New type of large-scale signature change in emergent modified gravity

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(Received 9 January 2024; accepted 13 March 2024; published 1 April 2024)

Emergent modified gravity presents a new class of gravitational theories in which the structure of space-time with Riemannian geometry of a certain signature is not presupposed. Relying on crucial features of a canonical formulation, the geometry of space-time is instead derived from the underlying dynamical equations for phase-space degrees of freedom together with a covariance condition. Here, a large class of spherically symmetric models is solved analytically for Schwarzschild-type black hole configurations with generic modification functions, using a variety of slicings that explicitly demonstrate general covariance. For some choices of the modification functions, a new type of signature change is found and evaluated. In contrast to previous versions discussed for instance in models of loop quantum gravity, signature change happens on timelike hypersurfaces in the exterior region of a black hole where it is not covered by a horizon. A large region between the horizon and the signature-change hypersurface may nevertheless be nearly classical, such that the presence of a signature-change boundary around Lorentzian space-time, or a Euclidean wall around the Universe, is consistent with observations provided signature change happens sufficiently far from the black hole.

DOI: 10.1103/PhysRevD.109.084001

I. INTRODUCTION

One of the motivations of modified or quantum gravity is that a consistent theory that goes beyond general relativity at large curvature may be able to solve important problems of the classical theory, such as the presence of singularities. Solutions to these problems may result from a classical alternative to general relativity, or from a quantization. In the former case, it is necessary to circumvent the rigid status of general relativity as a largely unique low-curvature theory of gravity and space-time geometry. In the latter, one needs good control on possible effects expected from quantization. Depending on the approach, there are various expectations as to what quantum gravity might entail, such as fundamental discreteness of space and time that could undo the usual continuum picture of a space-time manifold equipped with Riemannian geometry. Because a fully discrete description of space-time is rather intractable, it is preferable to proceed more carefully and see how the classical continuum theory could be modified by quantum or other corrections as the curvature scale is increased. Such a treatment has the advantage that Riemannian geometry (and therefore an unambiguous meaning of the curvature scale) remains applicable at least for some time on the approach to large curvature. It also makes it possible to retain a well-defined meaning of black holes through the usual definition of a horizon.

In this way, one is led to effective line elements that, on one hand, may include quantum or other modifications and, on the other hand, make it possible to apply the usual concepts of curvature and black holes. However, the application of line elements also means that covariance must be maintained strictly at this level, even if one expects that discrete or other fundamental space-time structures may ultimately break continuous symmetries. A line element, classical or effective, is well-defined only if its metric coefficients transform in such a way that any change of coordinate differentials is compensated for exactly, implying an invariant line element. This property cannot be tested if one works in a preferred gauge or slicing of space-time, as often done in such models.

Within spherically symmetric models, a complete set of covariant theories with second-order field equations has recently been derived as emergent modified gravity [1,2], building on previous explicit models in which covariance could be demonstrated [3,4]. Even in vacuum, these theories are more general than just the classical theory even though they have the same derivative order of equations of motion. The usual restrictions on invariant actions based on curvature invariants can be circumvented by exploiting subtle features of the canonical formulation of space-time theories, which turn out to be more general than action principles because they do not require assumptions about the space-time volume element. In particular,

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within emergent modified gravity it is possible to have a space-time geometry determined by a function of the fundamental fields that follows from an evaluation of the theory (and is in this sense emergent), in contrast to action principles that require a fundamental metric or tetrad field. Emergent modified gravity is formulated canonically through a Hamiltonian that may be Legendre transformed to a Lagrangian. However, the derivation of the emergent metric makes use of further properties, in particular of Poisson brackets of the canonical constraints, that are not available in a Lagrangian formulation or an action principle. (The Lagrangian formulation has gauge transformations that are equivalent to canonical gauge transformations generated by the constraints on shell, when the constraints and the equations of motion hold. However, a derivation of the emergent metric is possible only off shell, on which level the canonical and Lagrangian gauge transformations are not equivalent.) Therefore, to the best of our knowledge, it is not possible to reproduce the effects discussed here based on an action formulation, even if one restricts attention to regions of fixed metric signature.

The available set of models is also more general than the specific examples analyzed in [3,4], in particular because some of the parameter choices make it possible to have a covariant form of signature change. The rest of this paper presents a detailed analysis of such models and implications for black holes. Remarkably, we will find closed-form analytical solutions even in the presence of two generic modification functions. We therefore derive geometrical properties that are universally valid within a large class of emergent modified gravity. The underlying theory and equations are presented in Sec. II, followed by detailed derivations of solutions of Schwarzschild and Painlevé-Gullstrand type in Sec. III, and a discussion of causal structure focussing on the signature-change hypersurface in Sec. IV. Unlike in previous examples in models of loop quantum gravity [5,6], signature change in the present examples happens on a timelike hypersurface located at low curvature. It is therefore important that our solutions are valid for a sufficiently large class of modification function that may imply this new type of signature change and, at the same time, are consistent with observations in a large part of space-time outside of the black hole horizon. We will see that this is indeed possible.

II. EMERGENT MODIFIED GRAVITY OF SPHERICALLY SYMMETRIC MODELS

A generic spherically symmetric line element can be written as

$$\mathrm{d}s^2 = -N(t,x)^2\mathrm{d}t^2 + \frac{e_2(t,x)^2}{e_1(t,x)}(\mathrm{d}x + M(t,x)\mathrm{d}t)^2 + e_1(t,x)\mathrm{d}\Omega^2 \tag{1}$$

with the lapse function N, the shift vector M, and a spatial metric derived from components e_1 and e_2 of a densitized triad. (Without loss of generality, we assume $e_1 > 0$, fixing the orientation of the spatial triad.) On a phase space given by the fields (e_1, e_2) and canonically conjugate momenta (k_1, k_2), the dynamics is governed completely by the diffeomorphism constraint

$$D[M] = \int \mathrm{d}x\, M(k_2' e_2 - k_1 e_1') \tag{2}$$

and the Hamiltonian constraint [7,8]

$$H[N] = \int \mathrm{d}x N\left(\frac{(e_1')^2}{8\sqrt{e_1}e_2} - \frac{\sqrt{e_1}}{2e_2^2}e_1'e_2' + \frac{\sqrt{e_1}}{2e_2}e_1'' - \frac{e_2}{2\sqrt{e_1}} - \frac{e_2k_2^2}{2\sqrt{e_1}} - 2\sqrt{e_1}k_1k_2\right). \tag{3}$$

Classically, Hamilton's equations generated by $H[N] + D[M]$ for e_1 and e_2 show that k_1 and k_2 are related to components of extrinsic curvature of a constant-t slice in a space-time with line element (1).

A. Covariance conditions

At the same time, the constraints generate gauge transformations via Hamilton's equations of $H[\epsilon^0] + D[\epsilon]$ with gauge functions ϵ^0 and ϵ, whose geometrical role as hypersurface deformations in spherically symmetric space-time is determined by the Poisson brackets

$$\{D[\vec{M}_1], D[\vec{M}_2]\} = D[\mathcal{L}_{\vec{M}_1}\vec{M}_2], \tag{4}$$

$$\{H[N], D[\vec{M}]\} = -H[\mathcal{L}_{\vec{M}}N], \tag{5}$$

$$\{H[N_1], H[N_2]\} = D[e_1 e_2^{-2}(N_1N_2' - N_2N_1')]. \tag{6}$$

These gauge transformations make sure that the line element (1) describes a well-defined space-time geometry irrespective of the time coordinate t chosen to define constant-t hypersurfaces; when the constraints $D[M] = 0 = H[N]$ and the equations of motion they generate are satisfied, gauge transformations of the canonical theory are equivalent to coordinate transformations of (t, x) on spherically symmetric space-time [9,10]. This classic result makes it possible to interpret solutions of the canonical theory as space-time geometries.

Such an interpretation relies on several properties of the classical canonical theory that may easily be broken if the constraints are modified, for instance by possible quantum corrections. There are three broad conditions of relevant structures being preserved: (i) The modified constraints must remain first class, such that their mutual Poisson brackets still vanish on the constraint surface. If this is the case, the modification does not introduce gauge anomalies;

(ii) For gauge symmetries of a modified theory to correspond to hypersurface deformations in some space-time, the specific form of the brackets (4)–(6) must be preserved. This condition is stronger than just requiring first-class constraints because it prohibits modifications that could, for instance, add a Hamiltonian constraint to the right-hand side of (6). Such a modification would be first class, but it would not have the correct form required for hypersurface deformations. One modification of the brackets is nevertheless possible; the classical inverse radial metric $q^{xx} = e_1 e_2^{-2}$ in (6) could be replaced by a different phase-space function, $q^{xx}_{\rm em}$. The brackets would then be compatible with hypersurface deformations in a modified (or emergent) space-time in which the inverse of the new function $q^{\rm em}_{xx} = 1/q^{xx}_{\rm em}$ (assuming, for now, that it is positive) provides the radial metric component. A candidate space-time line element is then given by

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + q^{\rm em}_{xx} (\mathrm{d}x + M\mathrm{d}t)^2 + e_1 \mathrm{d}\Omega^2. \qquad (7)$$

However, the condition on constraint brackets does not guarantee that the phase-space function $q^{\rm em}_{xx}$ is subject to gauge transformations compatible with coordinate changes in an emergent space-time with line element (7). There is therefore a third covariance condition; (iii), that makes sure that gauge transformations of $q^{\rm em}_{xx}$ are equivalent to coordinate changes of a radial metric component if the constraint equations and equations of motion are satisfied.

The three conditions are strong, but it turns out that they leave room for modifications of the classical theory, even in vacuum without introducing extra fields or higher derivatives. (There are also compatible matter couplings to perfect fluids [11] and scalar fields [12].) As a new feature, they make it possible to describe signature change in a covariant manner within a single theory: If the classical $e_1 e_2^{-2}$ in (6) is replaced by a phase-space function γ that is not positive definite, it can define a radial metric only as $q^{\rm em}_{xx} = |\gamma|^{-1}$, while the compatibility of modified gauge transformations with coordinate changes then requires the sign of γ to multiply N^2 in the time component of the metric [13]. In general, the emergent space-time line element therefore reads,

$$\mathrm{d}s^2 = -\mathrm{sgn}(\gamma) N^2 \mathrm{d}t^2 + \frac{1}{|\gamma|} (\mathrm{d}x + M\mathrm{d}t)^2 + e_1 \mathrm{d}\Omega^2, \qquad (8)$$

if (6) is modified to

$$\{H[N_1], H[N_2]\} = D[\gamma (N_1 N_2' - N_2 N_1')]. \qquad (9)$$

These properties are valid in any gauge or slicing, but their specific evaluations depend on gauge choices (such as setting $M = 0$) as we will see in our explicit examples.

