodynamic existence of a black hole phase and the Buchdal limit coincide in this case is an interesting and unexpected property, and it can help to compare both processes, thermodynamic an dynamic, of forming a black hole. Then, for slightly higher temperatures, one can infer that when the Schwarzschild black hole phase dominates, it actually has sufficient energy to collapse itself to a black hole, i.e., when the two phases, hot flat space and Schwarzschild black hole, start to coexist, the black hole phase actually dominates completely, since the system has sufficient energy to collapse to a black hole. Thus, here, for $\frac{27}{32\pi} < RT < \infty$, the Schwarzschild black hole phase should be the only phase that exists, as the system must suffer total gravitational collapse.

For generic $\Lambda R^2$ in the range $\Lambda R^2 < 1$ a similar analysis can be made, but we do not do it here.

### D. Comments on the $\Lambda R^2 < 1$ case

We note that although the analysis made in Eqs. (19)–(22) is valid for a very small cosmological constant $\Lambda R^2$, and the analysis made in Eqs. (23)–(26) is valid for a cosmological constant $\Lambda R^2$ near unity from below, one can have a good idea of the behavior of the solutions as $\Lambda R^2$ is increased from zero up to a value less than unity. This is also done with the help with the results of the high temperature limit Eqs. (27)–(32).

For $\Lambda R^2 = 0$, i.e., for zero cosmological constant, we recover York's results for a heat reservoir in a Schwarzschild black hole space. In this case, when $RT < \frac{\sqrt{27}}{8\pi}$ there are no black hole solutions, only hot flat space, and when $RT \geq \frac{\sqrt{27}}{8\pi}$ there are two solutions, the small black hole $r_{+1}$ which is unstable, and the large black hole $r_{+2}$, which is stable, the two solutions are the same when $RT = \frac{\sqrt{27}}{8\pi}$ with horizon radius $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3}$. Still for $\Lambda R^2 = 0$, when $RT$ is very high, i.e., the temperature $T$ of the heat reservoir is very high, then $\frac{r_{+1}}{R}$ is small tending to zero and $\frac{r_{+2}}{R}$ is large tending to one. The behavior of the $\Lambda R^2 = 0$ can be heuristically explained through the thermal wavelength of the radiation $\lambda$ and the size of the reservoir $R$. For small $T$, one has that the corresponding thermal wavelength $\lambda \equiv \frac{1}{T}$ is high relative to $R$. In fact, $RT$ small means $\frac{R}{\lambda}$ small, i.e., $\frac{\lambda}{R}$ high, so that the wavelength of the thermal energy packets is stuck to the walls of the reservoir, and the packets cannot collapse to form a black hole. For higher $T$, $\lambda$ is small relative to $R$. Thus, $RT$ large means $\frac{R}{\lambda}$ large, i.e., $\frac{\lambda}{R}$ low, so that the wavelength of the thermal packets is sufficiently small, the packets are free inside the reservoir, and eventually collapse to form a black hole.

For $\Lambda R^2$ fixed and tiny we can spell the results as well. Now the space of black hole solutions is over a two-dimensional domain, specifically, $RT$ and $\Lambda R^2$. When $RT$ is less than some number, which itself is smaller than $\frac{\sqrt{27}}{8\pi}$, then there are no solutions, only hot de Sitter space, and when $RT$ is larger than this same number, which itself is smaller than $\frac{\sqrt{27}}{8\pi}$, then there are two solutions, the small black hole $r_{+1}$, which is unstable, and the large black hole $r_{+2}$, which is stable. When $RT$ is very high, i.e., when the temperature $T$ of the heat reservoir is very high, then $\frac{r_{+1}}{R}$ is small tending to zero and $\frac{r_{+2}}{R}$ is large tending to one. The two solutions are the same solution when $RT = \frac{\sqrt{27}}{8\pi}(1 - \frac{5}{54}\Lambda R^2)$, $\Lambda R^2$ being the fixed and tiny value, with coincident horizon radius given by $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3}(1 + \frac{17}{81}\Lambda R^2)$. Moreover, from the result that the coincident horizon radius has the value just given, one can deduce that as one goes along increasing $\Lambda$, specifically, increasing $\Lambda R^2$, for fixed $RT$, the radii $\frac{r_{+1}}{R}$ and $\frac{r_{+2}}{R}$ increase. This behavior for spaces with tiny $\Lambda R^2$ can be heuristically explained through the thermal wavelength, the size of the reservoir, and the cosmological length. For small $T$, one has that the corresponding thermal wavelength $\lambda \equiv \frac{1}{T}$ is high relative to $R$. In fact, $RT$ small means $\frac{R}{\lambda}$ small, i.e., $\frac{\lambda}{R}$ high, so that the wavelength of the thermal energy packets is stuck to the walls of the reservoir, and the packets cannot collapse to form a black hole. But now, due to the new cosmological length scale $\ell$ set by $\Lambda$, $\ell = \frac{1}{\sqrt{\Lambda}}$, the space inside the reservoir is more curved and, so to speak, a bit larger, and thus a lower $T$, i.e., a higher $\lambda$, is allowed so that they are free to collapse inside the reservoir and form a black hole in this case.

For $\Lambda R^2$ fixed, not tiny and less than one, we can deduce several results. Solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+2}}{R}$ start to appear at a certain $RT$ which is ever decreasing as $\Lambda R^2$ is increasing, and when $\Lambda R^2$ is close to one then one finds that $RT$ is close to and a bit higher than $\frac{1}{2\pi}$. Thus, when $RT$ is small one can find black hole solutions for spaces with cosmological constant near one, but there are no black holes for spaces with any other lower cosmological constant. Moreover, for $RT$ close to $\frac{1}{2\pi}$, then $\frac{r_{+1}}{R}$ and $\frac{r_{+2}}{R}$ are very near one and merge at $\frac{1}{2\pi}$. In addition, for fixed $RT$ as one goes along increasing $\Lambda$, i.e., increasing $\Lambda R^2$, then the radii $\frac{r_{+1}}{R}$ and $\frac{r_{+2}}{R}$ increase. The solution $\frac{r_{+1}}{R}$ is the small solution and the solution $\frac{r_{+2}}{R}$ is the large solution that for $\Lambda R^2$ close to one yields $\frac{r_{+2}}{R}$ close to one. When $RT$ is very high, i.e., when the temperature $T$ of the heat reservoir is very high, then $\frac{r_{+1}}{R}$ is small tending to zero and $\frac{r_{+2}}{R}$ is large tending to one. This behavior of $\Lambda R^2$ fixed, not tiny and less than one, can be heuristically explained through the thermal wavelength, the size of the reservoir, and the cosmological length. As the temperature gets lower and lower, the associated thermal wavelength $\lambda = \frac{1}{T}$ gets higher and higher, and to have black hole solutions the space needs to be more curved, and so larger, to accommodate those wavelengths $\lambda$ and allow the corresponding thermal energy packets to collapse. For very low temperatures, in the limit $RT \to \frac{1}{2\pi}$, i.e., $\frac{\lambda}{R} = 2\pi$, energy packets can only build a black hole horizon for sufficiently high $\Lambda$, i.e., for $\Lambda R^2 \to 1$. This first black hole that appears at $RT \to \frac{1}{2\pi}$ and $\Lambda R^2 \to 1$ has large horizon radius given by $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} \to 1$. For higher $T$, i.e., higher $RT$, then the wavelength of the energy packets is sufficiently small that allows for black hole solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+2}}{R}$.

To sum up, in the $\Lambda R^2 < 1$ case, for sufficiently high temperatures there are two solutions, the small mass branch with black hole horizon radius $r_{+1}$ which is unstable, and the massive branch with black hole horizon radius which is stable, a fact the holds for any temperature $RT$ and any $\Lambda R^2 < 1$. This is similar to what happens in the thermodynamics of pure Schwarzschild, i.e., $\Lambda R^2 = 0$. As $\Lambda R^2$ is increased from zero, black holes can form with less and less temperatures $RT$, and for $\Lambda R^2$ near one, can form black holes with the least temperature, namely, $RT$ tending to $\frac{1}{2\pi}$.

The case with $\Lambda R^2 = 1$ precisely has to be dealt as a separate case, as an intermediate value case for $\Lambda R^2$. As we will show it reserves interesting surprises.

## IV. THERMODYNAMICS OF SCHWARZSCHILD–DE SITTER BLACK HOLES IN THE CANONICAL ENSEMBLE: INTERMEDIATE VALUE OF THE COSMOLOGICAL CONSTANT, $\Lambda R^2 = 1$, THE NARIAI UNIVERSE INSIDE THE HEAT RESERVOIR

### A. Solutions and the Euclidean metric

We treat here the intermediate positive cosmological constant, $\Lambda R^2 = 1$, problem. Again, we put the boundary of a spherical cavity with a black hole in a positive cosmological constant background inside a heat reservoir, at some radius $R$, where it is also specified a fixed temperature $T$, see Fig. 1 anew. The intermediate value of the cosmological constant is precisely

$$\Lambda R^2 = 1. \tag{47}$$

For this value of $\Lambda R^2$, and for fixed $RT$ one can find solutions of Eq. (13) for black hole horizon radii $r_+$ analytically. However, when $\Lambda R^2 = 1$ the problem has to be treated with care. There are still two solutions, $r_{+1}$ and $r_{+2}$.

The solution $r_{+1}$, the small black hole solution, can be taken directly from Eq. (13) putting $\Lambda R^2 = 1$ which is then $4\pi RT = \frac{1}{\frac{r_+}{R}} \frac{1-(\frac{r_+}{R})^2}{\sqrt{1-\frac{r_+}{R}}\sqrt{\left(1-\frac{1}{3}\left(1+\frac{r_+}{R}+(\frac{r_+}{R})^2\right)\right)}}$. This equation can be transformed into a quartic equation in $\frac{r_+}{R}$ yielding then the solution $r_{+1}$, the small and unstable solution.

The solution $r_{+2}$, the large black hole solution, needs special attention. One cannot simply put $\Lambda R^2 = 1$ into Eq. (13), i.e., $4\pi RT = \frac{1}{\frac{r_+}{R}} \frac{1-\Lambda R^2(\frac{r_+}{R})^2}{\sqrt{1-\frac{r_+}{R}}\sqrt{\left(1-\frac{\Lambda R^2}{3}\left(1+\frac{r_+}{R}+(\frac{r_+}{R})^2\right)\right)}}$, which gives directly the solution $\frac{r_{+2}}{R} = 1$ with $RT = \frac{1}{2\pi}$. The correct way is to take the limit $\Lambda R^2 \to 1$ and $\frac{r_{+2}}{R} \to 1$. Then, $RT$ can have a broad range of values. One then also finds that, for $\frac{r_{+2}}{R} \to 1$, the cosmological radius solution can be taken from Eq. (5) to give $\frac{r_{c2}}{R} \to 1$. This limit takes us to the Nariai universe, which we now find. The Nariai universe is to be seen as Schwarzschild–de Sitter black hole with maximal mass, it is an extremal case.

The Nariai solution can be found from the Schwarzschild–de Sitter solution in the limit that the two horizons $r_{+2}$ and $r_{c2}$ coincide, see also Appendix A. Here, we have a heat reservoir at $R$ that acts as a reservoir for the inside region, the region containing a black hole. This heat reservoir, at $R$, is in between $r_{+2}$ and $r_{c2}$, and thus the limit we want to take is such that $r_{+2}$, $R$, and $r_{c2}$ coincide, see Appendix B for detail. We drop the subscript 2 in the following analysis. Now, if we do $\frac{r_+}{R} \to 1$, then Eq. (12), $T = \frac{T_+^{\rm H}}{\sqrt{V(R)}}$, together with Eq. (4), $V(r) = (1-\frac{r_+}{r})\left(1-\frac{\Lambda r^2}{3}\left(1+\frac{r_+}{r}+(\frac{r_+}{r})^2\right)\right)$, gives at face value that the heat reservoir is at very high temperature $T$. But, there is a way to have $T$ finite with $\frac{r_+}{R} \to 1$. From Eq. (12) we see that if we do $\frac{r_+}{R} \to 1$ concomitantly with $T_+^{\rm H} \to 0$ then $T$ is finite. Since $T_+^{\rm H} = \frac{1}{4\pi r_+}(1-\Lambda r_+^2)$, see Eq. (11), $T_+^{\rm H} \to 0$ means $1-\Lambda r_+^2 \to 0$, but since $\frac{r_+}{R} \to 1$ this also means $1-\Lambda R^2 \to 0$. In brief, in this limit we have $\frac{r_+}{R} \to 1$ and $\Lambda R^2 \to 1$, both from below and both of the same infinitesimal order. Then, Eq. (12), $T = \frac{T_+^{\rm H}}{\sqrt{V(R)}}$, gives in this limit that $T = \frac{1}{2\pi R}\frac{(1-\frac{r_+}{R})+(1-\sqrt{\Lambda R^2})}{\sqrt{1-\frac{r_+}{R}}\sqrt{(1-\frac{r_+}{R})+2(1-\sqrt{\Lambda R^2})}}$. Since $1-\sqrt{\Lambda R^2}$ and $1-\frac{r_+}{R}$ are infinitesimal in this limit, we see that $T$ is finite and can have a range of values depending on the precise infinitesimal values of $1-\sqrt{\Lambda R^2}$ and $1-\frac{r_+}{R}$. One more thing. We have deduced that in this limit $r_+ \to R \to \frac{1}{\sqrt{\Lambda}}$, so that from Eq. (5) one has $r_c \to \frac{1}{\sqrt{\Lambda}}$, and since $\frac{1}{\sqrt{\Lambda}} \to R$, then $r_c \to R$. So, in the limit we have $r_+ = R = r_c$. To see that this is the Nariai limit with a reservoir $R$ in the middle, we have to do some work on the original line element, Eqs. (1) and (2). We present the results below, see Appendix B for a detailed derivation.

Let us start. The reservoir temperature $T$ is also the local Tolman temperature $T$ at $R$ and has an associated expression given by $T = \frac{T_+^{\rm H}}{\sqrt{V(R)}}$, with $T_+^{\rm H}$ being the tiny Hawking temperature as we have just found. Expanding the metric potential $V(r)$ of Eq. (2) near $r_+$ in a Taylor series gives $V(r) = 4\pi T_+^{\rm H}(r-r_+) - \frac{1}{R^2}(r-r_+)^2$, plus higher order terms. Make now the transformations $(t,r) \to (\bar{t},z)$ as $r - r_+ = 4\pi T_+^{\rm H} R^2 \sin^2\left(\frac{1}{2}\arccos(\frac{z}{R})\right)$ and $t = \frac{\bar{t}}{2\pi T_+^{\rm H} R}$ with $0 \le t \le \frac{1}{T_+^{\rm H}}$ corresponding to $0 \le \bar{t} \le 2\pi R$ and $r_+ \le r \le R$ corresponding to $-R \le z \le Z$. Then, since $V(r)$ is actually a $V(r,r_+)$, we have here $V(r) = V(r,r_+) = V(r-r_+) = V(z) = (2\pi T_+^{\rm H} R)^2 \sin^2(\arccos(\frac{z}{R})) = (2\pi T_+^{\rm H} R)^2(1-\frac{z^2}{R^2})$. From the original Schwarzschild–de Sitter line element, Eqs. (1) and (2), together with Eq. (3), and dropping the bar in $\bar{t}$ which is now meaningless, we obtain then the Nariai line element, i.e.,

$$ds^2 = V(z)dt^2 + \frac{dz^2}{V(z)} + R^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$0 \le t \le 2\pi R, \qquad -R < z < Z, \tag{48}$$

where $Z$ is now the heat reservoir boundary in the $z$-direction, the other coordinates are in the range $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$, and the metric potential $V(z)$ being given by

$$V(z) = 1 - \frac{z^2}{R^2}. \tag{49}$$

The line element given in Eqs. (48) and (49) corresponds to the Nariai universe, which can be seen to be decomposable into a two-dimensional de Sitter space times a sphere. So, the ensemble with its boundary data, $T$ and $R$, provide automatically the range of coordinates of the solution. Note also that the range of values for the heat reservoir boundary $Z$ is $-R \le Z \le R$. From Eq. (49) we see that there are two horizons, one is $z_+ = -R$, the other is $z_c = R$, but the subscripts now are just names, since the two horizons are of the same type. The topology of the Nariai universe in Euclideanized form is $R^2 \times S^2$ and its boundary has $S^1 \times S^2$ topology, where the $S^1$ subspace has proper length $\frac{1}{T}$.

Now, the temperature $T$ is

$$T = \frac{T_+^{\rm H}}{\sqrt{V(Z)}}, \qquad (50)$$

with $T_+^{\rm H}$ being the Hawking temperature given by $T_+^{\rm H} = \frac{\kappa}{2\pi}$, and $\kappa$ being the surface gravity of the black hole horizon. For the metric (48) one has $\kappa = \frac{1}{2} V'(z_+)$, and so the Hawking temperature for the + horizon is $T_+^{\rm H} = \frac{1}{4\pi} (\frac{dV}{dz})_{z_+}$. Using Eq. (48) we get

$$T_+^{\rm H} = \frac{1}{2\pi R}. \qquad (51)$$

As well, from Eq. (49) we have that the metric potential at the heat reservoir boundary $Z$ is

$$V(Z) = 1 - \frac{Z^2}{R^2}. \qquad (52)$$

So, Eqs. (50)–(52) give that the reservoir temperature is given by $T = \frac{1}{2\pi R \sqrt{1 - \frac{Z^2}{R^2}}}$, where $Z$ is the boundary on the $z$ coordinate, see Fig. 2 for a representation of the Nariai universe in a heat reservoir. For a given $T$ at the boundary, for the ensemble, there are two solutions of this equation, in general. Namely, one solution is for $Z$ between $-R$ and 0 and the other for $Z$ between 0 and $R$. These two solutions yield different physical situations of course, as the reservoir boundary $Z$ is put in different positions relatively to $z_+$. Note that in Schwarzschild–de Sitter space, the two solutions were for the horizon $r_+$, one small $r_{+1}$ the other large $r_{+2}$, both relative to the reservoir $R$. Here, the two solutions are not for the horizons but instead for the boundary $Z$, one $Z_1$, the other $Z_2$, with

$$Z_1 = -R \sqrt{1 - \frac{1}{(2\pi R T)^2}}, \qquad (53)$$

and

$$Z_2 = R \sqrt{1 - \frac{1}{(2\pi R T)^2}}, \qquad (54)$$

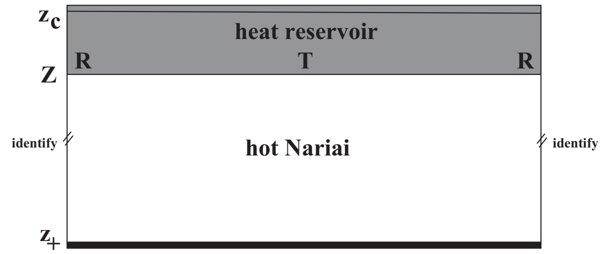


FIG. 2. A drawing of a Nariai horizon $z_+$ within a heat reservoir at temperature $T$, with cylindrical radius $R$, and situated at $Z$. The cosmological horizon $z_c$ is situated beyond the heat reservoir. The Euclideanized space and its boundary have $R^2 \times S^2$ and $S^1 \times S^2$ topologies, respectively, where the $S^1$ subspace with proper length $\beta = \frac{1}{T}$ is not displayed. See text for more details.

now both relative to the horizon $z_+$. Within the two choices for the boundary, $Z_1$ or $Z_2$, we can pick the one we wish. In addition, since $z_+$ and $z_c$ are indistinguishable, in the sense they are of the same type, we can also interchange $z_+$ with $z_c$, in which case the situation would be the same. The boundary has always area radius $R$, which together with $T$, forms the data for the canonical ensemble.

