Remarks on applying dispersion relations to amplitudes with infrared singularities

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Dispersion relations (DR) are known to be a powerful instrument for studying scattering amplitudes. In particular, they often apply to calculations in perturbative QCD and the Standard Model. We argue that applying DR to amplitudes with double-logarithmic (DL) contributions should be done with proper caution because DL terms are often infrared divergent. Ignoring this circumstance leads to incorrect results. As an example of such a situation, we consider applying DR to decays of on-shell W- and Z-bosons into fermion pairs in the double-logarithmic approximation.

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I. INTRODUCTION

Dispersion relations (DR) are often used in perturbative QCD and the Standard Model to simplify calculations of various reactions at high energies both in fixed order calculations and in approaches involving total resummations. In particular, they often apply to calculating scattering amplitudes in double-logarithmic approximation (DLA). In this case, calculation of amplitude $M(s)$, with s being the invariant total energy, consists of two steps: First, calculation of $\Im M$, and then calculation of M, using DR with one substraction:

$$M(s) = \frac{s}{\pi}\int_{s_0}^{\infty} ds' \frac{\Im M(s')}{s'(s'-s)} \approx -\frac{1}{\pi}\int_{s_0}^{s} ds' \frac{\Im M(s')}{s'}. \qquad (1)$$

Applying DR considerably simplifies calculations because (i) imaginary parts are free of the ultraviolet divergences, so the use of DR makes it possible to avoid the procedure of renormalization, (ii) calculating $\Im M$ is simpler technically because the cut propagators therein are replaced by the delta functions, and (iii) the graphs with zero imaginary parts can be neglected at once.

The latter item was frequently used by Lipatov with calculations in the leading logarithmic approximation; see e.g. Ref. [1]. However, we demonstrate in the present paper that applying it to calculations in DLA should be done with proper caution to avoid incorrect conclusions. To illustrate our point, we consider very simple examples. Namely, amplitudes of decays of electroweak bosons in fermion pairs—first, decay of a Z-boson and then decay of a W-boson. We demonstrate that total resummation of radiative corrections to the Born amplitudes in DLA leads to Sudakov-like expressions for those amplitudes and explain which mistakes could be made, when item (iii) above is applied in an inappropriate way.

Our paper is organized as follows: In Sec. II we remind the reader about the basic features of the Sudakov form factor, which we will use here. Section III is for considering decay of the on-shell Z-boson in DLA. Decay of the on-shell W-boson is considered in Sec. IV. Finally, Sec. V is for concluding remarks.

II. BASIC FORMULAS FOR THE SUDAKOV FORM FACTOR

Let us consider the vertex Γ_λ of the electromagnetic interaction of fermions. It includes two form factors, $f(q^2)$ and $g(q^2)$:

$$\Gamma_\lambda = \bar{u}(p_2)\left[\gamma_\lambda f(q^2) - \frac{\sigma_{\lambda\nu} q_\nu}{2m} g(q^2)\right] u(p_1), \qquad (2)$$

where q is the momentum of the virtual photon and $p_{1,2}$ are the fermion momenta, so that $q = p_2 - p_1$. When $q^2 < 0$, Eq. (2) describes scattering of a virtual photon γ^* with momentum q off a fermion with momentum p_1. When $q^2 > 0$, Γ_λ describes decays of a virtual photon γ^* into fermion-antifermion pairs $f\bar{f}$:

$$\gamma^*(q) \to f(p_1)\bar{f}(p_2), \qquad (3)$$

where $q = p_2 + p_1$, so q^2 is positive. Behavior of the form factor $f(q^2)$ at negative and large q^2, i.e. at

$$|q^2| \gg |p_{1,2}^2|, \tag{4}$$

was considered first in Ref. [2]. This resulted in the discovery of the double-logarithmic (DL) contributions in QED. Total resummation of DL terms leads to the famous Sudakov exponents which rapidly fall at both positive and negative q^2 providing $|q^2|$ is large.

Now we focus on the decay channel in Eq. (3). The fermions in Eq. (3) can be either off-shell or on-shell. The form factors for these situations are different. Keeping $f(q^2)$ as a generic notation, we introduce the notation $f'(q^2)$ for the form factor in the kinematics where the produced fermions are off-shell, with $|p_{1,2}^2| \gg m^2$ (m is the fermion mass), and denote the form factor $f''(q^2)$ when the fermions are on-shell. The form factor $f'(q^2)$ is IR stable:

$$f'(q^2) = \exp\left[-\frac{\alpha}{2\pi} L'\right], \tag{5}$$

with

$$L' = \ln(-q^2/|p_1^2|) \ln(-q^2/|p_2^2|). \tag{6}$$

On the contrary, the form factor $f''(q^2)$ is IR divergent, so an IR cutoff is needed. As a result, Eq. (5) is replaced by