An interesting (though not completely general) class of modified theories can be obtained by replacing $H[N]$ of the classical theory with a linear combination $H[\alpha N] + D[\beta N]$ where α and β are suitable phase-space functions. The diffeomorphism constraint is left unmodified such that the spatial structure remains classical. The fact that a linear combination of algebra generators can modify the resulting dynamics is somewhat counterintuitive but, as explained in detail in [1,14], it is possible because the Hamiltonian constraint $H[N]$, by definition, generates hypersurface deformations in the normal direction. Redefining the Hamiltonian constraint therefore changes the normal direction n^μ, and the latter together with the inverse spatial metric $q^{\mu\nu}$ determines the inverse space-time metric $g^{\mu\nu} = q^{\mu\nu} - n^\mu n^\nu$. A redefined Hamiltonian constraint may then change the compatible space-time geometry of solutions, even though it does not modify the constraint surface on which $H[N] = 0$ and $D[M] = 0$. However, there is a well-defined space-time geometry only if our conditions (i)–(iii) formulated above are satisfied.

These conditions, specialized to linear combinations of the Hamiltonian and diffeomorphism constraints, have been evaluated in [1]. Condition (i) is automatically satisfied in this case. Condition (ii), requiring the specific form of hypersurface-deformation brackets, implies that α and β are related by

$$\beta(e_1, k_2) = -\frac{\sqrt{e_1} e_1'}{2 e_2^2} \frac{\partial \alpha}{\partial k_2} - 2\sqrt{e_1} k_2 \frac{\partial \alpha}{\partial e_1'}. \qquad (10)$$

Condition (iii), imposing covariance in the sense that the resulting modified structure function γ transforms like an inverse radial metric, then requires that

$$\alpha(e_1, k_2) = \mu(e_1) \sqrt{1 - s\lambda(e_1)^2 k_2^2} \qquad (11)$$

with two free functions μ and λ, depending only on e_1. Moreover, for later convenience, a sign factor $s = \pm 1$ has been extracted explicitly in this equation.

Using all the conditions, the equation resulting from (ii) leads to

$$\beta(e_1, k_2) = \mu(e_1) \frac{\sqrt{e_1}}{2 e_2^2} \frac{\partial e_1}{\partial x} \frac{s\lambda(e_1)^2 k_2}{\sqrt{1 - s\lambda(e_1)^2 k_2^2}} \qquad (12)$$

while the modified structure function equals

$$\gamma = \mu(e_1)^2 \left(1 + \frac{1}{4 e_2^2} \frac{s\lambda(e_1)^2}{1 - s\lambda(e_1)^2 k_2^2} \left(\frac{\partial e_1}{\partial x}\right)^2\right) \frac{e_1}{e_2^2}. \qquad (13)$$

We obtain the emergent radial metric

$$q^{\rm em}_{xx} = \mu(e_1)^{-2} \left|1 + \frac{1}{4 e_2^2} \frac{s\lambda(e_1)^2}{1 - s\lambda(e_1)^2 k_2^2} \left(\frac{\partial e_1}{\partial x}\right)^2\right|^{-1} \frac{e_2^2}{e_1} \qquad (14)$$

and the signature factor

$$\epsilon = \mathrm{sgn}(\gamma) = \mathrm{sgn}\left(1 + \frac{1}{4e_2^2}\frac{s\lambda(e_1)^2}{1 - s\lambda(e_1)^2 k_2^2}\left(\frac{\partial e_1}{\partial x}\right)^2\right). \quad (15)$$

These two expressions define the emergent space-time line element

$$\mathrm{d}s_{\mathrm{em}}^2 = -\epsilon N^2 \mathrm{d}t^2 + q_{xx}^{\mathrm{em}}(\mathrm{d}x + M\mathrm{d}t)(\mathrm{d}x + M\mathrm{d}t) + e_1 \mathrm{d}\Omega^2. \quad (16)$$

The inverse space-time metric equals

$$g_{\mathrm{em}}^{\mu\nu} = \frac{1}{q_{xx}^{\mathrm{em}}} s_x^\mu s_x^\nu + \frac{1}{e_1}(s_\vartheta^\mu s_\vartheta^\nu + \csc(\vartheta) s_\varphi^\mu s_\varphi^\nu) - n^\mu n^\nu, \quad (17)$$

where

$$n^\mu = \frac{1}{N}(t^\mu - M s_x^\mu) \quad (18)$$

with a spatial basis $(s_x^\mu, s_\vartheta^\mu, s_\varphi^\mu)$.

B. The signature-change hypersurface

Models with $s = -1$ have not been studied in detail yet. While $s = 1$ implies a positive definite structure function, which then directly determines the inverse spatial metric, $s = -1$ may allow for ranges of x in which γ is negative. The emergent space-time then has Euclidean signature in such a region, separating it from Lorentzian signature at positive γ by a signature-change hypersurface in space-time. Such hypersurfaces are defined by $\gamma(t_{\mathrm{sc}}, x_{\mathrm{sc}}) = 0$, which may have disjoint solutions for $(t_{\mathrm{sc}}, x_{\mathrm{sc}})$, implying multiple signature-change hypersurfaces in general. A signature-change hypersurface may be timelike or space-like depending on where it appears relative to horizons. In black-hole models, a signature-change hypersurface that occurs in a static (exterior) region is timelike because γ can depend only on the spatial coordinate x, which lies in a discrete set of values determined by $\gamma(x_{\mathrm{sc}}) = 0$. In a spatially homogeneous model for a black-hole interior, γ depends only on the time coordinate t, such that a signature-change hypersurface in this region is spacelike, determined by $t = t_{\mathrm{sc}}$ with a solution t_{sc} of $\gamma(t_{\mathrm{sc}}) = 0$. We will see that $s = -1$ in our models can only lead to timelike signature-change hypersurfaces at a unique value of x_{sc}. The following discussion of general properties is based on this outcome, but similar statements can easily be made for spacelike signature-change hypersurfaces as well.

In a static region, the occurrence of a timelike signature-change hypersurface requires that the structure function vanishes at a certain value of $x = x_{\mathrm{sc}}$. (If the region is not static but the signature-change hypersurface remains time-like, it is always possible to introduce local coordinates such that the hypersurface is defined by a constant value of the radial coordinate.) Starting in a range of x-values for which the emergent space-time metric has Lorentzian signature, inspection of the inverse metric (17) reveals that near the signature-change hypersurface, space-time degenerates into a family of $(2+1)$-dimensional geometries with inverse metric,

$$g_{\mathrm{em}}^{\mu\nu} \approx \frac{1}{x_{\mathrm{sc}}^2}(s_\vartheta^\mu s_\vartheta^\nu + \csc^2(\vartheta) s_\varphi^\mu s_\varphi^\nu) - n^\mu n^\nu, \quad (19)$$

and topology $\mathbb{R} \times S^2$. Approaching the hypersurface from the Euclidean region, the signature-change hypersurface is the limiting case of a family of spacelike hypersurfaces.

It then follows that the radial component of the emergent metric (14) diverges at x_{sc}. This conclusion holds irrespective of the gauge or coordinate system used because $\det(g^{-1}) = 0$ is a coordinate invariant statement. Therefore, there is no coordinate choice that can remove this divergence of a metric component, implying a physical singularity according to the standard definition. However, invariant objects such as the Ricci scalar are not necessarily singular at a signature-change hypersurface. Our model in spherical symmetry derived below provides an example of a signature-change hypersurface with a nonsingular geometry.

The standard definition of geodesic incompleteness might also suggest a physical singularity because timelike geodesics from the Lorentzian region cannot be extended as timelike geodesics into the Euclidean region. However, there may be extensions to spacelike geodesics if it is possible to use the final values of a timelike geodesic in the Lorentzian region as initial conditions for a spacelike geodesic in the Euclidean region. An important question related to geodesic completeness is whether such an extension is unique, which requires a well-defined tangent vector at the hypersurface as well as a continuous set of coordinate transformations that can be applied in a region across the signature-change hypersurface. Details of such extensions require specific models, but the main challenging property can be inferred from the behavior of the space-time metric that gives rise to signature change on a timelike hypersurface.

Assuming a timelike signature-change hypersurface at $x = x_{\mathrm{sc}}$, the radial component q_{xx}^{em} of the metric diverges at this value. Normalization of the tangent vector of a geodesic approaching the hypersurface,

$$-1 = -N^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + q_{xx}^{\mathrm{em}}\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 + e_1\left(\frac{\mathrm{d}\vartheta}{\mathrm{d}\tau}\right)^2 + e_1 \sin^2\vartheta\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\tau}\right)^2, \quad (20)$$

then requires that $v^x = \mathrm{d}x/\mathrm{d}\tau$ approaches zero or that some of the other velocity components diverge at the signature-change hypersurface. In both cases, the geodesic is asymptotically tangent to the hypersurface and does not provide a unique final direction into the Euclidean region on the other side of the hypersurface. A similar argument follows from lightlike geodesics, which in the radial case require a divergent

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \sqrt{\frac{q_{xx}^{\mathrm{em}}}{N}} \tag{21}$$

at the hypersurface. In our specific models we will show that the hypersurface may nevertheless be reached at a finite distance from an interior point of the complete space-time manifold, including the Euclidean region.

C. Hamiltonian constraints

Using the explicit solutions for α and β, the new Hamiltonian constraint is given by the expression,

$$H^{(\mathrm{new})}[N] = \int \mathrm{d}x\, N\mu\sqrt{1-s\lambda^2k_2^2}\left(\left(\frac{1}{8\sqrt{e_1}e_2} - s\lambda^2\frac{\sqrt{e_1}}{2e_2^2}\frac{k_1k_2}{1-s\lambda^2k_2^2}\right)(e_1')^2 - \frac{\sqrt{e_1}}{2e_2^2}e_1'e_2' + \frac{\sqrt{e_1}}{2e_2}e_1'' + s\lambda^2\frac{\sqrt{e_1}}{2e_2^2}\frac{e_2k_2}{1-s\lambda^2k_2^2}e_1'k_2' - \frac{e_2}{2\sqrt{e_1}} - \frac{e_2k_2^2}{2\sqrt{e_1}} - 2\sqrt{e_1}k_1k_2\right), \tag{22}$$

for given s, μ, and λ.

The form of functions α, β, and γ makes use of the phase-space variables (e_1, e_2) and (k_1, k_2) initially obtained in the classical theory. However, modifications of equations of motion and of the structure function imply that (e_1, e_2) no longer are densitized-triad components of the emergent spatial metric, and (k_1, k_2) are no longer directly related to components of extrinsic curvature. It is therefore possible to apply canonical transformations, introducing further changes in the phase-space dependence of α, β, and γ. Such transformations do not change physical or geometrical implications, but they may sometimes be convenient for solving or interpreting equations. The specific versions (11), (12) and (13) are unique up to canonical transformations, provided modifications happen only by replacing the Hamiltonian constraint with a suitable linear combination of the classical constraints. As an example, the models analyzed in [3,4] are equivalent to our case of $s = +1$ and constant μ and λ with a specific relationship between these two constants, up to a canonical transformation of (e_2, k_2). Canonical transformations can be used to extend these models to nonconstant λ, and to describe the case of $s = -1$ by similar means.