From Eqs. (53) and (54) we see that for

$$RT < \frac{1}{2\pi}, \qquad (55)$$

there are no solutions for $Z_1$ or $Z_2$, so in this case the boundary $Z$ does not exist, a reservoir does not exist, one cannot define a temperature $T$ anywhere, and so there is no thermodynamic Nariai solution. Presumably one has simply hot de Sitter space inside $R$. In decimal notation Eq. (55) is $RT < 0.159$, approximately.

From Eqs. (53) and (54) we see that for

$$RT \ge \frac{1}{2\pi}, \qquad (56)$$

there are two Nariai solutions, one with one horizon $z_+ = -R$ and boundary $Z_1$, the other with one horizon $z_+ = -R$ and boundary $Z_2$, both boundaries can be picked up. When the equality sign holds in Eq. (56) there is one solution with $Z_1 = Z_2 = 0$, so in this case the boundary $Z$ pops up in the middle, at $Z = 0$, and so $-R \le z \le 0$. In this case, The reservoir is at $Z = 0$, has radius $R$, and the horizon is at $z_+ = -R$.

The generic Tolman temperature formula in the Nariai space is $T(z) = \frac{1}{2\pi R \sqrt{1 - \frac{z^2}{R^2}}}$ for $-R \le z < Z$, with the reservoir temperature $T$ being expressed as $T \equiv T(Z)$. So, $T(z = -R) = \infty$ as expected since $z = -R$ is a horizon, it is the horizon $z_+$. Increasing $z$ from $-R$ one sees that $T(z)$ decreases and stops if one picks $Z_1$, and if one picks $Z_2$ it decreases up to $z = 0$, and then increases back up to $Z_2$. In case $Z_2 = R$, then $T(Z_2 = R) = \infty$ as is expected since

$z = R$ is a horizon, it is the horizon $z_c$. Clearly, the horizons are given by $z_+ = -R$ and $z_c = R$, so they do not depend on $T$. The dependence on $T$ is transferred to the boundary $Z$, so the structure has changed from that of the Schwarzschild–de Sitter.

We now list the most relevant general formulas for the thermodynamics of Nariai. These can be taken directly from the equations provided in Sec. II. The action $I$ is now

$$I = \beta R - \pi R^2. \tag{57}$$

The action and the free energy are related by $I = \beta F$, so $F = R - T\pi R^2$. Now, $F = E - TS$, so the thermodynamic or quasilocal energy here also the thermal energy at $R$ is

$$E = R. \tag{58}$$

The entropy is

$$S = \pi R^2, \tag{59}$$

and is independent of $z_+$. The heat capacity $C_A = (\frac{dE}{dT})_A$ is

$$C_A = 0, \tag{60}$$

so there is neutral thermodynamic equilibrium in the Nariai universe. That $C_A = 0$ can be taken directly from Eq. (58) which shows that the energy $E$ has no dependence on the temperature $T$.

### B. High-temperature limit

For the value of the cosmological constant considered in this section, $\Lambda R^2 = 1$, one can work out the limit in which $RT$ goes to infinity, see Eqs. (53) and (54). Since $R$ is the quantity that we consider as the gauge, $RT$ going to infinity is the same in this context as $T$ going to infinity. Let us then find explicit results by taking the limit of high temperature.

For very high $T$, or very high $TR$, one has from Eq. (53) the possibility

$$\frac{Z_1}{R} = -1 + \frac{1}{2}\frac{1}{(2\pi RT)^2}, \tag{61}$$

with $\frac{1}{2}\frac{1}{(2\pi RT)^2} \ll 1$. In this case $Z_1$ is very near the horizon $z_+ = -R$ and one finds that the space in-between the horizon and the reservoir is essentially a Rindler space. To see this, note that in this limit $1 - z^2 \ll 1$ for $0 \le z \le Z_1$. So with a $\bar{z}$ coordinate defined by $1 - z^2 = \bar{z}^2$ one obtains from Eqs. (48) and (49) the line element $ds^2 = \bar{z}^2 dt^2 + d\bar{z}^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$, i.e., the Euclidean Rindler line element. For very high $T$, or very high $TR$, one has from Eq. (54) the other possibility

$$\frac{Z_2}{R} = 1 - \frac{1}{2}\frac{1}{(2\pi RT)^2}, \tag{62}$$

with $\frac{1}{2}\frac{1}{(2\pi RT)^2} \ll 1$. In this case the boundary $Z_2$ is very near the horizon $z_c = R$, but since $-R \le z \le Z_2 < z_c$ the space in-between the horizon and the reservoir is not generically a Rindler space, it approximates the Rindler line element only within the region near $Z_2$.

### C. Thermodynamic phases and phase transitions between hot Nariai and hot de Sitter in the $\Lambda R^2 = 1$ case

We now work out thermodynamic phases and phase transitions for the $\Lambda R^2 = 1$ case. The discussion is valid for the thermodynamically stable black hole, the black hole $r_{+2}$, since the unstable one $r_{+1}$ has at most a fleeting existence and could not count for a phase. Here the black hole $r_{+2}$ is in fact a Nariai universe.

From Eq. (57) we see that the action $I$ for Nariai is $I_{\text{Nariai}} = \beta R - \pi R^2$, where we have used that in Nariai one has $r_{+2} = R$. So the free energy $F = \frac{I}{\beta} = IT$ for a hot Nariai phase is

$$F_{\text{Nariai}} = R - \pi R^2 T, \quad \Lambda R^2 = 1. \tag{63}$$

Another phase that might exist is hot de Sitter, in which case $r_+ = 0$, $V(R) = 1 - \frac{\Lambda R^2}{3} = \frac{2}{3}$ as $\Lambda R^2 = 1$, and the action is $I_{\text{HdS}} = \beta R(1 - \sqrt{\frac{2}{3}})$. The free energy is then

$$F_{\text{HdS}} = \left(1 - \sqrt{\frac{2}{3}}\right)R, \qquad \Lambda R^2 = 1. \tag{64}$$

In the canonical ensemble the phase that has lowest $F$ is the phase that dominates. So, the Nariai universe dominates over hot de Sitter space, or the two phases coexist equally, when

$$F_{\text{Nariai}} \le F_{\text{HdS}}, \qquad \Lambda R^2 = 1. \tag{65}$$

One finds equality between the two actions when $R - \pi R^2 T = (1 - \sqrt{\frac{2}{3}})R$, i.e., when

$$(RT)_{\text{eq}} = \frac{\sqrt{2}}{\sqrt{3}\pi}, \qquad \Lambda R^2 = 1. \tag{66}$$

In decimal notation, this is $(RT)_{\text{eq}} = 0.260$, approximately. So Nariai prevails over hot de Sitter, or the two phases coexist equally, when

$$RT \ge \frac{\sqrt{2}}{\sqrt{3}\pi}, \qquad \Lambda R^2 = 1. \tag{67}$$

Recall that for $RT < \frac{1}{2\pi}$, there are no Nariai solutions only hot de Sitter, see Eq. (55), and for $RT \ge \frac{1}{2\pi}$, see

Eq. (56), two possible cases pop up, the unstable black hole which is of no interest here, and the Nariai universe which is neutrally stable and of interest here. So, for $\Lambda R^2 = 1$ we have the following picture. For $0 \le RT < \frac{1}{2\pi}$ hot de Sitter is mandatory. For $\frac{1}{2\pi} \le RT < \frac{\sqrt{2}}{\sqrt{3}\pi}$ hot de Sitter prevails as a thermodynamic phase over Nariai, so that if the phase is a Nariai one, it will probably transition to a hot the Sitter phase. For $\frac{\sqrt{2}}{\sqrt{3}\pi} = RT$ hot de Sitter and Nariai coexist as thermodynamic phases. For $\frac{\sqrt{2}}{\sqrt{3}\pi} \le RT < \infty$ Nariai prevails as a thermodynamic phase over hot de Sitter. Note that in the phase transition from hot de Sitter to hot Nariai or from hot Nariai to hot de Sitter there is topology change, since the Euclidean topology of hot de Sitter is $S^1 \times R^3$, and the Euclidean topology of Nariai is $R^2 \times S^2$.

### D. Comments on the $\Lambda R^2 = 1$ case

It is really interesting that the resulting metric inside the heat reservoir is described by the Nariai metric. The procedure of obtaining it in our context is completely different from the usual procedure. The heat reservoir radius $R$ and the temperature $T$ play a crucial role here, and so the limit to the Nariai universe is naturally related to thermodynamics. As we have just seen, the Nariai solution is of utmost importance in any analysis of the Schwarzschild–de Sitter space in the canonical ensemble.

A feature of great importance in the overall picture is the minimum temperature $T$, i.e., $RT$, of the ensemble above which there are Nariai solutions for a given $\Lambda$, i.e., $\Lambda R^2$. For $RT < \frac{1}{2\pi}$ there are no black hole solutions whatsoever for any $\Lambda R^2$, specifically, there are no small black hole solutions with horizon radius $\frac{r_{+1}}{R}$ neither Nariai universes with $\frac{r_{+2}}{R} = 1$. Indeed, from the equations found above for the heat reservoir boundary of the Nariai universe, namely, $Z_1 = -R\sqrt{1 - \frac{1}{(2\pi RT)^2}}$ and $Z_2 = R\sqrt{1 - \frac{1}{(2\pi RT)^2}}$, we see that for $RT < \frac{1}{2\pi}$, there are no solutions for $Z_1$ or $Z_2$. So in this case the boundary $Z$ does not exist, a reservoir does not exist, one cannot define a temperature $T$ anywhere, and so there is no thermodynamic Nariai solution. Presumably one has simply hot de Sitter space inside $R$. The reason is clear if one thinks in thermal wavelengths. For very small temperatures, the associated thermal wavelength is very long and there is no boundary $Z$ that can accommodate such corresponding thermal energy packets. For $RT = \frac{1}{2\pi}$ there is one solution only, it has $\Lambda R^2 = 1$ precisely. There are no solutions at this temperature of any other $\Lambda R^2$. So, a Nariai solution is the first solution to pop up as one increases the temperature from absolute zero. As one increases $RT$ above $\frac{1}{2\pi}$, solutions with $\Lambda R^2$ different from one start to appear, first for $\Lambda R^2$ near one, than for $\Lambda R^2$ far from one as $RT$ is further increased as we have discussed in the previous section for $\Lambda R^2 < 1$.

The high temperature limit for the Nariai universe, i.e., for $\Lambda R^2 = 1$, connects with the cases $\Lambda R^2 < 1$. Indeed, when $\Lambda R^2 < 1$, for $RT$ very large, there are two solutions, the very small one $\frac{r_{+1}}{R}$ tending to zero, and the very large one $\frac{r_{+2}}{R}$ tending to one. When $\Lambda R^2 = 1$ only the very large one, $\frac{r_{+2}}{R}$, satisfies the condition $\frac{r_{+2}}{R} = 1$, necessary to get a Nariai universe.

It remains to be found the spectrum of solutions for $\Lambda R^2 > 1$. We turn to this problem now, and it happens that there are unexpected results.

## V. THERMODYNAMICS OF SCHWARZSCHILD–DE SITTER BLACK HOLES IN THE CANONICAL ENSEMBLE: LARGE VALUES OF THE COSMOLOGICAL CONSTANT, $\Lambda R^2 > 1$

### A. Solutions

We treat here the large positive cosmological constant, $\Lambda R^2 > 1$, problem. Again, we put the boundary of a spherical cavity with a black hole in a positive cosmological constant background inside a heat reservoir, at some radius $R$, where it is also specified a fixed temperature $T$, see Fig. 1 anew. Large positive cosmological constant means exactly in this context that

$$1 < \Lambda R^2 \le 3. \tag{68}$$

Within this range, for fixed $RT$ and generic $\Lambda R^2$, we can draw the result that looking at Eq. (13) we can ascertain with surprise that for $\Lambda R^2 > 1$ there are solutions with $\frac{r_+}{R} < 1$. Nonetheless, it is hard to find analytic solutions of Eq. (13) for black hole horizon radii $r_+$. However, for $\Lambda R^2$ very near one from above, one can make some progress.

For $\Lambda R^2$ a tiny bit larger than one, i.e., for $(\Lambda R^2 - 1) \ll 1$, there are no black hole solutions for

$$RT < \frac{1}{2\pi}\left(1 + \left(\frac{3}{8}(\Lambda R^2 - 1)\right)^{\frac{1}{3}}\right), \quad (\Lambda R^2 - 1) \ll 1, \tag{69}$$

only hot de-Sitter space, valid in this order of approximation, as all equations in this context below are valid to this order. Still for small $\Lambda R^2 - 1$, i.e., for $(\Lambda R^2 - 1) \ll 1$, there are two black hole solutions for

$$RT \ge \frac{1}{2\pi}\left(1 + \left(\frac{3}{8}(\Lambda R^2 - 1)\right)^{\frac{1}{3}}\right), \quad (\Lambda R^2 - 1) \ll 1, \tag{70}$$

One of the two solutions is the small black hole $r_{+1}(R, \Lambda, T)$, and the other solution is the large black hole $r_{+2}(R, \Lambda, T)$. The plus sign inside the parenthesis in Eq. (70) is what one expects really. The two solutions merge into one sole solution when the equality sign in

Eq. (20) holds. In this case the coincident double solution has horizon radius given by

$$\frac{r_{+1}}{R}=\frac{r_{+2}}{R}=1-\left(\frac{3}{8}(\Lambda R^2-1)^2\right)^{\frac{1}{3}},\qquad (\Lambda R^2-1)\ll 1. \tag{71}$$

The corresponding cosmological radius can then be found from Eq. (5) to be given by

$$\frac{r_{c1}}{R}=\frac{r_{c2}}{R}=1+\left(\frac{3}{8}(\Lambda R^2-1)^2\right)^{\frac{1}{3}},\qquad (\Lambda R^2-1)\ll 1. \tag{72}$$

One can work out in this order in $\Lambda R^2-1$ the action $I$, the energy $E$, the entropy $S$, and the heat capacity $C_A$, given through Eqs. (6)–(10). Apart from the entropy $S=4\pi r_+^2$ for each of the two black hole solutions, the other quantities would not be particularly illuminating. An instance where these quantities can be worked out, in particular the heat capacity $C_A$, is the hight temperature limit to which we now turn.

### B. High temperature limit: Analytical solution

For the range of high values of the cosmological constant considered in this section, $1<\Lambda R^2\leq 3$, one can find solutions in the limit in which $RT$ goes to infinity, see Eqs. (12) and (13). Since $R$ is the quantity that we consider as the gauge, $RT$ going to infinity is the same in this context as $T$ going to infinity. Let us then find explicit results by taking the limit of high temperature. In this case the equations can be solved.

For a given reservoir temperature $T$ there are two black hole solutions, the small black hole solution $r_{+1}$ and the large black hole solution $r_{+2}$. We set $T$ fixed but very high, in the sense $T\to\infty$. From Eq. (12) there are two possibilities. Either $T_+^{\rm H}\to\infty$ which corresponds to the small black hole solution having a very small $r_{+1}$, or $V(R)\to 0$ which corresponds to the large black hole solution $r_{+2}$ but now not approaching the reservoir radius in this range of $\Lambda R^2$. Let us work one solution at a time for a fixed very high value of $T$, $T\to\infty$. Here we present the expressions at zeroth order, not displaying the corrections in $\frac{1}{T}$.

The first solution for a very high heat reservoir temperature, $T\to\infty$, is $r_+=r_{+1}\to 0$ with $T_+^{\rm H}\to\infty$. It is clear from Eq. (11), together with Eqs. (12) and (13), that this requires that the black hole solution is of the form $r_+=r_{+1}\to 0$. So, in this limit one has again $T_{+1}^{\rm H}=\frac{1}{4\pi r_{+1}}$, and then from Eq. (13) one finds the small black hole $r_{+1}$ solution to be of the form $\frac{r_{+1}}{R}=\frac{1}{4\pi RT\sqrt{1-\frac{\Lambda R^2}{3}}}$. The expression inside the square root is clearly positive. So, in the $T\to\infty$ limit we have,

$$\frac{r_{+1}}{R}=0, \tag{73}$$

plus higher order corrections. As a by-product, we also find from Eq. (3) that in this limit one has $m_1=\frac{r_{+1}}{2}$. For the heat capacity $C_A$, given in Eq. (10), i.e., $C_A=(\frac{dE}{dT})_A$, equivalently, $C_A=(\frac{dE}{dT})_R$, we find from Eq. (8) that $C_{A_{+1}}=\frac{1}{2\sqrt{V(R)}}(\frac{dr_{+1}}{dT})_R$, which upon using Eq. (28) yields $C_{A_{+1}}=-\frac{1}{8\pi T^2(1-\frac{\Lambda R^2}{3})}\leq 0$, so in the limit

$$C_{A_{+1}}=0_-, \tag{74}$$

here $0_-$ means that $C_{A_{+1}}$ tends to zero from negative values, so that $C_{A_{+1}}$ is nonpositive. Having negative heat capacity, the small black hole $r_{+1}$ solution is thus unstable.