$$f''(q^2) = \exp\left[-\frac{\alpha}{4\pi} L''\right], \tag{7}$$

where

$$L'' = \ln^2(q^2/\mu^2) - \Theta(m^2 - \mu^2) \ln^2(m^2/\mu^2), \tag{8}$$

with μ being the infrared (IR) cutoff. The exponential form of $f(q^2)$ was also obtained in the QCD context in Ref. [3] and in the Standard Model in Ref. [4]. A similar exponentiation of DL contributions to the form factor $g(q^2)$ of Eq. (2) was obtained in Ref. [5], which finalized studying the electromagnetic vertex Γ_λ in DLA:

$$\Gamma_\lambda = \bar{u}(p_2)\left[\gamma_\lambda f(q^2) + \frac{\sigma_{\lambda\nu} q_\nu}{m} \frac{df(q^2)}{d\rho}\right] u(p_1), \tag{9}$$

where $\rho = \ln(-q^2/\mu^2)$ and $f(q^2)$ can stand for either the off-shell or on-shell form factor. Denoting electric charges of e^+ and e^- as $e\bar{Q}$ and eQ respectively, with $Q = -1$ and $\bar{Q} = -Q = +1$, we can write Eqs. (5) and (7) at positive q^2 in the following form:

$$\begin{aligned} f'' &= \exp\left[\frac{\alpha}{4\pi} \bar{Q} Q L''\right] = \exp\left[-\frac{\alpha}{4\pi} Q^2 L''\right] \\ &= \exp\left[-\frac{\alpha}{4\pi} \bar{Q}^2 L''\right], \\ f' &= \exp\left[\frac{\alpha}{2\pi} \bar{Q} Q L'\right] = \exp\left[-\frac{\alpha}{2\pi} Q^2 L'\right] \\ &= \exp\left[-\frac{\alpha}{2\pi} \bar{Q}^2 L'\right]. \end{aligned} \tag{10}$$

Relations between on-shell and off-shell scattering amplitudes in DLA were considered in detail in Ref. [6].

III. DL CORRECTIONS TO THE DECAY $Z \to f\bar{f}$

In this section we consider the decay of a Z-boson into the fermion-antifermion pair $f\bar{f}$:

$$Z(q) \to f(p_1)\bar{f}(p_2), \tag{11}$$

presuming that both the Z-boson and produced fermions are on-shell and the fermion mass m is small compared to the mass m_Z of the Z-boson: $m_Z \gg m$, which makes it reasonable to calculate this graph in DLA. We also presume that the produced fermions have nonzero electric charges and denote electric charges of the produced fermion and antifermion Q_f and $\bar{Q}_f$ respectively. The first-loop contribution $A_Z^{(1)}$ corresponds to the graph in Fig. 1. Applying the dispersion relations to this graph and using Eq. (10), we obtain

$$A_Z^{(1)} = A_Z^{\mathrm{Born}}\left[\frac{\alpha Q_f \bar{Q}_f}{4\pi} L_Z\right], \tag{12}$$

with L_Z defined as follows:

$$L_Z = \ln^2(m_Z^2/\mu^2) - \Theta(m_f^2 - \mu^2) \ln^2(m_f^2/\mu^2). \tag{13}$$

Exponentiation of $A_Z^{(1)}$ allows us to obtain the expression for A_Z in DLA:

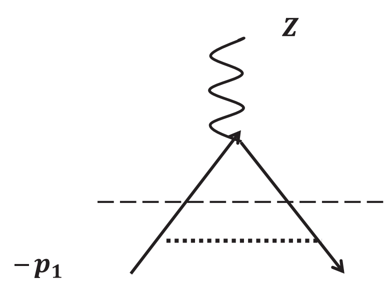

FIG. 1. First-loop contribution to $Z \to f\bar{f}$. The waved line is for the Z-boson, straight lines denote fermions, and the dotted line is for the virtual photon. The dashed line denotes the cut.

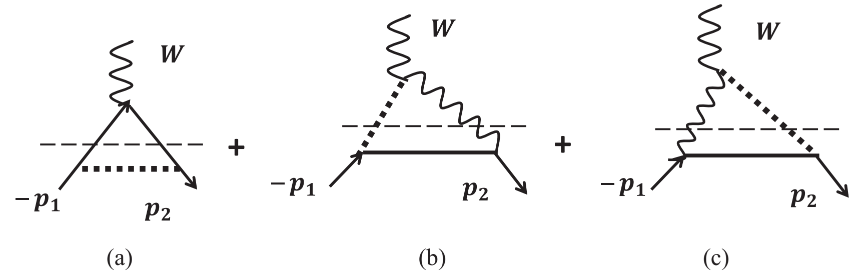

FIG. 2. First-loop contributions to the amplitude of W-decay into fermions. The waved lines are for the W-boson, straight lines denote the fermions, and the dotted lines are for the virtual photons. The dashed lines denote the cuts.

$$A_Z = A_Z^{\rm Born} \exp\left[\frac{\alpha Q_f \bar{Q}_f}{4\pi} L_Z\right] = A_Z^{\rm Born} \exp\left[-\frac{\alpha Q_f^2}{4\pi} L_Z\right]. \quad (14)$$

The exponent in Eq. (14) has the negative sign, so accounting for DL corrections suppresses $A_Z^{\rm Born}$, which is typical for logarithms of the Sudakov type.