1. Periodic variables: Lorentzian case

For the case of $s = 1$, we perform the canonical transformation

$$k_2 = \frac{\sin(\lambda\tilde{k}_2)}{\lambda}, \qquad e_2 = \frac{\tilde{e}_2}{\cos(\lambda\tilde{k}_2)},$$
$$k_1 = \tilde{k}_1 + \frac{\tilde{e}_2}{\cos(\lambda\tilde{k}_2)}\frac{\partial}{\partial e_1}\left(\frac{\sin(\lambda k_2)}{\lambda}\right), \qquad e_1 = \tilde{e}_1, \tag{23}$$

where the new variables are written with a tilde. The Hamiltonian constraint (22) then becomes,

$$H_+^{(\mathrm{c})}[N] = \int \mathrm{d}x\, N\mu\left(\left(\frac{\cos^2(\lambda k_2)}{8\sqrt{e_1}e_2} - \lambda^2\frac{\sqrt{e_1}}{2e_2^2}\left(k_1 + e_2k_2\frac{\partial\ln\lambda}{\partial e_1}\right)\frac{\sin(2\lambda k_2)}{2\lambda}\right)(e_1')^2 - \frac{\sqrt{e_1}}{2e_2^2}e_1'e_2'\cos^2(\lambda k_2) + \frac{\sqrt{e_1}}{2e_2}e_1''\cos^2(\lambda k_2) - \frac{e_2}{2\sqrt{e_1}} - \frac{e_2}{2\sqrt{e_1}}\frac{\sin^2(\lambda k_2)}{\lambda^2} - 2\sqrt{e_1}\left(k_1\frac{\sin(2\lambda k_2)}{2\lambda} + e_2\left(\frac{\sin(2\lambda k_2)}{2\lambda}k_2 - \frac{\sin^2(\lambda k_2)}{\lambda^2}\right)\frac{\partial\ln\lambda}{\partial e_1}\right)\right), \tag{24}$$

where we have dropped the tilde for the sake of convenience, with structure function,

$$q^{xx}_{(\mathrm{c})+} = \mu^2\cos^2(\lambda k_2)\left(1 + \lambda^2\left(\frac{e_1'}{2e_2}\right)^2\right)\frac{e_1}{e_2^2}. \tag{25}$$

This transformation replaces square roots by trigonometric functions, but the dependence on k_2 is not periodic unless λ does not depend on e_1. (Periodic dependence on k_2 is often desired in models of loop quantum gravity.) A second canonical transformation,

$$k_2 = \frac{\bar{\lambda}}{\lambda}\tilde{k}_2, \qquad e_2 = \frac{\lambda}{\bar{\lambda}}\tilde{e}_2, \qquad k_1 = \tilde{k}_1 - \tilde{e}_2\tilde{k}_2\frac{\partial\ln\lambda}{\partial\tilde{e}_1}, \qquad e_1 = \tilde{e}_1, \tag{26}$$

where $\bar{\lambda}$ is an arbitrary nonzero constant, can be used to make the Hamiltonian constraints strictly periodic in k_2,

$$H_{+}^{(\mathrm{cc})}[N] = \int \mathrm{d}x N \frac{\bar{\lambda}}{\lambda}\mu\left[\left(\left(\frac{1}{8\sqrt{e_1}e_2} - \frac{\sqrt{e_1}}{2e_2}\frac{\partial \ln\lambda}{\partial e_1}\right)\cos^2(\bar{\lambda}k_2) - \bar{\lambda}^2\frac{\sqrt{e_1}}{2e_2^2}k_1\frac{\sin(2\bar{\lambda}k_2)}{2\bar{\lambda}}\right)(e_1')^2 - \frac{\sqrt{e_1}}{2e_2^2}e_1'e_2'\cos^2(\bar{\lambda}k_2)\right.$$
$$\left. + \frac{\sqrt{e_1}}{2e_2}e_1''\cos^2(\bar{\lambda}k_2) - \frac{e_2}{2\sqrt{e_1}} - \frac{e_2}{2\sqrt{e_1}}\frac{\sin^2(\bar{\lambda}k_2)}{\bar{\lambda}^2} - 2\sqrt{e_1}\left(k_1 - e_2\frac{\tan(\bar{\lambda}k_2)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial e_1}\right)\frac{\sin(2\bar{\lambda}k_2)}{2\bar{\lambda}}\right], \tag{27}$$

where we have again dropped the tilde, with structure function

$$q^{xx}_{(\mathrm{cc})+} = \frac{\bar{\lambda}^2}{\lambda^2}\mu^2\cos^2(\bar{\lambda}k_2)\left(1 + \bar{\lambda}^2\left(\frac{e_1'}{2e_2}\right)^2\right)\frac{e_1}{e_2^2}. \tag{28}$$

The constraint (27) and its structure function (28) are periodic in the new k_2. In this sense, they are related to models of loop quantum gravity in which this periodicity is usually interpreted as a necessary requirement for gauge theories based on holonomies. However, the specific terms in (27) are different from most models that have been considered in this context. Moreover, in past developments of spherically symmetric loop quantum gravity it has been tacitly assumed that the classical e_2^2/e_1 still describes a meaningful radial metric. As shown by emergent modified gravity, this assumption and any deviations of a modified Hamiltonian constraint from (27) violate general covariance and do not imply reliable effective line elements.

2. Hyperbolic variables: Signature-change case

The constraint (22) with $s = -1$ is mathematically equivalent to the case $s = 1$ with the substitution $\lambda \to i\lambda$. Furthermore, in the canonical transformations used in the previous sections we may replace trigonometric functions with hyperbolic ones. Therefore, the case $s = -1$ with hyperbolic canonical transformations can simply be expressed as (27) with the substitutions $\lambda \to i\lambda$ and $\bar{\lambda} \to i\bar{\lambda}$,

$$H_{-}^{(\mathrm{cc})}[N] = \int \mathrm{d}x N \frac{\bar{\lambda}}{\lambda}\mu\left[\left(\left(\frac{1}{8\sqrt{e_1}} - \frac{\sqrt{e_1}}{2}\frac{\partial \ln\lambda}{\partial e_1}\right)\cosh^2(\bar{\lambda}k_2) + \bar{\lambda}^2\frac{\sqrt{e_1}}{2}\frac{k_1}{e_2}\frac{\sinh(2\bar{\lambda}k_2)}{2\bar{\lambda}}\right)\frac{(e_1')^2}{e_2}\right.$$
$$\left. + \frac{\sqrt{e_1}}{2}\left(\frac{e_1''}{e_2} - \frac{e_1'e_2'}{e_2}\right)\cosh^2(\bar{\lambda}k_2) - \frac{e_2}{2\sqrt{e_1}}\left(1 + \frac{\sinh^2(\bar{\lambda}k_2)}{\bar{\lambda}^2}\right) - 2\sqrt{e_1}\left(k_1 - e_2\frac{\tanh(\bar{\lambda}k_2)}{\bar{\lambda}}\frac{\partial\ln\lambda}{\partial e_1}\right)\frac{\sinh(2\bar{\lambda}k_2)}{2\bar{\lambda}}\right] \tag{29}$$

with structure function

$$q^{xx}_{(\mathrm{cc})-} = \frac{\bar{\lambda}^2}{\lambda^2}\mu^2\cosh^2(\bar{\lambda}k_2)\left(1 - \bar{\lambda}^2\left(\frac{e_1'}{2e_2}\right)^2\right)\frac{e_1}{e_2^2}. \tag{30}$$

The case of $s = -1$, studied as the main example in what follows, may therefore be interpreted as a hyperbolic version of covariant models for spherically symmetric loop quantum gravity.

3. Possible interpretations of the modification functions

Canonical transformations can be used to bring modifications closer to versions that have occurred in different approaches to quantum gravity, thereby helping to find suitable interpretations of modification functions. For instance, the transformation applied for $s = 1$, leading to periodic modifications in k_2, suggests that λ in this case may be viewed as a covariant implementation of the holonomy length in models of loop quantum gravity. This function would then be related to the fundamental discreteness scale of quantum space-time.

For $s = -1$, the k_2-dependence is not periodic and therefore does not correspond to models of standard loop quantum gravity. However, the modified constraint may be viewed as a model of loop quantization with a noncompact local gauge group SO(2, 1) instead of SO(3) (or their covering groups). Noncompact groups are not often considered in loop quantum gravity because they lead to unbounded basic operators and are therefore harder to implement. However, they might well play a role depending on how reality conditions are solved.

Alternatively, both $\lambda(e_1)$ and $\mu(e_1)$ may appear through Hamiltonian renormalization in which e_1 provides the running scale. The function $\mu(e_1)$ in particular would have a simple interpretation as a renormalization of Newton's constant. In contrast to standard renormalization, emergent modified gravity does not require the renormalization scale to be a Lorentz scalar, as observed in an application to modified Newtonian dynamics (MOND) [15].

III. SOLUTIONS

Canonical equations of gravitational theories provide unique solutions only if certain gauge choices are made that specify the coordinate frame or slicing in which the corresponding space-time geometry is expressed. The main classical slicing conditions used in spherically symmetric models can be generalized to emergent modified gravity, as shown in this section.

A. Schwarzschild-like exterior

We will be using space-time solutions in various gauges when we analyze geodesics and other properties of emergent modified gravity. For constant μ and λ, the case of $s=1$ has been analyzed in [3,4], while the case of $s=-1$ has been discussed briefly in [16]. In particular, the latter contribution shows the possibility of signature change for $s=-1$ at large x in the exterior region of a black hole. However, this is possible only if $\lambda>1$, implying significant modifications of gravity even in intermediate ranges of x that should be directly accessible by observations. It is therefore important to confirm that signature change is still possible if λ is no longer constant and may increase from small values in observationally accessible regimes to larger values at the outer fringes of the Universe. One of the main results of the present paper is that this is indeed possible. We will see that most of the calculations for constant λ go through with only minimal changes if λ is not constant. Our derivations in the remainder of this section are based on the form (22) of the Hamiltonian constraint.

We first compute the line element of a static region of space-time suitable for the exterior of a nonrotating black hole. We directly obtain $M=0$, $\dot{e}_1=0=\dot{e}_2$, and therefore $k_1=0=k_2$, which allows us to fix the spatial gauge by declaring that $e_1(x)=x^2$. All these equations take their classical form, and with vanishing k-terms for static configurations, the Hamiltonian constraint is classical too (up to an additional multiplier of μ). It implies $e_2(x)=x/\sqrt{1-2m/x}$ where, based on the position of the horizon, the integration constant m turns out to have the same interpretation as mass as in the classical solution.

There are additional consistency conditions from the requirement that $\dot{k}_1=0$ and $\dot{k}_2=0$ for static behavior are compatible with the equations of motion. They imply almost the same result as in the classical theory,

$$N\alpha\mu=\sqrt{1-\frac{2m}{x}}, \tag{31}$$

where α is a constant and can be considered a rescaling of the time coordinate. This parameter can be used to cancel μ only if the latter is constant, but not if it depends on e_1 and therefore on x. For nonconstant μ, m retains its interpretation of mass if the latter is defined via the Schwarzschild radius at $2m$. (If μ is asymptotically constant such that $\mu-\text{const}$ falls off faster than $1/x$, interpreting m as the mass would also be consistent with Newton's potential. However, depending on s and λ, the asymptotic limit can be very nonclassical and may no longer be a suitable indicator of the mass.)