The second solution for a very high heat reservoir temperature, $T\to\infty$, is some $r_{+2}$ but now not necessarily near $R$. Indeed, from Eq. (13), i.e., $4\pi RT=\frac{1}{\frac{r_+}{R}}\frac{1-\Lambda R^2(\frac{r_+}{R})^2}{\sqrt{1-\frac{r_+}{R}}\sqrt{(1-\frac{\Lambda R^2}{3}(1+\frac{r_+}{R}+(\frac{r_+}{R})^2))}}$, one sees that, when $\Lambda R^2>1$, $RT\to\infty$ for $1-\frac{\Lambda R^2}{3}(1+\frac{r_{+2}}{R}+(\frac{r_{+2}}{R})^2)=0$. This is a quadratic for $\frac{r_{+2}}{R}$ and it has as solution

$$\frac{r_{+2}}{R}=\frac{1}{2}\left(\sqrt{\frac{3}{\Lambda R^2}}\sqrt{4-\Lambda R^2}-1\right). \tag{75}$$

This is the curve traced by $\frac{r_{+2}}{R}$ as a function of $\Lambda R^2$ when $RT=\infty$. So, from Eq. (75) we find that when $\Lambda R^2=1$ one has $\frac{r_{+2}}{R}=1$ as it should, and when $\Lambda R^2=3$ one has $\frac{r_{+2}}{R}=0$. In this latter case, the solution $\frac{r_{+2}}{R}$ joins the small black hole solution $\frac{r_{+1}}{R}=0$, so that $\Lambda R^2=3$ is the turning point of $RT=\infty$, $\frac{r_{+1}}{R}=\frac{r_{+2}}{R}=0$. For the heat capacity $C_A$, given in Eq. (10), i.e., $C_A=(\frac{dE}{dT})_A$, equivalently, $C_A=(\frac{dE}{dT})_R$, we find that $C_{A_{+2}}=\frac{\sqrt{\Lambda R^2}}{8\pi T^2}(2+\sqrt{\Lambda R^2}-\sqrt{12-3\Lambda R^2})(2-\sqrt{\Lambda R^2}+\sqrt{12-3\Lambda R^2})\geq 0$, which after reworking can also be written as $C_{A_{+2}}=\frac{\sqrt{\Lambda R^2}}{4\pi T^2}(\Lambda R^2+\sqrt{\Lambda R^2}\sqrt{12-3\Lambda R^2}-4)\geq 0$, so in the infinite temperature limit

$$C_{A_{+2}}=0_+, \tag{76}$$

where $0_+$ means that $C_{A_{+2}}$ tends to zero from positive values, so that $C_{A_{+2}}$ is essentially positive, and the large black hole $r_{+2}$ solution is stable. For $\Lambda R^2=3$ the heat capacity is $C_{A_{+2}}=\frac{\sqrt{3}}{\pi T^2}$, and in the infinite temperature limit one recovers Eq. (76). The cosmological radius can also be found from Eq. (75), yielding for any $\Lambda R^2$ in the range in question that

$$\frac{r_{c2}}{R}=1, \tag{77}$$

where Eq. (5) has been used. Note from the Tolman formula that the whole region between $r_{+2}$ and $R$ is at infinite temperature. Indeed, $T(r) = \frac{T}{V(R)}$, for $\frac{r_{+2}}{R} \le \frac{r}{R} \le 1$. Since $T = \infty$ and $V(R)$ is finite one has that $T(r)$ is infinite in the region. The temperature at $r$ normalized to the heat reservoir temperature $T$ is $\frac{T(r)}{T} = \frac{1}{V(R)}$ which is finite everywhere except at $r_{+2}$ where it is infinite. So the radius $r_{+2}$ yields a doubly infinite temperature. For $\Lambda R^2 = 3$, the space is pure de Sitter at infinite temperature with $\Lambda R^2 = 3$ from the reservoir at $R$ up to the center where there is a singular black hole with horizon radius given by $\frac{r_{+2}}{R} = 0$.

### C. Thermodynamic phases and phase transitions between hot Schwarzschild–de Sitter and hot de Sitter in the $\Lambda R^2 > 1$ case

We now mention thermodynamic phases and phase transitions for the $\Lambda R^2 > 1$ case. The discussion is valid for the thermodynamically stable black hole, the black hole $r_{+2}$, since the unstable one $r_{+1}$ has at most a brief existence that could not count as a phase. From Eq. (6) we get that the action $I$ for a hot Schwarzschild–de Sitter $r_{+2}$ phase is $I_{\mathrm{SdS}} = \beta R(1 - \sqrt{V(R)}) - \pi r_{+2}^2$, where from Eq. (7) we have $V(R) = (1 - \frac{r_{+2}}{R})(1 - \frac{\Lambda R^2}{3}(1 + \frac{r_{+2}}{R} + (\frac{r_{+2}}{R})^2))$, with $1 < \Lambda R^2 \le 3$ here. The free energy $F = \frac{I}{\beta} = IT$ for hot Schwarzschild–de Sitter is then

$$F_{\mathrm{SdS}} = \left(1 - \sqrt{V(R)}\right) R - \pi T r_{+2}^2, \quad 1 < \Lambda R^2 \le 3, \tag{78}$$

Another phase that might exist is hot de Sitter, in which case $r_+ = 0$, $V(R) = 1 - \frac{\Lambda R^2}{3}$ and the action is $I_{\mathrm{HdS}} = \beta R(1 - \sqrt{1 - \frac{\Lambda R^2}{3}})$. The free energy is then

$$F_{\mathrm{HdS}} = \left(1 - \sqrt{1 - \frac{\Lambda R^2}{3}}\right) R, \qquad 1 < \Lambda R^2 \le 3. \tag{79}$$

In the canonical ensemble the phase that has lowest $F$ is the phase that dominates. So, the hot Schwarzschild–de Sitter black hole dominates over hot de Sitter, or the two phases coexist equally, when

$$F_{\mathrm{SdS}} \le F_{\mathrm{HdS}}, \qquad 1 < \Lambda R^2 \le 3, \tag{80}$$

i.e., $\left(1 - \sqrt{V(R)}\right) R - \pi T r_{+2}^2 \le \left(1 - \sqrt{1 - \frac{\Lambda R^2}{3}}\right) R$, i.e.,

$$RT \ge \frac{\sqrt{1 - \frac{\Lambda R^2}{3}} - \sqrt{V(R)}}{\pi \frac{r_+^2}{R^2}}, \quad 1 < \Lambda R^2 \le 3. \tag{81}$$

As before, Eq. (35) is an implicit equation because $r_{+2} = r_{+2}(R, T)$. For each $\Lambda R^2$ in the interval above, and for each $RT$ one gets an $r_{+2}$, which can then be put into the expression just found to see whether the Schwarzschild–de Sitter phase dominates over the de Sitter phase or not. In the case it dominates than a black hole can nucleate thermodynamically from hot de Sitter space.

Here, we just comment on the limiting case, $\Lambda R^2 = 3$, which can be done directly. In this case the only solution is $T = \infty$, with $r_+ = 0$, i.e., de Sitter space with a singularity at the center. Thus, as $T \to \infty$, one has $F_{\mathrm{SdS}} \to R$ from below. The de Sitter free energy for $\Lambda R^2 = 3$ is $F_{\mathrm{SdS}} = R$, exactly. Although both free energies are equal to $R$ in the limit, one free energy tends to zero, the other is identically zero. So for $\Lambda R^2 \to 3$, $F_{\mathrm{SdS}} \le F_{\mathrm{HdS}}$ and one can say that singular Schwarzschild de Sitter phase prevails. In the limit both phases have the same free energy and coexist in the ensemble in equal quantities. The difference between the two phases is that one has a singular black hole at the center, i.e., a naked massless singularity, and the other does not. Note that in the phase transition from hot de Sitter to the Schwarzschild–de Sitter black hole phase, and vice-versa, there is topology change, since the Euclidean topology of hot de Sitter is $S^1 \times R^3$, and the Euclidean topology of the Schwarzschild–de Sitter black hole is $R^2 \times S^2$.

### D. Comments on the $\Lambda R^2 > 1$ case

We note that although the analysis made for Eqs. (69)–(72) is valid for a very small value of $\Lambda R^2 - 1$, one can have a good idea of the behavior of the solutions as $\Lambda R^2$ is increased up to the value 3. This is also done with the help of the results of the high temperature limit Eqs. (73)–(77).

For $\Lambda R^2$ near one from above, we recover the previous result that for $RT < \frac{1}{2\pi}$ there are no solutions. So $RT = \frac{1}{2\pi}$ is the minimum temperature to have solutions at all. As in the case $\Lambda R^2 < 1$ which has solutions for $RT$ higher than $\frac{1}{2\pi}$, the case $\Lambda R^2 > 1$ also has solutions for $RT$ higher than $\frac{1}{2\pi}$. For fixed $RT$ greater than the minimum value, i.e., $RT \ge \frac{1}{2\pi}$, two solutions exist up to a maximum value of $\Lambda R^2$, where at this value the two solutions merge, $r_{+1} = r_{+2}$.

For $RT = \infty$ one could find the exact dependence of $\frac{r_{+2}}{R}$ in terms of $\Lambda R^2$. Here, the maximum value is $\Lambda R^2 = 3$, where the solutions merge with horizon radius $r_{+1} = r_{+2} = 0$. This behavior for $RT = \infty$ can be heuristically explained through the thermal wavelength, the size of the reservoir, and the cosmological length. $T$ going to infinity means that the associated thermal wavelength is zero and so the small black hole as radius $r_{+1} = 0$, i.e., a black hole solution of zero size can be formed. The understanding of the large black hole with horizon radius $r_{+2}$ very small when compared to $R$, indeed tending to zero, is here not so straightforward. The cosmological scale $\ell \equiv \frac{1}{\sqrt{\Lambda}}$ has now the minimum possible value, $\ell = \frac{R}{\sqrt{3}}$. $T$ going to infinity implies that the associated thermal wavelength is vanishingly small, and the result implies that this wavelength only fits within the scale allowed by $\ell$ so that $r_{+2}$ is also vanishingly small.

## VI. DIAGRAMS FOR THE SCHWARZSCHILD–DE SITTER AND NARIAI THERMODYNAMIC SOLUTIONS AND ANALYSIS

### A. Diagrams for the Schwarzschild–de Sitter and Nariai thermodynamic solutions

#### 1. Preliminaries

We now draw some diagrams that help in the understanding of the thermodynamic solution of the Schwarzschild–de Sitter and Nariai horizons in a cavity.

There are two different sets of diagrams. The first set contains six diagrams. In each diagram, it is plotted, for a fixed value of $4\pi RT$, the values of $\frac{r_+}{R}$ that are solution of the thermodynamic problem, as a function of $\sqrt{\Lambda R^2}$, see Fig. 3. The second set contains also six diagrams. In each diagram, it is plotted, for a fixed value of $\Lambda R^2$, the values of $\frac{r_+}{R}$ that are solution of the thermodynamic problem, as a function of $4\pi RT$, see Fig. 4. We use the variable $4\pi RT$ rather than $RT$ because it is in a sense more natural in this analysis.

#### 2. Diagrams with RT fixed

The first set of six diagrams is shown in Fig. 3. It gives a snapshot for each $4\pi RT$ of how the black hole horizon radii $\frac{r_+}{R}$ behave in relation to $\sqrt{\Lambda R^2}$.

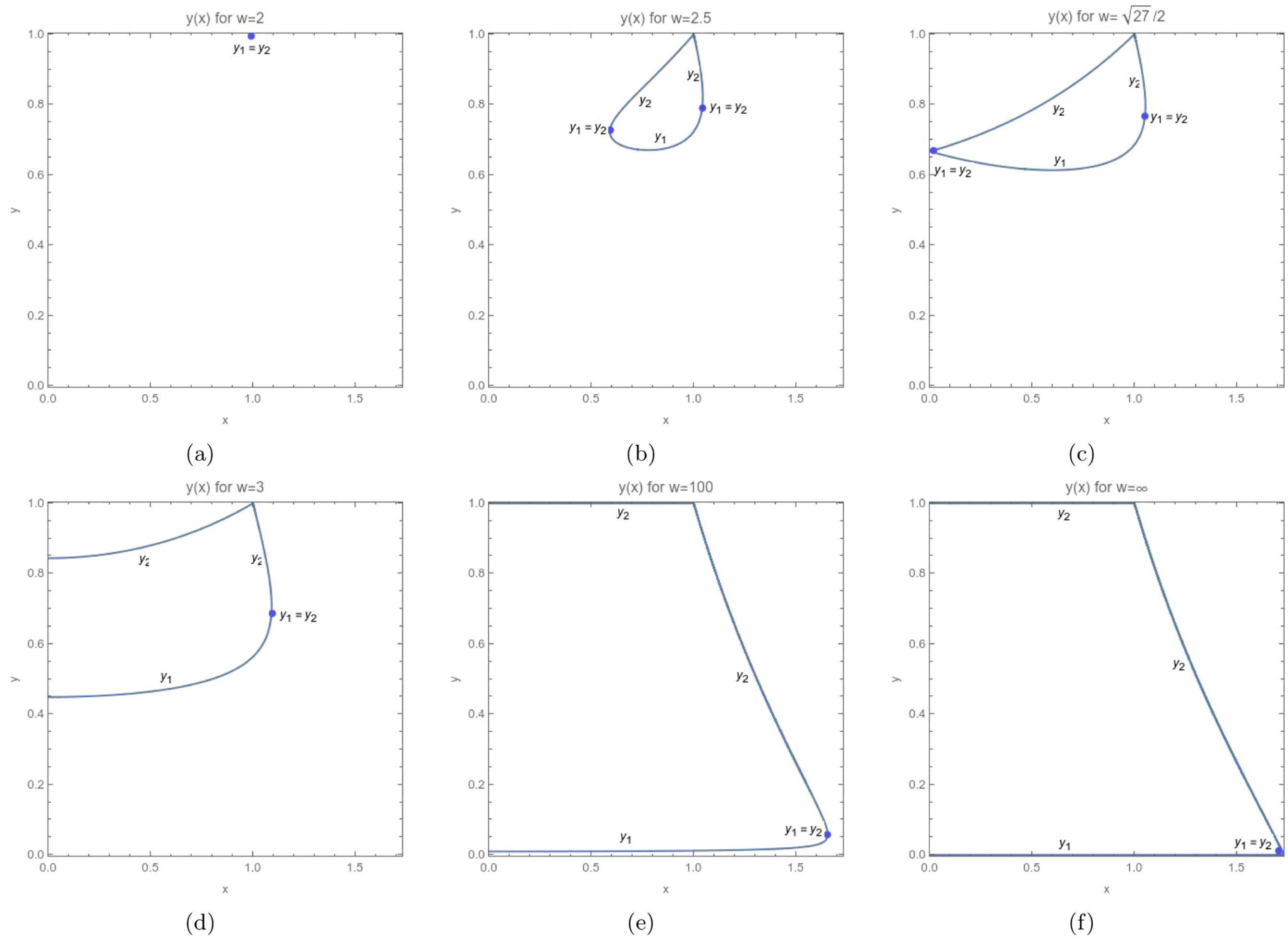


FIG. 3. Plots of $\frac{r_+}{R}$ as a function of $\sqrt{\Lambda R^2}$ for six different values of $4\pi RT$. For plotting purposes it is defined $x \equiv \sqrt{\Lambda R^2}$, $y \equiv \frac{r_+}{R}$, and $w \equiv 4\pi RT$. (a) A plot of the horizon solution for temperature $w = 2$, i.e., $RT = \frac{1}{2\pi} = 0.16$, the later equality being approximate. The only solution is the Nariai solution with $x = 1$ and $y_1 = y_2 = 1$. (b) A plot of the two horizon solutions $y_1$ and $y_2$ for temperature $w = 2.5$, i.e., $RT = 0.20$ approximately. (c) A plot of the two horizon solutions $y_1$ and $y_2$ for temperature $w = \frac{\sqrt{27}}{2} = 2.60$, i.e., $RT = \frac{\sqrt{27}}{8\pi} = 0.21$, the decimal equalities being approximate. (d) A plot of the two horizon solutions $y_1$ and $y_2$ for temperature $w = 3$, i.e., $RT = 0.24$ approximately. (e) A plot of the two horizon solutions $y_1$ and $y_2$ for temperature $w = 100$, i.e., $RT = 8.0$ approximately. (f) A plot of the two horizon solutions $y_1$ and $y_2$ for temperature $w = 10000$, i.e., $RT = 796$ approximately. Note that $10000 \to \infty$ in this context. See text for details.

The case $0 \le 4\pi RT < 2$ is not represented since there are no black hole solutions. One has either absolute zero pure de Sitter space when $4\pi RT = 0$, and hot, say, de Sitter space when $0 < 4\pi RT < 2$. For a heat reservoir radius $R$ of the order of the size of a neutron, typical temperatures would be of the order of $10^{11}$ K, so the term hot even when $4\pi RT < 2$ can be considered as appropriate.

The case $4\pi RT = 2$, i.e., $RT = \frac{1}{2\pi} = 0.159$, the last equality being approximate, see Fig. 3(a), is the case with minimum temperature that yields a solution $\frac{r_+}{R}$. This first solution is a solution with $\Lambda R^2 = 1$, and no other $\Lambda R^2$. It has $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 1$, and no other radius. It is a Nariai solution, the coldest one. Indeed, it is the first solution that pops out when $4\pi RT$, i.e., $T$, increases from zero. This means that Nariai is easier to manufacture thermodynamically than Schwarzschild–de Sitter. The corresponding cosmological horizon radius obey $\frac{r_{c1}}{R} = \frac{r_{c2}}{R} = 1$.

The case $4\pi RT = 2.5$, i.e., $RT = 0.199$ approximately, see Fig. 3(b), shows that for each $\Lambda R^2$ there are two solutions, the small one $\frac{r_{+1}}{R}$, unstable, and the large one $\frac{r_{+2}}{R}$, stable, but these solutions only exist for a narrow range of the abscissas, namely, $0.60 \le \Lambda R^2 \le 1.10$, where the numbers are approximate, i.e., there are solutions for $\Lambda R^2 < 1$ and for $\Lambda R^2 > 1$, but all solutions are still near $\Lambda R^2 = 1$. The Nariai solution is also included in this case, now with a temperature higher than the previous case. The cosmological horizon radii $\frac{r_{c1}}{R}$ and $\frac{r_{c2}}{R}$ can then be found directly from the corresponding horizon radii.

The case $4\pi RT = \frac{\sqrt{27}}{2} = 2.598$, i.e., $RT = \frac{\sqrt{27}}{8\pi} = 0.207$, the equalities in decimal notation being approximate, see Fig. 3(c), displays for the first time a solution with zero cosmological constant, $\Lambda R^2 = 0$, which is the pure Schwarzschild solution, the one with zero cosmological constant, found first by York. This solution is a coincident horizon solution with $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3}$. For other larger $\Lambda R^2$ there are two solutions, the small one $\frac{r_{+1}}{R}$, unstable, and the large one $\frac{r_{+2}}{R}$, stable, and these solutions only exist for some range of the abscissas, namely, $0 \le \Lambda R^2 \le 1.12$, where the latter number is approximate. The Nariai solution is also included in this case, now with a temperature higher than the previous case. The cosmological horizon radii $\frac{r_{c1}}{R}$ and $\frac{r_{c2}}{R}$ can then be found directly from the corresponding horizon radii using the appropriate equation.