IV. DL CORRECTIONS TO THE DECAY $W \to f'\bar{f}$

Now consider decay of the on-shell W-boson into the on-shell fermion couple:

$$W \to f'(p_1)\bar{f}(p_2). \quad (15)$$

To begin with, consider this process in the first loop. Denote the electric charges of the fermion (antifermion) Q'_f ($\bar{Q}_f$) and denote their masses m'_f and m_f respectively. We also assume for simplicity that $Q'_f \neq 0$ and $\bar{Q}_f \neq 0$, though the final answer will not be sensitive to this assumption. The first-loop calculation involves three graphs depicted in Fig. 2.

Dealing with the graph in Fig. 2(a) is quite similar to the calculation in the previous section save for the fact that the masses of the fermion and antifermion can differ from each other. It leads to the following result:

$$A_W^{(2a)} = A_W^{\rm Born}\left[\frac{\alpha Q'_f \bar{Q}_f}{4\pi} L_W\right], \quad (16)$$

where

$$L_W = \ln^2(m_W^2/\mu^2) - \frac{1}{2}\Theta(m_f'^2 - \mu^2)\ln^2(m_f'^2/\mu^2) - \frac{1}{2}\Theta(m_f^2 - \mu^2)\ln^2(m_f^2/\mu^2). \quad (17)$$

Let us notice that $Q'_f \bar{Q}_f > 0$, so had the graph in Fig. 2(a) been the only first-loop contribution to $A_W^{(1)}$, its exponentiation would lead to enhancement of $A_W^{\rm Born}$ by DL corrections instead of the standard Sudakov suppression as it was e.g. in Eq. (12).

Now proceed to graphs in Figs. 2(b) and 2(c) and denote their DL contributions $A_W^{(2b)}$ and $A_W^{(2c)}$ respectively. Obviously, the amplitudes above the cuts in Figs. 2(b) and 2(c) correspond to emission/absorption of on-shell photons, with $k^2 = 0$, by on-shell W-bosons. Such processes are forbidden by the momentum conservation and therefore $A_W^{(2b)} = A_W^{(2c)} = 0$. However, this argumentation is not quite correct: the point is that the photons in the expressions corresponding to the graphs in Figs. 2(b) and 2(c) cannot be on-shell because these expressions are singular at $k^2 = 0$. Such IR singularities should be regulated with introducing an IR cutoff, which prevents the photons from being on-shell. It means that there is no violation of the momentum conservation and justifies applying DR to the graphs in Figs. 2(b) and 2(c). As a result, we obtain

$$A_W^{(2b)} = A_W^{\rm Born}\left[-\frac{\alpha Q'_f Q_W}{8\pi} L_W\right],$$
$$A_W^{(2c)} = A_W^{\rm Born}\left[-\frac{\alpha \bar{Q}_f Q_W}{8\pi} L_W\right], \quad (18)$$

where Q_W is the electric charge of the W-boson. Adding up $A_W^{(2a)}, A_W^{(2b)}$, and $A_W^{(2c)}$, and using $Q_W = Q'_f + \bar{Q}_f$, we arrive at the first-loop contribution $A_W^{(1)}$:

$$A_W^{(1)} = A_W^{\rm Born}\left[-\frac{\alpha}{8\pi}(Q_f'^2 + \bar{Q}_f{}^2) L_W\right]. \quad (19)$$

Its exponentiation yields amplitude A_W in DLA:

$$A_W = A_W^{\rm Born} \exp\left[-\frac{\alpha}{8\pi}(Q_f'^2 + \bar{Q}_f^2) L_W\right]. \quad (20)$$

The amplitudes $A_W^{(1)}$ and A_W in Eqs. (19) and (20) are represented in the form where either Q'_f or $\bar{Q}_f$ can be equal to zero. It makes it possible to apply these formulas to the W-decays with the participation of neutrinos, i.e. $W \to l\bar{\nu}$, $W \to l\nu$, with l denoting leptons.

V. CONCLUSION

In the present paper we have examined practical use of the dispersions relations. There is a very important technical issue in this subject: the Feynman graphs with zero imaginary parts can be left out. Admitting correctness of this statement in general, we argued for its careful application. We have shown that there are cases when the imaginary parts (which formally should have vanished because of the momentum conservation) do not vanish in reality. In particular, it applies to the graphs, whose imaginary parts contain DL contributions induced by soft photons. When such photons are put on their mass shell, the DL contributions become IR divergent. Introducing the IR cutoff to regulate the divergences provides the photons with a fictitious mass, which blocks vanishing the imaginary parts. This observation should be taken into account before neglecting the graphs with zero imaginary parts, when dispersion relations are used.

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