None of the solutions for phase-space variables are significantly modified in this gauge. However, the emergent space-time metric, and therefore space-time geometry, does have nontrivial corrections. For covariance of this gauge within a well-defined space-time geometry, the emergent space-time metric must be compatible with the full modified constraint and not just with its static restriction in which all k-terms disappear. The resulting radial metric component, given in general by (14), evaluates to

$$q_{xx}^{\text{em}}=\frac{\epsilon e_2^2}{\mu^2 e_1(1+s\lambda^2x^2/e_2^2)} \tag{32}$$

in the present case, with $k_2=0$, $e_1=x^2$ and the signature factor

$$\epsilon=\text{sgn}(1+s\lambda^2x^2/e_2^2). \tag{33}$$

We obtain the emergent space-time line element,

$$\begin{aligned}\mathrm{d}s_{\text{em}}^2&=-\epsilon\left(1-\frac{2m}{x}\right)\frac{\mathrm{d}t^2}{\alpha^2\mu^2}\\&\quad+\frac{\epsilon\mathrm{d}x^2/\mu^2}{(1-2m/x)(1+s\lambda^2(1-2m/x))}+x^2\mathrm{d}\Omega^2\\&=-\epsilon\left(1-\frac{2m}{x}\right)\frac{\mathrm{d}t^2}{\alpha^2\mu^2}\\&\quad+\frac{-s\epsilon}{\alpha^2\mu^2(\lambda^2+s)}\left(1-\frac{2m}{x}\right)^{-1}\left(\frac{X_\lambda}{x}-1\right)^{-1}\mathrm{d}x^2\\&\quad+x^2\mathrm{d}\Omega^2,\end{aligned} \tag{34}$$

if we introduce the function

$$X_\lambda(x)=\frac{2m\lambda(x)^2}{\lambda(x)^2+s}. \tag{35}$$

The signature-change hypersurface is located at $x=x_\lambda$ with an implicit equation

$$x_\lambda=X_\lambda(x_\lambda)=\frac{2m\lambda(x_\lambda)^2}{\lambda(x_\lambda)^2+s}, \tag{36}$$

for x_λ, provided $\lambda^2+s\neq0$. [For constant λ, X_λ is constant and (36) directly defines x_λ in terms of λ.] The signature factor evaluates to

$$\epsilon\equiv\text{sgn}\left(-s(\lambda^2+s)\left(\frac{X_\lambda}{x}-1\right)\right). \tag{37}$$

For $s=1$, $x_\lambda<2m$ is not in the static exterior and $\epsilon=\text{sgn}(\mu^2(1+\lambda^2)(1-X_\lambda/x))=1$ for $x>2m$, such that there is no signature change in this case. The resulting line element

$$ \mathrm{d}s_{\mathrm{em}}^2 = -\left(1-\frac{2m}{x}\right)\frac{\mathrm{d}t^2}{\alpha^2\mu^2} + \frac{1}{\mu^2(1+\lambda^2)}\left(1-\frac{2m}{x}\right)^{-1}\left(1-\frac{X_\lambda}{x}\right)^{-1}\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2, \tag{38} $$

is asymptotically flat in a strict sense only if μ and λ are asymptotically constant for $x \gg 2m$. (If this condition is not satisfied, the line element is quasiasymptotically flat [17]; see also [18].)

If $s = -1$, the emergent spatial metric is positive definite if $\lambda < 1$. If $\lambda = 1$ in a region of space, $\epsilon = \mathrm{sgn}(-2s\lambda^2 m/x) = 1$ and the line element is given by

$$ \mathrm{d}s_{\mathrm{em}}^2 = -\left(1-\frac{2m}{x}\right)\frac{\mathrm{d}t^2}{\alpha^2\mu^2} + \frac{x/(2m)}{1-2m/x}\frac{\mathrm{d}x^2}{\mu^2\lambda^2} + x^2\mathrm{d}\Omega^2. \tag{39} $$

If $\lambda > 1$, we have $\epsilon = 1$ for $x < x_\lambda$, with a Loretzian-signature line element

$$ \mathrm{d}s_{\mathrm{em}}^2 = -\left(1-\frac{2m}{x}\right)\frac{\mathrm{d}t^2}{\alpha^2\mu^2} + \frac{\mathrm{d}x^2/(\mu^2(\lambda^2-1))}{(1-2m/x)(X_\lambda/x-1)} + x^2\mathrm{d}\Omega^2, \tag{40} $$

and $\epsilon = -1$ for $x > x_\lambda$ with a Euclidean-signature line element

$$ \mathrm{d}s_{\mathrm{em}}^2 = \left(1-\frac{2m}{x}\right)\frac{\mathrm{d}t^2}{\alpha^2\mu^2} + \frac{\mathrm{d}x^2/(\mu^2(\lambda^2-1))}{(1-2m/x)(1-X_\lambda/x)} + x^2\mathrm{d}\Omega^2. \tag{41} $$

There is therefore Euclidean-signature 4-dimensional space surrounding Lorentzian space-time containing a black hole. For nonconstant λ, the Lorentzian geometry can be compatible with observations if λ is sufficiently small for a suitable range of x. If λ grows beyond the value one for larger x, we enter the signature-change region.

The Ricci scalar of (40) is given by

$$ \begin{aligned} \mathcal{R} = {} & \left(\frac{2}{x^2} + \left(\frac{2}{x^2} - \frac{3mx_\lambda}{x^4}\right)\mu^2(\lambda^2-1)\right) + \mu^2(\lambda^2-1)\frac{(3m-2x)}{x^3}X_\lambda' + \frac{(\lambda^2-1)(x-2m)}{2x^2}(\mu^2)'X_\lambda' \\ & + \frac{(2m-x)(x-X_\lambda)}{2x^2}(\mu^2)'(\lambda^2-1)' + \frac{(\lambda^2-1)(2m-4x+3X_\lambda)}{2x^2}(\mu^2)' - \frac{(\lambda^2-1)(2m-x)(x-X_\lambda)}{\mu^2x^2}((\mu^2)')^2 \\ & + \frac{(\lambda^2-1)(2m-x)(x-X_\lambda)}{x^2}(\mu^2)'' - \frac{(3m-2x)(x-X_\lambda)}{x^3}(\mu^2(\lambda^2-1))', \end{aligned} \tag{42} $$

where the primes denote x-derivatives. The Ricci scalar remains finite at the signature-change hypersurface at $x = x_\lambda$. The Kretschmann scalar $K \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ evaluates to

$$ \begin{aligned} K = {} & \frac{\mu^4(\lambda^2-1)^2}{x^8}\left(\frac{4x^2}{\mu^4(\lambda^2-1)^2} + \frac{8x^2}{\mu^2(\lambda^2-1)}(x-2m)(x-X_\lambda) + 4x^2(x^2-4mx+12m^2)\right. \\ & \left. - 8x(x^2-5mx+15m^2)X_\lambda + (6x^2-32mx+81m^2)X_\lambda^2\right) \\ & + \frac{4\lambda(\lambda^2-1)\mu^4(x-X_\lambda)(3m^2(7X_\lambda-4x)+4mx(x-3X_\lambda)+2x^2X_\lambda)}{x^7}\lambda' + \frac{4\lambda^2\mu^4(9m^2-8mx+2x^2)(x-X_\lambda)^2}{x^6}(\lambda')^2 \\ & + \frac{8\lambda(\lambda^2-1)\mu^3(x-X_\lambda)(2m^2+m(5X_\lambda-6x)+2x(x-X_\lambda))}{x^5}\lambda'\mu' + \frac{8\lambda^2\mu^3 m(2m-x)(x-X_\lambda)^2}{x^5}(\lambda')^2\mu' \\ & - \frac{4\lambda(\lambda^2-1)\mu^2(2m-x)(x-X_\lambda)(6mx-8mX_\lambda+xX_\lambda)}{x^5}\lambda'(\mu')^2 \\ & + \frac{2(\lambda^2-1)^2\mu^3(2m^2(8x^2-14xX_\lambda+7X_\lambda^2)+mxX_\lambda(11x_\lambda-12x)+4x^2X_\lambda(x-X_\lambda))}{x^7}\mu' + \frac{K_1(x)}{x^6}(\mu')^2 + \frac{K_2(x)}{x^6}\mu'' \\ & + \frac{\mu^4(\lambda^2-1)^2(x-2m)^2}{4\alpha^4\mu^4x^4}((\mu^2(\lambda^2-1))')^2((X_\lambda)')^2 + \frac{\mu^4(\lambda^2-1)^2m(2m-x)}{\alpha^2\mu^2x^5}(\mu^2(\lambda^2-1))'((X_\lambda)')^2 \end{aligned} $$

$$+\frac{\mu^4(\lambda^2-1)^2(9m^2-8mx+2x^2)}{x^6}((X_\lambda)')^2-\frac{\mu^4(\lambda^2-1)^2(2m-x)(x-X_\lambda)(2\alpha^2\mu^2m+x(2m-x)(\alpha^2\mu^2)'(x))}{\alpha^2\mu^4x^5}X'_\lambda(\mu^2)''$$
$$+\frac{K_3(x)}{2\mu^2x^6}X'_\lambda(\mu^2(\lambda^2-1))'+\frac{3\mu^4(\lambda^2-1)^2(x-2m)^2(x-X_\lambda)}{2\mu^6x^4}X'_\lambda((\mu^2)')^3$$
$$+\frac{\mu^4(\lambda^2-1)^2(2m-x)(2m(6x-7X_\lambda)+xX_\lambda)}{2\mu^2x^5}X'_\lambda((\mu^2)')^2+\mu^4(\lambda^2-1)^2\frac{2m(7mx-9mX_\lambda-2x^2+3xX_\lambda)}{\mu^2x^6}X'_\lambda(\mu^2)'$$
$$+\frac{2\mu^4(\lambda^2-1)^2(3m^2(4x-7X_\lambda)-4mx(x-3X_\lambda)-2x^2X_\lambda)}{x^7}X'_\lambda \tag{43}$$

with

$$K_1(x)=(\lambda^2-1)^2\mu^2(X_\lambda^2(140m^2-96mx+17x^2)-4xX_\lambda(62m^2-43mx+8x^2)+16x^2(7m^2-5mx+x^2))$$
$$K_2(x)=-4(\lambda^2-1)^2\mu^3m(2m-x)(4x-5X_\lambda)(x-X_\lambda)+8\lambda(\lambda^2-1)\mu^3m(2m-x)(x-X_\lambda)^2x\lambda'$$
$$K_3(x)=(\lambda^2-1)(x-X_\lambda)(x(x-2m)(\mu^2)'(4\mu^2m+x(2m-x)(\mu^2)')-4\mu^4(9m^2-8mx+2x^2)). \tag{44}$$

This expression is also finite at the signature-change hypersurface where it takes the value

$$K|_{x=x_\lambda}=\frac{1}{4x_\lambda^6}\left(8\mu^4(\lambda^2-1)^2(x_\lambda-2m)^2(X'_\lambda(x_\lambda)-1)^2+16x_\lambda^2\frac{\mu^4(\lambda^2-1)^2(X'_\lambda(x_\lambda)-1)^2(x_\lambda(x_\lambda-2m)(\mu^2)'-2\mu^2m)^2}{\mu^4}\right). \tag{45}$$

Both scalars retain their classical divergence at $x=0$. (In the next subsection we will see that the exterior solution can be extended to $x<2m$ in the usual way by flipping the role of radial and time coordinates.)

We conclude that the singularity at $x=x_\lambda$ in the metric (40) may be consistent with an interpretation as a coordinate singularity.