The case $4\pi RT = 3$, i.e., $RT = 0.239$ approximately, see Fig. 3(d), shows that for zero cosmological constant, $\Lambda R^2 = 0$, there are two pure Schwarzschild solutions with $\frac{r_{+1}}{R} < \frac{2}{3}$ and $\frac{r_{+2}}{R} > \frac{2}{3}$, the first unstable, the second stable. For other larger $\Lambda R^2$ there are also two solutions, the small one $\frac{r_{+1}}{R}$, unstable, and the large one $\frac{r_{+2}}{R}$, stable, and these solutions only exist for some range of the abscissas, namely, $0 \le \Lambda R^2 \le 1.20$, the latter number being approximate. The Nariai solution is also included in this case, now with a temperature higher than the previous case. The cosmological horizon radii $\frac{r_{c1}}{R}$ and $\frac{r_{c2}}{R}$ can then be found directly from the corresponding horizon radii.

The case $4\pi RT = 100$, i.e., $RT = 7.958$ approximately, see Fig. 3(e), is a case where the temperature is high, but not divergingly high. For zero cosmological constant, $\Lambda R^2 = 0$, the two pure Schwarzschild solutions are, one with $\frac{r_{+1}}{R}$ now very small approaching zero, which is unstable, and the other $\frac{r_{+2}}{R}$ now large and approaching one, which is stable. For other larger $\Lambda R^2$ there are also two solutions, the small one approaching zero $\frac{r_{+1}}{R}$, unstable, and the large one approaching one $\frac{r_{+2}}{R}$, stable, and these solutions exist for a larger range of the abscissas, namely, $0 \le \Lambda R^2 \le 2.70$, the latter number being approximate. The Nariai solution is also included in this case, now with a temperature higher than the previous case. The cosmological horizon radii $\frac{r_{c1}}{R}$ and $\frac{r_{c2}}{R}$ can then be found directly from the corresponding horizon radii.

The case $4\pi RT = \infty$, precisely $4\pi RT = 10000$, i.e., $RT = 795.8$ approximately, see Fig. 3(f), displays the maximum spectrum of solutions. This case has been analyzed above in some detail and exact expressions for $\frac{r_{+1}}{R}$ and $\frac{r_{+2}}{R}$ have been found. There are solutions in the range $0 \le \Lambda R^2 \le 3$. The small horizon solution has $\frac{r_{+1}}{R} = 0$ all the way and is unstable. The large horizon solution has $\frac{r_{+2}}{R} = 1$ up to $\Lambda R^2 = 1$, and then decreases to zero at $\Lambda R^2 = 3$ where it joins the $\frac{r_{+1}}{R}$ solution, and it is a stable solution. The Nariai solution is also included in this case, now with a temperature tending to infinity, $4\pi RT \to \infty$. The cosmological horizon radii $\frac{r_{c1}}{R}$ and $\frac{r_{c2}}{R}$ can then be found directly from the corresponding horizon radii and there are exact expressions for them.

### 3. Diagrams with $\Lambda R^2$ fixed

The second set of six diagrams is shown in Fig. 4. It gives a snapshot for each $\Lambda R^2$ of how the black hole horizon radii $\frac{r_+}{R}$ behave in relation to $4\pi RT$.

The case $\Lambda R^2 = 0$, see Fig. 4(a), is the case with minimum $\Lambda R^2$ in this context. For this case, in the range $0 \le 4\pi RT < \frac{\sqrt{27}}{2}$, there are no radii $\frac{r_+}{R}$ that are solution of the thermodynamic problem, so presumably the inside of the reservoir is filled with hot de Sitter space. At $4\pi RT = \frac{\sqrt{27}}{2}$ the coincident solution $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3}$ appears. For larger $4\pi RT$, $\frac{r_{+1}}{R}$ tends to zero and $\frac{r_{+2}}{R}$ tends to one. Since $\Lambda R^2 = 0$ means zero cosmological constant, i.e., $\Lambda = 0$, this case is York's solution, specifically, the pure Schwarzchild case.

The case $\Lambda R^2 = 0.64$, see Fig. 4(b), is a case with an intermediate value of $\Lambda R^2$. At some definite value of $4\pi RT$, smaller than the value of the previous case, the coincident solution $\frac{r_{+1}}{R} = \frac{r_{+2}}{R}$ appears. For larger $4\pi RT$, $\frac{r_{+1}}{R}$ tends to zero and $\frac{r_{+2}}{R}$ tends to one.

The case $\Lambda R^2 = 1$, see Fig. 4(c), is the case where the Nariai universe exists. For this case, in the range

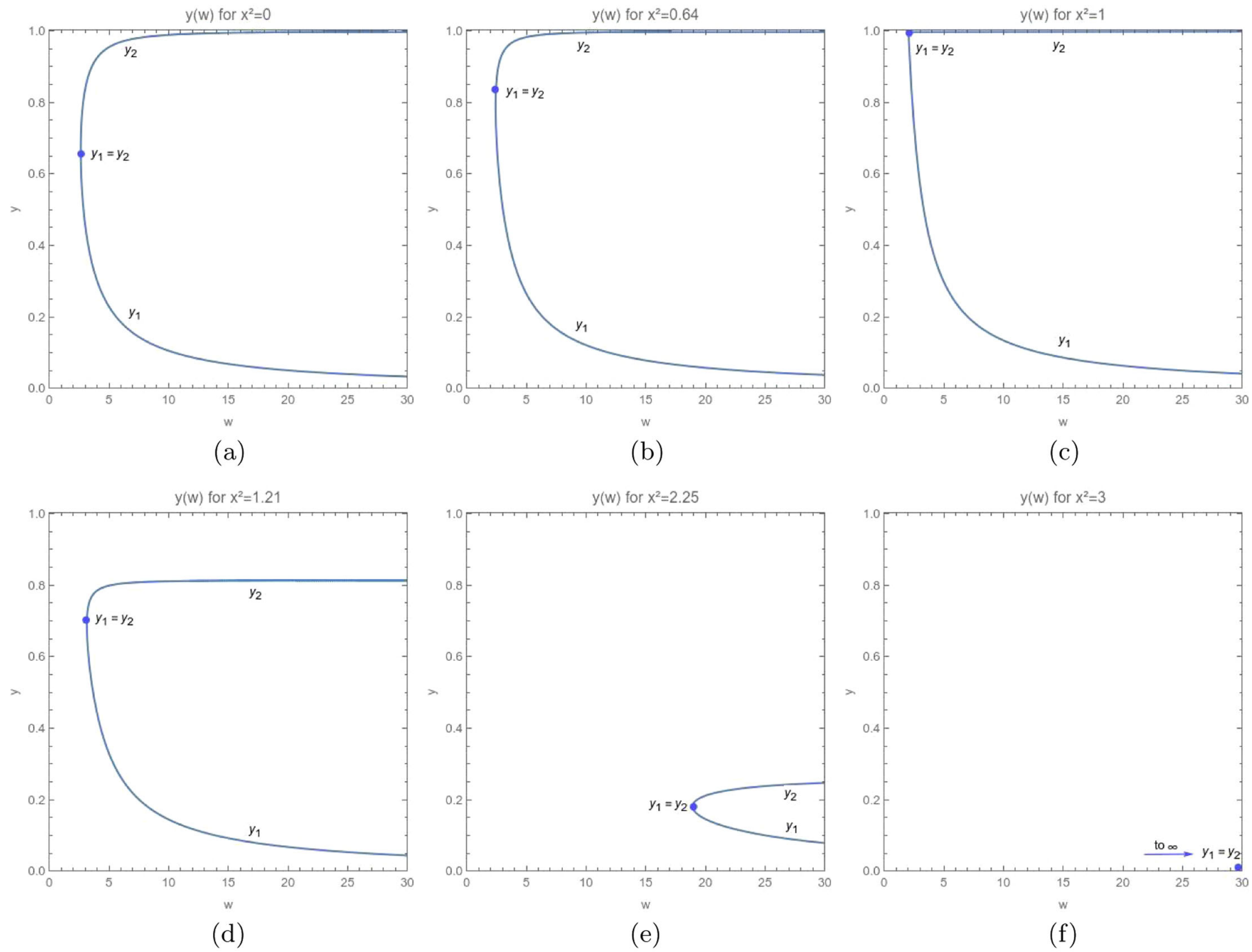


FIG. 4. Plots of $\frac{r_+}{R}$ as a function of $4\pi RT$ for six different values of $\Lambda R^2$. For plotting purposes it is defined $x \equiv \sqrt{\Lambda R^2}$, $y \equiv \frac{r_+}{R}$, and $w \equiv 4\pi RT$. (a) A plot of the two horizon solutions $y_1$ and $y_2$ for $x^2 = 0$. At temperature $w = \frac{\sqrt{27}}{2}$, i.e., $RT = \frac{\sqrt{27}}{8\pi}$, the coincident solution $y_1 = y_2 = \frac{2}{3}$ appears. For larger $w$, $y_1$ tends to zero and $y_2$ tends to one. (b) A plot of the two horizon solutions $y_1$ and $y_2$ for $x^2 = 0.64$. (c) A plot of the two horizon solutions $y_1$ and $y_2$ for $x^2 = 1$. At temperature $w = 2$, i.e., $RT = \frac{1}{2\pi}$, the solution $y_2 = 1$ appears, which in this case is a coincident solution, indeed $y_1 = y_2 = 1$. It is a Nariai solution. The solution $y_2 = 1$ for any higher temperature $w$ is Nariai. (d) A plot of the two horizon solutions $y_1$ and $y_2$ for $x^2 = 1.21$. (e) A plot of the two horizon solutions $y_1$ and $y_2$ for $x^2 = 2.25$. (f) A plot of the two horizon solutions $y_1$ and $y_2$ for $x^2 = 3$. Here, there is only the coincident solution, with values $y_1 = y_2 = 0$ and temperature $w = \infty$. See text for details.

$0 \le 4\pi RT < 2$, there are no radii $\frac{r_+}{R}$ that are solution of the thermodynamic problem, so presumably the inside of the reservoir is filled with hot de Sitter space. At $4\pi RT = 2$ the coincident solution appears with $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 1$. This is the coldest Nariai solution. For larger $4\pi RT$, $\frac{r_{+1}}{R}$ tends to zero and $\frac{r_{+2}}{R}$ tends to one, indeed the solution $\frac{r_{+2}}{R} = 1$ for any $4\pi RT \ge 2$ is a hot Nariai universe.

The case $\Lambda R^2 = 1.21$, see Fig. 4(d), is a case typical of $\Lambda R^2 > 1$. At some definite $4\pi RT$ the coincident solution $\frac{r_{+1}}{R} = \frac{r_{+2}}{R}$ appears. For larger $4\pi RT$, $\frac{r_{+1}}{R}$ tends to zero and $\frac{r_{+2}}{R}$ tends to some value that is less than one.

The case $\Lambda R^2 = 2.25$, see Fig. 4(e), is also a case typical of $\Lambda R^2 > 1$, but the new features are more evident. At some definite $4\pi RT$ the coincident solution $\frac{r_{+1}}{R} = \frac{r_{+2}}{R}$ appears, now with greater $4\pi RT$ than the previous case. For larger $4\pi RT$, $\frac{r_{+1}}{R}$ tends to zero and $\frac{r_{+2}}{R}$ tends to some value less than one. This value less than one decreases rapidly with increasing $\Lambda R^2$.

The case $\Lambda R^2 = 3$, see Fig. 4(f), is the last possible case, is a limit case. In this case, there is only one solution, which is the coincident solution $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 0$. In the plot this solution is represented as a point in the highest shown $4\pi RT$, which is meant to be $4\pi RT = \infty$. There is divergently hot de Sitter space in the cavity at radius $R$, apart from a central singularity at zero radius, $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 0$.

#### *4. Additions*

It is important to make additional comments to the plots that have been displayed in Figs. 3 and 4. In what follows the discussion will be qualitative.

Stacking with interpolation Figs. 3(a)–3(f), one can glimpse the correctness of Figs. 4(a)–4(f), and stacking with interpolation Figs. 4(a)–4(f), one can glimpse, in turn, the correctness of Figs. 3(a)–3(f).

One striking feature, that can be deduced from the plots, is that the space of black hole horizon radius solutions is enlarged as the reservoir temperature $T$, or rather $4\pi RT$, is increased. In fact, for very low temperatures there are no solutions for any $\Lambda$, or rather, for any $\Lambda R^2$. At the temperature $4\pi RT = 2$ there is only one solution, the coldest possible Nariai universe. For higher $4\pi RT$ there are solutions for some values of $\Lambda R^2$, but not all. For instance, for $4\pi RT = \frac{\sqrt{27}}{2}$, there appears a solution with $\Lambda = 0$, i.e., $\Lambda R^2 = 0$, and there are solutions up to $\Lambda R^2 = 1.12$, approximately, so that the range of $\Lambda R^2$ is $0 \leq \Lambda R^2 \leq 1.12$ for this temperature. Finally, for infinite temperature, $4\pi RT = \infty$, there are solutions for all possible $\Lambda R^2$, namely $0 \leq \Lambda R^2 \leq 3$. To understand other features of the solutions it is perhaps advisable to incorporate the case $\Lambda R^2 = 1$ in both the low and high cosmological constant cases and thus divide the whole range into $0 \leq \Lambda R^2 \leq 1$ and $1 \leq \Lambda R^2 \leq 3$.

In the range $0 \leq \Lambda R^2 \leq 1$, now with the help of Figs. 3 and 4, one can summarize the qualitative explanation for the reason of why black hole solutions with lower cosmological constant appear only for higher $T$, i.e., higher $RT$. Thus, let us start with $\Lambda = 0$, so that the cosmological length scale $\ell = \frac{1}{\sqrt{\Lambda}}$ is infinite, $\ell = \infty$. In this case there is no coupling of this length scale with the other two, $\lambda = \frac{1}{T}$ and $R$. In this situation, we see that for low $T$, high $\lambda$, one has $\lambda \gg R$, so that, since the thermal wavelength is very large compared to the reservoir radius $R$, then this wavelength is stuck to the reservoir and the corresponding energy cannot collapse to form a black hole in any circumstances. When $T$ is sufficiently increased, i.e., $RT$ is larger than about one, the wavelength is sufficiently small, and the corresponding energy can travel freely inside the reservoir and can collapse, so that formation of black holes is possible. The value $4\pi RT = \frac{\sqrt{27}}{2}$ divides no black hole from black hole solutions. Now, let us do $\ell$ finite. This is the third scale. For low enough $\ell$, say a bit larger than $R$, the space inside the reservoir gets higher curvature due to the high cosmological constant, and so in some manner this inner space has more length along the radius, so that although the reservoir area radius is still $R$, the radial length is large, and so the volume is also larger. This means that energy packets with higher $\lambda$, relatively to the cases with lower $\ell$, can continue to travel freely in the inside and can form black holes. The limiting situation is when $R$ and $\ell$ are equal, i.e., $\Lambda R^2 = 1$, or $\frac{R^2}{l^2} = 1$, so that energy packets with the highest possible $\lambda$, actually $\lambda = 2\pi R$, can give a solution, which in this case is a Nariai solution.

In the range $1 \leq \Lambda R^2 \leq 3$, now with the help of Figs. 3 and 4, one can also give a qualitative explanation for the reason of why now black hole solutions with higher cosmological constant, i.e., higher $\Lambda R^2$, appear only for higher $T$, i.e., higher $RT$. Within this range, the cosmological constant is very high, i.e., the cosmological length scale $\ell$ is very short, and so determines and dominates the processes. Indeed, now $\ell < R$. Let us now start with the situation when $R$ and $\ell$ are equal, i.e., $\Lambda R^2 = 1$, or $\frac{R^2}{l^2} = 1$, so that the energy packets with the highest possible $\lambda$, actually $\lambda = 2\pi R$, can give a solution, a Nariai solution in this case. Still, for $\frac{R^2}{l^2} = 1$ and higher temperatures, i.e., lower wavelengths $\lambda$, there is smaller unstable $r_{+1}$ black hole solution and the Nariai solution. For $\frac{R^2}{l^2}$ larger than one, i.e., $\ell$ a bit smaller, the temperatures have to be a higher, and so the wavelength $\lambda$ of the energy packets has to be a bit smaller, to have the two black black hole solutions, as in the $\Lambda R^2 < 1$ case. The small $r_{+1}$ solution forms in the same manner. The new feature is with the large black hole solution $r_{+2}$. Now, for fixed $\Lambda R^2 < 1$, the $r_{+2}$ solution is always less then $R$ even when $T$ is very large. This means that the corresponding small wavelengths $\lambda$ now are constrained by the scale $\ell$ so that the interplay is between $r_{+2}$, $\lambda$, and $\ell$ and not anymore between $r_{+2}$, $\lambda$, and $R$. The final case is when $\ell = \frac{R}{\sqrt{3}}$. The space inside is pure de Sitter, except for a singular black horizon at the center, with the energy packets having a zero thermal wavelength $\lambda = 0$, with the temperature $T$ of the reservoir being infinite. Only those $\lambda = 0$ energy packets can collapse to form a black hole, packets with a higher $\lambda$, corresponding to a lower reservoir $T$ cannot fit to the scale set by $\ell$.

Another characteristic radius that is part of the Schwarzschild–de Sitter solution is the cosmological radius $r_c$. In the setting we are working, where the heat reservoir is for the inside that harbors a possible black hole, the cosmological radius $r_c$ has only a secondary role. This characteristic radius $r_c$ can be calculated once the black hole horizon radius is found on thermodynamic grounds.

### B. Mathematical analysis of the plots: Black hole horizons

#### *1. Nomenclature*

We now obtain through a mathematical analysis some important features displayed in the plots above, Figs. 3 and 4. We repeat here Eq. (13), i.e., $4\pi RT = \frac{1}{\frac{r_+}{R}} \frac{1-\Lambda R^2 (\frac{r_+}{R})^2}{\sqrt{1-\frac{r_+}{R}}\sqrt{(1-\frac{\Lambda R^2}{3}(1+\frac{r_+}{R}+(\frac{r_+}{R})^2))}}$. The natural variables without units are $\Lambda R^2$ and $\frac{r_+}{R}$. In this context it is perhaps preferable to work with $\sqrt{\Lambda R^2}$ rather than with $\Lambda R^2$, so to shorten the notation we define the variables $x$ and $y$ as

$$x \equiv \sqrt{\Lambda R^2}, \tag{82}$$

$$y \equiv \frac{r_+}{R}, \tag{83}$$

with the range being $0 \leq x \leq \sqrt{3}$, or $0 \leq x^2 \leq 3$, and $0 \leq y \leq 1$. In these variables, Eq. (13) is $4\pi TR = \frac{1-x^2y^2}{y\sqrt{1-y}\sqrt{1-\frac{x^2}{3}(1+y+y^2)}}$. Define in addition the variable $w$ as

$$w \equiv 4\pi RT. \tag{84}$$

Then, with these definitions Eq. (13) is

$$w = \frac{1 - x^2y^2}{y\sqrt{1-y}\sqrt{1-\frac{x^2}{3}(1+y+y^2)}}, \tag{85}$$

with $2 \leq w < \infty$. Solutions exist only for $w \geq 2$.

Now, for a fixed temperature $T$, more properly for a fixed $RT$, i.e., fixed $w$, one has $dw = 0$, and so $\frac{dy}{dx} = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial y}}$. After some calculations we obtain

$$\frac{dy}{dx} = -\frac{2xy(1-y)Q(y)}{3R(y)}, \tag{86}$$

where

$$Q(y) \equiv (1 + y + y^2)(1 + x^2y^2) - 6y^2, \tag{87}$$

and

$$R(y) \equiv -\frac{2}{3}(1 + x^2y^2)(1 - y)[3 - x^2(1 + y + y^2)] + (1 - x^2y^2)^2 y. \tag{88}$$

In addition, we need in the analysis $\frac{\partial w}{\partial y}$. Obtain from Eq. (85) that

$$\frac{\partial w}{\partial y} = \frac{\sqrt{3}S}{2y^2(1-y)^{3/2}[3 - x^2(1+y+y^2)]^{3/2}}, \tag{89}$$

where we have defined

$$S(x,y) = x^4[2y^2(1-y^3) + 3y^5] + 2x^2(-y^3 - 3y^2 + 1) - 6 + 9y. \tag{90}$$

We have seen that the point $\Lambda R^2 = 1$, i.e., $x^2 = 1$, is important, as it gives the Nariai solution. So, let us consider below the limit when $x \to 1$ from below and from above. We will see now that the result depends on how the limit is taken. We recall that for each $\Lambda R^2$ there are two solutions, $r_{+1}$, the small solution, and $r_{+2}$, the large solution, which change as $RT$ is changed, i.e., for each $x$, there are $y_1$ and $y_2$, which change as $w$ is changed.

### 2. Analysis

There are general results here that we can mention. Let us analyze the three ranges separately, namely, the regimes $x^2 < 1$, $x^2 = 1$, and $x^2 > 1$.

$x^2 < 1$:

This range of $x^2$ is specifically $0 \leq x^2 < 1$. In this range of $x$, the range of $y$ is

$$0 \leq y < 1. \tag{91}$$

We now find $y_1$, then $y_2$, and then we analyze the coincident solutions $y_1 = y_2$.

First we find $y_1$, i.e., $r_{+1}$, when $x$ varies within this range, $0 \leq x^2 < 1$. For that, we fix $y < 1$ and move along the positive $x$ direction. In the space $(x, y)$ the corresponding point moves along a horizontal line. Then, from Eq. (85) one finds $w = \frac{1-x^2y^2}{y\sqrt{1-y}\sqrt{1-\frac{x^2}{3}(1+y+y^2)}}$, i.e., $y$ obeys an equation of the type

$$y_1\sqrt{1-y_1}\sqrt{1-\frac{x^2}{3}(1+y_1+y_1^2)}w = 1 - x^2y_1^2, \quad 0 \leq x^2 < 1, \tag{92}$$

for a given $x$ fixed. One of the solutions of this equation is $y_1$, with $w$ fixed and $w$ with a value obeying $w > 2$, and with $y_1(x) < 1$ always. The limit of $x \to 1$ for $y_1$ is smooth, see below. Another property is that for any $x$, $w \to \infty$ when $y_1 \to 0$. This is the solution $y_1 = 0$, i.e., $\frac{r_{+1}}{R} = 0$, when the temperature goes to infinity. Another interesting property is the point where $\frac{dy_1}{dx} = 0$, if there is one. This is when $r_{+1}$ attains a minimum value in relation to $\Lambda R^2$. This is the root of equation $Q(y) = 0$, and happens for the small solution, i.e., $y_1$. From Eq. (87), $Q(y_1) = 0$ gives

$$x^2 = \frac{5y_1^2 - y_1 - 1}{y_1^2(1 + y_1 + y_1^2)}. \tag{93}$$

Now, the lowest $y_1$ is given when $x = 0$ by the solution of $5y_1^2 - y_1 - 1 = 0$, which is $\frac{1+\sqrt{21}}{10}$. The highest $y_1$ is $y_1 \to 1$ which yields $x \to 1$. Thus, $\frac{dy_1}{dx} = 0$ in the range $0 \leq x^2 < 1$ happens in the range

$$\frac{1+\sqrt{21}}{10} \leq y_1 < 1, \tag{94}$$

i.e., $0.56 \leq y_1 < 1$, the first number in the left inequality being approximate, with the range of $x$ being $0 \leq x^2 < 1$.

The corresponding range of $w$ from Eq. (85) is $2 < w \le \frac{10\sqrt{10}}{(1+\sqrt{21})\sqrt{9-\sqrt{21}}}$, or in round numbers $2 < w \le 2.70$, the last number being approximate. Why there is an $x$ for which $\frac{dy}{dx} = 0$ in the $y_1$ solution is not clear on physical, heuristic, terms, but possibly is a nonlinear interplay between the length scale $\ell \equiv \frac{1}{\sqrt{\Lambda}}$ and the length scales $R$ and $\lambda = \frac{1}{T}$. Further properties for the small back hole depend on the specific $x$ and the specific $w$ that one picks, but there is nothing else more general that we can mention.

Second we find $y_2$, i.e., $r_{+2}$, when $x$ varies within this range, $0 \le x^2 < 1$. From Eq. (85), one has

$$y_2\sqrt{1-y_2}\sqrt{1-\frac{x^2}{3}(1+y_2+y_2^2)w} = 1-x^2y_2^2, \quad 0 \le x^2 < 1, \qquad (95)$$

for a given $x$ fixed, and one finds that there is a solution $y_2(x)$ for each $x$. The maximum of the curve is at $x = 1$ and $y_2 = 1$, and at this point the derivative can be any of those found above, so it is not well defined. It is important to find the behavior and the properties of $y_2$ when $x$ is near 1, i.e., $x \to 1$ with $y_2 \to 1$. A direct property is that for any $x$, $w \to \infty$ when $y_2 \to 1$. This is the solution $y_2 = 1$, i.e., $\frac{r_{+2}}{R} = 1$, when the temperature goes to infinity. Other important properties are related to the derivative of $y_2$, in particular, $\frac{dy_2}{dx}$ for $x \to 1$. The calculations of $\frac{dy_2}{dx}$ for $x \to 1$ are done in detail in the Appendix C, here we state the result, namely,

$$\frac{dy_2}{dx} = \frac{w}{\sqrt{w^2-4}} - 1, \quad w \ge 2, \quad x \to 1. \qquad (96)$$

Therefore, taking the two limits for the temperature, i.e., for $w$, we have

$$\frac{dy_2}{dx} \to \infty \ \text{ for } w \to 2, \quad \frac{dy_2}{dx} \to 0 \quad w \to \infty, \quad x \to 1. \qquad (97)$$

Thus, the derivative $\frac{dy_2}{dx}$ obeys $\frac{dy_2}{dx} \ge 0$, and can have values from 0 to infinity when one approaches $x = 1$ from $x < 1$. Another possible interesting property is the point where $\frac{dy_2}{dx} = 0$, if there is one. For the solution $y_2$ one finds that there is no point with zero derivative.

Now we find $y_1 = y_2$ for $w$ fixed, i.e., the point $x$ where $y_1 = y_2$. There can exist two points at which $\frac{dy}{dx} = \infty$ for fixed $w$, one for $x < 1$, the other for $x > 1$. Here we work out $x < 1$. It is given by the root of equation $R(y) = 0$, see Eq. (88), and it defines the bifurcation point where $y_1 = y_2$ for $w =$ constant. The equation is $\frac{2}{3}(1+x^2y^2)(1-y)[3-x^2(1+y+y^2)] - (1-x^2y^2)^2y = 0$, which can be put in the form $x^4[\frac{2}{3}y^2(1-y)(1+y+y^2)+y^5] - x^2[2y^2 - \frac{2}{3}(1-y)(1+y+y^2)] - (2-3y) = 0$. This is a quadratic in $x^2$ of the form $x^4a(y) - x^2b(y) - c(y) = 0$, where $a(y) = \frac{2}{3}y^2(1-y)(1+y+y^2)+y^5$, $b(y) = 2y^2 - \frac{2}{3}(1-y)(1+y+y^2)$, and $c(y) = 2–3y$. The solution is

$$x^2 = \frac{b(y) - \sqrt{b^2(y)+4a(y)c(y)}}{2a(y)}, \quad 0 < x^2 \le 1, \qquad (98)$$

Inverting Eq. (98) one finds $y(x)$, i.e., the solution $y_1 = y_2$. For $0 \le x^2 < 1$ there are solutions for $y > \frac{2}{3}$, which from Eq. (85) means in turn $w < \frac{\sqrt{27}}{2}$, i.e., for $RT < \frac{\sqrt{27}}{8\pi}$, with $RT = \frac{\sqrt{27}}{8\pi}$ at $x = 0$ being marginal. For higher $w$, i.e., higher $RT$, there are no solutions, $y_1$ and $y_2$ never coincide for any $x$ in this range $0 \le x^2 < 1$.

Now, we find $y_1 = y_2$, the coincident solution for fixed $x$, and $w$ varying. This is when $\frac{\partial w}{\partial y} = 0$, see Eq. (89), i.e., $S(x,y) = 0$. From Eq. (90) we have then that the bifurcation points where $y_1 = y_2$ for $x =$ constant are given by the equation $x^4[2y^2(1-y^3)+3y^5] + 2x^2(-y^3-3y^2+1) - 6 + 9y = 0$, i.e., $x^4[2y^2(1-y^3)+3y^5] - x^2[2(y^3+3y^2-1)] - 3(2-3y) = 0$. This is a quadratic in $x^2$, which can be written as $x^4d(y) - x^2e(y) - f(y) = 0$, with $d(y) = 2y^2+y^5$, $e(y) = 2(y^3+3y^2-1)$, $f(y) = 3(2-3y)$, and with solution

$$x^2 = \frac{e(y) - \sqrt{e^2(y)+4d(y)f(y)}}{2d(y)}, \quad 0 < x^2 \le 1. \qquad (99)$$

Inverting Eq. (99) one has $y(x)$, for which $y_1 = y_2$. To make progress and find an analytical result we take $x \to 1$ and $y \to 1$. The calculations are involved and we leave them to the Appendix C, the result is that the coincident solution takes the form

$$y_1 = y_2 = 1 - \left(\frac{3}{8}(1-x^2)^2\right)^{\frac{1}{3}}, \qquad (100)$$

for $(1-x^2) \ll 1$. Then, returning to the original variables one has $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 1 - (\frac{3}{2}(1-\Lambda R^2)^2)^{\frac{1}{3}}$ for $1 - \Lambda R^2$ tiny, which is Eq. (25).

$x^2 = 1$:

This specific value of $x$, $x = 1$, is important as it hides a considerable structure, indeed, the whole Nariai solution is inside it. For this value of $x$, the range of $y$ is

$$0 \le y \le 1. \qquad (101)$$

We now find $y_1$, then $y_2$, an then we analyze the coincident solutions $y_1 = y_2$.

First we find $y_1$, i.e., $r_{+1}$, when $x = 1$. For that, we fix $y < 1$ and move along the positive $x$ direction. In the space $(x,y)$ the corresponding point moves along a horizontal line. Then, from Eq. (85) one finds that for $x = 1$ one has

$w = \frac{\sqrt{3}\sqrt{1-y}(1+y)}{y\sqrt{2-y-y^2}}$. Since $2-y-y^2 = (1-y)(2+y)$, one has

$$(y_1\sqrt{2+y_1})w = \sqrt{3}(1+y_1), \qquad x^2 = 1, \tag{102}$$

and, from Eq. (89), one also finds $\frac{dw}{dy} < 0$. Now, Eq. (102) has one solution $y_1(x=1)$, with $w$ fixed and $w$ with a value obeying $w > 2$. For any $w$, with $w > 2$ one has $y_1(x=1) < 1$ always. It is clear that point $(x, y_1) = (1, y_1)$ lies on the branch of the curve that corresponds to the small root, to the small black hole.

Second, we find $y_2$, i.e., $r_{+2}$. From Eq. (85) one finds that for $x = 1$ one has

$$\left(y_2\sqrt{2+y_2}\right)w = \sqrt{3}(1+y_2), \qquad x^2 = 1. \tag{103}$$

For $x = 1$, the large black hole always has $y_2 = 1$ but $w$ can be any as we have seen. So, Eq. (103) in the context of $y_2$ is deceiving. Indeed, if we put $y_2 = 1$ into Eq. (102) one finds $w = 2$, i.e., $RT = \frac{1}{2\pi}$. But we know, from our calculation above that at $x = 1$, $w$ can be any, indeed $2 \le w < \infty$. So, Eq. (103) only gives one of the infinite number of solutions, which all correspond to the Nariai universe. Since this has been and will be further discussed we refrain to take further comments.

Now we find $y_1 = y_2$ for $w$ fixed. Here $x$ is fixed, $x = 1$, i.e., $x^2 = 1$. The solution is

$$x^2 = y_1 = y_2 = 1, \tag{104}$$

with $w = 2$. Then, in the original variables we have $\Lambda R^2 = \frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 1$ with $RT = \frac{1}{2\pi}$. It is the first Nariai universe solution as far as increasing temperature goes.

Now we find $y_1 = y_2$ for $x$ fixed and $w$ varying. It is given by Eq. (104) since $w$ is fixed as we saw.

$x^2 > 1$:

This range of $x^2$ is specifically $1 < x^2 \le 3$. In this range of $x$, the range of $y$ is

$$0 < y \le y_e, \qquad y_e = -\frac{1}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}}, \tag{105}$$

where $y_e = y_e(x)$ is the edge, or maximum, value that $y$ can have for each $x$ in this range. The expression for $y_e$ is given by the root of the equation $y^2 + y + 1 - \frac{3}{x^2} = 0$, see Eq. (85), with solution $y_e = -\frac{1}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}}$. Note that $y_e = -\frac{1}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}} < \frac{1}{x} \le 1$. When $y \to y_e$ we have from Eq. (85) that $w = \frac{1-x^2y_e^2}{y_e\sqrt{1-y_e}\sqrt{(y_e-y)(y_*+y)}}$ with $y_* \equiv \frac{1}{2} + \sqrt{3}\sqrt{\frac{1}{x^2} - \frac{1}{4}}$. Since $y_* = y_e + 1$ this can also be written as $w = \frac{1-x^2y_e^2}{y_e\sqrt{1-y_e}\sqrt{(y_e-y)(y_e+y+1)}}$. So, for each $x$ in the range $1 < x^2 \le 3$, one finds $y_e$ from Eq. (105), and then one finds the corresponding $w$, i.e., the corresponding $RT$, which for $y \to y_e$ is $w \to \infty$. We now find $y_1$, then $y_2$, and then we analyze the coincident solutions $y_1 = y_2$.

First we find $y_1$, i.e., $r_{+1}$, when $x$ varies within the range we are working, namely, $1 < x^2 \le 3$. For that, we give some $w$, then we fix $y < y_e$ and move along the positive $x$ direction. In the space $(x, y)$ the corresponding point moves along a horizontal line. Then, from Eq. (85) one finds $w = \frac{1-x^2y^2}{y\sqrt{1-y}\sqrt{1-\frac{x^2}{3}(1+y+y^2)}}$, i.e., $y$ obeys an equation of the type

$$y_1\sqrt{1-y_1}\sqrt{1-\frac{x^2}{3}(1+y_1+y_1^2)}w = 1 - x^2y_1^2, \qquad 1 < x^2 \le 3, \tag{106}$$

for a given $x$ fixed. One of the solutions of this equation is $y_1$, with $w$ fixed and $w$ with a value obeying $w > 2$, and with $y_1(x) < y_e$ always. The limit of $y_1$ for $x \to 1$ is smooth. Also, for any $x$ in the range $1 < x^2 \le 3$, when $w \to \infty$ one has $y_1 \to 0$. This is the solution $y_1 = 0$, i.e., $r_{+1} = 0$, when the temperature goes to infinity. Another possible interesting property is the point where $\frac{dy_1}{dx} = 0$. From Eq. (86) this happens when $Q(y_1) = 0$. Then, from Eq. (87) one finds that there are none in the range $1 < x^2 \le 3$. Moreover, $y_1(x)$ is a smooth curve, infinitely differentiable, in the whole range $0 \le x^2 \le 3$, there is no discontinuity in any derivative at $x = 1$. Further properties for the small back hole depend on the specific $x$ and the specific $w$ that one picks, but there is nothing else general that we can mention.

Second we find $y_2$, i.e., $r_{+2}$. From Eq. (85), one finds

$$y_2\sqrt{1-y_2}\sqrt{1-\frac{x^2}{3}(1+y_2+y_2^2)}w = 1 - x^2y_2^2, \qquad 1 < x^2 \le 3, \tag{107}$$

and one finds that there is a solution $y_2(x)$ for each $x$. The maximum of the curve is at $x = 1$, $y_2 = 1$, and the derivative can be any of those derived above, so it is not well defined in this sense. It is important to discuss the behavior and the properties of $y_2$, when $x \to 1$ from above. A direct property is that for any $x$, $w \to \infty$ when $y_2 \to y_e$. This is the solution $y_2 = y_e$ when the temperature goes to infinity. Other important properties are related to $\frac{dy_2}{dx}$, in particular, $\frac{dy_2}{dx}$ when $x \to 1$ from above. The calculations for finding $\frac{dy_2}{dx}$ when $x \to 1$ from above, are done in detail in the Appendix C, here we state the result, namely

$$\frac{dy_2}{dx} = -\frac{w}{\sqrt{w^2-4}} - 1, \qquad w \ge 2, \qquad x \to 1, \tag{108}$$

where the limit is from above. Therefore, taking the two limits for the temperature, i.e., for $w$, we have

$$\frac{dy_2}{dx} \to -\infty \quad \text{for } w \to 2, \qquad \frac{dy_2}{dx} \to -2 \quad \text{for } w \to \infty, \quad x \to 1, \tag{109}$$

where the limit is from above. Thus, the derivative $\frac{dy_2}{dx}$ obeys $\frac{dy}{dx} \leq 0$, and it can be from 0 to minus infinity when one approaches $x = 1$ from $x > 1$. We see again that the point $(x, y) = (1, 1)$ is very rich, in fact it corresponds to the Nariai limit, a universe full of structure as we have studied. Another possible interesting property is the point where $\frac{dy_2}{dx} = 0$, if there is one. For the solution $y_2$ one finds that there is no point with zero derivative.