B. Homogeneous interior

In the classical Schwarzschild solution, the interior geometry for $x<2m$ can be obtained from the exterior solution by flipping the roles of t and x as time and space, respectively. In emergent modified gravity, only the spatial part of the metric contains additional terms based on the covariance condition, while the time component, through the lapse function, may be modified only indirectly based on the equations it has to solve in a given gauge. It is therefore not clear that simply flipping t and x correctly transfers additional terms in the emergent spatial metric from the radial part to the time component. In the present case, an explicit independent derivation of the interior solution demonstrates that the classical procedure nevertheless applies.

As part of the gauge choice for a Schwarzschild-type interior, we assume that all fields, e_1, e_2, k_1, k_2, and N, depend only on a time coordinate T, and that $M=0$. The function e_1 was fixed by a simple gauge choice in the exterior. Flipping the coordinates is possible only if we have the same simple choice in the interior, but now as a dependence $e_1(T)=T^2$ on the new time coordinate T. The modification functions μ and λ may therefore be time dependent.

This e_1 has to be compatible with the equation of motion,

$$\dot{e}_1=-2\,\mu N\sqrt{e_1}k_2\sqrt{1-s\lambda^2k_2^2}, \tag{46}$$

which, using $\dot{e}_1=2T=2\sqrt{e_1}$, relates N to k_2 by

$$N=-\frac{1}{\mu k_2\sqrt{1-s\lambda^2k_2^2}}. \tag{47}$$

The equation of motion for k_2,

$$\dot{k}_2=\frac{\mu N}{2\sqrt{e_1}}\sqrt{1-s\lambda^2k_2^2}(1+k_2^2)=-\frac{1}{2T}\frac{1+k_2^2}{k_2}, \tag{48}$$

can then be solved directly by

$$k_2(T)=\sqrt{\frac{2m}{T}-1}. \tag{49}$$

Anticipating the final form of the line element, we identified an integration constant with (twice) the mass m. The other momentum, k_1, is determined by

$$k_1=-\frac{me_2}{2T^2(2m-T)}k_2=-\frac{m\sqrt{2m-T}}{2T^{5/2}}e_2 \tag{50}$$

using the Hamiltonian constraint. The final equation of motion then implies

$$\dot{e}_2 = -\mu N\sqrt{1 - s\lambda^2 k_2^2}\left(\frac{e_2 k_2}{\sqrt{e_1}} + 2\sqrt{e_1}k_1\right) = \frac{1}{2}e_2\left(\frac{1}{T} - \frac{1}{2m - T}\right) \tag{51}$$

which is solved by

$$e_2(T) = \alpha^{-1}\sqrt{T(2m - T)}, \tag{52}$$

with an integration constant α. Note that the free functions μ and λ canceled out in all the differential equations we had to solve. The solutions are therefore valid for any μ and λ depending on T through e_1.

We now have complete solutions for all phase-space functions and can compute the lapse function,

$$N = -\frac{1}{\mu\sqrt{(2m/T - 1)(1 - s\lambda^2(2m/T - 1))}}, \tag{53}$$

as well as the emergent radial metric component

$$q_{XX}^{\text{em}} = \frac{e_2^2}{\mu^2 e_1} = \frac{1}{\alpha^2\mu^2}\left(\frac{2m}{T} - 1\right). \tag{54}$$

The space-time metric equals

$$\mathrm{d}s_{\text{em}}^2 = -\frac{\mathrm{d}T^2/\mu^2}{(2m/T - 1)(1 + s\lambda^2 - 2ms\lambda^2/T)} + \left(\frac{2m}{T} - 1\right)\frac{\mathrm{d}X^2}{\alpha^2\mu^2} + T^2\mathrm{d}\Omega^2, \tag{55}$$

where, without loss of generality, we can absorb the constant α in the radial coordinate X, or we can keep it to cancel μ in the case where it is constant.

The range of the time coordinate T is determined by the condition that N is real. In the emergent space-time metric, N^2 is always positive and cannot be split into a sign factor and a lapse function squared as the radial metric component. To recall, the radial metric component is determined by the structure function γ which may be positive or negative (or zero). According to the structure of hypersurface deformations, this function determines the signature of space-time as well as the spatial metric. The lapse function does not appear in structure functions and therefore can only be used in the classical form, such that N^2 is positive and multiplies $\mathrm{sgn}(\gamma)\mathrm{d}t^2$.

Using this condition, T has the maximal value $T_{\max} = 2m$ at the horizon as a boundary of the interior region. Its minimum value is $T_{\min}^- = 0$ for $s = -1$, in which case $1 - s\lambda^2(2m/T - 1)$ in the lapse function remains positive for all T such that $2m/T > 1$. For $s = 1$, there is a positive lower bound on T determined by

$$T_{\min}^+ = \frac{2m\lambda^2}{1 + \lambda^2}. \tag{56}$$

If λ is not constant, this is an implicit equation for $T_{\min}^+$. The coordinate chart constructed here then ends, and at least in the case of constant λ it can be extended to an expanding interior solution as shown in [3,4]. There is no signature-change hypersurface in these models because the structure function γ remains positive in the allowed ranges of T. It is now easy to confirm that the line elements (55) and (34) are indeed related by a simple flip of space and time coordinates, $T = x$ and $X = t$, using the Lorentzian solution with $\epsilon = 1$ in the latter case.

C. Painlevé-Gullstrand line element

An interesting gauge choice that allows transitions through the horizon in classical general relativity is given by the Painlevé-Gullstrand solution. We will first derive a suitable form in emergent modified gravity by applying coordinate transformations from the exterior and interior solutions already found, and then confirm that the resulting metric components also follow uniquely from the constrained system.

1. Coordinate transformation from the static Schwarzschild gauge

In order to derive a suitable coordinate transformation, we first observe that the metric (41) has the Killing vector $\xi^\mu_{(t)}\partial_\mu = \partial_t$. A timelike geodesic with tangent vector u^μ then has the conserved quantity $e = -g_{\mu\nu}\xi^\mu_{(t)}u^\nu = -u_t$, where $g_{\mu\nu}$ refers to the emergent space-time metric. This equation tells us that u_t always remains finite even if we approach a signature-change hypersurface where some of the components of $g_{\mu\nu}$ may diverge.

Using the normalization condition $g_{\mu\nu}(\mathrm{d}x^\mu/\mathrm{d}\tau)(\mathrm{d}x^\nu/\mathrm{d}\tau) = -\epsilon$ for the tangent vector $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ of a geodesic, with the signature factor ϵ, a geodesic can be described by the 1-form,

$$\epsilon\mathrm{d}\tau = -u_\mu\mathrm{d}x^\mu = -u_t\mathrm{d}t - u_x\mathrm{d}x. \tag{57}$$

A generalization of the classical Painlevé-Gullstrand gauge can be defined by requiring that the new time coordinate t_{PG} equals proper time along infalling radial geodesics, while the spatial coordinate x remains unchanged compared with the Schwarzschild solution. If we compute the components u_t and u_x using normalization and conserved quantities, we can integrate the resulting $\mathrm{d}\tau$ and obtain t_{PG} as a function of t and x. Keeping track of the signature factor ϵ, this construction can be used in Lorentzian and Euclidean regions.

Normalization

$$-\epsilon = u_\mu u_\nu g^{\mu\nu} \tag{58}$$

with the inverse metric (17) and the signature factor ϵ implies,

$$\epsilon\left(-\frac{u_t^2}{N^2}+1\right)+2\epsilon\frac{M}{N^2}u_t u_x+\left(q^{xx}+\epsilon\frac{M^2}{N^2}\right)u_x^2=0, \tag{59}$$

in general. For zero shift in the original space-time, $M = 0$, this equation simplifies to

$$q^{xx}(u_x)^2 = \epsilon\left(\frac{u_t^2}{N^2}-1\right). \tag{60}$$

If there is signature change at large x, it is not meaningful to identify u_t with the conserved energy e as measured by an asymptotic observer. Instead, we can evaluate the normalization condition at some reference point such as x_λ of the signature-change hypersurface where $q^{xx} = 0$. Therefore, $u_t^2 = N(x_\lambda)^2$, or

$$u_t = \sqrt{N^2+\epsilon N^2 q^{xx}(u_x)^2}\Big|_{x_0}, \tag{61}$$

at a generic reference coordinate x_0. The different values for u_t parametrize the proper time of the different observers. If we choose an observer at rest at $x_0 < x_\lambda$ (where $q^{xx} \neq 0$) and use (41), we obtain

$$u_t = -\frac{\epsilon}{\alpha\mu(x_0)}\sqrt{1-\frac{2m}{x_0}}. \tag{62}$$

The sign choice is such that

$$u^t(x_0) = \frac{\alpha\mu(x_0)}{\sqrt{1-2m/x_0}} \tag{63}$$

is future-pointing. One can then take the limit $x_0 \to x_\lambda$ for timelike geodesics that are formally at rest at the signature-change surface. Only values in the range $x_0 \geq x_\lambda$ imply initial values for timelike geodesics in the Lorentzian region that reach the signature-change hypersurface. (There are no timelike geodesics starting at x_0 if this value is in the Euclidean region, but we may make a choice $x_0 > x_\lambda$ just to specify certain initial values of a timelike geodesic at $x < x_\lambda$ that is not at rest anywhere.) Equation (60) then implies,

$$\begin{aligned} u_x(x) &= \pm\sqrt{\epsilon q_{xx}(x)\left(\frac{u_t^2}{N(x)^2}-1\right)} \\ &= s_2\frac{1}{1-2m/x}\frac{1}{\sqrt{|\lambda(x)^2-1||1-X_\lambda/x|}}\sqrt{\epsilon\left(\frac{1-2m/x_0}{\mu(x_0)^2}-\frac{1-2m/x}{\mu(x)^2}\right)}, \end{aligned} \tag{64}$$

with $s_2 = \pm 1$. Since the signature factor ϵ changes at $x = x_\lambda$, the square root is real provided $(1-2m/x)/\mu(x)^2$ is increasing across x_λ, which is the case for any $\mu(x)$ that does not increase faster than $\sqrt{1-2m/x}$.

The coordinate transformation from t to $t_{\rm PG}(t,x)$ is now determined by

$$\begin{aligned} {\rm d}t_{\rm PG} &= -u_t{\rm d}t - u_x{\rm d}x \\ &= \frac{\epsilon}{\sqrt{\alpha}\mu(x_0)}\sqrt{1-\frac{2m}{x_0}}{\rm d}t - s_2\frac{1}{1-2m/x}\frac{1}{\sqrt{|\lambda(x)^2-1||1-X_\lambda(x)/x|}}\sqrt{\epsilon\left(\frac{1-2m/x_0}{\mu(x_0)^2}-\frac{1-2m/x}{\mu(x)^2}\right)}{\rm d}x. \end{aligned} \tag{65}$$

It is impossible to integrate this equation for generic $\lambda(x)$ and $\mu(x)$, but it is easy to check that the integrability condition $\partial^2 t_{\rm PG}/\partial t\partial x = \partial^2 t_{\rm PG}/\partial x\partial t$ is satisfied.

Writing

$${\rm d}t_{\rm PG} = \epsilon N(x_0){\rm d}t - s_2\sqrt{q_{xx}\epsilon\left(\frac{N(x_0)^2}{N(x)^2}-1\right)}{\rm d}x, \tag{66}$$

we obtain

$$
\begin{aligned}
-\epsilon N(x)^2\mathrm{d}t^2+\epsilon q_{xx}\mathrm{d}x^2 &= -\epsilon\frac{N(x)^2}{N(x_0)^2}\left(\mathrm{d}t_{\mathrm{PG}}+s_2\sqrt{\epsilon q_{xx}\left(\frac{N(x_0)^2}{N(x)^2}-1\right)}\mathrm{d}x\right)^2+\epsilon q_{xx}\mathrm{d}x^2\\
&= -\epsilon\mathrm{d}t_{\mathrm{PG}}^2-\epsilon\left(\frac{N(x)^2}{N(x_0)^2}-1\right)\mathrm{d}t_{\mathrm{PG}}^2-2\epsilon s_2\frac{N(x)^2}{N(x_0)^2}\sqrt{\epsilon q_{xx}\left(\frac{N(x_0)^2}{N(x)^2}-1\right)}\mathrm{d}t_{\mathrm{PG}}\mathrm{d}x+\epsilon q_{xx}\frac{N(x)^2}{N(x_0)^2}\mathrm{d}x^2\\
&= -\epsilon\mathrm{d}t_{\mathrm{PG}}^2+\epsilon q_{xx}\frac{N(x)^2}{N(x_0)^2}\left(\mathrm{d}x-\epsilon s_2\sqrt{\epsilon q^{xx}\left(\frac{N(x_0)^2}{N(x)^2}-1\right)}\mathrm{d}t_{\mathrm{PG}}\right)^2. \qquad (67)
\end{aligned}
$$

The emergent line element therefore equals,

$$
\begin{aligned}
\mathrm{d}s_{\mathrm{em}}^2 &= -\epsilon\mathrm{d}t_{\mathrm{PG}}^2+\epsilon\frac{\mu(x_0)^2}{\mu(x)^4(\lambda(x)^2-1)(1-2m/x_0)(X_\lambda(x)/x-1)}\\
&\times\left(\mathrm{d}x+\epsilon s_2\mu(x)^2\sqrt{(\lambda(x)^2-1)\left(\frac{X_\lambda(x)}{x}-1\right)\left(\frac{1-2m/x_0}{\mu(x_0)^2}-\frac{1-2m/x}{\mu(x)^2}\right)}\mathrm{d}t_{\mathrm{PG}}\right)^2+x^2\mathrm{d}\Omega^2, \qquad (68)
\end{aligned}
$$

in a gauge of Painlevé-Gullstrand type. Unlike the classical solution in this gauge, slices of constant t_{PG} are not flat, owing to a position-dependent factor of $(X_\lambda(x)/x-1)^{-1}$. The metric is degenerate at the signature-change hypersurface, $x=x_\lambda$ defined by $X_\lambda(x_\lambda)=x_\lambda$, but not at the horizon $x=2m$. It can be used in the interior as well as the exterior of the black hole. It is also well-defined in the Euclidean region $x>x_\lambda$, where the square root remains real (using $\lambda>1$, which is required for signature change to be possible).

2. Painlevé-Gullstrand slicing in the canonical theory

We have obtained the metric in a slicing analogous to the classical Painlevé-Gullstrand gauge by deriving a coordinate transformation from the exterior Schwarzschild region. Covariance of the underlying theory requires that the same metric coefficients can be obtained from the canonical equations with suitable gauge choices. In particular, while $e_1=x^2$ can still be used, staticity in the Schwarzschild gauge should be replaced by the condition of uniform lapse, $N=1$. The radial component of the emergent line element and the shift vector are then determined by the canonical equations of motion and constraints.

A nonvanishing shift vector makes it possible that k_1 and k_2 are nonzero even for time-independent e_1. The diffeomorphism constraint implies that these two phase-space functions are related by

$$k_1=\frac{e_2}{2x}k_2'. \qquad (69)$$

The Hamiltonian constraint then implies a first-order differential equation relating k_2 and e_2,

$$\frac{e_2}{2x}k_2^2+e_2k_2k_2'+\frac{e_2}{2x}-\frac{3x}{2e_2}+\frac{x^2}{e_2^2}e_2'=0. \qquad (70)$$

Using $\dot{e}_1=0$ according to one of the gauge conditions, the equation of motion for e_1 with $N=1$ implies,

$$Mx+x\mu k_2\sqrt{1-s\lambda^2k_2^2}+s\lambda^2\mu\frac{x^3}{e_2^2}\frac{k_2}{\sqrt{1-s\lambda^2k_2^2}}=0, \qquad (71)$$

from which we obtain the shift vector M as a function of e_2 and k_2.

We need one additional condition, supplied by the equation of motion for e_2. We assume that this function is time-independent, as in the classical Schwarzschild solution, but may differ from the classical expression, x. Using $e_1=x^2$, the Hamiltonian constraint (22) simplifies to

$$
\begin{aligned}
H[1]=\int\mathrm{d}x\mu\sqrt{1-s\lambda^2k_2^2}\Bigg(&s\lambda^2\frac{x^2}{e_2^2}\frac{k_2(e_2k_2'-2xk_1)}{1-s\lambda^2k_2^2}-\frac{e_2k_2^2}{2x}\\
&-2xk_1k_2-\frac{x^2e_2'}{e_2^2}+\frac{3x}{2e_2}-\frac{e_2}{2x}\Bigg) \qquad (72)
\end{aligned}
$$

and, with $\dot{e}_2=0$, implies

$$
\begin{aligned}
0&=s\left(\frac{\mu\lambda^2x^2k_2^2}{e_2\sqrt{1-s\lambda^2k_2^2}}\right)'+s\frac{\mu\lambda^2x^2(k_2'-2xk_1/e_2)}{e_2\sqrt{1-s\lambda^2k_2^2}}\\
&\quad+\frac{\mu e_2k_2}{x}\sqrt{1-s\lambda^2k_2^2}+2\mu xk_1\sqrt{1-s\lambda^2k_2^2}+(Me_2)'\\
&=s\left(\frac{\mu\lambda^2x^2k_2^2}{e_2\sqrt{1-s\lambda^2k_2^2}}\right)'+\frac{\mu e_2k_2}{x}\sqrt{1-s\lambda^2k_2^2}\\
&\quad+\mu e_2k_2'\sqrt{1-s\lambda^2k_2^2}+(Me_2)'. \qquad (73)
\end{aligned}
$$

Equations (71) and (73) contain the combination

$$f(x) = \frac{\mu e_2 \sqrt{1 - s\lambda^2 k_2^2}}{x} \tag{74}$$

in several places, which may be used instead of e_2. Doing so, we obtain

$$M e_2 + x k_2 f + \frac{s\lambda^2 \mu^2 x k_2}{f} = 0 \tag{75}$$

and

$$s\left(\frac{\mu^2 \lambda^2 x k_2}{f}\right)' + (k_2 + x k_2') f + (M e_2)' = 0. \tag{76}$$

Combining the last two equations, several terms cancel out and we arrive at $f' = 0$, such that $f = 1/C_2$ is constant and

$$e_2(x) = \frac{x}{C_2 \mu(x) \sqrt{1 - s\lambda(x)^2 k_2(x)^2}}. \tag{77}$$

In the Hamiltonian constraint, we then have the k_2-independent contribution

$$\begin{aligned} 1 - 3\frac{x^2}{e_2^2} + 2\frac{x^3}{e_2^3} e_2' &= 1 - C_2^2 \mu^2 (1 - s\lambda^2 k_2^2) \\ &\quad - 2C_2^2 x\mu \sqrt{1 - s\lambda^2 k_2^2}\left(\mu\sqrt{1 - s\lambda^2 k_2^2}\right)' \\ &= 1 - C_2^2 (x\mu^2 (1 - s\lambda^2 k_2^2))', \end{aligned} \tag{78}$$

which has to equal $-k_2^2 - 2x k_2 k_2' = -(x k_2^2)'$ for the Hamiltonian constraint to vanish. Therefore,

$$(x(k_2^2 - C_2^2 \mu^2 (1 - s\lambda^2 k_2^2)))' = -1 \tag{79}$$

or

$$k_2 = \pm\sqrt{\frac{C_2^2 \mu^2 - 1 + C_k/x}{1 + sC_2^2 \mu^2 \lambda^2}} \tag{80}$$

with an integration constant C_k, such that $(C_2^2\mu^2 - 1 + C_k/x)/(1 + sC_2^2\mu^2\lambda^2) \geq 0$ in a given range of x. We obtain

$$1 - s\lambda^2 k_2^2 = \frac{1 + s\lambda^2 (1 - C_k/x)}{1 + sC_2^2 \mu^2 \lambda^2} \tag{81}$$

and

$$e_2 = \frac{x}{C_2 \mu}\sqrt{\frac{1 + sC_2^2 \mu^2 \lambda^2}{1 + s\lambda^2 (1 - C_k/x)}} \tag{82}$$

provided x is such that $(1 + s\lambda^2 - s\lambda^2 C_k/x)/(1 + sC_2^2\mu^2\lambda^2) \geq 0$. The shift vector

$$\begin{aligned} M &= -\mu\left(1 + \frac{s\lambda^2 \mu^2}{f^2}\right) k_2 \sqrt{1 - s\lambda^2 k_2^2} \\ &= s_2 \mu \sqrt{1 + s\lambda^2 (1 - C_k/x)}\sqrt{C_2^2 \mu^2 - 1 + C_k/x} \end{aligned} \tag{83}$$

then follows from (75), where $s_2 = \pm 1$.

Finally, the emergent radial metric is given by

$$\begin{aligned} q_{xx}^{\mathrm{em}} &= \frac{1}{\mu^2}\left|1 + \frac{x^2}{e_2^2}\frac{s\lambda^2}{1 - s\lambda^2 k_2^2}\right|^{-1} \frac{e_2^2}{x^2} \\ &= \frac{1}{C_2^2 \mu^4}\frac{1}{|1 + s\lambda^2 - s\lambda^2 C_k/x|} \end{aligned} \tag{84}$$

with the signature factor

$$\epsilon = \mathrm{sgn}(1 + sC_2^2 \mu^2 \lambda^2), \tag{85}$$

Comparing the radial metric in the case of $s = -1$ with (68), we identify the free constants as

$$C_2 = \frac{\sqrt{1 - 2m/x_0}}{\mu(x_0)} \tag{86}$$

and

$$C_k = 2m. \tag{87}$$

We arrive at the emergent line element

$$\begin{aligned} \mathrm{d}s_{\mathrm{em}}^2 = -\epsilon \mathrm{d}t_{\mathrm{PG}}^2 &+ \frac{\mu(x_0)^2}{\mu(x)^4 |1 + s\lambda^2| (1 - 2m/x_0)}\frac{1}{|1 - X_\lambda(x)/x|} \\ &\times \left(\mathrm{d}x + s_2 \mu(x)^2 \sqrt{|1 + s\lambda^2|\left(1 - \frac{X_\lambda(x)}{x}\right)}\sqrt{\frac{1 - 2m/x_0}{\mu(x_0)^2} - \frac{1 - 2m/x}{\mu(x)^2}}\mathrm{d}t_{\mathrm{GP}}\right)^2 + x^2 \mathrm{d}\Omega^2. \end{aligned} \tag{88}$$

With this choice of constants, motivated by the previous Painlevé-Gullstrand gauge, the signature factor depends on the constant x_0,

$$\epsilon = \mathrm{sgn}\left(1 - \frac{\mu(x)^2}{\mu(x_0)^2}\lambda(x)^2\left(1 - \frac{2m}{x_0}\right)\right), \tag{89}$$

where we have taken $s = -1$ because only in this case does signature change occur. For the same reason, we may assume $\lambda^2 > 1$ in a region around the signature-change hypersurface. The conditions given by reality of k_2 and e_2 can now be rewritten simply as

$$\left(\frac{1 - 2m/x_0}{\mu(x_0)^2} - \frac{1 - 2m/x}{\mu(x)^2}\right)\epsilon \geq 0, \tag{90}$$

$$(\lambda^2 - 1)(X_\lambda/x - 1)\epsilon \geq 0. \tag{91}$$

Lorentzian signature $\epsilon = +1$ then requires that $x < x_0$ and $x < x_\lambda$, while Euclidean signature $\epsilon = -1$ requires that $x > x_0$ and $x > x_\lambda$, provided that μ does not increase faster than $1 - 2m/x$. The choice $x_0 = x_\lambda$ allows us to connect Lorentzian and Euclidean signature at the signature-change hypersurface.