Now we find $y_1 = y_2$ for $w$ fixed, i.e., the point $x$ where $y_1 = y_2$. There exist two points at which $\frac{dy}{dx} = \infty$ for fixed $w$, one for $x < 1$, the other for $x > 1$. Here we work out $x > 1$. It is given by the root of equation $R(y) = 0$, see Eq. (88), and it defines the bifurcation points where $y_1 = y_2$ for $w =$ constant. The equation is $\frac{2}{3}(1 + x^2y^2)(1 - y)[3 - x^2(1 + y + y^2)] - (1 - x^2y^2)^2 y = 0$, which can be put in the form $x^4[\frac{2}{3}y^2(1 - y)(1 + y + y^2) + y^5] - x^2[2y^2 - \frac{2}{3}(1 - y)(1 + y + y^2)] - (2 - 3y) = 0$. This is a quadratic in $x^2$. Writing it as $x^4 a(y) - x^2 b(y) - c(y) = 0$, where $a(y) = \frac{2}{3}y^2(1 - y)(1 + y + y^2) + y^5$, $b(y) = 2y^2 - \frac{2}{3}(1 - y)(1 + y + y^2)$, and $c(y) = 2–3y$, the solution is

$$x^2 = \frac{b(y) + \sqrt{b^2(y) + 4a(y)c(y)}}{2a(y)}, \qquad 1 < x^2 \leq 3. \tag{110}$$

Inverting Eq. (110) one obtains $y(x)$ for which $y_1 = y_2$. For $1 < x^2 \leq 3$ there are solutions for $y > 0$, i.e., $0 \leq y < y_e$, which means in turn $w > 2$, i.e., for $2 < w < \infty$. For $w = \infty$, one has $x^2 = 3$, and $y_1 = y_2 = 0$.

Now, we find $y_1 = y_2$, the coincident solution for fixed $x$, and $w$ varying. This is when $\frac{\partial w}{\partial y} = 0$, see Eq. (89), i.e., $S(x, y) = 0$. From Eq. (90) we have then that the bifurcation points where $y_1 = y_2$ for $x =$ constant are given by the equation $x^4[2y^2(1 - y^3) + 3y^5] + 2x^2(-y^3 - 3y^2 + 1) - 6 + 9y = 0$, i.e., $x^4[2y^2(1 - y^3) + 3y^5] - x^2[2(y^3 + 3y^2 - 1)] - 3(2 - 3y) = 0$. This is a quadratic in $x^2$, which can be written as $x^4 d(y) - x^2 e(y) - f(y) = 0$, with $d(y) = y^5 + 2y^2 > 0$, $e(y) = 2(y^3 + 3y^2 - 1)$, $f(y) = 3(2 - 3y)$, and with solution

$$x^2 = \frac{e(y) + \sqrt{e^2(y) + 4d(y)f(y)}}{2d(y)}, \qquad 1 < x^2 \leq 3. \tag{111}$$

One can show that the solution of the quadratic equation with the minus sign before radical is inconsistent with the condition $x > 1$, so it is not considered. Inverting Eq. (111) one has $y(x)$ for which $y_1 = y_2$. To make progress and find an analytical result we take $x \to 1$ and $y \to 1$. The calculations are involved and we leave them to the Appendix C, the result is that the coincident solution takes the form

$$y_1 = y_2 = 1 - \left(\frac{3}{8}(x^2 - 1)^2\right)^{\frac{1}{3}}, \tag{112}$$

for $x^2 - 1 \ll 1$. Then, returning to the original variables one has, $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 1 - (\frac{3}{8}(\Lambda R^2 - 1)^2)^{\frac{1}{3}}$ for $\Lambda R^2 - 1$ tiny, which is Eq. (71).

### 3. *Synopsis*

When looking for roots $y = y(x, w)$ of the equation $w(x, y) = w$, we have found that, depending on the fixed value of $w$, one can have no roots, one root, or two roots. The no root situation means that the space inside the heat reservoir is hot de Sitter space. The one root situation means that there is one possible black hole solution $y_1 = y_2$ inside the reservoir. The two roots situation means that there are two possible black hole solutions inside the reservoir, one small $y_1$ and unstable, the other large $y_2$ and stable. The case $x^2 = 1$ is important. When $x = 1$ there is the sole solution $y_1 = y_2 = 1$ for $w = 2$, but for any other $w$, $w > 2$, there are two solutions, one is $y_1$, corresponding to the small black hole, the other is $y_2 = 1$, corresponding to the larger black holes, which for $x^2 = 1$ have transubstantiated into the Nariai cosmological universes with all the allowed values of $w$, $w \geq 2$.

## C. Further analysis: Cosmological horizons

In the Schwarzschild–de Sitter space there is also the radius of the cosmological horizon $r_c$. However, since the radius of the heat reservoir $R$ is inside $r_c$, this latter has no role in the thermodynamics. The cosmological horizon radius, is just a parameter, which has only a coadjuvant role in the whole setting. It is found once $r_+$ has been found from the thermodynamics.

The radius of the cosmological horizon $r_c$ is the largest root of the equation $V(r) = 0$, see Eq. (2) for the expression of $V(r)$. As we have seen, it is related to the black hole horizon radius according to $r_c^2 + r_c r_+ + r_+^2 - \frac{3}{\Lambda} = 0$, whence $r_c = -\frac{r_+}{2} + \frac{r_+}{2}\sqrt{\frac{12 - 3\Lambda r_+^2}{\Lambda r_+^2}}$, see Eq. (5), or $r_c = -\frac{r_+}{2} + \sqrt{\frac{3}{\Lambda} - \frac{3}{4}r_+^2}$. Let us define

$$u \equiv \frac{r_c}{R}, \tag{113}$$

as the cosmological radius in units of $R$. Then, from Eq. (5) we have that $u$ is given in terms of $x$ and $y$ of Eqs. (82) and (83), respectively, by

$$u = -\frac{y}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}y^2}. \quad (114)$$

Calculating $\frac{du}{dy}$ one finds

$$\frac{du}{dy} \leq 0, \quad (115)$$

i.e., $\frac{dr_c}{dr_+} \leq 0$. This means that as the horizon radius $r_+$ increases the cosmological horizon decreases, in general. We restrict here to calculate some of the properties of the cosmological radius $u$.

We want to calculate the properties of $u$ at the neighborhood of the point $x = 1$, $y = 1$, $u = 1$, and find $\frac{du}{dx}$. Since this pertains to the large black hole, we use, as usual, the subscript 2 to refer to that solution, see Appendix C for details. One finds that

$$\left(\frac{du_2}{dx}\right)_{x=1_-} = \left(\frac{dy_2}{dx}\right)_{x=1_+} \quad (116)$$

and

$$\left(\frac{du_2}{dx}\right)_{x=1_+} = \left(\frac{dy_2}{dx}\right)_{x=1_-}, \quad (117)$$

where $1_-$ and $1_+$ mean that one is taking the limit $x \to 1$ from below, i.e., $x < 1$, and from above, i.e., $x > 1$, respectively. Using Eqs. (96), (108), (116), and (117) we can deduce some further specific properties of the cosmological horizon. They are that $(\frac{du_2}{dx})_{x=1_-} \to -\infty$ for $w \to 2$, and $(\frac{du_2}{dx})_{x=1_+} \to -2$ for $w \to \infty$, and that $(\frac{du_2}{dx})_{x=1_-} \to \infty$ for $w \to 2$, and $(\frac{du_2}{dx})_{x=1_+} \to 0$ for $w \to \infty$.

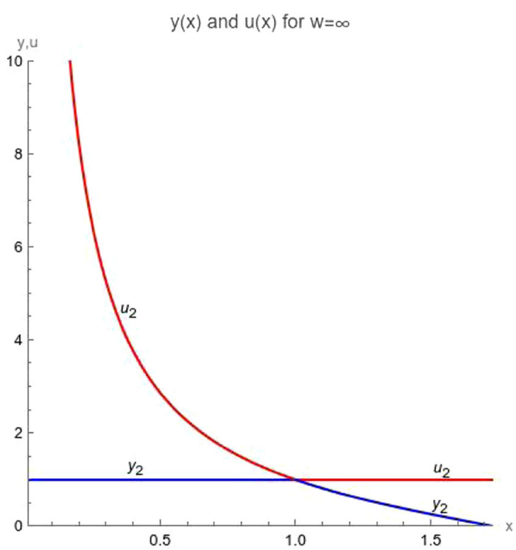


FIG. 5. Plots of $\frac{r_{+2}}{R}$ and $\frac{r_{c2}}{R}$ as a function of $\sqrt{\Lambda R^2}$ for $RT = \infty$, i.e., essentially infinite reservoir temperature. For plotting purposes it is defined $x \equiv \sqrt{\Lambda R^2}$, $y_2 \equiv \frac{r_{+2}}{R}$, $u_2 \equiv \frac{r_{c2}}{R}$, and $w \equiv 4\pi RT$. So this is the case $w = \infty$. Note that $y_2$ and $u_2$ exchange character at the bifurcation point $x = 1$. See text for details.

This interchange of equations as displayed in Eqs. (116) and (117), is mostly clearly seen in the case $w = \infty$, i.e., $RT \to \infty$. Indeed, the case $w \to \infty$ can be solved exactly as we have seen in Eq. (114). Thus, for $0 \leq x^2 \leq 1$, one has $y_2 = 1$ and $u_2 = -\frac{1}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}}$, whereas for $1 \leq x^2 \leq 3$, $y_2 = -\frac{1}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}}$ and $u_2 = 1$, see Fig. 5.

In brief, the black hole horizon equation for $\Lambda R^2 < 1$ turns into the cosmological horizon equation for $\Lambda R^2 > 1$, and the cosmological horizon equation for $\Lambda R^2 < 1$ turns into the black hole horizon equation for $\Lambda R^2 > 1$.

## VII. CONCLUSIONS

We have taken on the problem of understanding the Schwarzschild–de Sitter black hole thermodynamically. For that we have put the black hole in a cavity of radius $R$ surrounded by a heat reservoir and kept at temperature $T$. This structure allowed the thermodynamic problem to be solved within the canonical formalism and the Euclidean path integral to quantum gravity. Moreover, it led naturally into two other spaces, namely, hot de Sitter and Nariai cosmological spaces, besides the initial Schwarzschild–de Sitter black hole.

There are new results. One is that the range of the relevant parameter $\Lambda R^2$ is extendable up to 3, i.e., $0 \leq \Lambda R^2 \leq 3$. It was also found that to properly treat the problem one has to divide this range into three ranges, $0 \leq \Lambda R^2 < 1$, $\Lambda R^2 = 1$, $1 < \Lambda R^2 \leq 3$.

On the lower side of $\Lambda R^2$, $0 \leq \Lambda R^2 < 1$, York's solution for pure Schwarzschild is automatically incorporated when $\Lambda R^2 = 0$, appearing first for $RT = \frac{\sqrt{27}}{8\pi}$, with a coincident black hole horizon radius $r_{+1} = r_{+2} = \frac{2}{3}R$. For higher $\Lambda R^2$ the coincident black hole horizon radius gets increased values for some lower $RT$. A heuristic understanding of this behavior has been given. Changing the values and $\Lambda R^2$ and $RT$ one obtains either two thermodynamics solutions $r_{+1}$ small and $r_{+2}$ large, the first thermodynamically unstable and the second stable, or no solutions in which case one is in the presence of hot de Sitter.

At the intermediate value of $\Lambda R^2$, $\Lambda R^2 = 1$, the small $r_{+1}$ unstable black hole exists. More interestingly, the large $r_{+2}$ black hole is now the solution $r_{+2} = R$ and opens out into a spectrum of a beautiful set of Nariai universes that can have all temperatures in the range $\frac{1}{2\pi} \leq RT \leq \infty$. The ensemble data $R$ and $T$ now determine the location of the boundary $Z$ in the Nariai universe rather than the black hole radius solution. The Nariai universe is thermodynamically neutrally stable.

On the higher side of $\Lambda R^2$, $1 < \Lambda R^2 \leq 3$, unexpected black hole solutions also arise. The small $r_{+1}$ unstable black hole exists without changing character. The large $r_{+2}$ stable solutions interchange the role of black hole $r_{+2}$ and cosmological $r_{c2}$ horizons, and the maximum value $r_{+2}$ can have is attained for infinite temperature, i.e., $RT = \infty$, and is less than one changing as $\Lambda R^2$ changes up to 3. The case at the end of the range, $\Lambda R^2 = 3$, only exists for infinite temperature, and represents a reservoir filled with de Sitter space inside, except at the center, where there is a black hole with zero horizon radius $r_+ = 0$, i.e., a naked singularity.

Another important result is that increasing the temperature from zero, i.e., increasing $RT$ from zero, one finds that the first solution that appears for the whole range $0 \leq \Lambda R^2 \leq 3$ has temperature $RT = \frac{1}{2\pi}$ and is a solution with $\Lambda R^2 = 1$, and radius $r_{+1} = r_{+2} = R$, i.e., it is the coincident Nariai solution. As one increases $RT$ solutions with lower and higher $\Lambda R^2$ peal out.

Yet another interesting finding is related to phase transitions. In the $\Lambda R^2 < 1$ case, in particular for $\Lambda R^2 \ll 1$, the possible thermodynamic phases that appear as one increases the temperature from zero are hot de Sitter only, hot de Sitter favored in relation to the Schwarzschild–de Sitter black hole, hot de Sitter coexisting equally with the Schwarzschild–de Sitter black hole, the Schwarzschild–de Sitter black hole favored in relation to hot de Sitter, and the Schwarzschild–de Sitter black hole alone. This latter case comes out when one considers a high enough temperature so that there is sufficient energy in the cavity to surpass the Buchdahl bound and presumably the system collapses. There are topology changes when the system performs a phase transition from hot de Sitter to Schwarzschild–de Sitter and vice versa as it is allowed in this formalism, since it is in a semiclassical approximation to quantum gravity, and in quantum gravity topology changes of the space can happen. In the $\Lambda R^2 = 1$ case, i.e., the Nariai universe, one has that the possible thermodynamic phases that appear as one increases the temperature from zero are hot de Sitter only, hot de Sitter favored in relation to the Nariai universe, hot de Sitter coexisting equally with the Nariai universe, the Nariai universe favored in relation to hot de Sitter. Here there is no phase with only the Nariai universe. There are topology changes when the system performs a phase transition from hot de Sitter to the Nariai universe and vice versa as it is allowed in this formalism. In the $\Lambda R^2 > 1$ case, phase transitions between hot de Sitter and the Schwarzschild–de Sitter black hole, can also be explored.

Thus, we have given a full thermodynamic description of the Schwarzschild–de Sitter black hole space in a finite size cavity.

## ACKNOWLEDGMENTS

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## APPENDIX A: THE SCHWARZSCHILD–DE SITTER AND NARIAI SPACETIMES: BASICS

### 1. The Schwarzschild–de Sitter spacetime

The line element of the Schwarzschild–de Sitter spacetime in spherical coordinates $(t, r, \theta, \phi)$ is given by

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \text{(A1)}$$

where metric potential $V(r)$ has the form

$$V(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \quad \text{(A2)}$$

with $m$ being the spacetime mass and $\Lambda$ the cosmological constant which we consider positive, $\Lambda > 0$, see also Fig. 6. The coordinate ranges are $-\infty < t < \infty$, $r_+ < r < r_c$, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$, where $r_+$ and $r_c$ are the black hole and cosmological horizons of the spacetime, respectively. The spacetime has topology $R^4$. These coordinates can be further extended, e.g., through a Kruskal-Szekeres extension, but it is not necessary here to do so.

The equation $V(r) = 0$ is

$$r - 2m - \frac{\Lambda r^3}{3} = 0, \quad \text{(A3)}$$

which can be written as $m = \frac{r}{2}(1 - \frac{\Lambda r^2}{3})$. Note that Eq. (A3) has at most two positive real roots. When it has roots, one root corresponds to the black hole horizon $r_+$, with

$$r_+ = r_+(m, \Lambda). \quad \text{(A4)}$$


FIG. 6. A drawing of the black hole with its event horizon $r_+$ and of the cosmological horizon $r_c$ in the Schwarzschild–de Sitter spacetime. The topology of the 3-space is $R^3$.

The radius of the black hole horizon obeys the inequality $0 \leq r_+ \leq \frac{1}{\sqrt{\Lambda}}$. The explicit form of $r_+(m, \Lambda)$, can be given since Eq. (A3) is a cubic equation for $r$, but it is cumbersome and there is no need to present it explicitly.

Given $r_+$, then $V(r)$ in Eq. (A2) can be written as $V(r) = 1 - \frac{r_+(1-\frac{\Lambda r_+^2}{3})}{r} - \frac{\Lambda r^2}{3}$ or $V(r) = 1 - \frac{r_+}{r} - \frac{\Lambda}{3r}(r^3 - r_+^3)$, where use of Eq. (A3) has been made. The other root corresponds to the cosmological horizon $r_c$,

$$r_c = r_c(m, \Lambda), \tag{A5}$$

with $r_c \geq r_+$. The radius of the cosmological horizon obeys the inequality $\frac{1}{\sqrt{\Lambda}} \leq r_c \leq \sqrt{\frac{3}{\Lambda}}$. The explicit form of $r_c(m, \Lambda)$, can be given since Eq. (A3) is a cubic equation for $r$, but it is cumbersome and there is no need to present it explicitly.