IV. CAUSAL STRUCTURE NEAR THE SIGNATURE CHANGE HYPERSURFACE

We have obtained covariant models with timelike signature-change hypersurfaces for $s = -1$ and $\lambda > 1$ in a region around the hypersurface. Different methods such as limiting procedures and coordinate changes can be applied to elucidate the causal structure of such space(-time)s.

A. Asymptotic behavior

The metric (40) contains several x-dependent terms that modify some of the classical large-x behavior of the Schwarzschild solution. Moreover, signature change at $x = x_\lambda$ prevents us from taking the full limit of $x \to \infty$ within the Lorentzian region. A more interesting range is given by x asymptotically close to x_λ, in which some of the metric factors are approximately constant.

Introducing the coordinate transformation $x = x_\lambda - \rho$ from x to a positive ρ, the line element has the asymptotic form,

$$\mathrm{d}s^2_{\mathrm{em}} = -\epsilon\left(1 - \frac{2m}{x_\lambda}\right)\mathrm{d}t^2 + \epsilon\frac{x_\lambda/(\mu^2(\lambda^2 - 1))}{(1 - 2m/x_\lambda)}\frac{\mathrm{d}\rho^2}{\rho} + x_\lambda^2\mathrm{d}\Omega^2, \tag{92}$$

with leading-order terms in an expansion by ρ. The ρ-component of the metric remains nonconstant in these coordinate. However, the line element becomes manifestly Minkowskian (up to constant scalings of the coordinates) by defining a new coordinate

$$X = 2\sqrt{\rho} \tag{93}$$

such that

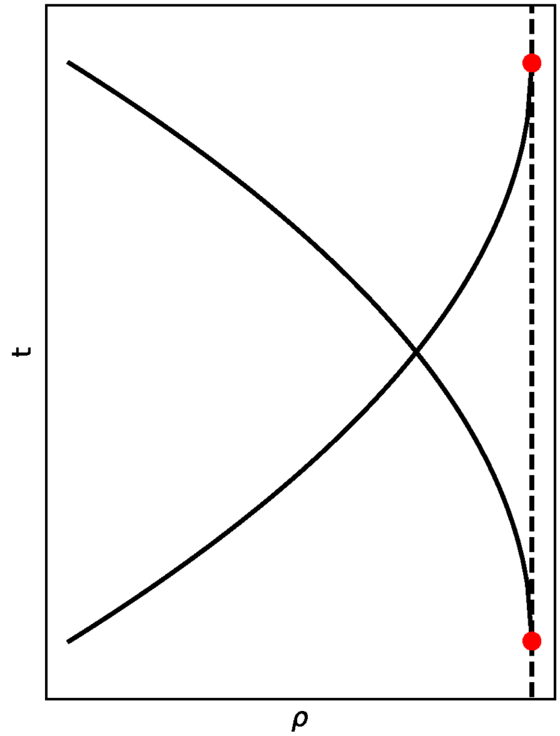

FIG. 1. A light cone asymptotically close to the signature-change hypersurface (dashed), where light rays end in a tangential direction (circles). Using the locally flat asymptotic line elements (94) and (95), the light rays can be unambiguously extended as spacelike geodesics in the Euclidean region.

$$\mathrm{d}s^2_{\mathrm{em}} = -\left(1 - \frac{2m}{x_\lambda}\right)\mathrm{d}t^2 + \frac{x_\lambda/(\mu^2(\lambda^2 - 1))}{(1 - 2m/x_\lambda)}\mathrm{d}X^2 + x_\lambda^2\mathrm{d}\Omega^2 \tag{94}$$

is asymptotically close to the signature-change hypersurface on the Lorentzian side, $\rho > 0$ and $\epsilon = 1$. These asymptotic geometries can be used to infer the light cone structure shown in Fig. 1.

The same procedure can be used on the other side, $\rho < 0$ and $\epsilon = -1$, instead defining the coordinate $X_{\mathrm{E}} = 2\sqrt{-\rho}$ of four-dimensional Euclidean space,

$$\mathrm{d}s^2_{\mathrm{em}} = \left(1 - \frac{2m}{x_\lambda}\right)\mathrm{d}t^2 + \frac{x_\lambda/(\mu^2(\lambda^2 - 1))}{(1 - 2m/x_\lambda)}\mathrm{d}X_{\mathrm{E}}^2 + x_\lambda^2\mathrm{d}\Omega^2. \tag{95}$$

Since the locally flat asymptotic line elements (94) and (95) are related by a single sign change, a geodesic asymptotically close to the hypersurface-deformation surface (given by a straight line in the coordinates of (94) and (95), respectively, has a unique limiting direction at the hypersurface that can be used to obtain an unambiguous extension across signature change. The signature-change hypersurface therefore does not imply geodesic incompleteness.

B. Timelike worldlines

The local Minkowski form asymptotically close to the signature-change hypersurface suggests that this hypersurface can be reached in finite proper time from the Lorentzian side. This expectation can be confirmed explicitly by a derivation of timelike geodesics, and in a similar manner for lightlike geodesics in the next subsection.

Using the results of Sec. III C, we obtain the covelocity of a radially infalling object at rest at x_0, with components

$$v_t = -\sqrt{1-\frac{2m}{x_0}},$$
$$v_x = \frac{1}{\mu\sqrt{\lambda^2-1}}\frac{2m}{x}\left(1-\frac{2m}{x}\right)^{-1}\left(\frac{X_\lambda}{x}-1\right)^{-1/2}\sqrt{1-\frac{x}{x_0}}. \tag{96}$$

The radial component v_x diverges at the signature-change hypersurface unless $x_0 = x_\lambda$, but $v^x = q^{xx}v_x$ is finite, and so is $\mathrm{d}x/\mathrm{d}t = v^x/v^t$ using $v^t = -v_t/N^2$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu\sqrt{\lambda^2-1}\left(1-\frac{2m}{x}\right)\sqrt{\frac{2m}{x}}\sqrt{\left(\frac{X_\lambda}{x}-1\right)\left(1-\frac{x}{x_0}\right)}. \tag{97}$$

This component vanishes at $x = x_\lambda$, in agrement with the tangential approach shown in Fig. 1. In particular, since the inverse $\mathrm{d}t/\mathrm{d}x$ diverges at $x = x_\lambda$, for a static observer at constant x with near-Schwarzschild time t it would take an infinite amount of time for a massive object or a light ray to reach the signature-change hypersurface. In this regard, this hypersurface appears as a horizon.

The proper-time distance along a geodesic starting at some point $x_i < x_\lambda$ and going up to x_λ, parametrized by coordinates as functions of x, is given by

$$\tau_\lambda = \int_{x_i}^{x_\lambda}\sqrt{-g_{\mu\nu}\frac{\partial x^\mu}{\partial x}\frac{\partial x^\nu}{\partial x}}\mathrm{d}x = \int_{x_i}^{x_\lambda}\sqrt{-g_{tt}\left(\frac{\partial t}{\partial x}\right)^2 - 2g_{tx}\frac{\partial t}{\partial x} - q_{xx}}\mathrm{d}x$$
$$= \int_{x_i}^{x_\lambda}\frac{1}{|v^x|}\sqrt{N^2 - q_{xx}(M+v^x)^2}\mathrm{d}x. \tag{98}$$

Using the Schwarzschild metric (40), this equation becomes

$$\tau_\lambda = \int_{x_i}^{x_\lambda}\frac{1}{\mu\sqrt{\lambda^2-1}}\sqrt{\frac{x}{2m}}\left(1-\frac{2m}{x}\right)^{-1/2}\left(\frac{X_\lambda}{x}-1\right)^{-1/2}\left(1-\frac{x}{x_0}\right)^{-1/2}\sqrt{1-\left(1-\frac{x}{x_0}\right)\frac{2m}{x}}\mathrm{d}x, \tag{99}$$

Consider now the coordinate expansions $x_i = x_\lambda - \rho_i$ and $x = x_\lambda - \rho$ with positive $\rho_i \ll x_\lambda$ and ρ, and $x_0 > x_\lambda$. To leading order, to which μ and λ may be treated as constants, we have

$$\tau_\lambda = \frac{1}{\mu\sqrt{\lambda^2-1}}\sqrt{\frac{x_\lambda}{2m}}\left(1-\frac{2m}{x_\lambda}\right)^{-1/2}\left(1-\frac{x_\lambda}{x_0}\right)^{-1/2}\sqrt{1-\left(1-\frac{x_\lambda}{x_0}\right)\frac{2m}{x_\lambda}}\int_0^{\rho_i}\sqrt{\frac{x_\lambda}{\rho}}\mathrm{d}\rho$$
$$= \frac{2x_\lambda}{\mu\sqrt{\lambda^2-1}}\sqrt{\frac{\rho_i}{2m}}\left(1-\frac{2m}{x_\lambda}\right)^{-1/2}\left(1-\frac{x_\lambda}{x_0}\right)^{-1/2}\sqrt{1-\left(1-\frac{x_\lambda}{x_0}\right)\frac{2m}{x_\lambda}}, \tag{100}$$

which is finite and real for $x_0 > x_\lambda$. (The value is complex if $x_0 < x_\lambda$ because geodesics with such an initial condition do not reach the signature-change hypersurface.) In the special case of the most energetic geodesic, formally given by $x_0 \to \infty$, this result simplifies to

$$\tau_\lambda = \frac{2x_\lambda}{\mu\sqrt{\lambda^2-1}}\sqrt{\frac{\rho_i}{2m}}. \tag{101}$$

The larger x_λ is, the larger this proper time.

C. Null worldlines

For radial null worldlines in (40), we find a relation between $\mathrm{d}t$ and $\mathrm{d}x$ given by

$$\mathrm{d}t = \pm\frac{1}{\sqrt{\mu^2(\lambda^2-1)(X_\lambda/x-1)}}\frac{\mathrm{d}x}{(1-2m/x)}. \tag{102}$$

We can use this result to simplify the covelocity,

$$\begin{aligned} \mathrm{d}\kappa &= -v_t \mathrm{d}t - v_x \mathrm{d}x \\ &= -v_t\left(\mathrm{d}t + \left(\sqrt{N^2 q^{xx}} + M\right)^{-1} \mathrm{d}x\right), \end{aligned} \tag{103}$$

with a constant $v_t < 0$ (such that $v^t > 0$) and choosing the negative sign of v_x in the second term for infalling light rays.

For a timelike worldline, we compute

$$\begin{aligned} \mathrm{d}\kappa &= -v_\nu \mathrm{d}x^\nu = -g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \mathrm{d}x^\nu \\ &= -\mathrm{d}\tau\left(g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) = \mathrm{d}\tau, \end{aligned} \tag{104}$$

and therefore κ along the worldline is proper time up to a constant shift. The expression $\mathrm{d}\kappa$ in (103), which uses the geodesic condition through constant v_t, can be locally integrated to a space-time function $\kappa(t,x)$ because it is closed since v_x depends only on x, and therefore locally exact. The result $\kappa(t,x)$ can be interpreted as a function that determines a foliation of a region of space-time into curves $\mathrm{d}\kappa = 0$ transversal to timelike geodesics, with a family of normal directions that integrate to timelike geodesics.

For null worldlines, the analog of the calculation (104) merely shows that $\mathrm{d}\kappa = 0$ and therefore κ is constant along any null worldline, without providing a relationship with the affine parameter. We do not obtain the affine parameter along null geodesics as an analog of proper time. Nevertheless, we may use the expression $\mathrm{d}\kappa$ in order to foliate space-time into null rays, given by constant κ which then plays the role of a null coordinate. This foliation allows us to estimate distances to the signature-change hypersurface as follows. We use one family for null worldlines, given by the infalling case, in order to introduce κ as a null coordinate constant along infalling null worldlines. (The same null coordinate will be used in order to transform to Eddington-Finkelstein form in the next subsection.) This coordinate then provides a certain distance measure along a single outgoing worldline that approaches the signature-change hypersurface as it crosses different infalling null worldlines, which corresponds to the distance one may use to draw a conformal diagram. If this null-coordinate distance is finite, an observer can send only a finite number of infalling light rays at regular intervals before reaching the signature-change hypersurface.

Along a geodesic,

$$\begin{aligned} \mathrm{d}\kappa = -v_t\Bigg(&\pm \frac{1}{\sqrt{(\lambda^2-1)(X_\lambda/x-1)}} \frac{1}{(1-2m/x)} \\ &+ \left(\sqrt{N^2 q^{xx}} + M\right)^{-1}\Bigg)\mathrm{d}x. \end{aligned} \tag{105}$$

As before, $N = \mu^{-1}\sqrt{1-2m/x}$ and $M = 0$ while

$$q_{xx} = \frac{1}{\mu^2(\lambda^2-1)(1-2m/x)(X_\lambda/x-1)} \tag{106}$$

and thus

$$q^{xx} = \mu^2(\lambda^2-1)(1-2m/x)(X_\lambda/x-1). \tag{107}$$

Therefore,

$$\mathrm{d}\kappa = \frac{-v_t(\pm 1 + 1)}{(1-2m/x)\sqrt{(\lambda^2-1)(X_\lambda/x-1)}} \mathrm{d}x \tag{108}$$

vanishes along infalling null wordlines, as expected, and gives us a nonzero null distance

$$\mathrm{d}\kappa = \frac{-2v_t}{(1-2m/x)\sqrt{(\lambda^2-1)(X_\lambda/x-1)}} \mathrm{d}x \tag{109}$$

when integrated along outgoing worldlines.

Asymptotically close to the signature-change hypersurface, $x = x_\lambda - \rho$ with $0 \le \rho \ll x_\lambda$, the leading-order expression,

$$\int \mathrm{d}\kappa = \int \frac{-2v_t}{(1-2m/x_\lambda)\sqrt{(\lambda^2-1)\rho/x_\lambda}} \mathrm{d}\rho \tag{110}$$

can be reduced to

$$\kappa = \frac{-2v_t}{(1-2m/x_\lambda)\sqrt{(\lambda^2-1)/x_\lambda}} \int_0^{\rho_i} \frac{\mathrm{d}\rho}{\sqrt{\rho}} \tag{111}$$

because λ is approximately constant. The integral

$$\kappa = \frac{-4v_t\sqrt{\rho_i}}{(1-2m/x_\lambda)\sqrt{(\lambda^2-1)/x_\lambda}} \tag{112}$$

is finite.

D. Null coordinates

Using a null coordinate v, the emergent metric in Eddington-Finkelstein form is given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{x}\right)\mathrm{d}u^2 + \frac{2}{|\mu|\sqrt{\lambda^2-1}} \frac{1}{\sqrt{X_\lambda/x-1}} \mathrm{d}u\mathrm{d}x. \tag{113}$$

(For a generic two-dimensional line element, the Eddington-Finkelstein form is given by

$$\mathrm{d}s^2 = -\frac{N^2 - q_{xx}M^2}{(v_t)^2} \mathrm{d}u^2 + 2q_{xx} \frac{\sqrt{N^2 q_{xx}}}{v_t} \mathrm{d}u\mathrm{d}x \tag{114}$$

with a null coordinate u.)

For constant μ and λ, the (outgoing) null coordinate is related to the original coordinates by a direct integration of $\mathrm{d}u = \mathrm{d}\kappa$ using equations from the preceding subsection,

$$u = t - \frac{s}{|\mu|\sqrt{|\lambda^2-1|}}\left(\sqrt{x}\sqrt{x_\lambda - x} + (4m + x_\lambda)\arctan(\sqrt{x_\lambda/x - 1}) + \frac{4m\sqrt{2m}}{\sqrt{x_\lambda - 2m}}\operatorname{arctanh}\left(\sqrt{\frac{2m}{x}\frac{x_\lambda - x}{x_\lambda - 2m}}\right)\right) + c, \tag{115}$$

with an integration constant c, where we have absorbed the constant v_t into the null coordinate. The coordinate becomes imaginary if we try to extend it to $x > x_\lambda$, where null worldlines no longer exist. The modified Eddington–Finkelstein metric (113) still has a coordinate singularity at the signature-change hypersurface.

The Eddington-Finkelstein form (114) can directly be transformed to double-null or Kruskal-Szekeres type variables by introducing

$$\mathrm{d}v = \frac{\mathrm{d}u}{(v_t)^2} + -\frac{2Nq_{xx}^{3/2}}{N^2 - q_{xx}M^2}\frac{\mathrm{d}x}{v_t}. \tag{116}$$

In the present case, we obtain

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{x}\right)\mathrm{d}u\mathrm{d}v \tag{117}$$

without modifications from the classical solution. However, the null coordinates have modified relationships with the Schwarzschild-type coordinates x and t. We have

$$u = t - x_*, \qquad v = t + x_*, \tag{118}$$

with

$$x_* = c + \frac{1}{|\mu|\sqrt{|\lambda^2-1|}}\left(\sqrt{x}\sqrt{x_\lambda - x} + (4m + x_\lambda)\arctan(\sqrt{x_\lambda/x - 1}) + \frac{4m\sqrt{2m}}{\sqrt{x_\lambda - 2m}}\operatorname{arctanh}\left(\sqrt{\frac{2m}{x}\frac{x_\lambda - x}{x_\lambda - 2m}}\right)\right) \tag{119}$$

for constant μ and λ. The null coordinates become imaginary for $x > x_\lambda$ and hence end at the signature-change hypersurface, even though (117) does not reveal a coordinate singularity at this place.

The metric (41) describing the Euclidean region does not allow null coordinates in the usual sense, but if we apply a Wick-like rotation $t \to i\bar{t}$, the metric becomes Lorentzian once again and is identical to that of the Lorentzian region (40) up to the change in time coordinate and retaining a positive radial component. Therefore, in these complex coordinates, one can perform the same procedure as used above in order to obtain null coordinates and a metric of the Kruskal-Szekeres form.

The result is almost identical, the metric now given by

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{x}\right)\mathrm{d}\bar{u}\mathrm{d}\bar{v}, \tag{120}$$

where the Schwarzschild coordinates are related to the null ones by

$$\bar{u} = \bar{t} - \bar{x}_*, \qquad \bar{v} = \bar{t} + \bar{x}_*, \tag{121}$$

with

$$\bar{x}_* = c + \frac{1}{|\mu|\sqrt{|\lambda^2-1|}}\left(\sqrt{x}\sqrt{x_\lambda - x} + (4m + x_\lambda)\arctan(\sqrt{1 - x_\lambda/x}) + \frac{4m\sqrt{2m}}{\sqrt{x_\lambda - 2m}}\operatorname{arctanh}\left(\sqrt{\frac{2m}{x}\frac{x - x_\lambda}{x_\lambda - 2m}}\right)\right). \tag{122}$$

It is therefore possible to draw a single Penrose diagram of the usual form, with both regions joined at the signature-change hypersurface.

V. CONCLUSIONS

We have obtained explicit analytical solutions for a large class of spherically symmetric black-hole models of emergent modified gravity with two generic modification functions. Focusing on a new type of signature change on timelike hypersurfaces at low curvature, we have analyzed the causal structure and confirmed that a Euclidean wall around the Universe may be consistent with astronomical and cosmological observations provided the modification function λ is small in a large range of the radial coordinate x but eventually crosses the threshold $\lambda = 1$ at large distances. The transition point is free within the general setting of emergent modified gravity and can easily be chosen to happen beyond the radius of the currently observable Universe. If a specific candidate for $\lambda(e_1)$ can be derived from a proposed quantum theory of gravity, for instance through Hamiltonian renormalization as a possible source of such running coefficients, the distance to the transition point would be a possible test of a given proposal.

Conceptually, our solutions in different gauges, related by explicit coordinate transformations, demonstrate full covariance even in regions that include a signature-change hypersurface. Moreover, we demonstrated that the Ricci and Kretschmann scalars remain finite there, and that any geodesic in the Lorentzian region allows a unique extension as a spacelike geodesic in the Euclidean region, using asymptotic locally flat line elements that are available in our covariant formulation. Signature change of this form is therefore nonsingular.

Signature change in these models requires a negative value of the sign parameter s, which always appears in the combination $s\lambda(e_1)^2$ in modified equations of motion and line elements. This combination of parameters allows us to distinguish between the two cases of $s = 1$, in which signature change does not happen, and $s = -1$, which allows signature change provided $\lambda > 1$ in some region. Another distinction is that $s = 1$ implies modifications of the k_2-dependence in the Hamiltonian constraint by bounded functions, while these functions are unbounded for $s = -1$. Accordingly, the large curvature behavior is also distinct in the two cases, with possible singularity avoidance for $s = 1$ (as shown for constant λ in [3,4]) but not for $s = -1$, as shown here.

The two cases can be combined if we view the full combination $s\lambda(e_1)^2$ of parameters as a single continuous modification function, $\nu(e_1)$, in addition to the old $\mu(e_1)$. If this function is negative with $|\nu(e_1)| > 1$ at large e_1 and turns positive at sufficiently small e_1, there can be signature change at large distances as well as a nonsingular black hole in the interior.

ACKNOWLEDGMENTS

This work was supported in part by NSF Grant No. PHY-2206591.

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