On the other hand, since there are two roots of Eq. (A3), $r_+$ and $r_c$, one can write

$$V(r) = \frac{\Lambda}{3r}(r - r_+)(r_c - r)(r + r_+ + r_c), \tag{A6}$$

with

$$r_c^2 + r_c r_+ + r_+^2 = \frac{3}{\Lambda}, \tag{A7}$$

and

$$r_c r_+(r_c + r_+) = \frac{6m}{\Lambda}. \tag{A8}$$

Thus $\Lambda$ and $m$ can be swapped for $r_+$ and $r_c$. Moreover, Eq. (A7) can be written as $r_c^2 + r_c r_+ + r_+^2 - \frac{3}{\Lambda} = 0$ which is a quadratic either for $r_c$ in terms of $r_+$ or vice versa. The solution is

$$r_c = -\frac{r_+}{2} + \frac{r_+}{2}\sqrt{\frac{12 - 3\Lambda r_+^2}{\Lambda r_+^2}}, \tag{A9}$$

or $r_c = -\frac{r_+}{2} + \sqrt{\frac{3}{\Lambda} - \frac{3}{4} r_+^2}$. Of course, the equation $r_c^2 + r_c r_+ + r_+^2 - \frac{3}{\Lambda} = 0$ is also a quadratic for $r_+$ which gives $r_+$ in terms of $r_c$ in the same form of Eq. (A9) with the roles reversed. Another way of obtaining this is that since there are two solutions of Eq. (A3), $r_+$ and $r_c$, one has from Eq. (A3) that $r_c - 2m - \frac{\Lambda r_c^3}{3} = 0$ and $r_+ - 2m - \frac{\Lambda r_+^3}{3} = 0$. Subtracting one equation from the other one eliminates $2m$ to get $(r_c - r_+) - \frac{\Lambda}{3}(r_c^3 - r_+^3) = 0$. Since $(r_c^3 - r_+^3) = (r_c - r_+)(r_c^2 + r_c r_+ + r_+^2)$, one finds that $1 - \frac{\Lambda}{3}(r_c^2 + r_c r_+ + r_+^2) = 0$, i.e., $r_c^2 + r_c r_+ + r_+^2 - \frac{3}{\Lambda} = 0$. One can then write the solution of $r_c$ in term of $r_+$ as $r_c = -\frac{r_+}{2} + \sqrt{\frac{3}{\Lambda} - \frac{3}{4} r_+^2}$, which is Eq. (A9). In addition, Eq. (A8) with the help of Eq. (A7) can be written as $r_c^2 + r_c r_+ - \frac{2m r_+^2}{r_+ - 2m} = 0$, which is a quadratic either for $r_c$ in terms of $r_+$ or vice versa. The solution is

$$r_c = -\frac{r_+}{2} + \frac{r_+}{2}\sqrt{\frac{r_+ + 6m}{r_+ - 2m}}, \tag{A10}$$

or $r_c = \frac{r_+}{2}(\sqrt{\frac{r_+ + 6m}{r_+ - 2m}} - 1)$. Another way of obtaining this is that since there are two solutions of Eq. (A3), $r_+$ and $r_c$, one has from Eq. (A3) that $r_c r_+^3 - 2m r_+^3 - \frac{\Lambda r_c^3 r_+^3}{3} = 0$ and $r_+ r_c^3 - 2m r_c^3 - \frac{\Lambda r_+^3 r_c^3}{3} = 0$. Subtracting one equation from the other one eliminates $\Lambda$ to get $-r_+ r_c(r_c^2 - r_+^2) + 2m(r_c^3 - r_+^3) = 0$. Thus, $-r_+ r_c(r_c - r_+)(r_c + r_+) + 2m(r_c - r_+)(r_c^2 + r_c r_+ + r_+^2) = 0$, i.e., $-r_+ r_c(r_c + r_+) + 2m(r_c^2 + r_c r_+ + r_+^2) = 0$. So, $r_c^2 + r_c r_+ - \frac{2m r_+^2}{r_+ - 2m} = 0$, with solution $r_c = -\frac{r_+}{2} + \frac{r_+}{2}\sqrt{\frac{r_+ + 6m}{r_+ - 2m}}$ which is Eq. (A10).

Note that $r_c = r_+$ for $r_c = r_+ = \frac{1}{\sqrt{\Lambda}} = 3m$, and this happens when $9m^2\Lambda = 1$. In this limit one can have either an extremal Schwarzschild–de Sitter spacetime, where the two regions off the extremal horizon, the inside and the outside regions, are time dependent, or the Nariai spacetime where the topology of the space changes, see below. For $9m^2\Lambda > 1$ there are no horizons, the spacetime is asymptotically de Sitter, with a massive naked singularity at the center.

### 2. The Nariai spacetime

The line element of the Nariai spacetime in spherical coordinates $(t, z, \theta, \phi)$, is given by

$$ds^2 = -V(z)dt^2 + \frac{dz^2}{V(z)} + \frac{1}{\Lambda}(d\theta^2 + \sin^2\theta d\phi^2), \tag{A11}$$

where the metric potential $V(z)$ has the form

$$V(z) = 1 - \Lambda z^2, \tag{A12}$$

with $\Lambda$ being the cosmological constant which we consider positive, $\Lambda > 0$, see also Fig. 7. The coordinate ranges are $-\infty < t < \infty$, $z_+ < z < z_c$, where $z_+$ and $z_c$ are the horizons of the spacetime, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$. Note that the spacetime has topology $R^2 \times S^2$. These coordinates can be further extended, e.g., through a Kruskal-Szekeres extension, but it is not necessary here to do so.

The equation $V(z) = 0$ is

$$1 = \Lambda z^2. \tag{A13}$$

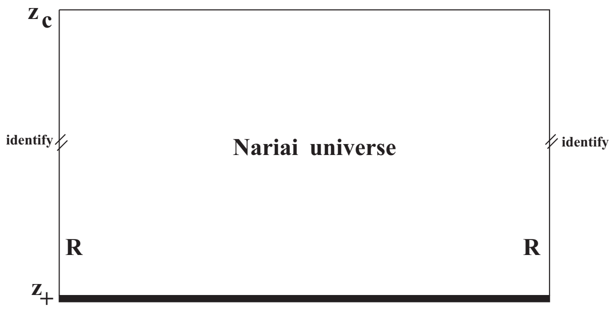


FIG. 7. A drawing of the Nariai universe with its event horizon $z_+$ and its cosmological horizon $z_c$. The cylindrical character of the Nariai space is made clear after identifying the two vertical end lines of the diagram. The radius $R$ is the radius of the cylinder. The labeling of the two horizons $z_+$ and $z_c$ is convention as they are of the same type. The topology of the 3-space is $R \times S^2$.

In general Eq. (A13) has two roots. One root corresponds to the black hole horizon $z_+$, by convention, with

$$z_+ = -\frac{1}{\sqrt{\Lambda}}. \tag{A14}$$

The other root corresponds to the cosmological horizon $z_c$, by convention, with

$$z_c = +\frac{1}{\sqrt{\Lambda}}, \tag{A15}$$

with $z_c \geq z_+$.

Note that when

$$\Lambda = \infty, \tag{A16}$$

both roots coincide,

$$z_+ = z_c = 0, \tag{A17}$$

and in this extremal limit the spacetime disappears.

Note also that when $\Lambda = 0$, the $z_+$ and $z_c$ have roots

$$z_+ = -\infty, \qquad z_c = +\infty, \tag{A18}$$

and in this limit there are no horizons, one is in the presence of simply a Minkowski space with two coordinates possibly wrapped around, i.e., one can have a torus, a cylinder, or an infinite plane. This can be seen from the angular part of Eq. (A11) by doing $\theta = \sqrt{\Lambda}x$, $\phi = \sqrt{\Lambda}y$, and then doing $\Lambda \to 0$ to give $ds^2 = -dt^2 + dz^2 + dx^2 + dy^2$, where the range of the coordinates $x$ and $y$ is $0 \leq x \leq x_1$ and $0 \leq y \leq y_1$, respectively, with $x_1$ and $y_1$ having any value one chooses from a finite value to infinite.

## APPENDIX B: THE NARIAI LIMIT FROM THE SCHWARZSCHILD–DE SITTER SPACE IN A THERMODYNAMIC SETTING

In order to understand the limiting thermodynamic process of obtaining a Nariai space from a Schwarzschild–de Sitter space in the limit $\Lambda R^2 = 1$, $\frac{r_+}{R} = 1$, and $\frac{r_c}{R} = 1$, it is useful to resort to Fig. 8.

To be complete we give again the Schwarzschild–de Sitter Euclidean line element

$$ds^2 = V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$0 \leq t < \beta_+^{\rm H}, \qquad r_+ \leq r \leq R, \tag{B1}$$

where the metric potential $V(r)$ has the form

$$V(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}. \tag{B2}$$

Note that the range of coordinates of Euclidean time $t$ is $0 \leq t < \beta_+^{\rm H}$, where $\beta_+^{\rm H}$ is the period of the coordinate such that the line element given by Eqs. (B1) and (B2) has no conical singularities. The relation with the Hawking temperature $T_+^{\rm H}$ is $\beta_+^{\rm H} = \frac{1}{T_+^{\rm H}}$. In addition, due to the reservoir at radius $R$ the range of coordinates of $r$ is now $r_+ \leq r \leq R$. Note now that this is Euclidean space and the topology of this space is $R^2 \times S^2$. There are two horizon for $V(r) = 0$, see Eq. (B2), the black hole horizon $r_+$ and the cosmological horizon $r_c$.

The local Tolman temperature $T$ of the heat reservoir at $R$ has to be treated with care. It is

$$T = \frac{T_+^{\rm H}}{\sqrt{V(R)}}, \tag{B3}$$

with $T_+^{\rm H}$ being the Hawking temperature given by $T_+^{\rm H} = \frac{\kappa_+}{2\pi}$, and $\kappa_+$ being the surface gravity of the black hole horizon and $V(R)$ is the potential at $R$. For the line element of Eq. (B1) one has $\kappa_+ = \frac{1}{2}V'(r_+)$, where a prime means derivative with respect to $r$, and so the Hawking temperature for the black hole is $T_+^{\rm H} = \frac{1}{4\pi}\left(\frac{dV}{dr}\right)_{r_+}$. Using Eq. (B2) we get

$$T_+^{\rm H} = \frac{1}{4\pi r_+}(1 - \Lambda r_+^2). \tag{B4}$$

The potential at $R$ is

$$V(R) = 1 - \frac{2m}{R} - \frac{\Lambda R^2}{3}. \tag{B5}$$

The Nariai solution can be found from the Schwarzschild–de Sitter solution in the limit that the two

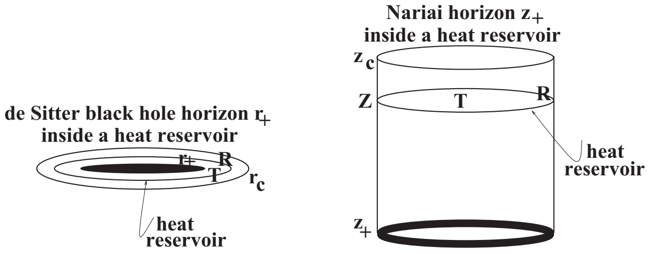


FIG. 8. Drawings of a black hole horizon inside a heat reservoir in the Schwarzschild–de Sitter black hole space on the left, and of the Nariai universe with one of its horizons inside a heat reservoir on the right. Specifically, on the left the drawing shows a slant view of a $t =$ constant and $\theta =$ constant space of the black hole horizon $r_+$ inside a heat reservoir at temperature $T$ and radius $R$ in the Schwarzschild–de Sitter geometry. Outside $R$ there is the cosmological horizon $r_c$. The Euclideanized space and its boundary have $R^2 \times S^2$ and $S^1 \times S^2$ topologies, respectively, with the $S^1$ subspace having proper length $\beta = \frac{1}{T}$. On the right, the drawing shows a slant view of a $t =$ constant and $\theta =$ constant space of the horizon $z_+$ inside a heat reservoir at temperature $T$, with cylindrical radius $R$, and situated at $Z$ in the Nariai universe geometry. Outside $Z$ there is the cosmological horizon $z_c$. The Euclideanized space and its boundary have $R^2 \times S^2$ and $S^1 \times S^2$ topologies, respectively, with the $S^1$ subspace having proper length $\beta = \frac{1}{T}$. In pictorial terms it is clear how the Schwarzschild–de Sitter black hole space originates the Nariai universe, with the slant view of the Schwarzschild–de Sitter space helping in the visualization of the process. Indeed, if the two Schwarzschild–de Sitter different horizon radii, $r_+$ and $r_c$, are squeezed into the heat reservoir radius $R$, then the two horizons pinch off to form the Nariai universe with the heat reservoir still at temperature $T$, with cylindrical radius $R$, and situated now at some $Z$ in-between the two displaced distinct horizons $z_+$ and $z_c$. See text for more details.

horizons $r_+$ and $r_c$ coincide. Here, we have a heat reservoir at $R$ that acts for the inside region which is a cavity with the black hole. This heat reservoir at $R$ is in-between $r_+$ and $r_c$, and thus the limit is such that $r_+$, $R$, and $r_c$ coincide. Now, if we do $r_+ \to R$ then Eq. (B3), $T = \frac{T_+^H}{\sqrt{V(R)}}$, gives at face value that the heat reservoir is at very high temperature since $V(R) \to 0$. But, there is a way to have $T$ finite with $r_+ \to R$, and this is to do concomitantly $T_+^H \to 0$. Since $T_+^H = \frac{1}{4\pi r_+}(1 - \Lambda r_+^2)$, see Eq. (B4), $T_+^H \to 0$ means $1 - \Lambda r_+^2 \to 0$. Since $r_+ \to R$, we put,

$$\frac{r_+}{R} = 1 - \varepsilon, \tag{B6}$$

$\varepsilon \ll 1$, which implies

$$\sqrt{\Lambda R^2} = 1 - \delta, \tag{B7}$$

where $\delta \equiv \frac{1}{2}(1 - \Lambda r_+^2) - \varepsilon$, $\delta \ll 1$, and $\frac{\varepsilon}{\delta}$ is of order one. Thus, $R$ and the length scale $\frac{1}{\sqrt{\Lambda}}$ are equal at zeroth order. Since $r_+ \to R \to \frac{1}{\sqrt{\Lambda}}$ then from Eq. (5) one has $\frac{r_c}{R} \to 1$, i.e., one has $r_c \to \frac{1}{\sqrt{\Lambda}}$, and since $\frac{1}{\sqrt{\Lambda}} \to R$, then $r_c \to R$. From the expansion of Eqs. (B6) and (B7) one has that Eq. (B4) gives

$$T_+^H = \frac{\delta + \varepsilon}{2\pi R}, \tag{B8}$$

in first order, as required. To understand the behavior of $V(r)$ near $R$ in this limit we use Eq. (B5). It gives

$$V(R) = \varepsilon(\varepsilon + 2\delta), \tag{B9}$$

in first order. Thus, Eq. (B3), i.e., $T = \frac{T_+^H}{\sqrt{V(R)}}$ gives

$$T = \frac{1}{2\pi R} \frac{\frac{\varepsilon}{\delta} + 1}{\sqrt{\frac{\varepsilon}{\delta}(\frac{\varepsilon}{\delta} + 2)}}, \tag{B10}$$

which given a $RT$, i.e., given a $T$, implies some $\frac{\varepsilon}{\delta}$, and so is finite and consistent.

But we have not finished. The metric potential $V(R)$ given in Eq. (B1) vanishes in these coordinates, and the line element of Eq. (B1) loses sense. So, we have to pay attention to this limit indeed with care. It is the Nariai limit with a reservoir $R$ in the middle, see Fig. 8, an interesting case. Expanding the metric potential $V(r)$ near $r_+$ in a Taylor series gives $V(r) = (\frac{dV}{dr})_{r=r_+}(r - r_+) + \frac{1}{2}(\frac{d^2V}{dr^2})_{r=r_+}(r - r_+)^2$ plus higher order terms. Recall that $(\frac{dV}{dr})_{r=r_+} = 4\pi T_+^H$ and find $(\frac{d^2V}{dr^2})_{r=r_+} = -\frac{2}{r_+^2}$. Now, in this limit $r_+ = R$, so $(\frac{d^2V}{dr^2})_{r=r_+} = -\frac{2}{R^2}$. So,

$$V(r) = 4\pi T_+^H(r - r_+) - \frac{1}{R^2}(r - r_+)^2, \tag{B11}$$

plus higher order terms. Make the coordinate transformations $(t, r) \to (\bar{t}, \bar{z})$ as

$$r - r_+ = 4\pi T_+^H R^2 \sin^2\left(\frac{1}{2}\frac{\bar{z}}{R}\right), \quad t = \frac{\bar{t}}{2\pi T_+^H R}, \tag{B12}$$

with $0 \leq t \leq \frac{1}{T_+^H}$ corresponding to $0 \leq \bar{t} \leq 2\pi R$, and $r_+ \leq r \leq R$ corresponding to $0 \leq \bar{z} \leq \bar{Z}$. Then obtain from Eq. (B2) that

$$V(r) = V(r - r_+) = V(\bar{z}) = (2\pi T_+^H R)^2 \sin^2\left(\frac{\bar{z}}{R}\right), \tag{B13}$$

and from Eq. (B1) obtain the line element,

$$ds^2 = \sin^2\left(\frac{\bar{z}}{R}\right) d\bar{t}^2 + d\bar{z}^2 + R^2 d\Omega^2, \tag{B14}$$

which is a form of the Nariai line element. Note now from Eq. (B13) that $V(R) = V(R - r_+) = V(\bar{Z}) = (2\pi T_+^{\rm H} R)^2 \sin^2(\frac{\bar{Z}}{R})$, and so from Eq. (B3), i.e., $T = \frac{T_+^{\rm H}}{\sqrt{V(R)}}$, one has

$$T = \frac{1}{2\pi R \sin(\frac{\bar{Z}}{R})}. \tag{B15}$$

This means that the ensemble boundary values $T$ and $R$ specify automatically the maximum value for $\bar{z}$, namely $\bar{Z}$. So, for the ensemble with boundary data one has $0 < t < 2\pi R$, $0 \le \bar{z} \le \bar{Z}$, $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$. In turn $0 \le \bar{Z} \le R\pi$. Note that $\bar{z} = 0$ and $\bar{z} = R\pi$ are horizons.

We can continue further. Let us make another coordinate transformation, $z = R\cos\frac{\bar{z}}{R}$, Then

$$ds^2 = V(z)dt^2 + \frac{dz^2}{V(z)} + R^2 d\Omega^2, \tag{B16}$$

where we have dropped the bar in $\bar{t}$. Then, the metric potential $V(z)$ is

$$V = 1 - \frac{z^2}{R^2}, \tag{B17}$$

the one given in Eq. (A12). So, for the ensemble with boundary data one has $0 < t < 2\pi R$, $-R < z < Z$, $0 \le \theta \le \pi, 0 \le \phi < 2\pi$. In turn $-R \le Z \le R$. Note that $z = -R$ and $z = R$ are horizons. The line element given in Eqs. (B16) and (B17) corresponds to a two-dimensional de Sitter space times a sphere, the topology is $R^2 \times S^2$. It has two horizons, we label $z_+ = -R$ and $z_{\rm c} = R$, but now they are just labels, since the two horizons have the same character.

We have that the heat reservoir temperature $T$ is now

$$T = \frac{T_{+\rm Nariai}^{\rm H}}{\sqrt{V(Z)}}, \tag{B18}$$

with $T_{+\rm Nariai}^{\rm H}$ being the Hawking temperature given by $T_{+\rm Nariai}^{\rm H} = \frac{\kappa_+}{2\pi}$, and $\kappa_+$ being the surface gravity of the black hole horizon. For the metric given in Eq. (B16) one has $\kappa_+ = \frac{1}{2}V'(z_+)$, and so the Hawking temperature for $+$ horizon is $T_{+\rm Nariai}^{\rm H} = \frac{1}{4\pi}(\frac{dV}{dz})_{z_+}$. Using Eqs. (B16) and (B17) we get

$$T_{+\rm Nariai}^{\rm H} = \frac{1}{2\pi R}. \tag{B19}$$

So, from Eqs. (B18) and (B19) one finds

$$T = \frac{1}{2\pi R\sqrt{1 - \frac{Z^2}{R^2}}}, \tag{B20}$$

where $Z$ is the boundary of $Z$. For a given $T$ at the boundary, for the ensemble, there are two solutions of Eq. (B20) in general, namely, one for $Z$ between $-R$ and 0 and the other for $Z$ between 0 and $R$. These two solutions may be thought of yielding different physical situations. Note also that in Schwarzschild–de Sitter the two solutions were for the horizon $r_+$, one small $r_{+1}$ the other large $r_{+2}$, here the two solutions are not for the horizons but instead for the boundary $Z$, one $Z_1$, the other $Z_2$, with

$$Z_1 = -R\sqrt{1 - \frac{1}{(2\pi RT)^2}}, \tag{B21}$$

and

$$Z_2 = R\sqrt{1 - \frac{1}{(2\pi RT)^2}}. \tag{B22}$$

We are free to choose where to put the boundary, we have two choices, either $Z_1$ or $Z_2$, noting that $z_+$ and $z_{\rm c}$ can also be exchanged if one wants. Exchanging $Z_1$ with $Z_2$ and concomitantly exchanging $z_+$ with $z_{\rm c}$ reverses to the original situation. The boundary is at $Z_1$ or $Z_2$, has cylindrical radius $R$, and acts as a reservoir for the inner region, i.e., $z \le Z_1$ or $z \le Z_2$ depending on where we choose to put the boundary. Note, in addition that the reservoir instead of being two-dimensional with space topology $S^2$ as in the de Sitter space, it is still two dimensional but now with topology $R \times S^1$, i.e., it is a cylinder.

From Eqs. (B21) and (B22) we see that for

$$RT < \frac{1}{2\pi}, \tag{B23}$$

there are no solutions for $Z_1$ or $Z_2$, so in this case the boundary in $Z$ does not exist, and so there is no Nariai solution for the thermodynamic problem in the canonical ensemble. The reason is clear if one thinks in thermal wavelengths, indeed when the temperatures are relatively very small, the associated thermal wavelengths are very long and there is no boundary Z that can accommodate them.

From Eqs. (B21) and (B22) we see that for

$$RT \ge \frac{1}{2\pi}, \tag{B24}$$

there are two Nariai solutions, one with horizon $z = -R$ and boundary $Z_1$, the other with the horizon still given by $z = -R$ but the boundary is given by $Z_2$, both boundaries to the same horizon can be picked up. When the equality sign holds there is one solution with $Z_1 = Z_2 = 0$, so in this case the boundary in $Z$ pops up in the middle, at

$z = 0$, and so $-R \leq z \leq 0$. The reservoir is at $Z_1 = Z_2 = 0$ and has cylindrical radius $R$ and the horizon is at $z = -R$.

Three comments are in order. The first comment is that the resulting metric inside the reservoir is in this case described by the Nariai universe as we have seen. The procedure of obtaining the Nariai metric as a limit from the Schwarzschild–de Sitter metric is known and has different versions, e.g., see [5,41]. However, we have done it here in a completely different manner and in a completely different context that is related naturally to thermodynamics. The procedure we used is similar to that described in [13] for the extremal limit of nonextremal electrically charged black holes. Here, in Eq. (B11), the potential expansion takes the form $V(r) = 4\pi T_+^{\rm H}(r - r_+) - \frac{1}{R^2}(r - r_+)^2$ instead of $V(r) = 4\pi T_+^{\rm H}(r - r_+) + \frac{1}{R^2}(r - r_+)^2$, i.e., a plus instead of the minus sign of [13] appears in the second term of the expansion. As a result, we have arrived at the Nariai metric instead of obtaining the Bertotti-Robinson metric as in [13]. The second comment, is that in the above considerations, we have discussed two completely different limits, one is the high-temperature limit and the other is the extremal Nariai limit. However, they can be combined if the boundary $Z_1 \to -R$. The third comment is that we have done the Nariai limit from below, i.e., from $\Lambda R^2 < 1$. If we do the limit $\Lambda R^2 \to 1$ from above, i.e., from $\Lambda R^2 > 1$, we get the same result, namely, the Nariai universe inside a heat reservoir in the canonical ensemble.

## APPENDIX C: DEDUCTION OF FORMULAS OF SEC. VI

Here, we deduce in detail some equations found in Sec. VI. Specifically, we want to deduce Eqs. (96), (100), (108), and (112), and study Eqs. (116) and (117).

For the derivation of Eqs. (96), (100), (108), and (112) it is convenient to shorten the notation and define the variables $\sqrt{\Lambda R^2}$ and $\frac{r_+}{R}$ as $x$ and $y$ variables, namely,

$$x \equiv \sqrt{\Lambda R^2}, \tag{C1}$$

$$y \equiv \frac{r_+}{R}. \tag{C2}$$

With these definitions we want to find Eqs. (96) and (100) of the $x^2 < 1$ case, and Eqs. (108) and (112) of the $x^2 > 1$ case.

$x^2 < 1$:

Let us deduce Eq. (96). We have to find the behavior of $y_2$, i.e., $r_{+2}$, at $x^2$ near 1, i.e., $\Lambda R^2$ near 1. From Eq. (85), or from Eq. (92), one finds that there is a solution $y_2(x)$ for each $x$. The maximum of the curve is at $x = 1$ and $y_2 = 1$, and the derivative there is not well defined. So, it is important to discuss the behavior and the properties of $y_2$ when $x \to 1$. For that we define infinitesimal quantities $\delta$ and $\varepsilon$, related to $x$ and $y$, respectively, by

$$x = 1 - \delta, \qquad y = 1 - \varepsilon, \tag{C3}$$

with $\delta \ll 1$ and $\varepsilon \ll 1$, both positive, and with $\frac{\varepsilon}{\delta}$ being a quantity of order one. Indeed, $\delta > 0$ for $x < 1$, and $\varepsilon \geq 0$ always. Then, from Eq. (85) one gets

$$w = \frac{2(\frac{\varepsilon}{\delta} + 1)}{\sqrt{\frac{\varepsilon}{\delta}(\frac{\varepsilon}{\delta} + 2)}}, \tag{C4}$$

valid in first order, in the order we are working. One sees that for $w$ finite $\frac{\varepsilon}{\delta}$ is finite, so generically here in this calculation $\varepsilon$ is indeed of the order $\delta$. From Eq. (86) one also gets

$$\frac{dy}{dx} = \frac{\varepsilon}{\delta}, \tag{C5}$$

also valid in the order we are working. We obtain from Eq. (C4) that $\frac{\varepsilon}{\delta} = \frac{w}{\sqrt{w^2-4}} - 1$ with $w \geq 2$. Thus, since here $\frac{dy}{dx} = \frac{\varepsilon}{\delta}$, Eq. (C5), one finds that

$$\frac{dy}{dx} = \frac{w}{\sqrt{w^2 - 4}} - 1, \quad w \geq 2, \tag{C6}$$

which is Eq. (96), the equation we wanted to deduce.

Let us deduce now Eq. (100). We want to find $y_1 = y_2$, the coincident solution for fixed $x$, and $w$ varying. This is when $\frac{dw}{dy} = 0$, see Eq. (89), i.e., $S(x, y) = 0$. From Eq. (90) we have then that the bifurcation points where $y_1 = y_2$ for $x =$ constant are given by Eq. (99). We are interested in finding $y_1 = y_2$ when $x \to 1$. Instead of working with Eq. (99) we work with Eq. (90) which is more direct. We find from $x = 1 - \delta$ of Eq. (C3), that Eq. (90) in this limit is $S = (y-1)^3(y^2+3y+4) - 4\delta(y-1)^2(y+1)(1+y+y^2) + 4\delta^2(2y^2+y^5)$. In addition, from $y = 1 - \varepsilon$ of Eq. (C3), Eq. (90) becomes then $S = 12\delta^2 - 24\delta\varepsilon^2 - 8\varepsilon^3$. As we have seen, the coincident root is given when $\frac{dw}{dy} = 0$, i.e., $S = 0$. Then, $12\delta^2 - 24\delta\varepsilon^2 - 8\varepsilon^3 = 0$, i.e., $\delta^2 - 2\delta\varepsilon^2 - \frac{2}{3}\varepsilon^3 = 0$. This is a quadratic in $\delta$, with solution given by $\delta = \sqrt{\frac{2}{3}\varepsilon^3 + \varepsilon^4} + \varepsilon^2$, i.e. $\delta = \varepsilon^{\frac{3}{2}}\sqrt{\frac{2}{3} + \varepsilon} + \varepsilon^2$. Since $\varepsilon \ll 1$ the dominant term gives

$$\delta = \left(\frac{2}{3}\varepsilon^3\right)^{\frac{1}{2}}. \tag{C7}$$

Thus, $\delta$ goes with $\varepsilon^{\frac{3}{2}}$ or, if one prefers, $\varepsilon$ goes with $\delta^{\frac{2}{3}}$, indeed $\varepsilon = (\frac{3}{2}\delta^2)^{\frac{1}{3}}$. Recall that $x = 1 - \delta$, and so $x^2 = 1–2\delta$ in this approximation, and that $y = 1 - \varepsilon$. Then, substituting $\delta$ for $x^2$ and $\varepsilon$ for $y$ in Eq. (C7), one has

$$x^2 = 1 - \left(\frac{8}{3}(1 - y)^3\right)^{\frac{1}{2}}. \tag{C8}$$

We see that Eq. (C8) is Eq. (99) in the limit $x \to 1$ and $y \to 1$. Inverting Eq. (C8) one has that the coincident solution takes the form

$$y_1 = y_2 = 1 - \left(\frac{3}{8}(1-x^2)^2\right)^{\frac{1}{3}}, \tag{C9}$$

valid for $1 - x^2 \ll 1$, which is Eq. (100), the equation we wanted to deduce.

$x^2 > 1$:

Let us deduce Eq. (108). We have to find $y_2$, i.e., $r_{+2}$. From Eq. (85), or from Eq. (106), one finds that there is a solution $y_2(x)$ for each $x$. The maximum of the curve is at $x = 1$ and $y_2 = 1$, and the derivative there it is not well defined. So, it is important to discuss the behavior and the properties of $y_2$ when $x \to 1$ from above. For that we define infinitesimal quantities $\bar{\delta}$ and $\bar{\varepsilon}$, related to $x$ and $y$, respectively, by

$$x = 1 + \bar{\delta}, \qquad y = 1 - \bar{\varepsilon}, \tag{C10}$$

with $\bar{\delta} \ll 1$ and $\bar{\varepsilon} \ll 1$, both positive, and with $\frac{\bar{\varepsilon}}{\bar{\delta}}$ being a quantity of order one. Indeed, $\bar{\delta} > 0$ for $x > 1$, and $\bar{\varepsilon} > 0$ always. Then, from Eq. (85) one gets

$$w = \frac{2(\frac{\bar{\varepsilon}}{\bar{\delta}} - 1)}{\sqrt{\frac{\bar{\varepsilon}}{\bar{\delta}}(\frac{\bar{\varepsilon}}{\bar{\delta}} - 2)}}, \tag{C11}$$

valid in first order, in the order we are working. One sees that for $w$ finite $\frac{\bar{\varepsilon}}{\bar{\delta}}$ is finite, so generically here in this calculation $\bar{\varepsilon}$ is of the order of $\bar{\delta}$. From Eq. (86) one also gets

$$\frac{dy}{dx} = -\frac{\bar{\varepsilon}}{\bar{\delta}}, \tag{C12}$$

also valid in the order we are working. Since $\bar{\delta} > 0$ and $\bar{\varepsilon} > 0$, we obtain from Eq. (C11) $\frac{\bar{\varepsilon}}{\bar{\delta}} = \frac{w}{\sqrt{w^2-4}} + 1$ with $w \geq 2$. Thus, we have from Eq. (C12) that

$$\frac{dy}{dx} = -\frac{w}{\sqrt{w^2 - 4}} - 1, \quad w \geq 2, \tag{C13}$$

which is Eq. (108), the equation we wanted to deduce.

Let us deduce now Eq. (112). We want to find $y_1 = y_2$, the coincident solution for fixed $x$, and $w$ varying. This is when $\frac{dw}{dy} = 0$, see Eq. (89), i.e., $S(x, y) = 0$. From Eq. (90) we have then that the bifurcation points where $y_1 = y_2$ for $x =$ constant are given by Eq. (111). We are interested in finding $y_1 = y_2$ when $x \to 1$ from above. Instead of working with Eq. (111) we work with Eq. (90) which is more direct. We find then, from Eq. (105) one has $y_\mathrm{e} = 1–2\bar{\delta}$, plus higher order. For that, let $x = 1 + \bar{\delta}$ as in Eq. (C10). Then, from Eq. (105) one has $y_\mathrm{e} = 1–2\bar{\delta}$, plus higher order terms. Now we have to work out the vicinity of $y_\mathrm{e}$. Then put $y = y_\mathrm{e} - \epsilon$ where again $y_\mathrm{e}$ is the root of the equation $y^2 + y + 1 - \frac{3}{x^2} = 0$, with solution given by Eq. (105), i.e., $y_\mathrm{e} = -\frac{1}{2} + \sqrt{3}\sqrt{\frac{1}{x^2} - \frac{1}{4}}$, and note that $\epsilon$ and $\bar{\varepsilon}$ are different quantities. Then, these two equations yield $y = 1–2\bar{\delta} - \epsilon$. From $x = 1 + \bar{\delta}$ of Eq. (C10) we find that Eq. (90) is $S = (y-1)^3(y^2+3y+4) + 4\bar{\delta}(y-1)^2(y+1)(1+y+y^2) + (2y^2+y^5)4\bar{\delta}^2$. From $y = 1–2\bar{\delta} - \epsilon$ we have just found, we find that Eq. (90) is now $S = 12\bar{\delta}^2 + 24\bar{\delta}\epsilon^2 - 8\epsilon^3$. The coincident root is given when $\frac{\partial w}{\partial y} = 0$, i.e., $S = 0$. Then, $12\bar{\delta}^2 + 24\bar{\delta}\epsilon^2 - 8\epsilon^3$, i.e., $\bar{\delta}^2 + 2\bar{\delta}\epsilon^2 - \frac{2}{3}\epsilon^3 = 0$. The solution is $\bar{\delta} = -\epsilon^2 + \sqrt{\epsilon^4 + \frac{2}{3}\epsilon^3}$. It turns out that equation $\frac{\partial w}{\partial y} = 0$ gives a self-consistent solution if $\bar{\delta} \ll \bar{\varepsilon}$, see below. Since $\epsilon \ll 1$ the dominant term gives $\bar{\delta} = (\frac{2}{3}\epsilon^3)^{\frac{1}{2}}$. We defined $\bar{\varepsilon}$ as $y = 1 - \bar{\varepsilon}$ and also had $y = 1–2\bar{\delta} - \epsilon$, so $\bar{\varepsilon} = 2\bar{\delta} + \epsilon$. But $\bar{\delta}$ goes with $\epsilon^{\frac{3}{2}}$, so in this order $\bar{\varepsilon} = \epsilon$, so that

$$\bar{\delta} = \left(\frac{2}{3}\bar{\varepsilon}^3\right)^{\frac{1}{2}}. \tag{C14}$$

Thus, $\bar{\delta}$ goes with $\bar{\varepsilon}^{\frac{3}{2}}$ or, if one prefers, $\bar{\varepsilon}$ goes with $\bar{\delta}^{\frac{2}{3}}$, indeed $\bar{\varepsilon} = (\frac{3}{2}\bar{\delta}^2)^{\frac{1}{3}}$. Recall that $x = 1 + \bar{\delta}$, and so $x^2 = 1 + 2\bar{\delta}$ in this approximation, and that $y = 1 - \bar{\varepsilon}$. Then, substituting $\bar{\delta}$ for $x^2$ and $\bar{\varepsilon}$ for $y$ in Eq. (C14), one has

$$x^2 = 1 + \left(\frac{8}{3}(1-y)^3\right)^{\frac{1}{2}}. \tag{C15}$$

We see that Eq. (C15) is Eq. (111) in the limit $x \to 1$ from above and $y \to 1$. Inverting Eq. (C15) one has that the coincident solution takes the form

$$y_1 = y_2 = 1 - \left(\frac{3}{8}(x^2-1)^2\right)^{\frac{1}{3}}, \tag{C16}$$

valid for $x^2 - 1 \ll 1$, which is Eq. (112), the equation we wanted to deduce.

Finally, we calculate the implications of Eqs. (116) and (117). We have defined $u$ as

$$u \equiv \frac{r_\mathrm{c}}{R}, \tag{C17}$$

as the cosmological radius in units of $R$, which from Eq. (A9) means that $u$ obeys

$$u = -\frac{y}{2} + \sqrt{\frac{3}{x^2} - \frac{3}{4}y^2}. \tag{C18}$$

Calculating $\frac{du}{dy}$ at the neighborhood of the point $x = 1$, $y = 1$, $u = 1$, and recalling that we are interested in the large black hole, i.e., the one with subscript 2, one finds from Eq. (C18), with the help of Eqs. (C5) and (C12), that

$$\left(\frac{du_2}{dx}\right)_{x=1_-} = \left(\frac{dy_2}{dx}\right)_{x=1_+} \tag{C19}$$

and

$$\left(\frac{du_2}{dx}\right)_{x=1_+} = \left(\frac{dy_2}{dx}\right)_{x=1_-}, \tag{C20}$$

where $1_-$ and $1_+$ means that one is taking the limit $x \to 1$ from below, i.e., $x < 1$, and from above, i.e., $x > 1$, respectively. We can now deduce some further specific properties of the cosmological horizon. In Eq. (C13) we found $(\frac{dy_2}{dx})_{x=1_+} = -\frac{w}{\sqrt{w^2-4}} - 1$ so that from Eq. (C19) we have $(\frac{du_2}{dx})_{x=1_-} = -\frac{w}{\sqrt{w^2-4}} - 1$. Thus, $(\frac{du_2}{dx})_{x=1_-} \to -\infty$ for $w \to 2$, and $(\frac{du_2}{dx})_{x=1_+} \to -2$ for $w \to \infty$. In Eq. (C6) we found $(\frac{dy_2}{dx})_{x=1_-} = \frac{w}{\sqrt{w^2-4}} - 1$ so that from Eq. (C20) we have $(\frac{du_2}{dx})_{x=1_+} = \frac{w}{\sqrt{w^2-4}} - 1$. Thus, $(\frac{du_2}{dx})_{x=1_+} \to \infty$ for $w \to 2$, and $(\frac{du_2}{dx})_{x=1_+} \to 0$ for $w \to \infty$. This was stated in the main text.


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*Correction:* A typographical error in Eq. (21) has been fixed.